# CONTENTS

## Chapter I  Manifolds and Vector Bundles  

1. Manifolds  
2. Vector Bundles  
3. Almost Complex Manifolds and the $\bar{\partial}$-Operator 

## Chapter II  Sheaf Theory  

1. Presheaves and Sheaves  
2. Resolutions of Sheaves  
3. Cohomology Theory  
4. Čech Cohomology with Coefficients in a Sheaf 

## Chapter III  Differential Geometry  

1. Hermitian Differential Geometry  
2. The Canonical Connection and Curvature of a Hermitian Holomorphic Vector Bundle  
3. Chern Classes of Differentiable Vector Bundles  
4. Complex Line Bundles  

---

**xi**
## Contents

### Chapter IV  Elliptic Operator Theory  108

1. Sobolev Spaces  108  
2. Differential Operators  113  
3. Pseudodifferential Operators  119  
5. Elliptic Complexes  144  

### Chapter V  Compact Complex Manifolds  154

1. Hermitian Exterior Algebra on a Hermitian Vector Space  154  
2. Harmonic Theory on Compact Manifolds  163  
3. Representations of $\mathfrak{sl}(2, \mathbb{C})$ on Hermitian Exterior Algebras  170  
4. Differential Operators on a Kähler Manifold  188  
5. The Hodge Decomposition Theorem on Compact Kähler Manifolds  197  
6. The Hodge-Riemann Bilinear Relations on a Kähler Manifold  201  

### Chapter VI  Kodaira’s Projective Embedding Theorem  217

1. Hodge Manifolds  217  
2. Kodaira’s Vanishing Theorem  222  
3. Quadratic Transformations  229  
4. Kodaira’s Embedding Theorem  234  

### Appendix (by Oscar García-Prada)  
**Moduli Spaces and Geometric Structures**  241

1. Introduction  241  
2. Vector Bundles on Riemann Surfaces  243  
3. Higgs Bundles on Riemann Surfaces  253  
4. Representations of the Fundamental Group  258  
5. Non-abelian Hodge Theory  261  
6. Representations in $\mathbf{U}(p, q)$ and Higgs Bundles  265
# Contents

7. Moment Maps and Geometry of Moduli Spaces 269  
8. Higher Dimensional Generalizations 276

<table>
<thead>
<tr>
<th>References</th>
<th>284</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author Index</td>
<td>291</td>
</tr>
<tr>
<td>Subject Index</td>
<td>293</td>
</tr>
</tbody>
</table>
There are many classes of manifolds which are under rather intense investigation in various fields of mathematics and from various points of view. In this book we are primarily interested in differentiable manifolds and complex manifolds. We want to study (a) the “geometry” of manifolds, (b) the analysis of functions (or more general objects) which are defined on manifolds, and (c) the interaction of (a) and (b). Our basic interest will be the application of techniques of real analysis (such as differential geometry and differential equations) to problems arising in the study of complex manifolds. In this chapter we shall summarize some of the basic definitions and results (including various examples) of the elementary theory of manifolds and vector bundles. We shall mention some nontrivial embedding theorems for differentiable and real-analytic manifolds as motivation for Kodaira’s characterization of projective algebraic manifolds, one of the principal results which will be proved in this book (see Chap. VI). The “geometry” of a manifold is, from our point of view, represented by the behavior of the tangent bundle of a given manifold. In Sec. 2 we shall develop the concept of the tangent bundle (and derived bundles) from, more or less, first principles. We shall also discuss the continuous and $C^\infty$ classification of vector bundles, which we shall not use in any real sense but which we shall meet a version of in Chap. III, when we study Chern classes. In Sec. 3 we shall introduce almost complex structures and the calculus of differential forms of type $(p, q)$, including a discussion of integrability and the Newlander-Nirenberg theorem.

