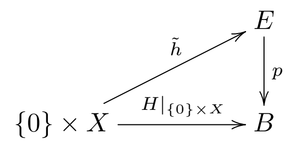
Geometry and Topology – hand-in sheet 2

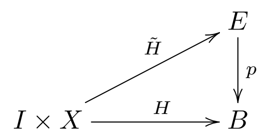
Hand in by 2/June

Prelude to the exercise

Definition 1. A map $p : E \longrightarrow B$ has the *homotopy lifting property with respect to a space* X if given a homotopy $H : I \times X \longrightarrow B$ and a map $\tilde{h} : \{0\} \times X \longrightarrow E$ such that the following diagram commutes



then there is a map $\tilde{H} : I \times X \longrightarrow E$ which lifts H and agrees with $\tilde{h}$ on $\{0\} \times X$, i.e., the following diagram commutes:



Exercise 2. Assume that $p : E \longrightarrow B$ has the homotopy lifting property with respect to $D^2 \cong I \times I$ and $D^1 = I$. Assume further that E and B are path connected and any map

$$\varphi : (I \times I, \partial(I \times I)) \longrightarrow (B, b_0)$$

is homotopy equivalent to the constant one via a relative homotopy $H : I \times I \times I \longrightarrow B$, i.e.,

$$H(0, s, t) = \varphi(s, t), \qquad \forall s, t \in I \times I;$$

$$H(1, s, t) = b_0, \qquad \forall s, t \in I \times I;$$

$$H(u, \partial(I \times I)) = b_0, \qquad \forall u \in I.$$

Let $x_0 \in E$, $b_0 = p(x_0)$ and $F = p^{-1}(b_0)$. Show that the corresponding sequence of fundamental groups

$$\{0\} \longrightarrow \pi_1(F, x_0) \xrightarrow{\iota_*} \pi_1(E, x_0) \xrightarrow{p_*} \pi_1(B, b_0)$$

is exact, that is, the kernel of each map is the image of the previous map. Show further that the cardinality of $\pi_1(B, b_0)/p_*(\pi_1(E, x_0))$ is the number of path connected components of F.

Hint: Often you will have a homotopy, say $H : I \times I \times I \longrightarrow B$ for which you can find by hand a lift $\tilde{h}$ to all but one of the faces of $\partial(I \times I \times I)$. Argue that since the sides of the cube with one face removed is homotopic to the disc, the homotopy lifting property property can be used to find a lift $\tilde{H}$ of H to the whole cube which agrees with $\tilde{h}$ where both maps are defined.

Remark: To give some idea of when this result can be used we remark on the properties being required of E, B and $p : E \longrightarrow B$.

1. The condition that $p : E \longrightarrow B$ satisfies the homotopy lifting property is satisfied for example if E and B are manifolds and $p : E \longrightarrow B$ is a proper smooth submersion.

2. The condition that any map $\varphi : (I \times I, \partial(I \times I)) \longrightarrow (B, b_0)$ is homotopic to the constant map via relative homotopies is equivalent to saying that the second homotopy group of B, $\pi_2(B)$, is trivial.