A SURGERY FOR GENERALIZED COMPLEX STRUCTURES ON 4-MANIFOLDS

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Abstract. We introduce a surgery for generalized complex manifolds whose input is a symplectic 4-manifold containing a symplectic 2-torus with trivial normal bundle and whose output is a 4-manifold endowed with a generalized complex structure exhibiting type change along a 2-torus. Performing this surgery on a $K3$ surface, we obtain a generalized complex structure on $3\mathbb{C}P^2\#19\overline{\mathbb{C}P^2}$, which has vanishing Seiberg–Witten invariants and hence does not admit complex or symplectic structure.

Introduction

Generalized complex structures, introduced by Hitchin [3] and developed by the second author in [2], are a simultaneous generalization of complex and symplectic structures. In this paper we answer, in the affirmative, the question of whether there exist manifolds which are neither complex nor symplectic yet do admit generalized complex structure.

Since generalized complex manifolds must be almost complex, this question becomes nontrivial first in dimension 4, where we are fortunate to have obstructions to the existence of complex and symplectic structures coming from Seiberg–Witten theory. For example, a simply-connected complex or symplectic 4-manifold with $b_+ \geq 3$ must have a nonzero Seiberg–Witten invariant [7].

Each tangent space of a generalized complex manifold has a distinguished subspace equipped with a symplectic form and transverse complex structure; the transverse complex dimension is called the *type*, a local invariant of the geometry which may vary along the manifold. We show that in 4 dimensions, a connected and nondegenerate type change locus must be a smooth 2-torus, which also inherits a complex structure, i.e. it must be a nonsingular elliptic curve.

We then introduce a surgery for generalized complex manifolds which is a particular case of the C^∞ *logarithmic transformation* introduced by Gompf and Mrowka [1]. This surgery modifies a neighbourhood of a symplectic 2-torus with trivial normal bundle in a symplectic 4-manifold, producing a new manifold endowed with a generalized complex structure with type change along a 2-torus. Performing this surgery along a fiber of an elliptic $K3$ surface, we obtain a generalized complex structure on $3\mathbb{C}P^2\#19\overline{\mathbb{C}P^2}$, a manifold with vanishing Seiberg–Witten invariants [9, 7].

We thank Tomasz Mrowka for advice which led to our final example. We also thank Nigel Hitchin for helpful conversations.

1. Generalized complex structures

In this section we recall the definition and basic examples of generalized complex structures, following [2].

Given a closed 3-form H on a manifold M, we define the *Courant bracket* of sections of the sum $T \oplus T^*$ of the tangent and cotangent bundles by

$$[X+\xi, Y+\eta]_H = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(\eta(X) - \xi(Y)) + i_Y i_X H.$$

The bundle $T \oplus T^*$ is also endowed with a natural symmetric pairing of signature (n, n):

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\eta(X) + \xi(Y)).$$

Definition. A *generalized complex structure* on a manifold with closed 3-form (M, H) is a complex structure on the bundle $T \oplus T^*$ which preserves the natural pairing and whose $+i$-eigenspace is closed under the Courant bracket.

A generalized complex structure can be fully described in terms of its $+i$-eigenspace L, which is a maximal isotropic subspace of $T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$ satisfying $L \cap \overline{L} = \{0\}$. Alternatively, it can be described using differential forms. Recall that the exterior algebra $\wedge^\bullet T^*$ carries a natural spin representation for the metric bundle $T \oplus T^*$; the Clifford action of $X + \xi \in T \oplus T^*$ on $\rho \in \wedge^\bullet T^*$ is

$$(X+\xi)\cdot\rho = i_X\rho + \xi\wedge\rho.$$

The subspace $L \subset T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$ annihilating a spinor $\rho \in \wedge^\bullet T^*_{\mathbb{C}}$ is always isotropic. If L is maximal isotropic, then ρ is called a *pure spinor* and it must have the following algebraic form at every point:

$$\rho = e^{B+i\omega}\wedge\Omega, \tag{1.1}$$

where B and ω are real 2-forms and Ω is a decomposable complex form. Pure spinors annihilating the same space must be equal up to rescaling, hence a maximal isotropic $L \subset T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$ may be uniquely described by a line bundle $K \subset \wedge^\bullet T^*_{\mathbb{C}}$.

