Group theory – Sheet 10 & 11

Exercises from the book: all exercises from chapter 20.

In what follows, p, q and r are prime numbers with $r < q < p$ and G is a finite group.

1*) Let G be a group of order pqr.
a) Show that either the p-Sylow or the q-Sylow must be normal.
b) In either case, show that G has a subgroup H of order pq. Show that H is normal.
c) Conclude that if $p \neq 1 \mod q$, then both the p-Sylow and the q-Sylow are normal subgroups.
d) Show that if q and r do not divide $p-1$, then the p-Sylow is contained in the center of G.

2) Let G be a group of order np^k, with $k > 0$, $p > 2$, $n > 1$ and p coprimes.
a) Show that if $n < p$ then G is not simple,
b) Show that if $n < 2p$ and $k > 1$, then G is not simple,
c) Show that if $k > n/p$ and $n < p^2$, then G is not simple.

3) Show that the intersection of all p-Sylows is a normal subgroup.

4) Let $p > 2$. What is the order of a p-Sylow of S_{2p}? Give an example of one such group. Finally, find all p-Sylows of S_{2p}.

5) Let $p > 2$. Find generators for a p-Sylow of S_{p^2}. Show that this is a non-Abelian group of order p^{p+1}.

6) Let $H < G$ be a subgroup and K be a p-Sylow subgroup of G.
a) Is it true that $H \cap K$ is a p-Sylow subgroup of H?
b) If H has a unique p-Sylow, is it true that it must be $H \cap K$?
c) If H is normal, is it true that $H \cap K$ is a p-Sylow subgroup of H?
d) If K is the only p-Sylow subgroup of G, is $H \cap K$ a p-Sylow of H?