Analysis in one complex variable
Lecture 16 – Mock exam Q2

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Jun 2020
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Exercise (2)

Let f be analytic on a closed disc $\bar{D}$ of radius $b > 0$, centered at z_0.

- *Show that the value of f at z_0 can be computed as either of the following two averages:*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta, \text{ where } 0 < r < b$$

$$f(z_0) = \frac{1}{\pi b^2} \int_D f(x + iy) dy dx.$$

- *Is the converse true? That is, if a continuous function $f : U \to \mathbb{C}$ satisfies*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta,$$

for all $z \in U$ and all r such that $\bar{D}_r(z) \subset U$, is f holomorphic?

Proof.

Since f is holomorphic, we have, by the Cauchy formula that

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(\xi)}{\xi - z_0} d\xi$$

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To compute the path integral, we parametrize $\partial D_r(z_0)$ as

$$\theta \mapsto \xi(\theta) := re^{i\theta} + z_0, \qquad \text{with } \theta \in [0, frm{-}epi].$$

Then $d\xi = ire^{i\theta} d\theta$

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Then $d\xi = ire^{i\theta} d\theta$ and the integral becomes

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(re^{i\theta} + z_0)}{re^{i\theta}} ire^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta} + z_0) d\theta \end{aligned}$$

□

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$$
\begin{aligned}
\frac{1}{\pi b^2}\int_{D_b(z_0)} f(z)dxdy &= \frac{1}{\pi b^2}\int_0^b\int_0^{2\pi} f(z)rd\theta dr \\
&= \frac{2\pi}{\pi b^2}\int_0^b f(z_0)rdr \\
&= \frac{2\pi f(z_0)}{\pi b^2}\frac{1}{2}r^2\Big|_{r=0}^b \\
&= \frac{f(z_0)}{b^2}b^2 \\
&= f(z_0).
\end{aligned}
$$

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Proof.

The converse is not true. Indeed, if f satisfies the average property (e.g., if f is holomorphic), then so do $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}\,(f)$.

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The converse is not true. Indeed, if f satisfies the average property (e.g., if f is holomorphic), then so do $u = \text{Re}(f)$ and $v = \text{Im}\,(f)$.
Notice that since u is real, by the Cauchy Riemman relations, it is only holomorphic if it is constant.
So the real part of any nonconstant holomorphic function provides a counter-example.
Concretely, take, for example

$$u(x, y) = x = \text{Re}(x + iy).$$

□