Definition. Given a generalized complex structure $\mathcal{J}$, the line bundle $K \subset \wedge^\bullet T^*_{\mathbb{C}}$ annihilating its $+i$-eigenspace is the *canonical bundle* of $\mathcal{J}$.

Note that the condition $L \cap \overline{L} = \{0\}$ at E over a point $p \in M$, is equivalent to the requirement that

$$\Omega\wedge\overline{\Omega}\wedge\omega^{n-k} \neq 0 \tag{1.2}$$

for any generator $\rho = e^{B+i\omega}\wedge\Omega$ of K over p, where $k = \deg(\Omega)$ and $2n = \dim(M)$. Therefore at each point of a generalized complex manifold, $\ker\Omega\wedge\overline{\Omega}$ is a subspace of the real tangent space with induced symplectic structure and transverse complex structure.

Definition. Let $\mathcal{J}$ be a generalized complex structure and $e^{B+i\omega}\wedge\Omega$ a generator of its canonical bundle at a point p. The *type* of $\mathcal{J}$ at p is the degree of Ω.

We remark that while the type of a generalized complex structure may jump along loci in the manifold, its parity must remain constant on connected components of M (see [2]).

Finally, the Courant integrability of L is equivalent to the requirement that, for any local generator $\rho \in C^\infty(K)$, one has

$$d\rho + H \wedge \rho = v \cdot \rho \tag{1.3}$$

for some section $v \in C^\infty(T_{\mathbb{C}} \oplus T^*_{\mathbb{C}})$. In summary, a generalized complex structure may be specified by a line sub-bundle $K \subset \wedge^\bullet T^*_{\mathbb{C}}$ whose local generators satisfy (1.1), (1.2) and (1.3).

Example 1.1. Let (M^{2n}, I) be a complex manifold. Then the following operator on $T \oplus T^*$ is a generalized complex structure:

$$\mathcal{J}_I = \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}$$

The $+i$-eigenspace of $\mathcal{J}_I$ is $T^{0,1} \oplus T^{*1,0}$, which annihilates the canonical bundle $K = \wedge^{n,0}T^*$ and is therefore of type n.

Example 1.2. Let (M, ω) be a symplectic manifold. Then

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

is a generalized complex structure with $+i$-eigenspace $\{X - i\omega(X) : X \in T_{\mathbb{C}}M\}$ and canonical bundle generated by the differential form $e^{i\omega}$. Symplectic structures, therefore, have type zero.

Example 1.3. A real closed 2-form B gives rise to an orthogonal transformation of $T \oplus T^*$ via $X + \xi \mapsto X + \xi + i_X B$. This transformation, called a *B-field transform*, preserves the Courant bracket, and hence it acts by conjugation on any given generalized complex structure $\mathcal{J}$ on M, producing a new one. The induced action on the canonical bundle is simply $K \mapsto e^B \wedge K$.

If B is not closed, then it induces an isomophism between the H-Courant bracket and the $H+dB$-Courant bracket. In particular, if $[H] = 0 \in H^3(M, \mathbb{R})$, the bracket $[,]_H$ is isomorphic to $[,]_0$ by the action of a nonclosed 2-form.

In the next example, we demonstrate that the type of a generalized complex structure may not be constant; it jumps from type 0 to type 2 along a codimension 2 submanifold.

Example 1.4. *(Local model)* Consider $\mathbb{C}^2$ with complex coordinates z_1, z_2. The differential form

$$\rho = z_1 + dz_1 \wedge dz_2$$

is equal to $dz_1 \wedge dz_2$ along the locus $z_1 = 0$, while away from this locus it can be written as

$$\rho = z_1 \exp\left(\tfrac{dz_1 \wedge dz_2}{z_1}\right). \tag{1.4}$$

Since it also satisfies $d\rho = -\partial_2 \cdot \rho$, we see that it generates a canonical bundle K for a generalized complex structure which has type 2 along $z_1 = 0$ and type 0 elsewhere.

