Differentiable manifolds – exercise sheet 3

Whenever necessary, you can assume that the Čech cohomology $\check{H}^k(M;\mathbb{R};\mathfrak{U})$ is independent of $\mathfrak{U}$ as long as $\mathfrak{U}$ is a good cover of M.

Exercise 1 (Euler characteristic)**.** Let $\{V^k : 0 \leq k \leq n\}$ be a family of finite dimensional vector spaces where $n \in \mathbb{N}$ is some fixed number. Whenever necessary, let $V_{-1} = V_{n+1} = \{0\}$. Let $d_k : V^k \longrightarrow V^{k+1}$ be linear maps such that $d_{k+1} \circ d_k = 0$ for all i and define

$$H^k = \frac{\ker(d_k)}{\mathrm{Im}(d_{k-1})}.$$

Show that

$$\sum (-1)^k \dim(V^k) = \sum (-1)^k \dim(H^k).$$

Conclude that if $\mathfrak{U}$ is a finite open cover of a manifold then

$$\sum (-1)^k \dim(\check{C}^k(M;\mathbb{R};\mathfrak{U})) = \sum (-1)^k \dim(\check{H}^k(M;\mathbb{R};\mathfrak{U})).$$

Hint: Use the rank nullity theorem from linear algebra, namely, if $A : V \longrightarrow W$ is a linear map,

$$\dim(V) = \dim(\mathrm{Im}(A)) + \dim(\ker(A)).$$

Definition 2. For a cover $\mathfrak{U}$ of M, the *Euler characteristic of M with respect to the cover* $\mathfrak{U}$ is the number

$$\chi(M;\mathfrak{U}) = \sum (-1)^k \dim(\check{H}^k(M;\mathbb{R};\mathfrak{U}))$$

The *Euler characteristic of M*, denoted by $\chi(M)$, is $\chi(M;\mathfrak{U})$ where $\mathfrak{U}$ is any finite good cover of M.

Exercise 3. Cover the sphere S^2 with four open sets obtained by slightly enlarging the tetrahedral triagulation of the sphere (see Figure 1). Compute the Euler characteristic of S^2 with respect to this open cover.

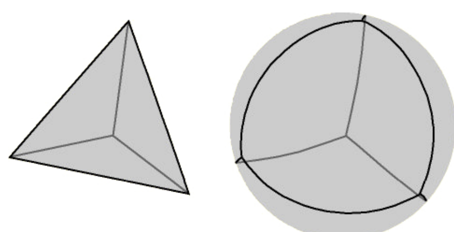

Figure 1: Tetrahedral decomposition of the sphere.

Exercise 4. Consider S^1 as the interval $[0,1]$ with the ends identified. Cover S^1 by the open sets $U_0 = (0, 2/3)$, $U_1 = (1/3, 1)$ and $U_2 = (2/3, 1) \cup (0, 1/3)$. Compute the Euler characteristic of S^1 from this cover.

Exercise 5. Let M and N be manifolds and $\mathfrak{U}$ and $\mathfrak{V}$ be finite covers of M and N respectively. Then the product $M \times N$ is covered by

$$\mathfrak{U} \times \mathfrak{V} := \{U \times V : U \in \mathfrak{U}, \quad V \in \mathfrak{V}\}.$$

Show that

$$\chi(M \times N) = \chi(M) \times \chi(N).$$

Exercise 6. Assuming that the Čech cohomology $\check{H}^k(M;\mathbb{R};\mathfrak{U})$ is independent of $\mathfrak{U}$ as long as $\mathfrak{U}$ is a good cover of M show that S^2 is not diffeomorphic to $S^1 \times S^1$.

Exercise 7. Let G be an Abelian group, $K < G$ be a subgroup and $H = G/K$ be the quotient so that we have natural maps

$$\{e\} \longrightarrow K \xrightarrow{\iota} G \xrightarrow{\pi} H \longrightarrow \{e\}$$

with the property that the image of each map is the kernel of the next.

Let M be a manifold and $\mathfrak{U}$ be a finite open cover of M. Show that we get a corresponding sequence of maps

$$\check{C}^k(M;K;\mathfrak{U}) \xrightarrow{\iota} \check{C}^k(M;G;\mathfrak{U}) \xrightarrow{\pi} \check{C}^k(M;H;\mathfrak{U})$$

For which the kernel of π is the image of ι. Further, show that these maps commute with differentials, that is, the following diagram commutes:

$$\begin{array}{ccccc}
\check{C}^{k+1}(M;K;\mathfrak{U}) & \longrightarrow & \check{C}^{k+1}(M;G;\mathfrak{U}) & \longrightarrow & \check{C}^{k+1}(M;H;\mathfrak{U}) \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow \\
\check{C}^{k}(M;K;\mathfrak{U}) & \longrightarrow & \check{C}^{k}(M;G;\mathfrak{U}) & \longrightarrow & \check{C}^{k}(M;H;\mathfrak{U}) \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow \\
\check{C}^{k-1}(M;K;\mathfrak{U}) & \longrightarrow & \check{C}^{k-1}(M;G;\mathfrak{U}) & \longrightarrow & \check{C}^{k-1}(M;H;\mathfrak{U})
\end{array}$$

Conclude that we obtain corresponding maps between the different cohomology groups:

$$\check{H}^k(M;K;\mathfrak{U}) \xrightarrow{\iota^*} \check{H}^k(M;G;\mathfrak{U}) \xrightarrow{\pi^*} \check{H}^k(M;H;\mathfrak{U}).$$

Show that $\ker(\pi^*) = \mathrm{Im}(\iota^*)$.