Differentiable manifolds – hand-in sheet 3

Hand in by 14/Dec

Before solving the exercise below, see the last hand-in exercise sheet to recall the definitions of terms below.

Exercise

1) Let M be a manifold $\Omega^\bullet(M)$ be the (infinite dimensional) vector space of smooth forms on M and $\mathcal{A}$ be the set of all $\mathbb{R}$-linear endomorphisms of $\Omega^\bullet(M)$, i.e., elements of $\mathcal{A}$ are $\mathbb{R}$-linear maps which send forms to forms. Examples of elements of $\mathcal{A}$ are

- Given a vector field $X \in \mathfrak{X}(M)$, interior product by X, $i_X \in \mathcal{A}$, $i_X : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ for all k;
- Given a 1-form ξ, exterior product by ξ is in $\mathcal{A}$, $\xi\wedge : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, for all k;
- The exterior derivative d is an element of $\mathcal{A}$, $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, for all k.

We can introduce a grading in $\mathcal{A}$. Namely, we declare that an element $\alpha \in \mathcal{A}$ has degree l if $\alpha : \Omega^k(M) \longrightarrow \Omega^{k+l}(M)$ for all k, so the elements introduced above have degree -1, 1 and 1, respectively.

We introduce a bracket in $\mathcal{A}$ as follows. For $\alpha \in \mathcal{A}^l$, $\beta \in \mathcal{A}^m$, we define

$$[\alpha, \beta] = \alpha\beta + (-1)^{lm+1}\beta\alpha.$$

This is called the graded commutator of α and β.

1. Show that $(\mathcal{A}^\bullet, [\cdot,\cdot])$ is a graded Lie algebra with a bracket of degree 0.
2. Show that $[d, d] = 0$ and hence (from hand-in sheet 2) the bracket

   $$[\![\alpha, \beta]\!] := [[\alpha, d], \beta] \tag{1}$$

   satisfies Jacobi.
3. Compute the bracket above for $\alpha = X + \xi$ and $Y + \eta$, where $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$ and vector field act on forms by interior product and 1-forms act on forms by exterior product.
4. Compute

   $$[\![X + \xi, Y + \eta]\!] + [\![Y + \eta, X + \xi]\!].$$
5. Let L be an isotropic subbundle of $TM \oplus T^*M$. Conclude that if L is involutive with respect to the bracket (1), then the space of sections of L is a Lie algebra.