Blowing up generalized Kähler 4-manifolds

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Abstract

We show that the blow-up of a generalized Kähler 4-manifold in a nondegenerate complex point admits a generalized Kähler metric. As with the blow-up of complex surfaces, this metric may be chosen to coincide with the original outside a tubular neighbourhood of the exceptional divisor. To accomplish this, we develop a blow-up operation for bi-Hermitian manifolds.

Contents

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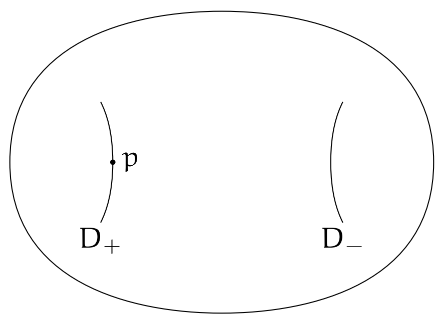
1 Introduction

Let $(M, \mathbb{J}_+, \mathbb{J}_-)$ be a generalized Kähler 4-manifold such that both generalized complex structures $\mathbb{J}_+, \mathbb{J}_-$ have even type, meaning that they are equivalent to either a complex or symplectic structure at every point. In other words, their underlying real Poisson structures P_+, P_- have either rank 0 (at complex points) or 4 (at symplectic points). The structure $\mathbb{J}_\pm$ is equipped with a canonical section $s_\pm$ of its anticanonical line bundle, vanishing on the locus $D_\pm$ of complex points, where $P_\pm$ has rank zero. From [6], it follows that the symplectic leaves of P_+ and P_- must be everywhere transverse, so that D_+, D_- are disjoint.

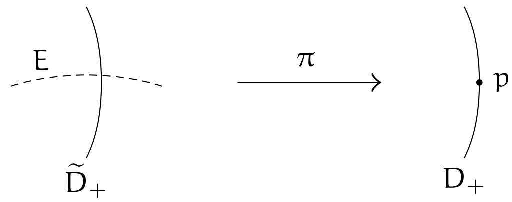


It was shown in [2] that in a neighbourhood of a complex point $p \in D_+$ which is nondegenerate, in the sense of being a nondegenerate zero of s_+, there are complex coordinates (w, z) such that the generalized complex structure $\mathbb{J}_+$ is equivalent to that defined by the differential form

$$\rho_+ = w + dw \wedge dz. \tag{1.1}$$

Note that $D_+ = w^{-1}(0)$, along which $\rho_+|_{D_+} = dw \wedge dz$ defines a complex structure, whereas for $w \neq 0$, we have $\rho_+ = w \exp(B + i\omega)$, for $B + i\omega = d \log w \wedge dz$, defining a symplectic form ω away from D_+, as required.

It was then shown [2, Theorem 3.3] that the complex blow-up at p using the coordinates (z, w) inherits a generalized complex structure. We detail in Section 2 why this structure is independent of the chosen coordinates. Thus we obtain a canonical blow-up $(\widetilde{M}, \widetilde{\mathbb{J}}_+)$ of $(M, \mathbb{J}_+)$ at p, equipped with a generalized holomorphic map $\pi : \widetilde{M} \longrightarrow M$ which is an isomorphism outside the exceptional divisor $E = \pi^{-1}(p)$. The complex locus $\widetilde{D}_+$ of the blow-up is the proper transform of D_+, and the exceptional divisor E is a 2-sphere which intersects $\widetilde{D}_+$ transversely at one point and is Lagrangian with respect to ω elsewhere; this makes E a generalized complex brane [2].



In Section 5.1, we use the bi-Hermitian tools developed in Section 3 to construct a *degenerate* generalized Kähler structure on the blow-up, in the sense that the metric degenerates along the exceptional divisor E. Finally, in Section 5.2, we use a deformation procedure detailed in Section 4 to obtain a positive-definite metric, defining a generalized Kähler structure such that π is an isomorphism away from a tubular neighbourhood of the exceptional divisor E. The generalized complex structure $\mathbb{J}_-$ does not lift uniquely to the blow-up, as there is no preferred choice of symplectic area for E; this degree of freedom inherent in the generalized Kähler blow-up is familiar from the usual Kähler blow-up operation.

2 Generalized complex blow-up

Let (w, z) be standard coordinates for $M = \mathbb{C}^2$, and consider the generalized complex structure $\mathbb{J}$ defined by the form ρ_+ given in (1.1). This structure extends uniquely to a generalized complex structure $\widetilde{\mathbb{J}}$ the blow-up $\widetilde{M} = [\mathbb{C}^2 : 0]$ of the plane in the origin, simply because the anticanonical section $\sigma = w\partial_w \wedge \partial_z$ does. That is, the line generated by ρ_+ may be written

$$\langle \rho_+ \rangle = e^{\sigma}\Omega^{2,0}(M),$$

and in the two blow-up charts $(w_0, z_0) = (w/z, z)$ and $(w_1, z_1) = (w, z/w)$, this pulls back to the line $e^{\widetilde{\sigma}}\Omega^{2,0}(\widetilde{M})$, where

$$\widetilde{\sigma} = w_0\partial_{w_0} \wedge \partial_{z_0} = \partial_{w_1} \wedge \partial_{z_1}.$$

Clearly, $\widetilde{\sigma}$ drops rank along the proper transform of $w^{-1}(0)$, namely $w_0^{-1}(0)$.

