Group theory – Hand in sheet 3

(1) *Let $H < G$ be a subgroup. Show that*

$$g_1 H g_2 H = g_1 g_2 H \qquad \forall g_1, g_2 \in G \tag{1}$$

if and only if H is a normal subgroup, where the first set above is defined as

$$g_1 H g_2 H = \{g_1 h_1 g_2 h_2 : h_1, h_2 \in H\}.$$

Conclude that $\mathcal{H}$, the set of left H-cosets, inherits a product from G whenever H is normal.

(2) *Let G be a finite group and $H < G$ a subgroup. Show that the number of subgroups of G conjugated to H divides $[G : H]$, the index of H in G.* (Hint: consider the set $S = \{gHg^{-1} : g \in G\}$, i.e., the set of subgroups of G conjugated to H. There is a natural transitive G-action on this set. The result then follows from the orbit–stabilizer theorem).

(3) *Show that the map from G to automorphisms of G, $G \xrightarrow{Ad} Aut(G)$, given by*

$$g \mapsto Ad_g; \qquad Ad_g(h) = ghg^{-1}$$

is a group homomorphism. Compute its kernel.

(4) *Let S be a set and G a group acting on S. Let $S_0 \subset S$ be a subset of S. Consider the following subsets of G*

$$H = \{g \in G : gs = s \quad \forall s \in S_0\}$$
$$K = \{g \in G : gs \in S_0 \quad \forall s \in S_0\}$$

Show that $H \subset K$ and that H and K are subgroups of G. Show that H is a normal subgroup of K.

(5) *Show that if $n < p$, for some prime p then any S_n action on a set with p elements has more than one orbit, i.e., there is no transitive action of S_n on a set with p elements.*

(6) *Show that if a finite group G has only two conjugacy classes, then $G \cong \mathbb{Z}_2$.*