General background references for the material in this chapter are Bishop and Crittenden [1], Lang [1], Narasimhan [1], and Spivak [1], to name a few relatively recent texts. More specific references are given in the individual sections. The classical reference for calculus on manifolds is de Rham [1]. Such concepts as differential forms on differentiable manifolds, integration on chains, orientation, Stokes’ theorem, and partition of unity are all covered adequately in the above references, as well as elsewhere, and in this book we shall assume familiarity with these concepts, although we may review some specific concept in a given context.
1. Manifolds

We shall begin this section with some basic definitions in which we shall use the following standard notations. Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively, with their usual topologies, and let $K$ denote either of these fields. If $D$ is an open subset of $K^n$, we shall be concerned with the following function spaces on $D$:

(a) $K = \mathbb{R}$:

1. $\mathcal{E}(D)$ will denote the real-valued \textit{indefinitely differentiable} functions on $D$, which we shall simply call $C^\infty$ functions on $D$; i.e., $f \in \mathcal{E}(D)$ if and only if $f$ is a real-valued function such that partial derivatives of all orders exist and are continuous at all points of $D$ [$\mathcal{E}(D)$ is often denoted by $C^\infty(D)$].

2. $\mathcal{A}(D)$ will denote the real-valued \textit{real-analytic} functions on $D$; i.e., $\mathcal{A}(D) \subset \mathcal{E}(D)$, and $f \in \mathcal{A}(D)$ if and only if the Taylor expansion of $f$ converges to $f$ in a neighborhood of any point of $D$.

(b) $K = \mathbb{C}$:

1. $\mathcal{O}(D)$ will denote the complex-valued \textit{holomorphic} functions on $D$, i.e., if $(z_1, \ldots, z_n)$ are coordinates in $\mathbb{C}^n$, then $f \in \mathcal{O}(D)$ if and only if near each point $z^0 \in D$, $f$ can be represented by a convergent power series of the form

$$f(z) = f(z_1, \ldots, z_n) = \sum_{a_1, \ldots, a_n = 0}^{\infty} a_{a_1, \ldots, a_n} \left( z_1 - z^0_1 \right)^{a_1} \cdots \left( z_n - z^0_n \right)^{a_n}.$$  

(See, e.g., Gunning and Rossi [1], Chap. I, or Hörmander [2], Chap. II, for the elementary properties of holomorphic functions on an open set in $\mathbb{C}^n$).

These particular classes of functions will be used to define the particular classes of manifolds that we shall be interested in.

A \textit{topological $n$-manifold} is a Hausdorff topological space with a countable basis† which is locally homeomorphic to an open subset of $\mathbb{R}^n$. The integer $n$ is called the \textit{topological dimension} of the manifold. Suppose that $\mathcal{S}$ is one of the three $K$-valued families of functions defined on the open subsets of $K^n$ described above, where we let $\mathcal{S}(D)$ denote the functions of $\mathcal{S}$ defined on $D$, an open set in $K^n$. [That is, $\mathcal{S}(D)$ is either $\mathcal{E}(D)$, $\mathcal{A}(D)$, or $\mathcal{O}(D)$]. We shall only consider these three examples in this chapter. The concept of a family of functions is formalized by the notion of a \textit{presheaf} in Chap. II.

\textbf{Definition 1.1:} An $\mathcal{S}$-\textit{structure}, $\mathcal{S}_M$, on a $K$-manifold $M$ is a family of $K$-valued continuous functions defined on the open sets of $M$ such that

†The additional assumption of a countable basis (“countable at infinity”) is important for doing analysis on manifolds, and we incorporate it into the definition, as we are less interested in this book in the larger class of manifolds.
(a) For every \( p \in M \), there exists an open neighborhood \( U \) of \( p \) and a homeomorphism \( h : U \to U' \), where \( U' \) is open in \( K^n \), such that for any open set \( V \subset U \)

\[
f : V \to K \in S_M \text{ if and only if } f \circ h^{-1} \in S(h(V)).
\]

(b) If \( f : U \to K \), where \( U = \bigcup_i U_i \) and \( U_i \) is open in \( M \), then \( f \in S_M \) if and only if \( f|_{U_i} \in S_M \) for each \( i \).

It follows clearly from (a) that if \( K = \mathbb{R} \), the dimension, \( k \), of the topological manifold is equal to \( n \), and if \( K = \mathbb{C} \), then \( k = 2n \). In either case \( n \) will be called the \textit{K-dimension} of \( M \), denoted by \( \dim_K M = n \) (which we shall call \textit{real-dimension} and \textit{complex-dimension}, respectively). A manifold with an \( S \)-structure is called an \textit{S-manifold}, denoted by \((M, S_M)\), and the elements of \( S_M \) are called \textit{S-functions} on \( M \). An open subset \( U \subset M \) and a homeomorphism \( h : U \to U' \subset K^n \) as in (a) above is called an \textit{S-coordinate system}.

