Differentiable manifolds – hand-in sheet 3

Hand in by 09/Oct

The first Chern class

Definition 1. A *complex vector bundle of rank k* over a manifold M is a manifold E together with

1. a smooth map $\pi : E \longrightarrow M$;
2. a cover $\{U_\alpha : \alpha \in A\}$ of M and diffeomorphisms $\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}^k$ for which the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\Phi_\alpha"] \arrow[d, "\pi"] & U_\alpha \times \mathbb{C}^k \arrow[d, "\pi_1"'] \\ U_\alpha \arrow[r, "\mathrm{Id}"] & U_\alpha, \end{tikzcd}$$

   where π_1 is projection onto the first factor;
3. if $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\Phi_\beta \circ \Phi_\alpha^{-1}(x, \cdot) : \mathbb{C}^k \longrightarrow \mathbb{C}^k$$

   is complex linear for all x.

A complex vector bundle of rank 1 is also referred to as a *complex line bundle*.

Exercise 1.

1. Following the argument used for real line bundles, show that (strong isomorphism classes of) complex line bundles over M are classified by $\check{H}^1(M; C^\infty(M; \mathbb{C}^*); \mathfrak{U})$ for any good cover $\mathfrak{U}$ (you can use that every complex vector bundle over a ball has a trivialization).
2. Use the fact that the complex logarithm can be defined on balls to show that for any open U homeomorphic to a ball the following sequence is exact

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} C^\infty(U; \mathbb{C}) \xrightarrow{exp} C^\infty(U; \mathbb{C}^*) \longrightarrow 0.$$

   Use this and the results from the first exercise sheet to conclude that (strong isomorphism classes of) complex line bundles over M are classified by $\check{H}^2(M; \mathbb{Z}; \mathfrak{U})$.