Observe that this structure is invariant under translations in the z_2 direction, hence we can take a quotient by the standard $\mathbb{Z}^2$ action to obtain a generalized complex structure on the torus fibration $D^2 \times T^2$, where D^2 is the unit disc in the

z_1-plane. Using polar coordinates, $z_1 = re^{2\pi i\theta_1}$, the canonical bundle is generated, away from the central fibre, by

$$\begin{aligned}\exp(B + i\omega) &= \exp(d\log r + id\theta_1) \wedge (d\theta_2 + id\theta_3)\\ &= \exp(d\log r \wedge d\theta_2 - d\theta_1 \wedge d\theta_3 + i(d\log r \wedge d\theta_3 + d\theta_1 \wedge d\theta_2)),\end{aligned}$$

where θ_2 and θ_3 are coordinates for the 2-torus with unit periods. Away from $r = 0$, therefore, the structure is a B-field transform of a symplectic structure ω, where

$$\begin{aligned}B &= d\log r \wedge d\theta_2 - d\theta_1 \wedge d\theta_3\\ \omega &= d\log r \wedge d\theta_3 + d\theta_1 \wedge d\theta_2.\end{aligned} \tag{1.5}$$

The type jumps from 0 to 2 along the central fibre $r = 0$, inducing a complex structure on the restricted tangent bundle, for which the tangent bundle to the fibre is a complex sub-bundle. Hence the type change locus inherits the structure of a smooth elliptic curve with Teichmüller parameter $\tau = i$.

Example 1.5. Endow $D^2 \times T^2$ with the generalized complex structure of Example 1.4 and consider the action of $\mathbb{Z}_m$ given in polar coordinates by

$$(r, \theta_1, \theta_2, \theta_3) \mapsto (r, \theta_1 + 1/m, \theta_2 + k/m, \theta_3),$$

where k is co-prime with m. This action extends to the fiber over $r = 0$, has no fixed points and preserves the generalized complex structure. Hence the quotient, which is a singular T^2 fibration with multiple central fibre, has a generalized complex structure. Away from the central fibre, the coordinates $(r', \theta_1', \theta_2', \theta_3') = (r^m, m\theta_1, \theta_2 - k\theta_1, \theta_3)$ are well-defined, and the generalized complex structure is generated by $\exp(B + i\omega)$, where

$$\begin{aligned}B &= d\log r' \wedge (d\theta_2' + \tfrac{k}{m} d\theta_1') - \tfrac{1}{m} d\theta_1' \wedge d\theta_3'\\ \omega &= \tfrac{1}{m}(d\log r' \wedge d\theta_3' + d\theta_1' \wedge d\theta_2').\end{aligned} \tag{1.6}$$

Note that the symplectic form is a rescaling of that in equation (1.5). As in the previous example, the central fibre obtains a complex structure.

2. The type-changing locus

In the last two examples, the type of the generalized complex structure jumped from 0 to 2 along a 2-torus, which then inherited a complex structure. We now show that this happens generically in four dimensions.

Recall that a generalized complex manifold has a canonical bundle $K \subset \wedge^\bullet T^*_{\mathbb{C}}$, so the projection from $\wedge^\bullet T^*_{\mathbb{C}}$ onto $\wedge^0 T^*_{\mathbb{C}} = \mathbb{C}$ determines a canonical section s of K^*. For a 4-dimensional manifold, the type of a generalized complex structure jumps from 0 to 2 precisely when this section vanishes.

Definition. A point p in the type-changing locus of a generalized complex structure on a 4-manifold is *nondegenerate* if it is a nondegenerate zero of $s \in C^\infty(K^*)$.

Theorem 2.1. *The following hold for a 4-dimensional generalized complex manifold:*

1. *A nondegenerate point in the type-changing locus has a neighbourhood in which the type changes along a smooth 2-manifold with induced complex structure.*
2. *A compact connected component of the type-changing locus whose points are nondegenerate must be a smooth elliptic curve.*

Proof. To prove the first claim, let $\rho = \rho_0 + \rho_2 + \rho_4$, with $\deg(\rho_i) = i$, be a nonvanishing local section of K around a type-changing point p. Then $s(\rho) = \rho_0$ and nondegeneracy implies that $d\rho_0 : T_pM \longrightarrow \mathbb{C}$ is onto. The implicit function theorem implies that the zeros of ρ_0 near p form a 2 dimensional manifold. According to equations (1.1) and (1.2), ρ_2 induces a complex structure on T_pM for which it generates the canonical line $\wedge^{2,0}T^*_pM$. The integrability condition (1.3) states that

$$d\rho_0 = i_X\rho_2 \in T^{*1,0}_pM,$$

for some $X \in C^\infty(T_{\mathbb{C}}M)$. Therefore $d\rho_0(T^{0,1}_pM) = 0$, showing that the zero set of ρ_0 has a complex structure.

