Analysis in one complex variable
Lecture 15 – Harmonic vs Holomorphic II

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Recall

Theorem

Let $U \subset C$ be a simply connected domain let $u\colon U \to \mathbb{R}$ be a harmonic function, then there is a holomorphic function $f\colon U \to \mathbb{C}$ with $\mathrm{Re}(f) = u$.

Question

What can we say if U is not simply-connected?

Harmonic vs Holomorphic

Theorem

Let $U \subset \mathbb{C}$ be a simply connected domain, let $z_1, \dots, z_n \in U$ and let $U^* = U \backslash \{z_1, \dots, z_n\}$. Given a harmonic function $u \colon U^* \to \mathbb{R}$, there are constants $a_1, \dots, a_n$ and a holomorphic function $f \colon U^* \to \mathbb{C}$ such that

$$u = \mathrm{Re}(f) + \sum_i a_i \log|z - z_i|.$$

Harmonic vs Holomorphic

Lemma

Let $U \subset \mathbb{C}$ *be a connected open subset and* $h\colon U \to \mathbb{C}$ *be continuous. If for every loop* $\gamma\colon [0,1] \to U$

$$\int_\gamma h(z)\,dz = 0,$$

then h *has a (holomorphic) primitive.*

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Remark

$$\begin{aligned}\frac{\partial \log|z|}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\frac{1}{2}\log(x^2+y^2) \\ &= \frac{1}{4}\left(\frac{2x-2iy}{x^2+y^2}\right) \\ &= \frac{1}{2}\left(\frac{\bar{z}}{z\bar{z}}\right) \\ &= \frac{1}{2z}\end{aligned}$$

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Proof.

Given u, let $g = 2\frac{\partial u}{\partial z}$. Then

$$\frac{\partial g}{\partial \bar{z}} = 2\frac{\partial^2 u}{\partial \bar{z}\partial z} = \frac{1}{2}\triangle u = 0,$$

so g is holomorphic.
Let

$$a_i = \frac{1}{2\pi i}\int_{\partial D_i} g(z)\,dz,$$

where D_i is a small disc centered on z_i (so that $z_j \notin D_i$ for $i \neq j$).

Harmonic vs Holomorphic

Proof.

Consider $h = g - \sum a_i \frac{1}{z-z_i}$ and let $\gamma \colon [0,1] \to U^*$ be a loop. Then

$$\begin{aligned}\int_\gamma h\,dz &= \sum_i W(\gamma, z_i) \int_{\partial D_i} h\,dz \\ &= \sum_i W(\gamma, z_i)\Big(\int_{\partial D_i} g\,dz - a_i \int_{\partial D_i} \frac{1}{z-z_i} dz\Big) \\ &= \sum_i W(\gamma, z_i)(2\pi i a_i - a_i 2\pi i) \\ &= 0.\end{aligned}$$

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Proof.

Let f be a primitive of h and let $u_1 = \mathrm{Re}(f)$, then

$$2\frac{\partial u_1}{\partial z} = \frac{\partial f}{\partial z} = h = g - \sum a_i \frac{1}{z - z_i} = 2\frac{\partial u}{\partial z} - 2\frac{\partial a_i \log|z - z_i|}{\partial z}$$

Therefore

$$\frac{\partial}{\partial z}(u - u_1 - \sum a_i \log|z - z_i|) = 0$$

$$u - u_1 - \sum a_i \log|z - z_i| = c.$$

Hence

$$u = u_1 + c + \sum a_i \log|z - z_i| = \mathrm{Re}(f + c) + \sum a_i \log|z - z_i|.$$

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Remark

The same result, with the same proof holds if we remove discs instead of points.

Harmonic vs Holomorphic

Corollary

If u is harmonic in the annulus, $D_{r_2} \backslash D_{r_1}$, then there are constants a and b such that

$$\int_0^{2\pi} u(r,\theta)d\theta = a \log r + b.$$

Harmonic vs Holomorphic

Proof.

Let f and a be such that $u = \mathrm{Re}(f) + a \log |z|$ and let $b = \mathrm{Re}\left(\int_{\partial D_{r_0}(0)} \frac{f(z)}{iz}\, dz\right)$. Then

$$\begin{aligned}
\int_0^{2\pi} u(r,\theta)d\theta &= \mathrm{Re}\left(\int_{\partial D_r(0)} f(re^{i\theta})\, d\theta\right) + \int_0^{2\pi} a \log |z| \\
&= \mathrm{Re}\left(\int_{\partial D_r(0)} \frac{f(z)}{iz}\, dz\right) + \int_0^{2\pi} a \log r \\
&= \mathrm{Re}\left(\int_{\partial D_{r_0}(0)} \frac{f(z)}{iz}\, dz\right) + 2\pi a \log r \\
&= b + 2\pi a \log r
\end{aligned}$$

□

Harmonic vs Holomorphic

Remark

In $\mathbb{C}^n$ *we let* $\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right)$.
It remains the case that $4\sum \frac{\partial}{\partial z_j}\frac{\partial}{\partial \bar{z}_j} = \triangle$.
But the natural operators to consider are

$$\partial = \sum \frac{\partial}{\partial z_j} dz_j, \qquad \overline{\partial} = \sum \frac{\partial}{\partial \bar{z}_j} d\bar{z}_j, \qquad \triangle = d^*d + dd^*$$

The relationship between ∂, $\overline{\partial}$ *and* $\triangle$ *is not as obvious and culminates with Hodge's Theorem for Kähler manifolds.*