Differentiable manifolds – exercise sheet 4

Whenever necessary, you can assume that the Čech cohomology $\check{H}^k(M;G;\mathfrak{U})$ is independent of $\mathfrak{U}$ as long as $\mathfrak{U}$ is a good cover of M.

Exercise 1. Show that there are natural identifications

$$\check{H}^0(M;C^\infty(M);\mathfrak{U}) = \{\text{globally defined, smooth functions on } M\}.$$

$$\check{H}^0(M;C^\infty(M;S^1);\mathfrak{U}) = \{\text{globally defined, smooth functions on } M \text{ with values in } S^1\}.$$

Exercise 2. Show that if G and G' are isomorphic Abelian groups, then $\check{H}^k(M;G;\mathfrak{U})$ is isomorphic to $\check{H}^k(M;G';\mathfrak{U})$.

Exercise 3.

1. Show that if $G = H \times K$ then $\check{H}^k(M;G;\mathfrak{U}) \cong \check{H}^k(M;H;\mathfrak{U}) \times \check{H}^k(M;K;\mathfrak{U})$.
2. Using the isomorphism

$$C^\infty(M;\mathbb{R}^*) \cong C^\infty(M;\mathbb{R}) \times C^\infty(M;\mathbb{Z}_2); \qquad g \mapsto (\log|g|, \mathrm{sign}(g))$$

   conclude that

$$\check{H}^k(M;C^\infty(M;\mathbb{R}^*);\mathfrak{U}) \cong \check{H}^k(M;\mathbb{Z}_2;\mathfrak{U})$$

   for $k > 0$.

Exercise 4. Let G be an Abelian group, $K < G$ be a subgroup and $H = G/K$ be the quotient so that we have natural maps

$$\{e\} \longrightarrow K \xrightarrow{\iota} G \xrightarrow{\pi} H \longrightarrow \{e\}$$

with the property that the image of each map is the kernel of the next.

Let M be a manifold and $\mathfrak{U}$ be a finite open cover of M. Show that we get a corresponding sequence of maps

$$\check{C}^k(M;K;\mathfrak{U}) \xrightarrow{\iota} \check{C}^k(M;G;\mathfrak{U}) \xrightarrow{\pi} \check{C}^k(M;H;\mathfrak{U})$$

For which the kernel of π is the image of ι. Further, show that these maps commute with differentials, that is, the following diagram commutes:

$$\begin{tikzcd}
\check{C}^{k+1}(M;K;\mathfrak{U}) \arrow[r] & \check{C}^{k+1}(M;G;\mathfrak{U}) \arrow[r] & \check{C}^{k+1}(M;H;\mathfrak{U}) \\
\check{C}^{k}(M;K;\mathfrak{U}) \arrow[r] \arrow[u, "\delta"] & \check{C}^{k}(M;G;\mathfrak{U}) \arrow[r] \arrow[u, "\delta"] & \check{C}^{k}(M;H;\mathfrak{U}) \arrow[u, "\delta"] \\
\check{C}^{k-1}(M;K;\mathfrak{U}) \arrow[r] \arrow[u, "\delta"] & \check{C}^{k-1}(M;G;\mathfrak{U}) \arrow[r] \arrow[u, "\delta"] & \check{C}^{k-1}(M;H;\mathfrak{U}) \arrow[u, "\delta"]
\end{tikzcd}$$

Conclude that we obtain corresponding maps between the different cohomology groups:

$$\check{H}^k(M;K;\mathfrak{U}) \xrightarrow{\iota^*} \check{H}^k(M;G;\mathfrak{U}) \xrightarrow{\pi^*} \check{H}^k(M;H;\mathfrak{U}).$$

Show that $\ker(\pi^*) = \mathrm{Im}(\iota^*)$.

Exercise 5. Let V be a vector space. The dual of V is the space

$$V^* = \{f : V \longrightarrow \mathbb{R} : f \text{ linear}\}.$$

1. Show that the map

$$V \longrightarrow V^{**}; \qquad v \mapsto v^{**} : V^* \longrightarrow \mathbb{R}, \qquad v^{**}(f) = f(v).$$

   is linear and that if V is finite dimensional, it is an isomorphism.

2. Given a linear map $A : V \longrightarrow W$, show that

$$A^* : W^* \longrightarrow V^*; \qquad A^*w^* : V \longrightarrow \mathbb{R}; \qquad A^*w^*(v) = w^*(Av).$$

   is a linear map from W^* to V^*.

3. Show that if V and W are finite dimensional $A^{**} = A$.

4. Show that if $A : V \longrightarrow W$ is an injection A^* is a surjection.

5. Show that if $A : V \longrightarrow W$ is a surjection A^* is an injection.

Definition 6. A *Lie group* is a manifold G endowed with a group structure for which group multiplication and inversion

$$G \times G \longrightarrow G; \qquad (g, h) \mapsto g \cdot h;$$
$$G \longrightarrow G; \qquad g \mapsto g^{-1};$$

are smooth maps.

Exercise 7. Show that $S^1 \subset \mathbb{C}$ is a Lie group if we endow it with multiplication of complex numbers as group operation.