Group theory – Hand in sheet 5

(1- 2 pt) *Prove that the group* $S^3/\{\pm \mathrm{Id}\}$ *is isomorphic to* $\mathrm{SO}(3)$*. Check that* S^3 *and* $\mathrm{SO}(3)$ *are not isomorphic to one another. Hint: see example viii in chapter 16.*

(2- 4 pt) (**Upper central series**) *Given a group* G*, let* $Z_0 = \{e\}$ *and define inductively*

$$Z_i = \{g \in G : ghg^{-1}h^{-1} \in Z_{i-1}, \quad \textit{for all} \;\; h \in G\}.$$

1. (1 pt) *Show that* Z_1 *is the center of* G*, that* $Z_i \subset Z_{i+1}$ *and that* Z_i *is a normal subgroup of* G *for every* i*. Finally, Show that* Z_{i+1}/Z_i *is the center of* G/Z_i*.*

   Remark: The series

   $$\{e\} \subset Z_1 \lhd Z_2 \lhd \cdots \lhd Z_i \lhd Z_{i+1} \cdots$$

   is called the upper central series*.*

   A group G *is called* **nilpotent** *if there is an* $n \in \mathbb{N}$ *for which* $Z_n = G$*. The first* n *for which this happens is called the* nilpotency class *of* G*.*

2. (1 pt) *Compute the upper central series for* G*, the group of real upper triangular* 3 *by* 3 *matrices whose entries along the diagonal are 1, i.e., the elements in* G *look like*

   $$\begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

3. (1 pt) *Can you guess what the upper central series is for be the group of real upper triangular* n *by* n *matrices whose entries along the diagonal are 1?*

4. (1 pt) *Show that if the center of* G *is trivial, then the upper central series is given by* $Z_i = \{e\}$*. Compute the upper central series for* D_7*,* D_{28} *and* D_8*.*

(3- 4 pt) (**Lower central series**) *Given a group* G*, let* $G_0 = G$ *and define inductively*

$$G_i = \langle ghg^{-1}h^{-1} : g \in G_{i-1}, h \in G \rangle,$$

where $\langle \cdot \rangle$ *denotes "the group generated by". So, for example,* G_1 *is the commutator subgroup of* G*.*

1. (1 pt) *Show that* $G_{i+1} < G_i$*. Further, show that* G_{i+1} *is a normal subgroup of* G_i *and that the quotient* G_i/G_{i+1} *is Abelian.*

   Remark: The series

   $$G = G_0 \rhd G_1 \cdots \rhd G_i \rhd G_{i+1} \rhd \cdots$$

   is called the **lower central series***.*

2. (1 pt) *Compute the lower central series for* G*, the group of real upper triangular* 3 *by* 3 *matrices whose entries along the diagonal are 1.*

3. (1 pt) Compute the lower central series of D_7, D_{28} and D_8.

4. (1 pt) *Show that if G is nilpotent with nilpotency class n, then $G_n = \{e\}$. Further, if there is an n for which $G_n = \{e\}$, then G is nilpotent.*

(4- 2 pt) *Recall from last sheet: Given a finite group G and $n \geq 0$ a natural number, a* representation *of G is a group homomorphism*

$$\varphi : G \longrightarrow U(n).$$

A representation is irreducible *if there is no subspace $V \subset \mathbb{C}^n$, with $V \neq \{0\}, V \neq \mathbb{C}^n$, such that*

$$\varphi(g)(V) = V, \qquad \forall g \in G.$$

Now new stuff: Given two representations of G, say $\varphi_1 : G \longrightarrow U(n)$ and $\varphi_2 : G \longrightarrow U(m)$, we say that a linear map $A : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ is equivariant *if*

$$A(\varphi_1(g)v) = \varphi_2(g)(Av) \qquad \forall v \in \mathbb{C}^n.$$

We say that φ_1 and φ_2 are equivalent if there is an equivariant linear map $A : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ which is an isomorphism of vector spaces.

1. (1 pt) *In the situation above, show that* $\ker(A) \subset \mathbb{C}^n$ *and* $\mathrm{Im}(A) \subset \mathbb{C}^m$ *are invariant under the action of G. Conclude that if $\varphi_1 : G \longrightarrow U(n)$ and $\varphi_2 : G \longrightarrow U(m)$ are irreducible, then either they are equivalent or the only equivariant map between $\mathbb{C}^n$ and $\mathbb{C}^m$ is the trivial one.*

2. (1 pt) *Show that if $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is equivariant for φ_1, i.e., $\varphi_1(g)A = A\varphi_1(g)$ for all $g \in G$ and φ_1 is irreducible, then A must be a multiple of the identity. (You may assume that A is diagonalizable)*