Differentiable manifolds – exercise sheet 12

Exercise 1. Let $A : V \longrightarrow V$ be a linear automorphism of an n-dimensional vector space. Show that $A^* : \wedge^n V^* \longrightarrow \wedge^n V^*$ is just multiplication by $\det A$.

Exercise 2. Let α be a closed form and β be an exact form. Show that $\alpha \wedge \beta$ is exact.

Exercise 3 (Cup product)**.** Let α and β be closed forms on M. Show that $\alpha \wedge \beta$ is closed. Further, show that the cohomology class of $\alpha \wedge \beta$ only depends on the cohomology classes of α and β and not on the particular representatives of these classes. Conclude that we have a product operation:

$$H^k(M) \cup H^l(M) \longrightarrow H^{k+l}(M)$$

induced by the wedge product.

Exercise 4. Let $f : M \longrightarrow N$ be smooth

- For $\alpha, \beta \in \Omega(N)$ Show that $f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta)$.
- Using the fact above or otherwise, show that $f^*(d\alpha) = d(f^*\alpha)$.
- Conclude that f induces a map $f^* : H^k(N) \longrightarrow H^k(M)$ and that this map preserves the cup product introduced in exercise 3.

Exercise 5. Compute the exterior derivative of the following forms

- xdy
- $xdx + ydy$
- $\frac{xdy - ydx}{x^2+y^2}$
- $\frac{xdy\wedge dz + ydz\wedge dx + zdx\wedge dy}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

Exercise 6. For $X \in \mathfrak{X}(M)$ we define the interior product with X as a map

$$\iota_X : \Omega^k(M) \longrightarrow \Omega^{k-1}(M); \qquad \iota_X\alpha(X_1, \cdots, X_{k-1}) = \alpha(X, X_1, \cdots, X_{k-1}).$$

Show that

$$\iota_X\iota_X\alpha = 0 \text{ for all } \alpha \in \Omega^k(M)$$

and that

$$\iota_X(\alpha \wedge \beta) = (\iota_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (\iota_X\beta),$$

where $\alpha \in \Omega^k(M)$.

Exercise 7. For $\xi \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$, define

$$X + \xi : \Omega(M) \longrightarrow \Omega(M); \qquad (X+\xi)\cdot\alpha = \iota_X\alpha + \xi \wedge \alpha.$$

Show that $(X+\xi)\cdot((X+\xi)\cdot\alpha) = \xi(X)\alpha$