For our three classes of functions we have defined

(a) \( S = \mathcal{E} \): differentiable (or \( C^\infty \)) manifold, and the functions in \( \mathcal{E}_M \) are called \( C^\infty \) functions on open subsets of \( M \).

(b) \( S = \mathcal{A} \): real-analytic manifold, and the functions in \( \mathcal{A}_M \) are called real-analytic functions on open subsets of \( M \).

(c) \( S = \mathcal{O} \): complex-analytic (or simply complex) manifold, and the functions in \( \mathcal{O}_M \) are called holomorphic (or complex-analytic functions) on open subsets of \( M \).

We shall refer to \( \mathcal{E}_M, \mathcal{A}_M, \) and \( \mathcal{O}_M \) as differentiable, real-analytic, and complex structures respectively.

**Definition 1.2:**

(a) An \( S \)-\textit{morphism} \( F : (M, S_M) \to (N, S_N) \) is a continuous map, \( F : M \to N \), such that

\[
f \in S_N \text{ implies } f \circ F \in S_M.
\]

(b) An \( S \)-\textit{isomorphism} is an \( S \)-morphism \( F : (M, S_M) \to (N, S_N) \) such that \( F : M \to N \) is a homeomorphism, and

\[
F^{-1} : (N, S_N) \to (M, S_M) \text{ is an } S \text{-morphism.}
\]

It follows from the above definitions that if on an \( S \)-manifold \((M, S_M)\) we have two coordinate systems \( h_1 : U_1 \to K^n \) and \( h_2 : U_2 \to K^n \) such that \( U_1 \cap U_2 \neq \emptyset \), then

\[
h_2 \circ h_1^{-1} : h_1(U_1 \cap U_2) \to h_2(U_1 \cap U_2) \text{ is an } S \text{-isomorphism on open subsets of } (K^n, S_{K^n}).
\]
Conversely, if we have an open covering \(\{U_\alpha\}_{\alpha \in A}\) of \(M\), a topological manifold, and a family of homeomorphisms \(\{h_\alpha: U_\alpha \to U'_\alpha \subset K^n\}_{\alpha \in A}\) satisfying (1.1), then this defines an \(S\)-structure on \(M\) by setting \(S_M = \{f: U \to K\}\) such that \(U\) is open in \(M\) and \(f \circ h_\alpha^{-1} \in S(h_\alpha(U \cap U_\alpha))\) for all \(\alpha \in A\); i.e., the functions in \(S_M\) are pullbacks of functions in \(S\) by the homeomorphisms \(\{h_\alpha\}_{\alpha \in A}\). The collection \(\{(U_\alpha, h_\alpha)\}_{\alpha \in A}\) is called an atlas for \((M, S_M)\).

In our three classes of functions, the concept of an \(S\)-morphism and \(S\)-isomorphism have special names:

(a) \(S = E\): differentiable mapping and diffeomorphism of \(M\) to \(N\).
(b) \(S = A\): real-analytic mapping and real-analytic isomorphism (or bianalytic mapping) of \(M\) to \(N\).
(c) \(S = O\): holomorphic mapping and biholomorphism (biholomorphic mapping) of \(M\) to \(N\).

It follows immediately from the definition above that a differentiable mapping

\[ f: M \longrightarrow N, \]

where \(M\) and \(N\) are differentiable manifolds, is a continuous mapping of the underlying topological space which has the property that in local coordinate systems on \(M\) and \(N\), \(f\) can be represented as a matrix of \(C^\infty\) functions. This could also be taken as the definition of a differentiable mapping. A similar remark holds for the other two categories.

Let \(N\) be an arbitrary subset of an \(S\)-manifold \(M\); then an \(S\)-function on \(N\) is defined to be the restriction to \(N\) of an \(S\)-function defined in some open set containing \(N\), and \(S_M|N\) consists of all the functions defined on relatively open subsets of \(N\) which are restrictions of \(S\)-functions on the open subsets of \(M\).

**Definition 1.3:** Let \(N\) be a closed subset of an \(S\)-manifold \(M\); then \(N\) is called an \(S\)-submanifold of \(M\) if for each point \(x_0 \in N\), there is a coordinate system \(h: U \to U' \subset K^n\), where \(x_0 \in U\), with the property that \(U \cap N\) is mapped onto \(U' \cap K^k\), where \(0 \leq k \leq n\). Here \(K^k \subset K^n\) is the standard embedding of the linear subspace \(K^k\) into \(K^n\), and \(k\) is called the \(K\)-dimension of \(N\), and \(n - k\) is called the \(K\)-codimension of \(N\).

