Mock exam 1 – Group theory

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) *For each list of groups a) and b) below, decide which of the groups within each list are isomorphic, if any:*

a) $\mathbb{Z}_{20}$, $\mathbb{Z}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_{10}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$.

b) $\mathbb{Z}_2 \times D_7$, $\mathbb{Z}_2 \times \mathbb{Z}_{14}$, D_{14}.

Remarks useful for the solution. a) We have seen several times in this course that if m and n are coprimes then $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{nm}$, and if m and n are not coprime, then these group are not isomorphic (see Armstrong theorem 10.5). Using this you can easily see that $\mathbb{Z}_{20} \cong \mathbb{Z}_4 \times \mathbb{Z}_5$ and $\mathbb{Z}_2 \times \mathbb{Z}_{10} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$, but $\mathbb{Z}_2 \times \mathbb{Z}_{10} \ncong, \mathbb{Z}_{20}$.

b) $\mathbb{Z}_2 \times \mathbb{Z}_{14}$ is Abelian, while the others are not, so it is not isomorphic to any other group in the list. The remaining 2 are isomorphic. This was done in lectures and is in Armstrong.

$\square$

2) *Let G be the set of sequences of integers endowed with the following product operation $+ : G \times G \longrightarrow G$*

$$(a_1, a_2, \cdots, a_n, \cdots) + (b_1, b_2, \cdots, b_n, \cdots) = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n, \cdots).$$

Show that this operation makes G into a group. Show that $\mathbb{Z} \times G \cong G$ and hence conclude that, for groups, it may be the case that $A \times C \cong B \times C$ even though $A \ncong B$.[1]

Remarks useful for the solution. This is exercise 10.10 from Armstrong, without the typo.

$\square$

[1] I'd never ask this in an exam, but at home you may try to prove that *for finite groups* it is true that $A \times C \cong B \times C$ implies $A \cong B$. If you just want to see a proof, take a look at Hirshon's paper On cancellation in groups.

3) *Let $n > m$ be natural numbers, $n > 4$, let X be a set with m elements. Show that the orbits of any action of S_n on X have size 1 or 2.*

Remarks useful for the solution. This is exercise 2 from exercise sheet 7. the important remark here is that an action of S_n on X is equivalent to a group homomorphism

$$S_n \longrightarrow S_m$$

since $m < n$, we see that this map has a nontrivial kernel, but the only normal subgroups of S_n and $\{e\}$, A_n and S_n, so the kernel must be A_n or S_n. In the second case the action is trivial. In the first case, the stabilizer of each point is must contain A_n, hence is either A_n or ${}_Sn$. By the orbit-stabilizer theorem, this means that the orbits have size 2 or 1, respectively.

□

4) *Let G be a group, S_G be group of bijections from G into itself and $\mathrm{Aut}(G) \subset S_G$ be the group of automorphisms of G. Consider the map $\mathrm{Ad} : G \longrightarrow S_G$, given by*

$$\mathrm{Ad}(g) : G \longrightarrow G \qquad \mathrm{Ad}(g)(x) = gxg^{-1}.$$

a) Show that $\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(G)$, i.e., for every $g \in G$, $\mathrm{Ad}(g) : G \longrightarrow G$ is an automorphism;

b) Show that $\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(G)$ is a group homomorphism and that the image of Ad is a normal subgroup of $\mathrm{Aut}(G)$. The image of Ad is called the group of inner automorphisms.

c) Show that the kernel of $\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(G)$ is the center of G and conclude that the group of inner automorphisms is isomorphic to the quotient G/Z_G.

d) Give an example of a group which has an automorphism which is not an inner automorphism.

Remarks useful for the solution. Done in lectures, in the book and partly solved in exercises 3 and 4 in the hand-in sheet 1.

□

5) *Classify all groups or order $2009 = 7^2 \cdot 41$.*

Remarks useful for the solution. Particular case of Armstrong exercise 20.7. and is a straightforward application of Sylow theorems.

□

6) *Let G be a group and $n \in \mathbb{N}$*

a) Let $H_i < G$ be subgroups, for $i \in \{1, \cdots, n\}$, show that

$$\cap_{i=1}^n H_i$$

is a subgroup of G.

b) If G is finite and p be a prime. Show that the intersection of all p-Sylows of G is a normal subgroup.

Remarks useful for the solution. a) See exercise 2, from hand-in sheet 2.

b) See exercise 4, from hand-in sheet 6.

$\square$

7) *Let G be a finite group and $K, H < G$. Prove or give a counter-example to the following claims.*

a) If $K \lhd H$ and $H \lhd G$ then $K \lhd G$.

b) If K is the only p-Sylow of G, then $K \cap H$ is a p-Sylow of H.

Remarks useful for the solution. a) This was covered in the exercise classes (Armstrong, exercise 15.5). The counter-example is $\mathbb{Z}_2 \lhd \mathbb{Z}_2 \times \mathbb{Z}_2 \lhd A_4$.

b) See exercise 7d), from hand-in sheet 6.

$\square$