The above construction of $\widetilde{\mathbb{J}}$ uses the complex structure defined by (w, z), but this complex structure is not determined canonically by $\mathbb{J}$. That is, there are automorphisms $\Phi = (\varphi, B) \in \mathrm{Diff}(M) \ltimes \Omega^{2,cl}(M, \mathbb{R})$ of $\mathbb{J}$ for which φ is not a holomorphic automorphism of $\mathbb{C}^2$. To show that $\widetilde{\mathbb{J}}$ is independent of the particular complex structure used to perform the blow-up, we must show that any such automorphism $\Phi \in \mathrm{Aut}(\mathbb{J})$ with $\varphi(0) = 0$ lifts to the blow-up $[\mathbb{C}^2 : 0]$.

Theorem 2.1. *Any automorphism of $\mathbb{J}$ on $M = \mathbb{C}^2$ fixing the origin lifts to the blow-up $\widetilde{M}$ of M in the origin.*

Proof. Let $\Phi = (\varphi, B) \in \mathrm{Aut}(\mathbb{J})$, meaning that

$$e^B \varphi^*(w + dw \wedge dz) = e^\lambda (w + dw \wedge dz), \tag{2.1}$$

for some $\lambda \in C^\infty(M, \mathbb{C})$. Also, assume $\varphi(0) = 0$. Let $p : \widetilde{M} \to M$ be the blow-down map. We will show that φ lifts to $\widetilde{\varphi} \in \mathrm{Diff}(\widetilde{M})$ such that $p \circ \widetilde{\varphi} = \varphi \circ p$, and then $(\widetilde{\varphi}, p^*B) \in \mathrm{Aut}(\widetilde{\mathbb{J}})$ is the required lift of the automorphism. The lift $\widetilde{\varphi}$ exists if and only if the functions $\widetilde{w} = \varphi^* w$, $\widetilde{z} = \varphi^* z$ are in the ideal generated by w and z in $C^\infty(M, \mathbb{C})$. By a theorem of Malgrange [10], this is equivalent to the following constraints: $\widetilde{w}(0) = 0$, $\widetilde{z}(0) = 0$, and

$$\left.\frac{\partial^{p+q}\widetilde{w}}{\partial^p \overline{w}\, \partial^q \overline{z}}\right|_{(0,0)} = 0 \quad \text{and} \quad \left.\frac{\partial^{p+q}\widetilde{z}}{\partial^p \overline{w}\, \partial^q \overline{z}}\right|_{(0,0)} = 0, \text{ for all } p, q \in \mathbb{N}. \tag{2.2}$$

To verify (2.2), we rewrite (2.1) as follows:

$$\widetilde{w} + d\widetilde{w} \wedge d\widetilde{z} = e^\lambda e^{-B}(w + dw \wedge dz) = e^\lambda (w + dw \wedge dz - wB), \tag{2.3}$$

where the summand of degree four is omitted from the last term since it vanishes. From this we immediately conclude that $\widetilde{w} = e^\lambda w$, so that $\widetilde{w}$ satisfies (2.2). But then

$$\begin{aligned} d\widetilde{w} \wedge d\widetilde{z} &= d(e^\lambda w) \wedge d\widetilde{z} \\ &= e^\lambda (dw + w d\lambda) \wedge \left(\tfrac{\partial \widetilde{z}}{\partial \overline{w}} d\overline{w} + \tfrac{\partial \widetilde{z}}{\partial w} dw + \tfrac{\partial \widetilde{z}}{\partial \overline{z}} d\overline{z} + \tfrac{\partial \widetilde{z}}{\partial z} dz\right). \end{aligned}$$

By (2.3), this coincides with $e^\lambda(dw \wedge dz - wB)$, and equating $dw \wedge d\overline{z}$ components we obtain

$$\left(1 + w\tfrac{\partial \lambda}{\partial w}\right)\tfrac{\partial \widetilde{z}}{\partial \overline{z}} - w\tfrac{\partial \lambda}{\partial \overline{z}}\tfrac{\partial \widetilde{z}}{\partial w} = -wB_{w\overline{z}}.$$

Solving for $\frac{\partial \widetilde{z}}{\partial \overline{z}}$ we obtain, near $(0,0)$,

$$\frac{\partial \widetilde{z}}{\partial \overline{z}} = \frac{w\left(\frac{\partial \lambda}{\partial \overline{z}}\frac{\partial \widetilde{z}}{\partial w} - B_{w\overline{z}}\right)}{1 + w\frac{\partial \lambda}{\partial w}}. \tag{2.4}$$

Similarly, equating $dw \wedge d\overline{w}$ components yields, near $(0,0)$,

$$\frac{\partial \widetilde{z}}{\partial \overline{w}} = \frac{w\left(\frac{\partial \lambda}{\partial \overline{w}}\frac{\partial \widetilde{z}}{\partial w} - B_{w\overline{w}}\right)}{1 + w\frac{\partial \lambda}{\partial w}}. \tag{2.5}$$

