Talk 1: Recap of Differential Geometry

February 16, 2016

We have made a rather large selection of exercises, containing what we think some points/insights that are useful in the rest of the course. If you have seen the material, feel free to skip through.

1 Vector bundles

Let K denote $\mathbb{R}$ or $\mathbb{C}$.

1.1 The product of two vector bundles is a vector bundle over the product.

For this exercise, let B_1, B_2 be smooth manifolds. Let $p_i : E_i \to B_i$ be K vector bundles of rank m_i for $i \in \{0, 1\}$. Show that $(p_1, p_2) : E_1 \times E_2 \to B_1 \times B_2$ is a vector bundle.

1.2 The restriction of a vector bundle is a vector bundle

Let $p : E_1 \to B$ be a vector bundle. Let $A \subset B$ be a submanifold. Show that the restriction bundle $p : p^{-1}(A) \to A$ is a vector bundle.

1.3 The direct sum of two vector bundles is again a vector bundle

Let B be a smooth manifold. Let $p_i : E_i \to B$ be vector bundles $i \in \{1, 2\}$(note no lowerscript i at B).

$$E_1 \oplus E_2 := \{(v_1, v_2) \in E_1 \times E_2 | p_1(v_1) = p_2(v_2)\}$$

(Hint: use previous exercises and restrict over the diagonal map)

1.4 Building a smooth vector bundle from fibers

[Exercise 1 from Marius Crainic - Differential geometry diktaat] Let $E = \{E_x\}_{x \in M}$ be a discrete vector bundle over M(I.e. just a collection of fibers without any topology). Assume we are given an open cover $\mathcal{U}$ of M, and, for each open $U \in \mathcal{U}$, a discrete local frame S_U of E over U. Assume that for any $U, V \in \mathcal{U}$, the two frames S_U and s_V are smoothly compatible. Then E admits a unique smooth structure which makes it into a vector bundle over M with the property that all S_U become (smooth) local frames. Moreover, the sections of E can be recognized as those discrete sections s with the property that they are smooth with respect to the given data $\{S_U\}_{U \in \mathcal{U}}$ in the following sense: for any $U \in \mathcal{U}$ writing

$$u(x) = \sum_{i=1}^{r} f_i(x) s_U^i(X) \quad (x \in U)$$

all the functions f_i are smooth functions $f_i : B \to E$.

1.5 The dual of a vector bundle is again a vector bundle

Let $p : E \to B$ be a vector bundle. Then we want to define the dual vector space. Point-wise it is clear what it should be. For a frame $e = (e_1, \cdots e_k)$ also know its dual, namely $e^* = (e_1^*, \cdots, e_k^*)$. Now we want to fit this information together to get a vector bundle. Show that this defines a unique smooth structure on E^*

1.6 Tensor products of vector bundles (Hard)

[Adapted from Hatcher - Vector bundles and K-theory] Given two vector bundles $p_1 : E_1 \to B$ of rank N_1 and $p_2 : E_2 \to B$ of rank N_2, we want to define the tensor product E_3 of two vector bundles. Point-wise it is clear what we want to do: we identify the fibers of E_3 by $E_{3,x} = p_1^{-1}(x) \otimes p_2^{-1}(x)$. Now we want to define the topology on E_3 which makes it a vector bundle.

a) Show that there exists an open cover $\{U_i\}$ of B such that we have trivialisations $\varphi_1 : p_1^{-1}(U_i) \to U_i \times K^{N_1}$ and $\varphi_2 : p_2^{-1}(U_i) \to U_i \times K^{N_2}$.

We can topologize $p_1^{-1}(U_i) \otimes p_2^{-1}(U_i)$ by letting the fiberwise tensor product map $\varphi_1 \otimes \varphi_2 : p_1^{-1}(U_i) \otimes p_2^{-1}(U_i) \to U \times (K^{N_1} \otimes K^{N_2})$ be a homeomorphism. If we also take $\varphi_1 \otimes \varphi_2$ be a diffeomorphism this also induces a smooth structure on $p_1^{-1}(U_i) \otimes p_2^{-1}(U_i)$.

b) Show that the topology and the smooth structure do not depend on our choice of trivialisations. (Hint: If we have an open neighborhood U of $x \in B$ with two trivialisations. How can we change from one trivialisation to another? What happens fiberwise? What happens on the $p_1^{-1}(U?)$

c) If V is an open subset of U_i, then the topology of $p_1^{-1}(V) \otimes p_2^{-1}(V)$ induced by the trivialisations over V is the same as the induced topology by the trivialisations over U. Show this.

d) Conclude that we get a well defined topology and smooth structure on $E_1 \otimes E_2$ which makes this into a vector bundle, which is fiberwise the tensor of the fibers of E_1 and E_2.

2 Orientation

(See also Lee prop 15.3/15.5) Let M be a smooth n-manifold.

a) Show that any non-vanishing n-form ω on M determines a unique orientation of M for which ω is positively oriented at each point.

b) Show that if M is given an orientation, then there exists a smooth non-vanishing n-form on M that is positively oriented at each point.

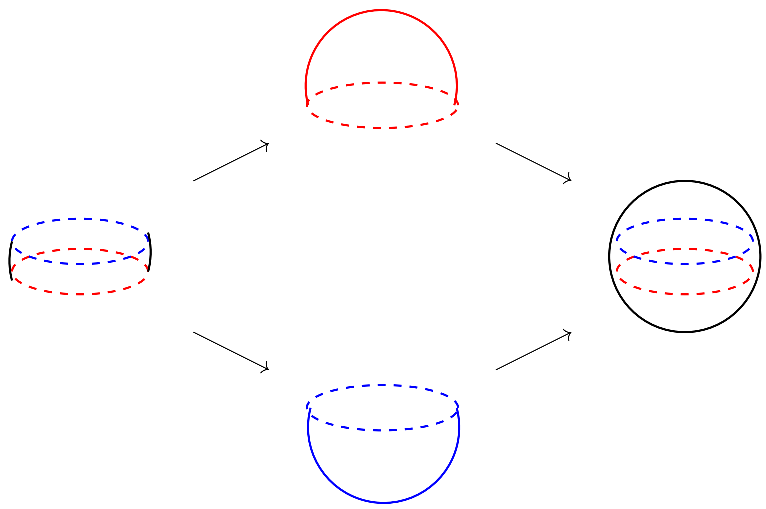
c) Show either using charts and orientation, or using forms, that the open Möbius band is not orientable.

3 Mayer-Vietoris

If you are not familiar with the zigzag lemma, we highly recommend Gil Cavalcanti's on long exact sequences from short ones. A hyperling is provided here, and also is appended in the email.

`http://www.staff.science.uu.nl/~caval101/homepage/Differential_geometry_2014_files/hisheet1.pdf`

Let M be the n-sphere. Using the cover of $\{U, V\}$ of two overlapping (slightly bigger than)-hemispheres



a) Construct the long exact Mayer-Vietoris sequence.

b) Show that U and V are diffeomorphic to $\mathbb{R}^n$

c) Show that $U \cap V$ is diffeomorphic to $\mathbb{R}^n \setminus \{0\}$, which is homotopy equivalent to $\mathbb{S}^{n-1}$.

d) Conclude that for $p > 1$ we have $H^p_{dR}(\mathbb{S}^n) = H^{p-1}_{dR}(\mathbb{S}^{n-1})$.