Differentiable manifolds – exercise sheet 14

Exercise 1. Let $\alpha \in \Omega^1(\mathbb{R}^2\backslash\{0\})$ be given by

$$\alpha = \frac{xdy - ydx}{x^2+y^2}.$$

Compute $d\alpha$. Compute the integral of α over

- the unit circle oriented counterclockwise.
- the circle of radius 1 centered at $(0,2)$ oriented counterclockwise.
- the circle of radius 2 centered at $(1,0)$ oriented counterclockwise.

Exercise 2. Let $\varphi : S^2\backslash\{N\} \longrightarrow \mathbb{R}^2$ be the stereographic projection. Show that $\varphi^*(\frac{dx\wedge dy}{1+(x^2+y^2)^2})$ extends to the north pole to give rise to a smooth 2-form on S^2. Compute its integral over S^2.

Exercise 3. Using the Poincaré lemma and integration, show that $H^1(S^2) = \{0\}$.

Exercise 4. A k-form α is *harmonic* if α and $\star\alpha$ are closed. Show that if M is compact and a harmonic form is not everywhere zero, it represents a nontrivial cohomology class.

Exercise 5. Let $\omega \in \Omega^2(\mathbb{R}^{2n})$ be given by

$$\omega = dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}.$$

Let Σ be a compact 2-dimensional manifold without boundary and let $\varphi : \Sigma \longrightarrow \mathbb{R}^{2n}$ be a smooth map. Compute

$$\int_\Sigma \varphi^*\omega.$$

Convention: in a compact oriented Riemannian manifold for $f \in C^\infty(M)$ we define

$$\int_M f := \int_M \star f.$$

Exercise 6 (The divergent)**.** Let M be an oriented Riemannian manifold and let $X \in \mathfrak{X}(M)$. Define the divergent of X to be

$$\nabla \cdot X = \star^{-1} d \star g(X).$$

Show that if M is $\mathbb{R}^n$ with usual metric and orientation

$$\nabla \cdot (X_i \frac{\partial}{\partial x_i}) = \sum \frac{\partial X_i}{\partial x_i}.$$

Exercise 7. Let $X \in \mathfrak{X}(M)$ be a vector field on an oriented compact Riemannian manifold with boundary. Let N be the unit outward pointing normal vector to boundary. Show that

$$\int_{\partial M} g(X,N) = \int_M \nabla \cdot X.$$