Finally, (2.4), (2.5) imply that (2.2) holds for $\widetilde{z}$, as required. □

3 Bi-Hermitian approach

Our main tool for describing the geometry of the blow-up will be the bi-Hermitian approach to generalized Kähler geometry [6], which we describe briefly here. Since we are interested in a neighbourhood of a point, we may assume that the torsion 3-form H of our generalized Kähler structure is cohomologically trivial. Such a generalized Kähler structure determines and is determined by a Riemannian metric g, a 2-form b, and a pair of complex structures I_+, I_- which are compatible with g and satisfy the condition

$$\pm\, d^c_\pm \omega_\pm = db, \tag{3.1}$$

where $\omega_\pm = gI_\pm$ are the usual Hermitian 2-forms and $d^c_\pm = [d, I^*_\pm]$ are the real Dolbeault operators associated to $I_\pm$. The correspondence between the generalized Kähler pair $\mathbb{J}_+, \mathbb{J}_-$ and the above bi-Hermitian data is as follows:

$$\mathbb{J}_\pm = \tfrac{1}{2}\begin{pmatrix} 1 & \\ -b & 1 \end{pmatrix}\begin{pmatrix} I_+ \pm I_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(I_+^* \pm I_-^*) \end{pmatrix}\begin{pmatrix} 1 & \\ b & 1 \end{pmatrix}. \tag{3.2}$$

It was observed in [7] that the bi-Hermitian condition endows the complex structure $I_\pm$ with a holomorphic Poisson structure $\sigma_\pm$ with real part

$$Q = \mathrm{Re}(\sigma_+) = \mathrm{Re}(\sigma_-) = \tfrac{1}{8}[I_+, I_-]g^{-1}. \tag{3.3}$$

Indeed, $\sigma_\pm$ derives from a pair of transverse holomorphic Dirac structures as described in [6], though we shall not make use of this here.

Any pair of complex structures satisfies the following identity for the commutator:

$$[I_+, I_-] = (I_+ - I_-)(I_- + I_+). \tag{3.4}$$

Therefore, the zeros of Q coincide with the loci where $I_+ = I_-$ or $I_+ = -I_-$. From (3.2), we see that the real Poisson structures $P_\pm$ underlying $\mathbb{J}_\pm$ are given by

$$P_\pm = -\tfrac{1}{2}(\omega_+^{-1} \mp \omega_-^{-1}) = \tfrac{1}{2}(I_+ \mp I_-)g^{-1}. \tag{3.5}$$

Therefore, we conclude that the zero locus of Q, and hence $\sigma_\pm$, is the union of the zero loci for P_+, P_-, namely the subsets D_+, D_- discussed in section 1.

The holomorphic Poisson structure $(I_\pm, \sigma_\pm)$ provides an economical means to describe the full generalized Kähler structure, as observed in [5].

Theorem 3.1 ([5], Theorem 6.2)**.** *Let (I_0, σ_0) be a holomorphic Poisson structure with $\mathrm{Re}(\sigma_0) = Q$. Any closed 2-form F satisfying the equation*

$$FI_0 + I_0^*F + FQF = 0 \tag{3.6}$$

defines an integrable complex structure $I_1 = I_0 + QF$*, a symmetric tensor* $g = -\frac{1}{2}F(I_0 + I_1)$*, and a 2-form* $b = -\frac{1}{2}F(-I_0 + I_1)$ *such that*

$$d^c_{I_0}\omega_{I_0} = -d^c_{I_1}\omega_{I_1} = db.$$

If g *is positive-definite, then* (g, I_0, I_1) *defines a bi-Hermitian structure satisfying* (3.1)*, and hence a generalized Kähler structure, where* $\mathbb{J}_-$ *is the symplectic structure* F.

As is hinted at in Theorem 3.1, in which g need not be positive-definite, it will be useful in studying the blowup for us to relax the generalized Kähler condition, allowing degenerations of the Riemannian metric while maintaining the remaining constraints.

Definition 3.2. A *degenerate bi-Hermitian* structure (g, b, I_+, I_-) consists of a possibly degenerate tensor $g \in \Gamma^\infty(\mathrm{Sym}^2 T^*)$, a 2-form $b \in \Omega^2$, and two integrable complex structures I_+, I_-, such that $gI_\pm + I_\pm^* g = 0$ and

$$d^c_+\omega_+ = -d^c_-\omega_- = db,$$

where $\omega_\pm = gI_\pm$. Informally, it is a generalized Kähler structure where g may be degenerate.

Degenerate bi-Hermitian structures arising from the construction in Theorem 3.1 as solutions to (3.6) enjoy a composition operation which we now review (see [5] for details).

If F_{01} is a closed 2-form solving

$$F_{01}I_0 + I_0^* F_{01} + F_{01}QF_{01} = 0, \tag{3.7}$$

for a holomorphic Poisson structure (I_0, σ_0) with $\mathrm{Re}(\sigma_0) = Q$, then it determines a second holomorphic Poisson structure (I_1, σ_1) with $\mathrm{Re}(\sigma_1) = Q$, via $I_1 = I_0 + QF_{01}$. If we then have another closed 2-form F_{12}, such that

$$F_{12}I_1 + I_1^* F_{12} + F_{12}QF_{12} = 0, \tag{3.8}$$

then it determines a third holomorphic Poisson structure (I_2, σ_2) with $\mathrm{Re}(\sigma_2) = Q$, via $I_2 = I_1 + QF_{12}$. Rewriting (3.7) and (3.8) as the pair

$$F_{01}I_0 + I_1^* F_{01} = 0, \qquad F_{12}I_1 + I_2^* F_{12} = 0,$$

we see that the closed 2-form $F_{02} = F_{01} + F_{12}$ satisfies

$$\begin{aligned} F_{02}I_0 + I_0^* F_{02} + F_{02}QF_{02} &= F_{02}I_0 + I_2^* F_{02} \\ &= F_{01}(I_2 - I_1) - (I_1^* - I_0^*)F_{12} \\ &= F_{01}QF_{12} - F_{01}QF_{12} = 0. \end{aligned}$$

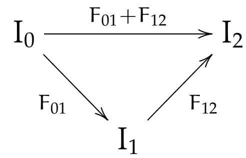


Figure 1: The composition of solutions to (3.7), (3.8).

