Differentiable manifolds – exercise sheet 9

Exercise 1. Let $\alpha \in \mathbb{R}$ be an irrational number and let $f : \mathbb{R} \longrightarrow T^2$ be defined by

$$f(t) = (e^{2\pi it}, e^{2\pi\alpha it}).$$

Show that $(f, \mathbb{R})$ is a submanifold of T^2. Show that the image of f is dense in T^2.

Exercise 2. Consider the map $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ defined by

$$f(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y)$$

Show that $(0, 1)$ is a regular value of f, and that the level set $f^{-1}\{(0, 1)\}$ is diffeomorphic to S^2.

Exercise 3. For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by the equation

$$M_a = \{(x, y) : y^2 = x(x - 1)(x - a)\}.$$

For which values of a is M_a the image of an embedding? For which values of a is M_a the image of an immersion?

Exercise 4. Define a map $f : S^2 \longrightarrow \mathbb{R}^4$ by

$$f(x, y, z) = (x^2 - y^2, xy, xz, yz).$$

Show that f is an immersion. Further show that if $f(\mathbf{x}) = f(\mathbf{y})$ then $\mathbf{x} = \pm\mathbf{y}$. Prove that f induces am embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$.

Exercise 5. Let $\alpha : M \longrightarrow T^*M$ be a map (not necessarily smooth). Show that α is smooth if and only if for all smooth vector fields X, $\alpha(X)$ is a smooth function.

Exercise 6. In exercise 6 of sheet 8 we showed that any vector bundle can be given a fiberwise metric $\langle\cdot,\cdot\rangle$ which is smooth in the sense that if X and Y are smooth sections of E then $\langle X, Y\rangle$ is a smooth function. In particular TM can be endowed with such a metric in which case we call the pair $(M, \langle\cdot,\cdot\rangle)$ a *Riemannian manifold*.

1. Given $f : M \longrightarrow \mathbb{R}$, show that there is a smooth vector field $\nabla f : M \longrightarrow TM$ such that for all $X \in \mathcal{X}(M)$, $\langle\nabla f, X\rangle = df(X)$. The vector field ∇f is called the gradient of f.
2. Show that if $y \in \mathbb{R}$ is a regular value of f then ∇f is orthogonal to the tangent space of the submanifold $f^{-1}(y)$, i.e., if $f(p) = y$ and X is tangent to $f^{-1}(y)$ at p, then $\langle\nabla f, X\rangle = 0$.
3. Show that if $\gamma : [0, 1] \longrightarrow M$ is such that $\langle\gamma'(t), \nabla f|_{\gamma(t)}\rangle > 0$ then $f \circ \gamma : [0, 1] \longrightarrow \mathbb{R}$ is an increasing function.