Geometry and Topology – hand-in sheet 3

Hand in by 20/June

Exercise 1 (The birth of long exact sequences). In what follows we let (U, ∂_U), (V, ∂_V) and (W, ∂_W) be chain complexes, that is, for example, U is a collection of Abelian groups U_i, $i \geq 0$, $\partial_U : U_i \longrightarrow U_{i-1}$ is a group homomorphism for all i and $\partial_U^2 = 0$.

1. Let $f : U \longrightarrow V$ be a group homomorphism (of degree zero), that is $f : U_i \longrightarrow V_i$, is a group homomorphism for all i. Show that if f commutes with the boundary operators, i.e., the diagram ,

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial_U} & U_i & \xrightarrow{\partial_U} & U_{i-1} & \xrightarrow{\partial_U} & \cdots \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\ \cdots & \xrightarrow{\partial_V} & V_i & \xrightarrow{\partial_V} & V_{i-1} & \xrightarrow{\partial_V} & \cdots \end{array}$$

   commutes. Then f sends boundaries to boundaries and cycles to cycles. In particular, f induces a map in homology:
$$f_* : H_i(U) \longrightarrow H_i(V), \qquad f_*[u] = [f(u)], \quad \text{for all } i.$$

2. Let $f : U \longrightarrow V$ and $g : V \longrightarrow W$ be group homomorphisms (of degree zero) which commute with the boundary operators. Assume that f is injective, g is surjective and $\mathrm{im}(f) = \ker(g)$, that is, we have a short exact sequence of chain complexes
$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0.$$
   Show that the induced maps
$$H_i(U) \xrightarrow{f_*} H_i(V) \xrightarrow{g_*} H_i(W)$$
   satisfy $\mathrm{im}(f_*) = \ker(g_*)$.

3. Next we try to define a map $\delta : H_i(W) \longrightarrow H_{i-1}(U)$ as follows. Given a cycle $w \in W_i$ ($\partial_W w = 0$), let $v \in V_i$ be such that $g(v) = w$. Then
$$0 = \partial_W w = \partial_W g(v) = g(\partial_V v),$$
   that is $\partial_V v \in \ker(g) = \mathrm{im}(f)$, hence there is $u \in U_{i-1}$ such that $f(u) = \partial_V v$. Show that $\partial_U u = 0$ and set $\delta[w] = [u]$. Show that δ is well defined.

4. Show that the sequence
$$\cdots \xrightarrow{\delta} H_i(U) \xrightarrow{f_*} H_i(V) \xrightarrow{g_*} H_i(W) \xrightarrow{\delta} H_{i-1}(U) \xrightarrow{f_*} \cdots$$
   is exact at every point.