Analysis in one complex variable
Lecture 16 – Mock exam Q4

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Exercise (4)

Consider the group homomorphism

$$\Phi\colon \mathrm{Sl}(2;\mathbb{C}) \to \textit{Möb}, \qquad \Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{az+b}{cz+d}.$$

- *Show that if* $v = (z_1, z_2)$ *is an eigenvector of* A*, that is* $Av = \lambda v$ *for some* $\lambda \in \mathbb{C}$*, then* $z = \frac{z_1}{z_2}$ *is a fixed point for* $\Phi(A)$*.*
- *Show that if* z *is a fixed point for* $\Phi(A)$*, then* $(z, 1)$ *is an eigenvector for* A*.*

Proof.

- I will first consider the the case in which $v = (z_1, z_2)$ is an eigenvector and $z_2 \neq 0$.
- Spelling out $Av = \lambda v$ we have

$$\begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

  Taking the quotient of the first coordinate by the second we have

$$\frac{az_1 + bz_2}{cz_1 + dz_2} = \frac{z_1}{z_2} \tag{1}$$

□

Proof.

Finally we compute:

$$\begin{aligned}\Phi(A)(z_1/z_2) &= \frac{az_1/z_2 + b}{cz_1/z_2 + d} \\ &= \frac{az_1 + bz_2}{cz_1 + dz_2} \\ &= \frac{z_1}{z_2}.\end{aligned}$$

showing that z_1/z_2 is a fixed point (the last equality follows from (1)).

Proof.

- Now I will consider the the case in which $v = (z_1, z_2)$ is an eigenvector and $z_2 = 0$ (hence $z_1/z_2 = \infty$).
- Then
$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$
And
$$\Phi(A)(z) = \frac{az + b}{d}$$
And $\Phi(A)(\infty) = \frac{a}{0} = \infty$.

□

Proof.

- For the converse, assuming that z is a fixed point, we have

$$\frac{az+b}{cz+d} = z$$

Let $\lambda := cz+d$, then $az+b=\lambda z$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = \begin{pmatrix} \lambda z \\ \lambda \end{pmatrix}.$$

□