Differentiable manifolds – exercise sheet 6

Exercise 1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f(x, y) = x^3 + xy + y^3 + 1.$$

Find the critical points and the critical values of f.

Exercise 2. Show that if $\varphi : M \longrightarrow N$ is a diffeomorphism, then $\varphi_* : T_pM \longrightarrow T_{\varphi(p)}N$ is an isomorphism of vector spaces for all $p \in M$.

Exercise 3. Show that a vector bundle of rank k, $E \xrightarrow{\pi} M$, is trivial if and only if it has k sections which are linearly independent at every fiber.

Exercise 4. Show that TS^1 is isomorphic to the trivial vector bundle $S^1 \times \mathbb{R}$.

Exercise 5. Define a line bundle over the circle, $S^1 \subset \mathbb{C}$ as follows. The bundle has a nonvanishing section s_1 defined on $S^1 \backslash \{-1\}$ and a nonvanishing section s_2 defined on $S^1 \backslash \{1\}$. On the points in S^1 lying on the upper half plane, $s_1 = s_2$ (that is, the transition function is identical to 1) and on the lower half plane $s_1 = -s_2$ (that is, the transition function is identical to -1). Show that this line bundle is not trivial.

Exercise 6. Let $f : M \longrightarrow N$ be smooth and let $E \xrightarrow{\pi} N$ be a vector bundle. Show that if E is trivial, then f^*E is trivial.

Definition 7. A *Riemannian inner product* on a vector space V is a symmetric bilinear map

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}$$

such that $\langle v, v \rangle > 0$ if $v \neq 0$.

A *Riemannian inner product* on a vector bundle $E \xrightarrow{\pi} M$ is a choice of a Riemannian inner product $\langle \cdot, \cdot \rangle_p$ on each fiber E_p which is smooth in the sense that whenever X and Y are smooth (local) sections of E, $\langle X, Y \rangle$ is a smooth function on its domain of definition.

Exercise 8. Prove that every vector bundle admits a Riemannian inner product:

1. Show that if $\langle \cdot, \cdot \rangle_i$ are inner products on a vector space V, $i = 1, \cdots, l$, and ψ_i (again $i = 1, \cdots, l$) are nonnegative numbers at least one of which is not zero then the map

   $$\langle \cdot, \cdot \rangle = \sum_i \psi_i \langle \cdot, \cdot \rangle_i, \qquad \langle v, w \rangle = \sum_i \psi_i \langle v, w \rangle_i,$$

   is a Riemmanian inner product.

2. Show that locally every bundle admits a Riemannian inner product (show that it exists on the domain of any trivialization).

3. Use the previous two items and partitions of unity to conclude that every bundle admits a Riemannian inner product.

Exercise 9. Let $E \xrightarrow{\pi} M$ be a vector bundle of rank k over M and let $\langle \cdot, \cdot \rangle$ be a Riemannian inner product on E. Show that if one can trivialize E over an open set U, i.e., there are k everywhere linearly independent sections $\{s_1, \cdots, s_k\}$, $s_i : U \longrightarrow E$, then one can trivialize E by orthonormal sections, i.e., there are sections $\sigma_i : U \longrightarrow E$, $i = 1, \cdots, k$ such that $\langle \sigma_i, \sigma_j \rangle = \delta_{ij}$ (Hint: Use Gram-Schmidt).

Conclude that given a vector bundle, one may choose local trivializations which take values in the group of orthogonal transformations $O(k) \subset GL(k; \mathbb{R})$. What does this mean for line bundles?

Definition 10. A *short exact sequence* of Abelian groups is a sequence of three Abelian groups K, G and H with maps

$$\{e\} \longrightarrow K \xrightarrow{\iota} G \xrightarrow{\pi} H \longrightarrow \{e\},$$

for which the kernel and image of consecutive maps agree. In more words, ι is an injection, π is a surjection and $\mathrm{Im}(\iota) = \ker(\pi)$. All of this together means that K is a subgroup of G and H is the quotient group G/K.

Definition 11. A *long exact sequence* of Abelian groups is a sequence of Abelian groups G_i together with maps $\varphi_i : G_i \longrightarrow G_{i+1}$:

$$\cdots \longrightarrow G_{i-1} \xrightarrow{\varphi_{i-1}} G_i \xrightarrow{\varphi_i} G_{i+1} \xrightarrow{\varphi_{i+1}} \cdots$$

such that $\mathrm{Im}(\varphi_{i-1}) = \ker(\varphi_i)$ for all i.

The result from the first hand-in exercise sheet then reads as "A short exact sequence of Abelian groups, induces a long exact sequence of Čech cohomologies".

Exercise 12. Using the result from the first hand-in exercise or otherwise, show that the following is a short exact sequence

$$\{0\} \longrightarrow C^\infty(M;\mathbb{R}) \xrightarrow{\exp} C^\infty(M;\mathbb{R}^*) \xrightarrow{\mathrm{sign}} \mathbb{Z}_2 \longrightarrow \{e\};$$

where the first map is the exponential map and the second is the sign map, $\mathrm{sign}(x) = x/|x| \in \mathbb{Z}_2$.

Conclude that for $\check{H}^i(M; C^\infty(M;\mathbb{R}^*); \mathfrak{U}) = \check{H}^i(M; \mathbb{Z}_2; \mathfrak{U})$ for $i > 0$.