Analysis in one complex variable
Lecture 16 – Mock exam Q5

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Exercise (5)

Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function and $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Prove or disprove the following statements.

- $u \circ f$ *is harmonic,*
- $f \circ u$ *is holomorphic.*

Proof.

- The first statement is true.
- Indeed, since u is harmonic it is the real part of a holomorphic function, say $u = \mathrm{Re}(g)$.
- Then $u \circ f = \mathrm{Re}(g \circ f)$ is the the real part of a holomorphic function (composition of holomorphic functions is holomorphic).
- Hence $u \circ f$ is harmonic.

$\square$

Proof.

- The second statement is false and to prove that this is the case it is enough to produce a counter example.
- We take $u(x, y) = x$, which is harmonic, since its second derivative vanishes and $f(z) = z$ which is holomorphic as we saw in lectures.
- Then

$$f \circ u(z) = f(u(x, y)) = f(x) = x,$$

- This is not holomorphic as it does not satisfy Cauchy–Riemann.

□