It is easy to see that an \(S\)-submanifold of an \(S\)-manifold \(M\) is itself an \(S\)-manifold with the \(S\)-structure given by \(S_M|N\). Since the implicit function theorem is valid in each of our three categories, it is easy to verify that the above definition of submanifold coincides with the more common one that an \(S\)-submanifold (of \(k\) dimensions) is a closed subset of an \(S\)-manifold \(M\) which is locally the common set of zeros of \(n - k\) \(S\)-functions whose Jacobian matrix has maximal rank.

It is clear that an \(n\)-dimensional complex structure on a manifold induces a \(2n\)-dimensional real-analytic structure, which, likewise, induces a \(2n\)-dimensional differentiable structure on the manifold. One of the questions
we shall be concerned with is how many different (i.e., nonisomorphic) complex-analytic structures induce the same differentiable structure on a given manifold? The analogous question of how many different differentiable structures exist on a given topological manifold is an important problem in differential topology.

What we have actually defined is a category wherein the objects are $S$-manifolds and the morphisms are $S$-morphisms. We leave to the reader the proof that this actually is a category, since it follows directly from the definitions. In the course of what follows, then, we shall use three categories—the differentiable ($S = E$), the real-analytic ($S = A$), and the holomorphic ($S = O$) categories—and the above remark states that each is a subcategory of the former.

We now want to give some examples of various types of manifolds.

**Example 1.4 (Euclidean space):** $K^n$, $(\mathbb{R}^n, \mathbb{C}^n)$. For every $p \in K^n$, $U = K^n$ and $h =$ identity. Then $\mathbb{R}^n$ becomes a real-analytic (hence differentiable) manifold and $\mathbb{C}^n$ is a complex-analytic manifold.

**Example 1.5:** If $(M, S_M)$ is an $S$-manifold, then any open subset $U$ of $M$ has an $S$-structure, $S_U = \{ f|_U : f \in S_M \}$.

**Example 1.6 (Projective space):** If $V$ is a finite dimensional vector space over $K$, then† $P(V) := \{ \text{the set of one-dimensional subspaces of } V \}$ is called the projective space of $V$. We shall study certain special projective spaces, namely

\[ P_n(\mathbb{R}) := P(\mathbb{R}^{n+1}) \]
\[ P_n(\mathbb{C}) := P(\mathbb{C}^{n+1}). \]

We shall show how $P_n(\mathbb{R})$ can be made into a differentiable manifold.

There is a natural map $\pi : \mathbb{R}^{n+1} - \{0\} \to P_n(\mathbb{R})$ given by

\[ \pi(x) = \pi(x_0, \ldots, x_n) := \{ \text{subspace spanned by } x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \}. \]

The mapping $\pi$ is onto; in fact, $\pi|_{S^n} = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \}$ is onto. Let $P_n(\mathbb{R})$ have the quotient topology induced by the map $\pi$; i.e., $U \subset P_n(\mathbb{R})$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1} - \{0\}$. Hence $\pi$ is continuous and $P_n(\mathbb{R})$ is a Hausdorff space with a countable basis. Also, since

\[ \pi|_{S^n} : S^n \to P_n(\mathbb{R}) \]

is continuous and surjective, $P_n(\mathbb{R})$ is compact.

If $x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} - \{0\}$, then set

\[ \pi(x) = [x_0, \ldots, x_n]. \]

We say that $(x_0, \ldots, x_n)$ are homogeneous coordinates of $[x_0, \ldots, x_n]$. If $(x'_0, \ldots, x'_n)$ is another set of homogeneous coordinates of $[x_0, \ldots, x_n]$,

†:= means that the object on the left is defined to be equal to the object on the right.
then $x_i = tx'_i$ for some $t \in \mathbb{R} - \{0\}$, since $[x_0, \ldots, x_n]$ is the one-dimensional subspace spanned by $(x_0, \ldots, x_n)$ or $(x'_0, \ldots, x'_n)$. Hence also $\pi(x) = \pi(tx)$ for $t \in \mathbb{R} - \{0\}$. Using homogeneous coordinates, we can define a differentiable structure (in fact, real-analytic) on $P_n(\mathbb{R})$ as follows. Let

$$U_\alpha = \{ S \in P_n(\mathbb{R}) : S = [x_0, \ldots, x_n] \text{ and } x_\alpha \neq 0 \}.$$  

