Differentiable manifolds – homework 10

Definition 1. A *complex structure* on a manifold is a choice of atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ such that $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{C}^n$ and the change of coordinates $\varphi_\beta \circ \varphi_\alpha^{-1} : V \subset \mathbb{C}^n \longrightarrow \mathbb{C}^n$ are holomorphic maps.

In all the exercises below V is an n-dimensional vector space.

1) Show that $\mathbb{C}P^n$, the set of complex lines through the origin in $\mathbb{C}^{n+1}$ is a complex manifold.

2) Solve exercise 2 from Chapter 2 in Warner.

3) Let $g \in \mathrm{Sym}^2 V^*$. Show that

$$g(X,Y) = \frac{1}{2}(g(X+Y, X+Y) - g(X,X) - g(Y,Y)),$$

i.e., g is determined by the values it takes in elements of the forms $X \otimes X \in \otimes^2 V$.

4) Let $\mathcal{V} = V \oplus V^*$. Then $\mathcal{V}$ is endowed with a symmetric pairing:

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y)) \qquad \forall X, Y \in V \text{ and } \xi, \eta \in V^*.$$

i) Show that this pairing has signature (n, n), i.e., there are vector subspaces V_+ and V_-, both of dimension n such that the pairing is positive definite on V_+ and negative definite on V_-.

ii) Let $\mathcal{V}$ act on $\wedge^\bullet V^*$ via

$$(X + \xi) \cdot \varphi = i_X \varphi + \xi \wedge \varphi \qquad \forall X \in V, \xi \in V^* \text{ and } \varphi \in \wedge^\bullet V^*.$$

Show that

$$(X + \xi) \cdot ((X + \xi) \cdot \varphi) = \xi(X)\varphi = \langle X + \xi, X + \xi \rangle \varphi.$$

iii) Given $\varphi \in \wedge^\bullet V^*$, define

$$L_\varphi = \{X + \xi \in \mathcal{V} : (X + \xi) \cdot \varphi = 0\}.$$

Show that if $\varphi \neq 0$, then L_φ is *isotropic*, i.e.,

$$\langle X + \xi, Y + \eta \rangle = 0 \qquad \forall X + \xi, Y + \eta \in L_\varphi.$$

5) Let $A : V \longrightarrow V$ be a linear map. Then A induces two linear maps $A : \wedge^k V \longrightarrow \wedge^k V$, and $e^A : \wedge^k V \longrightarrow \wedge^k V$ which can be described for a fixed choice of basis $\{e_1, \cdots, e_n\}$ for V by

$$A(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_j e_{i_1} \wedge \cdots \wedge A(e_{i_j}) \wedge \cdots \wedge e_{i_k}.$$

$$A_*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = A(e_{i_1}) \wedge \cdots \wedge A(e_{i_k}).$$

Since $\wedge^n V$ is a one dimensional vector space, any linear endomorphism of $\wedge^n V$ corresponds to multiplication by a scalar. Show that $A : \wedge^n V \longrightarrow \wedge^n V$ corresponds to multiplication by the trace of A and that $A_* : \wedge^n V \longrightarrow \wedge^n V$ corresponds to multiplication by the determinant of A.

6) Let $A : V \longrightarrow V$ be a linear map and let e^A denote its formal exponential

$$e^A = Id + A + \frac{A^2}{2} + \frac{A^3}{3!} + \cdots$$

Show that

$$e^{\mathrm{tr} A} = \det(e^A).$$

7*) Let $\alpha \in \wedge^2 V^*$. Show that if $\alpha \wedge \alpha = 0$ then α is decomposable.