Differentiable manifolds – exercise sheet 5

Exercise 1. Show that $T_{p,q}(M \times N) = T_pM \oplus T_qN$.

Exercise 2. Check that the definition of f_* given in lectures does not depend on the path chosen to represent a tangent vector.

Exercise 3. Let $f : M \longrightarrow N$ be smooth. Show that $f_* : T_pM \longrightarrow T_{f(p)}N$ is a linear map.

Exercise 4. Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be smooth. Compute $f_* : T_p\mathbb{R}^m \longrightarrow T_{f(p)}\mathbb{R}^n$ in the basis given by the coordinate vectors.

Exercise 5. Let $f : V \longrightarrow W$ be a linear map between vector spaces. Compute $f_* : T_0V \longrightarrow T_0W$.

Exercise 6. Let $f : M \longrightarrow N$ be a smooth map for which $f_* \equiv 0$. Show that if M is connected then f is the constant map.

Exercise 7. Consider $S^3 \subset \mathbb{C}^2$. For each $z = (z_1, z_2) \in S^3$, define a curve $\gamma_z : \mathbb{R} \longrightarrow S^3$ by

$$\gamma_z(t) = e^{it}z = (e^{it}z_1, e^{it}z_2).$$

Show that γ_z is a smooth curve whose velocity is never zero. Show further that $\gamma(t) = \gamma(t')$ if and only if there is $k \in \mathbb{Z}$ for which $t = t' + 2\pi k$.

Exercise 8. Let M be a compact manifold of dimension bigger than zero and let $f : M \longrightarrow \mathbb{R}$ be smooth. Show that there are at least two points at which $f_* : T_pM \longrightarrow \mathbb{R}$ is the zero map.

Exercise 9. Let M be an n-dimensional compact manifold, $n > 0$, and let $f : M \longrightarrow \mathbb{R}^n$ be smooth. Show that $f_* : T_pM \longrightarrow T_{f(p)}\mathbb{R}^n$ can not be an isomorphism of vector spaces for every $p \in M$.

Exercise 10. Read the section "Alternative definitions of tangent space" describing tangent vectors as derivations of germs of functions.

Exercise 11. In exercise 1 of last exercise sheet we saw that if there is a smooth surjection $f : M \longrightarrow N$ such that there is a good cover of N for which $f^{-1}\mathfrak{U}$ is a good cover of M, then M and N have the same Čech cohomology.

1. Use this to conclude that M and $M \times \mathbb{R}^k$ have the same Čech cohomology for any $k \geq 0$. Hint: argue that if U is homeomorphic to a disc, then so is $U \times \mathbb{R}^k$.

2. Let TM be the manifold corresponding to the tangent space of M. Show that M and TM have isomorphic Čech cohomologies.