Analysis in one complex variable
Lecture 16 – Mock exam Q3

Gil Cavalcanti

Utrecht University

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Utrecht

Exercise (3)

Let $f : \mathbb{C} \to \mathbb{C}$ be the holomorphic function with singularities given by

$$f(z) = \frac{e^{-2\pi i z}}{z^3 + i}.$$

- *Determine the singularities of f and for each of them, determine what type of singularity it is (removable, pole or essential).*
- *Compute the residue of f at each of its singularities.*
- *Compute the integrals*

$$\int_{-\infty}^{\infty} \frac{x^3 \cos 2\pi x - \sin 2\pi x}{x^6 + 1} dx.$$

$$\int_{-\infty}^{\infty} \frac{x^3 \sin 2\pi x - \cos 2\pi x}{x^6 + 1} dx.$$

Proof.

- The function f is the quotient of two holomorphic functions, hence it is a meromorphic function.
- Since the numerator is a nowhere vanishing function, f will have poles at the zeros of the denominator and the order of the poles of f is the order of the zeros of the denominator.
- If we denote by $\omega = e^{2\pi i/3}$ (a cubic root of 1) and pick $\alpha = e^{-\pi i/6}$ (one of the cubic roots of $-i$) we have that the denominator is $z^3 + i = (z - \alpha)(z - \alpha\omega)(z - \alpha\omega^2)$, that is, it has three simple zeros.
- **The function f has three simple poles at α, $\alpha\omega$ and $\alpha\omega^2$.**

Proof.

- $\omega = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ (a cubic root of 1)
- $\alpha = e^{-\pi i/6} = \cos\pi/6 - i\sin\pi/6 = \frac{\sqrt{3}}{2} - \frac{i}{2}$ (a cubic root of $-i$)
- For the computations that follow, it is convenient to have at hand

$$\alpha^2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i = -\omega,$$

$$1 + \omega + \omega^2 = 0, \qquad \omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Proof.

$$\mathrm{Res}_{\alpha}(f) = \frac{e^{-2\pi i\alpha}}{\alpha^2(1-\omega)(1-\omega^2)}$$

$$\mathrm{Res}_{\alpha\omega}(f) = \frac{e^{-2\pi i\alpha\omega}}{-\omega\alpha^2(1-\omega)^2}$$

$$\mathrm{Res}_{\alpha\omega^2}(f) = \frac{e^{-2\pi i\alpha\omega^2}}{\alpha^2\omega(\omega^2-1)(\omega-1)}$$

I will fill out specific values as needed to compute the integrals.

Proof.

- Since for both integrals the integrand is continuous (the denominator has no zeros) and the function being integrated goes to infinity as $1/x^3$, the integrals converge absolutely.
- Letting $g = \frac{x^3 \cos 2\pi x - \sin 2\pi x}{x^6+1}$, we have $g(-x) = -g(x)$, hence

$$\int_{-\infty}^{\infty} g(x)dx = 0.$$

Proof.

- For the second integral we observe that

$$f(x) = \frac{e^{-2\pi ix}}{x^3+i} = \frac{e^{-2\pi ix}(x^3-i)}{(x^3+i)(x^3-i)}$$
$$= \frac{(\cos(2\pi x) - i\sin(2\pi x))(x^3-i)}{x^6+1}$$

- Hence there was a sign cock up...
- We surely were meant to compute the integral of the real and imaginary parts of f and I will do that now.
- $$\mathrm{Im}\,(f(x)) = -\frac{\cos(2\pi x) + x^3\sin(2\pi x)}{x^6+1}$$

Proof.

- Observe that for z in the semi-circle in the *lower half plane* centered at 0 of radius R, $z = x + iy$ with $y \leq 0$ and the numerator in f is bounded by

$$|e^{-2\pi i(x+iy)}| = |e^{-2\pi ix}e^{2\pi y}| = |e^{2\pi y}| \leq 1.$$

$$f(x) = \frac{e^{-2\pi ix}}{x^3+i} = \frac{e^{-2\pi ix}(x^3-i)}{(x^3+i)(x^3-i)}$$
$$= \frac{(\cos(2\pi x) - i\sin(2\pi x))(x^3-i)}{x^6+1}$$

- Hence there was a sign cock up...

Proof.

- We surely were meant to compute the integral of the real and imaginary parts of f and I will do that now.
- $$\text{Im}\,(f(x)) = -\frac{\cos(2\pi x) + x^3 \sin(2\pi x)}{x^6 + 1}$$
- Since $1/z^3$ goes faster than $O(1/z^2)$ to zero as z goes to infinite, we conclude that
$$I := \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{\text{Im}\,(z_i)<0} \text{Res}_{z_i} f$$
- Everything from now is purely algebraic manipulations with complex numbers.

Proof.

- The poles of f in the lower half plane are at α and $\alpha\omega^2$, hence we have (recall $\alpha^2 = -\omega$ and $\omega^2 = -1 - \omega$)

$$
\begin{aligned}
I &= 2\pi i \left(\frac{e^{-2\pi i \alpha}}{\alpha^2(1-\omega)(1-\omega^2)} + \frac{e^{-2\pi i \alpha\omega^2}}{\alpha^2\omega(\omega^2-1)(\omega-1)} \right) \\
&= \frac{2\pi i}{\alpha^2(\omega^2-1)(\omega-1)} \left(e^{-2\pi i \alpha} + \omega^2 e^{-2\pi i \alpha\omega^2} \right) \\
&= -\frac{2\pi i}{\omega(\omega+1)(\omega-1)(\omega-1)} \left(e^{-2\pi i \alpha} + \omega^2 e^{2\pi i \bar{\alpha}} \right) \\
&= \frac{2\pi i}{\omega\omega^2(\omega-1)^2} \left(e^{-2\pi i \alpha} + \omega^2 e^{2\pi i \bar{\alpha}} \right) \\
&= \frac{2\pi i}{\omega^2 - 2\omega + 1} \left(e^{-2\pi i \alpha} + \omega^2 e^{2\pi i \bar{\alpha}} \right)
\end{aligned}
$$

Proof.

$$
\begin{aligned}
I &= -\frac{2\pi i}{3\omega}\left(e^{-2\pi i\alpha} + \omega^2 e^{2\pi i\bar{\alpha}}\right)\\
&= -\frac{2\pi i}{3}\left(\omega^{-1} e^{-2\pi i\alpha} + \omega e^{2\pi i\bar{\alpha}}\right)\\
&= -\frac{2\pi i}{3}\left(e^{-2\pi i\alpha - 2\pi i/3} + e^{2\pi i\bar{\alpha} + 2\pi i/3}\right)\\
&= -\frac{2\pi i}{3}\left(e^{-2\pi i(\sqrt{3}/2 - i/2) - 2\pi i/3} + e^{2\pi i(\sqrt{3}/2 + i/2) + 2\pi i/3}\right)\\
&= -\frac{2\pi i e^{-\pi}}{3}\left(e^{-2\pi i(\sqrt{3}/2 + 1/3)} + e^{2\pi i(\sqrt{3}/2 + 1/3)}\right)\\
&= -\frac{4\pi i e^{-\pi}}{3}\cos(2\pi i(\sqrt{3}/2 + 1/3))
\end{aligned}
$$

or something else ridiculous like this.