Geometry and Topology – hand-in sheet 2

Hand in by 26/February

Exercise 1. A *semigroup* is a set X endowed with a map $m : X \times X \longrightarrow X$ and an element $e \in X$ such that $m(e, x) = m(x, e) = x$ for all x in X. A *topological semigroup* is topological space which is a semigroup and for which the multiplication m is continuous. Following the steps below or otherwise prove that if X is a topological semigroup, then $\pi_1(X, e)$ is an Abelian group.

- Define an operation on loops based at e by

  $$\gamma_1 \,\tilde{\star}\, \gamma_2(t) := m(\gamma_1(t), \gamma_2(t)), \qquad \text{for all } \gamma_i : (I, \partial I) \longrightarrow (X, e).$$

  Show that if γ_i' is homotopic to γ_i as loops based at e then $\gamma_1 \,\tilde{\star}\, \gamma_2$ is homotopic to $\gamma_1' \,\tilde{\star}\, \gamma_2'$. Conclude that $\tilde{\star}$ defines an operation on $\pi_1(X, e)$:

  $$\tilde{\star} : \pi_1(X, e) \times \pi_1(X, e) \longrightarrow \pi_1(X, e).$$

- Letting $\star$ denote concatenation of paths and e denote the constant loop, use that $\gamma_1 \simeq \gamma_1 \star e$ and $\gamma_2 \simeq e \star \gamma_2$ to conclude that $\tilde{\star}$ agrees with the usual product on $\pi_1(X, e)$.

- Using that $\gamma_1 \simeq e \star \gamma_1$ and $\gamma_2 \simeq \gamma_2 \star e$, conclude that $\pi_1(X, e)$ is Abelian.