We may interpret this in the following way: a solution to (3.7) defines a degenerate bi-Hermitian structure with constitutent complex structures (I_0, I_1), and a solution to (3.8) does the same, but with complex structures (I_1, I_2). These two degenerate bi-Hermitian structures may be composed in the sense that the sum $F_{02} = F_{01} + F_{12}$ defines a new degenerate bi-Hermitian structure with constituent complex structures (I_0, I_2). This composition may be viewed as a groupoid (see Figure 1).

Definition 3.3 ([5])**.** Fix a real manifold M with real Poisson structure Q. Then we may define a groupoid whose objects are holomorphic Poisson structures (I_i, σ_i) on M with $\mathrm{Re}(\sigma_i) = Q$ and whose morphisms $\mathrm{Hom}(i, j)$ are real closed 2-forms F_{ij} such that the following two equations hold.

$$I_j - I_i = QF_{ij}$$
$$F_{ij}I_j + I_i^* F_{ij} = 0.$$

The composition of morphisms is then simply addition of 2-forms $F_{ij} + F_{jk}$.

Remark 3.4. Combined with Theorem 3.1, this definition provides a composition operation for the degenerate bi-Hermitian structures determined by the 2-forms F_{ij}.

4 Flow construction

We now review a method, introduced in [8] and developed in [5], for modifying a bi-Hermitian structure of the kind studied in the previous section using a smooth real-valued function. The method proceeds essentially by solving (3.8) using the flow of a suitably-chosen vector field, and then composing this solution with the given bi-Hermitian structure viewed as a solution to (3.7). This is a direct analog of the well-known modification of a Kähler form by adding f to the Kähler potential.

Theorem 4.1 ([8, 5])**.** *Let* (I_0, σ_0) *be a holomorphic Poisson structure with* $Q = \mathrm{Re}(\sigma_0)$*, and let* f *be a smooth real-valued function. Let* φ_t *be the time-*t *flow of the*

Hamiltonian vector field $X = Q(df)$. *Then, so far as the flow is well-defined, the closed 2-form*

$$F_t = \int_0^t \varphi_s^*(dd^c_{I_0} f)ds \tag{4.1}$$

satisfies Equation 3.6, i.e.

$$F_t I_0 + I_0^* F_t + F_t Q F_t = 0.$$

Remark 4.2. The above flow generates a family of integrable complex structures $I_t = I_0 + QF_t$, which are all equivalent, since $I_t = \varphi_t(I_0)$. If f is strictly plurisubharmonic for I_0, i.e. defines a Riemannian metric $h = -(dd^c_{I_0} f)I_0$, then from (4.1) we have

$$\lim_{t\to 0} t^{-1} F_t = dd^c_{I_0} f,$$

implying that the symmetric tensor

$$g_t = -\tfrac{1}{2} F_t(I_0 + I_t)$$

satisfies $\lim_{t\to 0} t^{-1} g_t = h$, so that g_t defines a Riemannian metric for sufficiently small $t \neq 0$, and so by Theorem 4.1, we obtain a generalized Kähler structure (g_t, I_0, I_t, b_t).

5 Generalized Kähler blow-up

We now apply the machinery of the preceding sections to the problem of blowing up the generalized Kähler 4-manifold $(M, \mathbb{J}_+, \mathbb{J}_-)$ introduced in Section 1 at a nondegenerate point $p \in D_+$ in the complex locus of $\mathbb{J}_+$. The first step (§ 5.1) is to blow up the generalized complex structure $\mathbb{J}_+$ and obtain a degenerate bi-Hermitian structure. In the second step (§ 5.2) we deform the degenerate bi-Hermitian structure by composing it with another degenerate bi-Hermitian structure obtained from the flow construction (§ 4). Finally (§ 5.3), we prove that the resulting deformation is positive-definite, defining a generalized Kähler structure on the blow-up.

5.1 Simultaneous blow-up

Lemma 5.1. *In a neighbourhood of the nondegenerate point* $p \in D_+$, *there exist complex coordinates* $(u_\pm, v_\pm)$ *such that the holomorphic Poisson structure* $(I_\pm, \sigma_\pm)$ *is given by* $u_\pm \partial_{u_\pm} \wedge \partial_{v_\pm}$.

Proof. From the normal form for $\mathbb{J}_+$ near p given by Equation 1.1, it follows that P_+ is isomorphic to $\mathrm{Im}(w\partial_w \wedge \partial_z)$. In particular, P_+ vanishes linearly along D_+. By Equations 3.3, 3.4, and 3.5, and since D_- is disjoint from D_+, it follows that $Q = -\frac{1}{2}[I_+, I_-]g^{-1}$ has linear vanishing along D_- as well. This means that the holomorphic Poisson structure $\sigma_\pm$ is a section of a holomorphic line bundle $\wedge^2 T_{1,0}$ with a nondegenerate zero at p. Hence we may choose $I_\pm$-complex coordinates $(u_\pm, v_\pm)$ near p such that $\sigma_\pm = u_\pm \partial_{u_\pm} \wedge \partial_{v_\pm}$, as required. □

We now demonstrate that the coordinates $(u_\pm, v_\pm)$ placing $\sigma_\pm$ into standard form are closely related to the coordinates (w, z) placing $\mathbb{J}_+$ into the standard form 1.1.

