Differentiable manifolds – exercise sheet 9

Exercise 1. Let $\alpha \in \mathbb{R}$ be an irrational number and let $f : \mathbb{R} \longrightarrow T^2$ be defined by

$$f(t) = (e^{2\pi it}, e^{2\pi\alpha it}).$$

Show that $(f, \mathbb{R})$ is a submanifold of T^2. Show that the image of f is dense in T^2.

Exercise 2. Consider the map $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ defined by

$$f(x, y, s, t) = (x^2 + y, x^2 + y2 + s^2 + t^2 + y)$$

Show that $(0, 1)$ is a regular value of f, and that the level set $f^{-1}\{(0, 1)\}$ is diffeomorphic to S^2.

Exercise 3. For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by the equation

$$M_a = \{(x, y) : y^2 = x(x - 1)(x - a)\}.$$

For which values of a is M_a the image of an embedding? For which values of a is M_a the image of an immersion?

Exercise 4. Define a map $f : S^2 \longrightarrow \mathbb{R}^4$ by

$$f(x, y, z) = (x^2 - y^2, xy, xz, yz).$$

Show that f is an immersion. Further show that if $f(\mathbf{x}) = f(\mathbf{y})$ then $\mathbf{x} = \pm\mathbf{y}$. Prove that f induces am embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$.

Exercise 5. Let $p : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a homogeneous polynomial of degree m in three variables, i.e.,

$$p(X_0, X_1, X_2) = \sum_{i+j+k=m} a_{ijk} X_0^i X_1^j X_2^k.$$

Let $\Sigma \subset \mathbb{R}P^n$ be the set defined by the zeros of p, i.e.,

$$\Sigma = \{[X_0, X_1, X_2] \in \mathbb{R}P^2 : p(X_0, X_1, X_2) = 0\}.$$

1. Show that Σ is indeed a well defined subset of $\mathbb{R}P^2$, i.e., if two points are in the same line through the origin, then either both are zeros of p or neither is a zero of p.

2. Show that if the system
$$\begin{cases} p(X_0, X_1, X_2) & = 0, \\ \frac{\partial p}{\partial X_0}(X_0, X_1, X_2) & = 0, \\ \frac{\partial p}{\partial X_1}(X_0, X_1, X_2) & = 0, \\ \frac{\partial p}{\partial X_2}(X_0, X_1, X_2) & = 0, \end{cases}$$
has no solutions, then Σ is an embedded submanifold of $\mathbb{R}P^2$.

3. Extend this result to $n + 1$ variables, namely, define $\Sigma \subset \mathbb{R}P^n$ as the zero set of a degree m homogeneous polynomial and find a condition for smoothness of Σ.

4. Extend this result to $n + 1$ complex variables, namely, define $\Sigma \subset \mathbb{C}P^n$ as the zero set of a degree m homogeneous polynomial with complex coefficients and find a condition for smoothness of Σ