Group theory – Last Sheet
Sylow groups

As we have mentioned in lectures, simple groups, i.e., groups which have no proper nontrivial subgroups, are the building blocks for finite groups. For that reason, there was a great interest in finding all simple groups. In these exercises we see that Sylow theorems are a very good tool to prove that groups of certain orders are not simple.

1. Prove that a group of order 200 has a normal 5-Sylow
2. Prove that if G has order 132, then G is nor simple,
3. Prove that if G has order 462, then G is not simple,
4. Prove that if G has order 231, then the center of G contains a 11-Sylow subgroup and a 7-Sylow is normal
5. Prove that if G has order 385, then the center of G contains a 7-Sylow subgroup and a 11-Sylow is normal
6. If G has order 105, then a 5-Sylow and a 7-Sylow subgroup are normal.
7. If G has order $3 \cdot 2^n$, then G is not simple. Similarly, if G has order $5 \cdot 2^n$ and $n > 3$, then G is not simple. The same is also true if G has order $15 \cdot 7^n$ and $n > 2$.
8. If you have time left at the end of the class: In each of the previous exercises there was an underlying principle which can be stated in terms of prime numbers p, q and r. For example, exercise 6, actually means that if a group has order pqr with $p > q > r$ prime numbers, then there is only one p- and only one q-Sylow. Try and identify the generic principle behind each exercise.
9. If you have time left over the Xmas break... For this exercise you can assume the following result: If a group G of order 60 is simple, then G is isomorphic to A_5. Knowing this, prove that if a non-Abelian simple group has order less than 100, then it is isomorphic to A_5. [hint: eliminate all orders but 60]