Lemma 5.2. *In a sufficiently small neighbourhood* U *of* p *where the following coordinates are defined, the functions* $u_\pm, v_\pm$ *lie in the ideal of* $C^\infty(U, \mathbb{C})$ *generated by* w, z.

Proof. Let ρ_+ be the generator (1.1) defined by $\mathbb{J}_+$ in U, and let $\rho_- = e^\beta$ be the generator defined by $\mathbb{J}_-$, which has symplectic type in U, so that $\beta = B + i\omega$ is a complex 2-form such that ω is symplectic.

The holomorphic Poisson structures $\sigma_\pm = u_\pm \partial_{u_\pm} \wedge \partial_{v_\pm}$ define generalized complex structures in U via the differential forms

$$u_\pm + du_\pm \wedge dv_\pm \in e^{\sigma_\pm} \Omega^{2,0}_\pm.$$

In [6], it is shown that these holomorphic Poisson structures may be expressed as a certain "wedge product" of the underlying generalized complex structures $(\mathbb{J}_+, \mathbb{J}_-)$. Explicitly, this provides the following identities[1]:

$$e^{\overline{\beta}}(w - dw \wedge dz) = e^{\lambda_-}(u_- + du_- \wedge dv_-)$$
$$e^{\beta}(w - dw \wedge dz) = e^{\lambda_+}(u_+ + du_+ \wedge dv_+),$$

for smooth functions $\lambda_+, \lambda_- \in C^\infty(U, \mathbb{C})$. Comparing these equations to (2.3), we see that the argument in the proof of Theorem 2.1 implies the required constraint on $u_\pm, v_\pm$. □

Theorem 5.3. *The complex structures* I_-, I_+ *underlying a generalized Kähler 4-manifold* $(M, \mathbb{J}_+, \mathbb{J}_-)$ *both lift to the blow-up of* $(M, \mathbb{J}_+)$ *at a nondegenerate complex point* $p \in D_+$.

Proof. Let $\psi : U \to \mathbb{C}^2$ be the chart defined by (w, z) in the normal form (1.1) and let $\varphi_\pm : U \to \mathbb{C}^2$ be the chart defined by $(u_\pm, v_\pm)$ in the normal form given by Lemma 5.1 . Then $\chi_\pm = \psi \circ \varphi_\pm^{-1}$ is a diffeomorphism and $\chi_\pm(0) = 0$.

[1]In general, if $\rho_\pm$ generate the canonical line bundles of $\mathbb{J}_\pm$, then $\rho_+^\top \wedge \rho_-$ generates $e^{\sigma_+}\Omega^{n,0}(M, I_+)$ and $\rho_+^\top \wedge \overline{\rho}_-$ generates $e^{\sigma_-}\Omega^{n,0}(M, I_-)$. Here $\rho^\top$ is the reversal anti-automorphism of forms.

The complex structure $I_\pm$ lifts to the blow-up $\widetilde{M}$ precisely when the diffeomorphism $\chi_\pm$ lifts to a diffeomorphism of blow-ups $\widetilde{\chi}_\pm : [\varphi_\pm(U) : 0] \to [\psi(U) : 0]$. This occurs if and only if $u_\pm$ and $v_\pm$ are contained in the ideal generated by w, z, which is itself guaranteed by Lemma 5.2. □

Remark 5.4. It follows from the theorem that the complex structure $\widetilde{I}_\pm$ we obtain on the blow-up of $(M, \mathbb{J}_+)$ may be identified with the usual complex blow-up of $(M, I_\pm)$ at p. Furthermore, since the holomorphic Poisson structure $\sigma_\pm$ vanishes at p, it follows that $\sigma_\pm$ lifts to a holomorphic Poisson structure on the blow-up.

We now apply Theorem 5.3 to obtain a degenerate bi-Hermitian structure on the blow-up of $(M, \mathbb{J}_+, \mathbb{J}_-)$ at $p \in D_+$. Let (g, I_+, I_-, b) be the bi-Hermitian structure on M defined by the generalized Kähler structure.

Corollary 5.5. *Let $(\widetilde{M}, \widetilde{\mathbb{J}}_+)$ be the blow-up of the generalized complex 4-manifold $(M, \mathbb{J}_+)$ at the nondegenerate point $p \in D_+$, with blow-down map π. Then $\widetilde{M}$ inherits a degenerate bi-Hermitian structure $(\widetilde{g}, \widetilde{b}, \widetilde{I}_+, \widetilde{I}_-)$ such that $\pi : (\widetilde{M}, \widetilde{I}_\pm) \to (M, I_\pm)$ is a usual holomorphic blow-down and $\widetilde{g} + \widetilde{b} = \pi^*(g + b)$.*

5.2 Deformation of degenerate bi-Hermitian structure

The degenerate bi-Hermitian structure on $\widetilde{M}$ obtained in Corollary 5.5 fails to define a generalized Kähler structure because $\widetilde{g}$ is not positive-definite along the exceptional divisor E. We now apply Theorem 4.1 to obtain[2] a second degenerate bi-Hermitian structure, which we use to modify $(\widetilde{g}, \widetilde{b}, \widetilde{I}_+, \widetilde{I}_-)$. The modification will leave the structures on $\widetilde{M}$ unchanged outside a tubular neighbourhood V_E of E which blows down to a neighbourhood of p in which $\mathbb{J}_-$ has symplectic type and is given by a complex 2-form with imaginary part ω. Let $\pi : \widetilde{M} \to M$ denote the blow-down map, and write $\widetilde{\omega} = \pi^*\omega$ for the pull-back of the symplectic form to V_E.

