Mock exam 2 – Group theory

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) *Let D_n be the dihedral group given by*

$$D_n = \langle a, b : a^n = b^2 = e; bab^{-1} = a^{-1} \rangle.$$

- *a) Compute Z_{D_n}, the center of D_n, for $n > 1$. Analyse carefully the cases $n = 2$, n even and greater than 2 and n odd.*
- *b) Show that if $n > 1$, then $D_{2n}/Z_{D_{2n}}$ is isomorphic to D_n.*

Remarks useful for the solution. a) This has been covered in exercises and examples in the book. The center of D_{2n} is given by $\{e, a^n\}$ for $n > 2$, while D_2 is Abelian an isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. D_{2n+1} has no center (see exercise 14.10 in Armstrong).

b) Is done for $n = 3$ in Armstrong (example 15.i) and is totally analogous for generic $n > 2$. Observe that since D_2 is Abelian, $D_2 = Z_{D_2}$ and here the quotient is trivial.

□

2) *For each list of groups a) and b) below, decide which of the groups within that list are isomorphic, if any:*

- *a) D_3, S_3 and the group generated by*

$$\langle a, b : a^3 = b^2 = e; aba^{-1} = ba \rangle.$$

- *b) D_{12}, $\mathbb{Z}_4 \times D_3$ and S_4.*

Remarks useful for the solution. a) We have proved and used a few times in class that D_3 and S_3 are isomorphic. About the third group, we see the relation $aba^{-1} = ba$ is equivalent to $bab = a^2 = a^{-1}$, so again thisis the dihedral group.

Alternatively, you might point out that there are only two groups of order 6 (we have classified those): $\mathbb{Z}_6$ and D_3. Given that the groups above are not Abelian, they must be isomorphic to D_3. This second solution is right and would give you full marks for the exercise, however it is far less satisfactory because in order to prove the theorem being used to solve the exercise we had to solve an even harder exercise, namely that every non-Abelian group of order 6 is isomorphic to D_3 and not just the three listed in this exercise.

b) The three groups listed here have different centers: $Z_{D_12} = \mathbb{Z}_2$, $Z_{\mathbb{Z}_4 \times D_3} \cong Z_{\mathbb{Z}_4} \times Z_{D_3} \cong \mathbb{Z}_4$ and $Z_{S_4} = \{e\}$, so no two are isomorphic.

□

3) *Let G be a finite group. We define a sequence (G_i) of subgroups of G as follows. We let $G_0 = G$ and define inductively G_i as the group generated by*

$$G_i = \langle ghg^{-1}h^{-1} : g \in G \text{ and } h \in G_{i-1} \rangle$$

So, for example, G_1 is the commutator subgroup of G.

a) Show that each G_i is subgroup of G_{i-1}. Further, show that $G_i \lhd G_{i-1}$ and that the quotient G_{i-1}/G_i is Abelian.

b) Show that if, for some i_0, $G_{i_0} = G_{i_0+1}$ then $G_n = G_{i_0}$ for all $n > i_0$.

c) Compute the sequence of subgroups G_i above for $G = D_8$, D_{10} and A_5.

Remarks useful for the solution. See exercise 3 (lower central series) in the hand-in sheet 5. There you will find the solution to this question and more facts about this sequence of groups.

The sequence for A_5 is not computed there, but is rather simple. Since A_5 is a simple non-Abelian group, and G_1 is non-trivial and a normal subgroup of A_5, it must be the whole A_5. Then according to *b*) the whole sequence is given by $G_i = A_5$.

□

4) *Show that if G has order $p_1 p_2 \cdots p_n$, for p_i primes with $p_i \le p_{i+1}$ and $H < G$ is a subgroup of order $p_2 \cdots p_n$, then H is normal.*

Remarks useful for the solution. This is an exercise from exercise sheet on Sylow groups (Armstrong exercise 20.12), was also present in the hand in sheet 6 (exercise 2) and you can find a solution there.

□

5) *Let G be a group of order np^k, with $n > 1$, $k > 0$, $p > 2$ and n and p coprimes.*

a) Show that if $n < p$ then G is not simple,

b) Show that if $n < 2p$ and $k > 1$, then G is not simple,

c) *Show that if* $k > n/p$ *and* $n < p^2$*, then* G *is not simple.*

Remarks useful for the solution. This is an exercise from hand-in sheet 6 (exercise 3) and you can find a solution there.

□

6) *In what follows let* G *be a finite group and* $K, H < G$*. Prove or give counter-examples to the following claims.*

a) *If* $K \lhd G$*, then* $K \cap H \lhd H$*.*

b) *If* K *is a* p*-Sylow of* G *then* $K \cap H$ *is a* p*-Sylow of* H*.*

Remarks useful for the solution. a) This is exercise 1 from hand-in sheet 2 and you can find a solution there (with H and K swapped) .
b) This is a small part of exercise 7 from hand-in sheet 6 and you can find a solution there.

□

7) *Let* $p > 2$*. What is the order of a* p*-Sylow of* S_{2p}*? Give an example of one such group. Finally, find all* p*-Sylows of* S_{2p}*.*

Remarks useful for the solution. This is an exercise from hand in sheet 6 (exercise 5) and you can find a solution there. □