Differentiable manifolds – exercise sheet 13

Exercise 1. Let V and W be vector spaces. Show that $V^* \otimes W \cong \mathcal{L}(V; W)$.

Exercise 2. Let $\varphi : M \longrightarrow N$ be a diffeomorphism. Show that the de Rham cohomology of M and N are isomorphic as algebras.

Exercise 3. Show that the 1-forms $\omega_1, \cdots, \omega_k$ are linearly independent if and only if $\omega_1 \wedge \cdots \wedge \omega_k \neq 0$.

Exercise 4 (The gradient vector field)**.** Let $g \in \Gamma(\mathrm{Sym}^2 T^*M)$ be a Riemannian metric on M, i.e., g is a smooth section and for $X \in T_pM \backslash \{0\}$ $g(X, X) > 0$.

1. Show that g gives rise to a bundle map (which we still denote by g) $g : TM \longrightarrow T^*M$, $X \mapsto g(X, \cdot)$. Show that this bundle map is a bundle isomorphism $g : TM \xrightarrow{\cong} T^*M$
2. Given $f \in \Omega^0(M)$ define the gradient of f,$\nabla f \in \mathfrak{X}(M)$, as $\nabla f = g^{-1}(df)$. Show that if $df_p \neq 0$ then ∇f is orthogonal to the level set $f^{-1}(p)$.
3. Show that f increases along the flow of ∇f.

Exercise 5 (Hodge star)**.** Let V be an n-dimensional vector space with a metric g and an orientation. The Hodge star operator is a linear operator on the exterior algebra $\wedge V^*$,

$$\star : \wedge^k V^* \longrightarrow \wedge^{n-k} V^*$$

defined as follows: Let $\{e_1, \cdots, e_n\}$ be a positive orthonormal basis of V^* and for $I \subset \{1, \cdots, n\}$, using multiindex notation, define

$$\star e_I = \pm e_{I^c},$$

where the sign is such that

$$e_I \wedge \star e_I = e_1 \wedge \cdots \wedge e_n.$$

1. (Try to) prove that this definition is independent of the (positive orthonormal) basis chosen;
2. Show that $e_I \wedge \star e_J = 0$ if $I \neq J$;
3. Show that for any $\alpha \in \wedge^k V^* \backslash \{0\}$, $\alpha \wedge \star\alpha$ is a positive multiple of $e_1 \wedge \cdots \wedge e_n$
4. Compute $\star^2$.

Exercise 6 (The curl)**.** Let M^3 be a 3-dimensional manifold with a Riemannian metric g define the curl of a vector field X as the composition

$$\nabla \times X := g^{-1} \star d(g(X)),$$

where, as in exercise 4, we regard g as bundle isomorphism $G : TM \longrightarrow T^*M$.

Compute $\nabla \times X$ for a vector field X in $\mathbb{R}^3$ with $\mathbb{R}^3$ endowed with the Euclidean metric. Show that in general $\nabla \times (\nabla f = 0)$ for all $f \in \Omega^0(M)$. Finally show that if $H^1(M) \neq \{0\}$, there are vector fields X such that $\nabla \times X = 0$ but X is not a gradient vector field of any function.

Exercise 7. Let $A \in \Gamma(\otimes^k T^*M)$ and $X, Y \in \mathfrak{X}(M)$. Show that

$$\mathcal{L}_X \mathcal{L}_Y A - \mathcal{L}_Y \mathcal{L}_X A = \mathcal{L}_{[X,Y]} A.$$