First we describe the degenerate bi-Hermitian structure using the formalism of Theorem 3.1. The complex structure $\widetilde{I}_-$ and the 2-form $\widetilde{\omega}$ satisfy (3.6), and so in V_E we have

$$\widetilde{I}_+ = \widetilde{I}_- + \widetilde{Q}\widetilde{\omega},$$

where $\widetilde{Q} = \mathrm{Re}(\widetilde{\sigma}_-) = \mathrm{Re}(\widetilde{\sigma}_+)$, as in (3.3), and $\widetilde{\sigma}_\pm$ is the blown up holomorphic Poisson structure. In the following, we construct a closed 2-form F_t in a possibly smaller tubular neighbourhood such that

$$\widetilde{I}_+^t = \widetilde{I}_+ + \widetilde{Q}F_t$$

[2]The flow construction may be applied equally well to degenerate bi-Hermitian structures.

defines a new complex structure $\widetilde{I}_+^t$. The final task, completed in Section 5.3, will be to show that the composition (5.1), in the sense of Definition 3.3, defines a generalized Kähler structure.

$$\widetilde{I}_- \xrightarrow{\widetilde{\omega}} \widetilde{I}_+ \xrightarrow{F_t} \widetilde{I}_+^t \tag{5.1}$$

We now construct F_t. Let (u, v) be I_+-holomorphic coordinates near p such that $\sigma_+ = u\partial_u \wedge \partial_v$, and let $(u_0, v_0) = (u/v, v)$ and $(u_1, v_1) = (u, v/u)$ be the two affine charts covering a tubular neighbourhood V_E of the exceptional divisor $E = u_1^{-1}(0) \cup v_0^{-1}(0)$. Using u_0, v_1 as affine coordinates on $E \cong \mathbb{C}P^1$, we may describe the Fubini-Study metric ω_E in terms of the Kähler potential

$$f_0 = \log(\tfrac{u_0\overline{u}_0}{1+u_0\overline{u}_0}) = \log(\tfrac{1}{1+v_1\overline{v}_1}),$$

which is smooth away from $u_0 = 0$ and satisfies $i\partial\overline{\partial}f_0 = \omega_E$. Although f_0 is singular, we observe that its Hamiltonian vector field is smooth:

$$\begin{aligned} Q(df_0) &= \mathrm{Re}(u_0\partial_{u_0} \wedge \partial_{v_0})d\log(\tfrac{u_0\overline{u}_0}{1+u_0\overline{u}_0}) \\ &= \tfrac{1}{1+u_0\overline{u}_0}\mathrm{Re}(\partial_{v_0}). \end{aligned}$$

Hence $Q(df_0)$ defines a smooth Poisson vector field on V_E.

Now choose a bump function $\epsilon \in C^\infty(V_E, [0, 1])$ which vanishes on a smaller tubular neighbourhood $U_E \subset V_E$ and is such that $1 - \epsilon$ has compact support in a closed disc bundle K over E, with $U_E \subset K \subset V_E$. Consider the smooth function $f_\epsilon \in C^\infty(V_E, \mathbb{R})$ given by

$$f_\epsilon = \epsilon\log(u\overline{u} + v\overline{v}) = \epsilon\log(v_0\overline{v}_0(1 + u_0\overline{u}_0)).$$

Since $i\partial\overline{\partial}\log(v_0\overline{v}_0(1 + u_0\overline{u}_0)) = i\partial\overline{\partial}\log(1 + u_0\overline{u}_0) = -i\partial\overline{\partial}f_0$, it follows that

$$f = c(f_0 + f_\epsilon), \;\; c \in \mathbb{R}_{>0} \tag{5.2}$$

has the property that $X = Q(df)$ is a smooth Poisson vector field in V_E and

$$i\partial\overline{\partial}f = \begin{cases} c\omega_E & \text{in } U_E \\ 0 & \text{outside } K \end{cases} \tag{5.3}$$

For sufficiently small $\delta > 0$, there exists an open neighbourhood V_E', with $K \subset V_E' \subset V_E$, on which the flow φ_t of X is well-defined for all $t \in (-\delta, \delta)$. Also, choose δ small enough so that there is a neighbourhood V_E'' with $\overline{V_E''} \subset V_E'$, with $\varphi_t(K) \subset V_E''$ for $t \in (-\delta, \delta)$. Using (5.3), we see that $\varphi_t^*(i\partial\overline{\partial}f)$ is smooth on V_E', with compact support contained in V_E'', for all $t \in (-\delta, \delta)$.

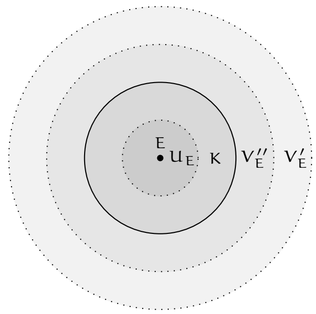


Figure 2: Normal cross-section of neighbourhoods of E contained in V_E

We now apply Theorem 4.1[3] to the flow φ_t on V'_E. This provides a solution

$$F_t = \int_0^t \varphi_s^*(dd^c_{\widetilde{I}_+} f)ds$$

to Equation 3.6 for all $t \in (-\delta, \delta)$, with compact support in V'_E. Therefore, we obtain a family of complex structures on V'_E given by

$$\widetilde{I}^t_+ = \widetilde{I}_+ + QF_t. \tag{5.4}$$

Since F_t has compact support contained in V'_E, the complex structure $\widetilde{I}^t_+$ may be extended to all of $\widetilde{M}$ by setting it equal to $\widetilde{I}_+$ outside V'_E. We summarize the above procedure in the following result.