To prove (2), let $\Sigma \longrightarrow M$ be a compact connected component of the type-changing locus with its induced complex structure. Then, since $ds \in C^\infty(T^*_{\mathbb{C}}M|_\Sigma \otimes K^*|_\Sigma)$ vanishes on vectors tangent to Σ, nondegeneracy implies that ds is a nowhere vanishing section of $N^* \otimes K|^*_\Sigma$, where N^* is the conormal bundle. In particular, $N^* \cong K|_\Sigma$. Since $\mathcal{J}$ is complex over Σ, we have an adjunction formula relating the canonical bundle K_Σ of the complex curve Σ, with the canonical bundle K restricted to Σ:

$$K|_\Sigma \cong K_\Sigma \otimes N^*,$$

showing that K_Σ is trivial and Σ is an elliptic curve, as required. □

3. The surgery

In this section we introduce a surgery for 4-manifolds with generalized complex structure which removes a neighborhood of a symplectic 2-torus and replaces it by a neighborhood of a torus where the generalized complex structure changes type, as in Example 1.4. This surgery is an example of a C^∞ *logarithmic transformation* as defined by Gompf and Mrowka [1], which we now recall.

Let $T \hookrightarrow M$ be a 2-torus with trivial normal bundle in a 4-manifold, and let $U \cong D^2 \times T^2$ be a tubular neighborhood. A C^∞ *logarithmic transform* of M is a manifold $\tilde{M}$ obtained by removing U and replacing it with $D^2 \times T^2$, glued in by a diffeomorphism $\psi : S^1 \times T^2 \longrightarrow \partial U$:

$$\tilde{M} = (M\backslash U) \cup_\psi (D^2 \times T^2).$$

The *multiplicity* of this transformation is the degree of the map $\pi\circ\psi : S^1 \times point \longrightarrow \partial D^2$, where $\pi : U \longrightarrow D^2$ is the first projection.

Theorem 3.1. *Let (M, σ) be a symplectic 4-manifold, $T \hookrightarrow M$ be a symplectic 2-torus with trivial normal bundle and tubular neighbourhood U. Let $\psi : S^1 \times T^2 \longrightarrow \partial U \cong S^1 \times T^2$ be the map given on standard coordinates by*

$$\psi(\theta_1, \theta_2, \theta_3) = (\theta_3, \theta_2, -\theta_1).$$

Then the multiplicity zero C^∞ logarithmic transform of M along T,

$$\tilde{M} = M\backslash U \cup_\psi D^2 \times T^2,$$

admits a generalized complex structure with type change along a 2-torus, and which is integrable with respect to a 3-form H, such that $[H]$ is the Poincaré dual to the circle in $S^1 \times T^2$ preserved by ψ. If M is simply connected and $[T] \in H^2(M, \mathbb{Z})$ is k times a primitive class, then $\pi_1(\tilde{M}) = \mathbb{Z}_k$.

Proof. By Weinstein's neighbourhood theorem [8], the neighborhood U is symplectomorphic to $D^2 \times T^2$ with standard sympletic form:

$$\sigma = \frac{1}{2} d\tilde{r}^2 \wedge d\tilde{\theta}_1 + d\tilde{\theta}_2 \wedge d\tilde{\theta}_3.$$

Now consider the symplectic form ω on $D^2 \backslash \{0\} \times T^2$ from example 1.4:

$$\omega = d \log r \wedge d\theta_3 + d\theta_1 \wedge d\theta_2.$$

The map $\psi : (D^2 \backslash D^2_{1/\sqrt{e}} \times T^2, \omega) \longrightarrow (D^2 \backslash \{0\} \times T^2, \sigma)$ given by

$$\psi(r, \theta_1, \theta_2, \theta_3) = (\sqrt{\log er^2}, \theta_3, \theta_2, -\theta_1)$$

is a symplectomorphism.

