Group theory – Mock Exam 2

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. If you are not sure about some definition of notation you encounter in the exam, please ask.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) Let H and J be subgroups of a finite group G. Show that if the orders of H and J have no common factors then their intersection is the trivial group, i. e., $H \cap J = \{e\}$.

2) Decide which of the following groups are isomorphic, if any:

a) $A_5 \times \mathbb{Z}_2$, S_5, D_{60} and $D_{20} \times \mathbb{Z}_3$.

b) $D_{10} \times Z_3$, $D_5 \times \mathbb{Z}_6$ and D_{30}.

3) Let G be a finite group. Show that if G has only two conjugacy classes then G is isomorphic to $\mathbb{Z}_2$.

4) Let G be a group of order $5 \cdot 11 \cdot 13$. Show that the 11 and the 13 Sylows are normal. Show that the 13-Sylow is in the center of G.

5) For this exercise, let G be a group of order $2n$, where n is an odd number greater than 1. Following the steps below or otherwise, prove that G is not simple.

a) Consider the action of G on itself by left multiplication. This furnishes a group homomorphism

$$\varphi : G \longrightarrow S_{2n}.$$

Show that φ is an injection;

b) According to Cauchy's theorem there is $x \in G$ of order 2. Show that the image of x is an odd permutation, hence $\varphi(G)$ is not contained in A_{2n};

c) Identifying G with its image in S_{2n} show that $G \cap A_{2n}$ is a nontrivial normal subgroup of G, hence G is not simple.