Group theory – Sheet 3

The exercises from the book are 11.4, 11.6, 11.7, 11.9, 11.10, 11.11, 11.12, 12.1, 12.3.

Recall the following results from lectures.

Theorem: *If G is a finite group and $H < G$ is a subgroup of index 2, then H is normal.*

Theorem (Lagrange): *Let G be a finite group. Then the order of any subgroup divides the order of G.*

Theorem (Cauchy): *If a prime p divides the order of a finite group G, then there is an element $g \in G$ of order p.*

Below, p is a prime number.

1) Show that a group of order p^2 is isomorphic to Z_{p^2} or $Z_p \times Z_p$. In particular, any such group is abelian.

2) Show that a group of order $2p$ is isomorphic to either Z_{2p} or D_p.

3*) The center of a group G is defined as the set

$$Z = \{g \in G : gh = hg \text{ for all } h \in G\}.$$

Show that if G has order p^n for some n, then its center is a nontrivial subgroup.