Geometry and Topology – Exam 2

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (2.0 pt)**.** In each list of spaces below, decide which spaces are homotopy equivalent to each other (remember to justify your answer)

a)
$$S^1 \times S^n \qquad \text{and} \qquad S^1 \vee S^n \vee S^{n+1} \qquad \text{for } n > 1,$$

b)
$$\mathbb{R}P^2 \# \mathbb{R}P^2 \# T^2, \qquad \mathbb{K} \# \mathbb{K}, \qquad T^2 \# T^2.$$

where $\mathbb{K}$ denotes the Klein bottle and T^2 is the 2-dimensional torus.

c)
$$S^{2n}, \qquad \mathbb{R}P^{2n}, \qquad \mathbb{C}P^n, \qquad \text{for } n > 1.$$

Exercise 2 (2.0 pt)**.** Let $p : \tilde{X} \longrightarrow X$ be a simply connected cover of the space X and let $A \subset X$ be a path connected and locally path connected subspace. Let $\tilde{A} \subset \tilde{X}$ be a path component of $p^{-1}(A)$. Show that $p : \tilde{A} \longrightarrow A$ is the covering space corresponding to the kernel of the map $\pi_1(A) \longrightarrow \pi_1(X)$.

Exercise 3 (1.0 pt)**.** The suspension of a set X is the quotient of $I \times X$ by the equivalence relation $(0, x) \sim (0, x')$ and $(1, x) \sim (1, x')$ for all $x, x' \in X$. Denoting by SX be the suspension of X, show that $\tilde{H}_n(X) = \tilde{H}_{n+1}(SX)$.

Exercise 4 (2.0 pt)**.** Let $\mathcal{U} = \{U_1, \cdots U_k\}$ be an open cover of a space X with the following properties

- All the intersections of the form $U_{i_0} \cap \cdots \cap U_{i_l}$ are either contractible or empty (in particular, each U_i is contractible);
- There is an $n > 0$ for which $U_{i_0} \cap \cdots \cap U_{i_n} = \emptyset$ for all possible choices of distinct indices.

Show that $H_i(X) = \{0\}$ for all $i \geq n$.

Note: you are *not allowed* to use Čech cohomology to prove this claim.

Exercise 5 (3.0 pt)**.** For each statement below, prove or give a counter example.

a) If $f : X \longrightarrow Y$ is a homotopy equivalence and $x \in X$ then $X \backslash \{x\}$ and $Y \backslash \{f(x)\}$ are homotopy equivalent.

b) If $\pi_1(X)$ is finite and X is compact, then every path connected covering space of X is compact.

c) Let X be path connected and locally path connected and $\tilde{X}$ be a path connected cover of X with covering map $p : \tilde{X} \longrightarrow X$. Then $p_* : H_k(\tilde{X}) \longrightarrow H_k(X)$ is an injection for all k.