Proposition 5.6. *The flow construction of Theorem 4.1, applied to the singular function f given in (5.2), produces a smooth family of solutions $(F_t)_{t\in(-\delta,\delta)}$ to (3.6) with compact support in a tubular neighbourhood of the exceptional divisor, and hence we obtain a degenerate bi-Hermitian structure*

$$(\widetilde{g}'_t, \widetilde{b}'_t, \widetilde{I}^t_+, \widetilde{I}_+)$$

on $\widetilde{M}$, where $\widetilde{I}^t_+$ is given by (5.4) and $\widetilde{g}'_t, \widetilde{b}'_t$ are as in Theorem 3.1, yielding

$$\widetilde{g}'_t = -\tfrac{1}{2}F_t(\widetilde{I}_+ + \widetilde{I}^t_+). \tag{5.5}$$

In Section 5.3, we compose the above degenerate bi-Hermitian structure with that from Corollary 5.5 and show the resulting structure is positive-definite.

[3] The fact that f is not smooth does not affect the validity of Theorem 4.1 in this case, as the vector field $X = Q(df)$ is a smooth Poisson vector field, and hence locally Hamiltonian.

Remark 5.7. The family of complex structures $\widetilde{I}^t_+$ on $\widetilde{M}$ constructed above defines a deformation of the blow-up complex structure $\widetilde{I}_+$ in the direction given by the class in $H^1(\mathcal{T})$ defined by the vector field $Z = Q(df)$, which is a holomorphic vector field on the annular neighbourhood of E defined by $V_E \backslash K$. The $(1,0)$ part of Z in this annular neighbourhood is (in the (u_0, v_0) chart)

$$\begin{aligned} Z^{1,0} &= c\widetilde{\sigma}_+(d(\log(\tfrac{u_0\overline{u}_0}{1+u_0\overline{u}_0}) + \log(v_0\overline{v}_0(1+u_0\overline{u}_0))) \\ &= c(u_0\partial_{u_0} \wedge \partial_{v_0})(u_0^{-1}du_0 + v_0^{-1}dv_0) \\ &= c(\partial_{v_0} - \tfrac{u_0}{v_0}\partial_{u_0}). \end{aligned}$$

This deformation class has a geometric interpretation: since $p \in D_+$ and $\sigma_+|_{D_+} = 0$, the contraction

$$\mathrm{Tr}(d\sigma_+|_{D_+})$$

defines a holomorphic vector field χ on D_+. The flow of $c\chi$ then provides a path $p(t)$ of points on D_+. The family of blow-ups of (M, I_+) at $p(t)$ provides a deformation of complex structure with derivative $[Z^{(1,0)}]$ at $t = 0$.

5.3 Positivity

Now that we have constructed the two degenerate bi-Hermitian structures on $\widetilde{M}$ occurring in (5.1), we must argue that their composition in the sense of Definition 3.3 is positive-definite. The composition is the (a priori degenerate) bi-Hermitian structure $(\widetilde{g}_t, \widetilde{b}_t, \widetilde{I}_-, \widetilde{I}^t_+)$, where

$$\begin{aligned} \widetilde{g}_t &= -\tfrac{1}{2}(\widetilde{\omega} + F_t)(\widetilde{I}_- + \widetilde{I}^t_+) \\ \widetilde{b}_t &= -\tfrac{1}{2}(\widetilde{\omega} + F_t)(-\widetilde{I}_- + \widetilde{I}^t_+). \end{aligned}$$

Rewriting this, we obtain

$$\begin{aligned} \widetilde{g}_t &= -\tfrac{1}{2}\left(\widetilde{\omega}(\widetilde{I}_- + \widetilde{I}_+) + \widetilde{\omega}(\widetilde{I}^t_+ - \widetilde{I}_+) + F_t(\widetilde{I}_- - \widetilde{I}_+) + F_t(\widetilde{I}_+ + \widetilde{I}^t_+)\right) \\ &= \widetilde{g} + \widetilde{g}'_t - \tfrac{1}{2}(\widetilde{\omega}\widetilde{Q}F_t - F_t\widetilde{Q}\widetilde{\omega}), \end{aligned} \tag{5.6}$$

where we use the fact that $\widetilde{I}_+ - \widetilde{I}_- = \widetilde{Q}\widetilde{\omega}$ and $\widetilde{I}^t_+ - \widetilde{I}_+ = \widetilde{Q}F_t$.