Let B be the 2-form defined by (1.5) on $D^2 \backslash \{0\} \times T^2$, and choose an extension $\tilde{B}$ of $\psi^{-1^*} B$ to $M \backslash T$. Therefore $(M \backslash T, \tilde{B} + i\sigma)$ is a generalized complex manifold of type 0, integrable with respect to the $d\tilde{B}$-Courant bracket.

Now the surgery $\tilde{M} = M \backslash T \cup_\psi D^2 \times T^2$ obtains a generalized complex structure since the gluing map ψ satisfies $\psi^*(\tilde{B} + i\sigma) = B + i\omega$, and this generalized complex structure exhibits type change along the 2-torus coming from the central fibre of $D^2 \times T^2$. This structure is integrable with respect to $H = d\tilde{B}$, which is a globally defined closed 3-form on $\tilde{M}$.

The 2-form $\tilde{B}$ can be chosen so that it vanishes outside a larger tubular neighbourhood U' of T, so that $H = d\tilde{B}$ has support in $U' \backslash U$ and has the form

$$H = f'(r) dr \wedge d\theta_1 \wedge d\theta_3,$$

for a smooth bump function f such that $f|_U = 1$ and vanishes outside U'. Therefore, we see that H represents the Poincaré dual of the circle parametrized by θ_2, as required.

The last claim is a consequence of van Kampen's theorem and that $H^2(M, \mathbb{Z})$ is spherical, as M is simply connected. □

Corollary 1. Since $\tilde{B}$ can be chosen to have support in a neighbourhood of the symplectic 2-torus T, the surgery above may be performed simultaneously on a collection of disjoint symplectic 2-tori in M.

Observe that the crucial property of the type-changing generalized complex structure on $D^2 \times T^2$ which allows us to perform the surgery is the behaviour of its symplectic form. As we saw, this is the same symplectic form, up to rescaling, as in Example 1.5. Hence we could, alternatively, use the generalized complex structure on $(D^2 \times T^2)/\mathbb{Z}_m$ described there as a model for the piece being glued in.

4. EXAMPLES

Example 4.1. Consider a symplectic 4-manifold $M = \Sigma \times T^2$, where Σ is a symplectic surface and T^2 a symplectic 2-torus. Performing the surgery from Theorem 3.1 along one of the T^2 fibers, we obtain a type-changing generalized complex structure on $X^3 \times S^1$, where X^3 is the *twisted connected sum* of the S^1-bundles $\Sigma \times S^1$ and $S^2 \times S^1$, in the language of [5].

For example, if $\Sigma = S^2$, we obtain a generalized complex structure on $S^3 \times S^1$, integrable with respect to a 3-form H representing a generator for $H^3(S^3, \mathbb{Z})$. Note that this manifold does not admit symplectic structure.

In the final example, we produce a generalized complex 4-manifold which admits neither symplectic nor complex structure. The non-existence of symplectic structure follows from a result in Seiberg–Witten theory.

Example 4.2. (Generalized complex structure on $3\mathbb{C}P^2\#19\overline{\mathbb{C}P^2}$) Consider an elliptically fibred $K3$ surface M. Any smooth elliptic fibre is a symplectic 2-torus with respect to a Kähler symplectic form, and has trivial normal bundle. Therefore we may perform our surgery along such a fiber to obtain a generalized complex manifold.

The effect of C^∞ logarithmic transformations on the $K3$ surface was studied by Gompf and Mrowka in [1]. Using a trick of Moishezon [6], they show that the differentiable manifold obtained through a transformation of multiplicity zero is $\tilde{M} = 3\mathbb{C}P^2\#19\overline{\mathbb{C}P^2}$. Since $H^3(\tilde{M}) = \{0\}$, the generalized complex structure on $\tilde{M}$ given by Theorem 3.1 has $[H] = 0$.

Since $3\mathbb{C}P^2\#19\overline{\mathbb{C}P^2}$ can be expressed as a connected sum of terms whose intersection forms are not negative definite, its Seiberg–Witten invariants vanish [9]. Therefore Taubes' theorem implies it does not admit symplectic structure [7]. This, in turn, obstructs the existence of a complex structure, since Kodaira's theorem [4] states that any complex surface with even first Betti number is a deformation of an algebraic surface.

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