Group theory – Sheet 1

The exercises from the book are 1.3, 1.4, 1.5, 2.3, 2.5, 2.7, 2.8.

(1) Determine which of the following subsets of $\mathrm{M}(n,\mathbb{R})$, the set of n by n real matrices, are groups under matrix multiplication:

- $\mathrm{M}(n,\mathbb{R})$,
- $\mathrm{Gl}(n,\mathbb{R})$, the set of n by n matrices with nonzero determinant,
- $\mathrm{Sl}(n,\mathbb{R})$, the set of n by n matrices with determinant 1,
- upper triangular matrices with nonzero determinant,
- $\mathrm{O}(n)$, the set of orthogonal matrices,
- symmetric matrices,
- skew symmetric matrices.

(2) Let G be a group and $x \in G$. Show that $x^n x^m = x^m x^n = x^{n+m}$ and that $(x^n)^m = x^{nm}$.

(3) Show that if every element g in a group G satisfies $g^2 = e$ then G is Abelian.

(4) Let $\{\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{k}\}$ denote a basis for $\mathbb{R}^4$ as a vector space and define an associative $\mathbb{R}$-bilinear product on $\mathbb{R}^4$ by the rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1} \qquad \mathbf{1p} = \mathbf{p} \qquad \forall \mathbf{p} \in \mathbb{R}^4.$$

- Show that $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$ and $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$,
- For $\mathbf{p} = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{R}^4$, with $a, b, c, d \in \mathbb{R}$, let $\overline{\mathbf{p}} = a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k} \in \mathbb{R}^4$. Show that

  $$\mathbf{p}\overline{\mathbf{p}} = a^2 + b^2 + c^2 + d^2.$$

  Conclude that every element in $\mathbb{R}^4\backslash\{0\}$ has a multiplicative inverse and hence $\mathbb{R}^4\backslash\{0\}$ is a (non-commutative) group.
- Show that for $\mathbf{p},\mathbf{q} \in \mathbb{R}^4$, $\overline{\mathbf{pq}} = \overline{\mathbf{q}}\,\overline{\mathbf{p}}$. Conclude that $\mathbf{pq}\overline{\mathbf{pq}} = \mathbf{p}\overline{\mathbf{p}}\mathbf{q}\overline{\mathbf{q}}$. Finally, show that $S^3 \subset \mathbb{R}^4$ is a group.

The vector space $\mathbb{R}^4$ with this group structure is known as the *quaternions*.

(5) In lectures we studied the group of symmetries of a regular hexagon. What is the group of symmetries of a regular n-gon?