Theorem 5.8. *Provided that* c *in* (5.2) *is chosen small enough, the symmetric tensor* $\widetilde{g}_t$ *defined by* (5.6) *is positive-definite on* $\widetilde{M}$ *for sufficiently small* $t \neq 0$*, defining a generalized Kähler structure on the blow-up.*

Proof. Since $F_t \to 0$ as $t \to 0$, it follows that $\widetilde{I}^t_+ \to \widetilde{I}_+$ as $t \to 0$. By Equation 5.5, therefore, we see that

$$\lim_{t\to 0} \tfrac{1}{t}\widetilde{g}'_t = -(dd^c_{\widetilde{I}_+} f)(\widetilde{I}_+) = \begin{cases} c\omega_E & \text{in } U_E \\ 0 & \text{outside } K \end{cases}$$

where ω_E is the Fubini-Study metric. This implies that $\widetilde{g}'_t$ is positive-definite when restricted to TE for sufficiently small nonzero t, and hence $\widetilde{g} + \widetilde{g}'_t$ is positive-definite in a neighbourhood of E for sufficiently small nonzero t. Also, the third summand in (5.6) is proportional to $\widetilde{\omega}$, which vanishes along E.

Fix $c = c_0 \in \mathbb{R}_{>0}$ in the definition (5.2) of f, and let $U \subset U_E$ be a tubular neighbourhood of E where the third summand in (5.6) is so small that $\widetilde{g}_t$ is positive-definite in U for sufficiently small nonzero t. Note that $\widetilde{g}_t$ is certainly positive-definite outside K (where it coincides with $\widetilde{g}$), hence it remains to show that $\widetilde{g}_t$ is positive in the intermediate region $K\backslash U$.

We have chosen U so that the third term in (5.6) is dominated there by the first two terms. This means that at each point in U and for each vector $v \neq 0$ (and for suficiently small nonzero t), we have

$$\begin{aligned} |\widetilde{Q}(F_t v, \widetilde{\omega} v)| &< \widetilde{g}(v,v) + \widetilde{g}'_t(v,v) \\ &= \widetilde{g}(v,v) - \tfrac{1}{2} F_t((\widetilde{I}_+ + \widetilde{I}^t_+)v, v) \\ &= \widetilde{g}(v,v) - F_t(\widetilde{I}_+ v, v) - \tfrac{1}{2}\widetilde{Q}(F_t v, F_t v) \\ &= \widetilde{g}(v,v) - F_t(\widetilde{I}_+ v, v). \end{aligned} \tag{5.7}$$

Since (5.7) holds for $c = c_0$, it will also hold in U for $c = \lambda c_0$, for any $\lambda \in (0,1)$, since for $x, y \in \mathbb{R}_{\geqslant 0}$ and $z \in \mathbb{R}$, we have the implication

$$(x < y + z) \Rightarrow (\lambda x < \lambda(y+z) \leqslant y + \lambda z)\,.$$

Therefore we have shown positivity of $\widetilde{g}_t$ in U for any $0 < c \leqslant c_0$, for sufficiently small nonzero t.

Now observe that the first term of (5.6), i.e. $\widetilde{g}$, is positive-definite on $K\backslash U$ and independent of c, whereas the second and third terms are each proportional to c. Hence by choosing $c \neq 0$ sufficiently small, we ensure that $\widetilde{g}_t$ is positive-definite on $K\backslash U$, in addition to U and outside K, for sufficiently small $t \neq 0$. This completes the proof. □

6 Examples

By the work of Goto [4], we know that the choice of a holomorphic Poisson structure on a compact Kähler manifold gives rise to a family of generalized Kähler structures deforming the initial Kähler structure. In this way, one obtains nontrivial generalized Kähler structures on any compact Kähler surface with effective anti-canonical divisor D. Performing a Kähler blow-up of such a surface at a point lying on D, we obtain a new Kähler surface with effective anti-canonical divisor given by the proper transform of D. Hence we may apply the Goto deformation and obtain a generalized Kähler structure on the blow-up. We believe that our construction gives an explicit realization of Goto's existence result in this case, as evidenced by Remark 5.7.

In the non-algebraic case, or for noncompact surfaces, our construction provides new generalized Kähler structures. For example, a result of Apostolov [1] states that for surfaces with odd first Betti number, a bi-Hermitian structure which is not strongly bi-Hermitian may only exist on blow-ups of minimal class VII surfaces with curves. If the minimal surface has a generalized Kähler structure, therefore, we may employ our result to obtain structures on the appropriate blow-ups.

Example 6.1 (Diagonal Hopf surfaces)**.** $X = S^3 \times S^1$ admits a family of generalized Kähler structures with bi-Hermitian structure (g, I_+, I_-) given by viewing X as a Lie group, taking g to be a bi-invariant metric, and (I_+, I_-) to be left and right-invariant complex structures compatible with g (see [6] for details). In these examples, D_+ and D_- are nonempty disjoint curves which sum to the anti-canonical divisor. We may therefore blow up any number of points lying on $D_+ \cup D_-$ and obtain generalized Kähler structures on these manifolds, which are diffeomorphic to $(S^3 \times S^1)\#k\overline{\mathbb{C}P^2}$. This provides another construction of bi-Hermitian structures on non-minimal Hopf surfaces, besides those discovered in [11, 9].

In a remarkable recent work [3], Fujiki and Pontecorvo obtained bi-Hermitian structures on hyperbolic and parabolic Inoue surfaces as well as Hopf surfaces, by carefully studying the twistor space of the underlying conformal 4-manifold. They then obtained bi-Hermitian structures when these surfaces are properly blown up, meaning that the surface is blown up at nodal singularities of the anti-canonical divisor. Finally, they obtained bi-Hermitian structures on a family of deformations of such blowups. We may of course blow up their minimal examples at smooth points of the anti-canonical divisor, using our procedure. It remains to determine how the various bi-Hermitian structures now known on $(S^3 \times S^1)\#k\overline{\mathbb{C}P^2}$ are related.

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