Differentiable manifolds – hand-in sheet 2

Hand in by 06/Dec

Prelude to the exercise

Definition 1. A *graded vector space* is a vector space which decomposes as a direct sum

$$V = \oplus_{n=-\infty}^{\infty} V_n,$$

where each V_n is a vector space and the elements of V_n are said to be the *homogenous* elements of *degree* n.

This means that an element of V does not have a degree associated to it, but it can be written as a sum of homogeneous elements of different degrees.

An example of graded vector space is given by polynomials in n variables with real, complex or even matrix coefficients. In this case, V_n is the set of homogeneous polynomials if degree n.

Definition 2. A *graded algebra* over the real numbers with a product of degree n is a real graded vector space, $A = \oplus A_i$, endowed with a bilinear operation

$$A \times A \xrightarrow{\cdot} A \qquad (X, Y) \mapsto X \cdot Y.$$

Such that for $X \in A_i$ and $Y \in A_j$, $X \cdot Y \in A_{i+j+n}$.

Using polynomial multiplication as algebra operation, polynomials of several variables are an example of a graded algebra with a multiplication of degree zero. Any algebra A is an example of a graded algebra with a product of degree zero by setting $A_0 = A$ and $A_i = \{0\}$ for $i > 0$.

Definition 3. A *graded Lie algebra* over the real numbers with a bracket of degree n is a real graded algebra, $A = \oplus A_i$ where the algebra operation is given by the bracket

$$A \times A \xrightarrow{[\cdot,\cdot]} A \qquad (X, Y) \mapsto [X, Y].$$

Such that for $X \in A_i$, $Y \in A_j$ and $Z \in A_k$

1. $[X, Y] \in A_{i+j+n}$, (degree n),
2. $[X, Y] = (-1)^{(i+n)\cdot(j+n)+1}[Y, X]$ (graded skew),
3. $[X, [Y, Z]] = [[X, Y], Z] + (-1)^{(n+i)(n+j)}[Y, [X, Z]]$ (graded Jacobi).

Exercise

Exercise 1. Let $V = \oplus_{n=0}^{\infty} V_n$ be a graded vector space such that only finitely many V_i are nontrivial and let A be the space of linear maps on V:

$$A = \{L : V \longrightarrow V |\ L \text{ is linear}\}$$

The vector space A admits a grading as follows, the space of elements of degree n is

$$A_n = \{L : V \longrightarrow V |\ L \text{ is linear and } L : V_i \longrightarrow V_{i+n} \text{ for all } i\}.$$

Define a degree zero bracket on A by

$$[\cdot,\cdot] : A_i \times A_j \longrightarrow A_{i+j}; \qquad [L_1, L_2] = L_1 L_2 + (-1)^{ij+1} L_2 L_1.$$

Show that with this bracket A is a graded Lie algebra.

Exercise 2. Let $(A, [\cdot,\cdot])$ be a graded Lie algebra with a bracket of degree n and assume that $d \in A_l$, with $n + l = 1 \bmod 2$, satisfies $[d, d] = 0$. Define a new bracket, $[\cdot,\cdot]_d$ on A by

$$[X, Y]_d = [[X, d], Y]].$$

Show that for $X \in A_i$, $Y \in A_j$ and $Z \in A_k$

- $[X, Y]_d \in A_{i+j+2n+l}$ (degree $2n + l$);
- $[X, [Y, Z]_d]_d = [[X, Y]_d, Z]_d + (-1)^{(2n+l+i)(2n+l+j)}[Y, [X, Z]_d]_d$ (graded Jacobi)

Hint: It may simplify your computations to define first the linear operator

$$D : A \longrightarrow A \qquad D(X) := DX := [X, d],$$

so that

$$[X, Y]_d = [DX, Y]$$

and check that

$$D^2 = 0 \qquad \textit{and} \qquad D[X, Y] = (-1)^{n+j}[DX, Y] + [X, DY].$$

and then apply D to the Jacobi identity

$$[DX, [Y, Z]] = [[DX, Y], Z] + (-1)^{(n+i+1)(n+j)}[Y, [DX, Z]]$$

Remark: The bracket $[\cdot,\cdot]_d$ defined above is not necessarily graded skew, hence it is not in general a graded Lie bracket.