Geometry and Topology – Mock Exam 1

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1. Let $F : D^2 \longrightarrow D^2$ be a continuous map defined on the closed 2-dimensional disc and such that $F(S^1) \subset S^1$. Define $f : S^1 \longrightarrow S^1$ by $f(z) = F(z)$. Show that if F is not surjective then f is null homotopic .

Exercise 2. Given a continuous map between topological spaces $f : X \longrightarrow Y$ show that if there are continuous maps $g, h : Y \longrightarrow X$ such that $f \circ g : Y \longrightarrow Y$ and $h \circ f : X \longrightarrow X$ are homotopic to the identity maps 1_Y and 1_X, then f is a homotopy equivalence

Exercise 3. Let S^2_n be the space obtained by removing n points from the sphere S^2.

1. Compute $\pi_1(S^2_n)$.
2. Compute $\pi_1(S^2_n)/[\pi_1(S^2_n), \pi_1(S^2_n)]$ and show that for $n = 3$

$$\pi_1(S^2_3)/[\pi_1(S^2_3), \pi_1(S^2_3)] = \pi_1(S^1 \times S^1).$$

Exercise 4. Let X be a topological space. Show that the following are equivalent:

- Every map $\gamma : S^1 \longrightarrow X$ is null homotopic;
- Every map $\gamma : S^1 \longrightarrow X$ extends to a map $G : D^2 \longrightarrow X$;
- $\pi_1(X, x_0) = \{e\}$ for all $x_0 \in X$.

Exercise 5.

1. Let X be a set with n points and the discrete topology and let SX be its suspension. Show that $\pi_1(SX)$ is the free group in n generators.
2. Let X be path connected. Show that SX is simply connected.

Exercise 6. Show that if a path connected and locally path connected space X has finite fundamental group then every continuous map $f : X \longrightarrow S^1$ is null homotopic.