Each $U_\alpha$ is open and $P_n(\mathbb{R}) = \bigcup_{\alpha=0}^n U_\alpha$ since $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} - \{0\}$. Also, define the map $h_\alpha : U_\alpha \to \mathbb{R}^n$ by setting

$$h_\alpha ([x_0, \ldots, x_n]) = \left( \frac{x_0}{x_\alpha}, \ldots, \frac{x_{\alpha-1}}{x_\alpha}, \frac{x_{\alpha+1}}{x_\alpha}, \ldots, \frac{x_n}{x_\alpha} \right) \in \mathbb{R}^n.$$

Note that both $U_\alpha$ and $h_\alpha$ are well defined by the relation between different choices of homogeneous coordinates. One shows easily that $h_\alpha$ is a homeomorphism and that $h_\alpha \circ h_\beta^{-1}$ is a diffeomorphism; therefore, this defines a differentiable structure on $P_n(\mathbb{R})$. In exactly this same fashion we can define a differentiable structure on $P(V)$ for any finite dimensional $\mathbb{R}$-vector space $V$ and a complex-analytic structure on $P(V)$ for any finite dimensional $\mathbb{C}$-vector space $V$.

**Example 1.7 (Matrices of fixed rank):** Let $\mathcal{M}_{k,n}(\mathbb{R})$ be the $k \times n$ matrices with real coefficients. Let $M_{k,n}(\mathbb{R})$ be the $k \times n$ matrices of rank $k (k \leq n)$. Let $M^m_{k,n}(\mathbb{R})$ be the elements of $\mathcal{M}_{k,n}(\mathbb{R})$ of rank $m (m \leq k)$. First, $\mathcal{M}_{k,n}(\mathbb{R})$ can be identified with $\mathbb{R}^{kn}$, and hence it is a differentiable manifold. We know that $M_{k,n}(\mathbb{R})$ consists of those $k \times n$ matrices for which at least one $k \times k$ minor is nonsingular; i.e.,

$$M_{k,n}(\mathbb{R}) = \bigcup_{i=1}^n \{ A \in \mathcal{M}_{k,n}(\mathbb{R}) : \det A_i \neq 0 \},$$

where for each $A \in \mathcal{M}_{k,n}(\mathbb{R})$ we let $\{ A_1, \ldots, A_l \}$ be a fixed ordering of the $k \times k$ minors of $A$. Since the determinant function is continuous, we see that $M_{k,n}(\mathbb{R})$ is an open subset of $\mathcal{M}_{k,n}(\mathbb{R})$ and hence has a differentiable structure induced on it by the differentiable structure on $\mathcal{M}_{k,n}(\mathbb{R})$ (see Example 1.5). We can also define a differentiable structure on $M^m_{k,n}(\mathbb{R})$. For convenience we delete the $R$ and refer to $M^m_{k,n}(\mathbb{R})$. For $X_0 \in M^m_{k,n}(\mathbb{R})$, we define a coordinate neighborhood at $X_0$ as follows. Since the rank of $X$ is $m$, there exist permutation matrices $P, Q$ such that

$$PX_0Q = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix},$$

where $A_0$ is a nonsingular $m \times m$ matrix. Hence there exists an $\epsilon > 0$ such that $\|A - A_0\| < \epsilon$ implies $A$ is nonsingular, where $\|A\| = \max_{ij} |a_{ij}|$, for $A = [a_{ij}]$. Therefore let

$$W = \{ X \in \mathcal{M}_{k,n} : PXQ = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ and } \|A - A_0\| < \epsilon \}.$$

Then $W$ is an open subset of $\mathcal{M}_{k,n}$. Since this is true, $U := W \cap M^m_{k,n}$ is an
open neighborhood of $X_0$ in $M^m_{k,n}$ and will be the necessary coordinate neighborhood of $X_0$. Note that

$$X \in U \text{ if and only if } D = CA^{-1}B,$$

where $PXQ = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

This follows from the fact that

$$\begin{bmatrix} I_m & 0 \\ -CA^{-1} & I_{k-m} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D - CA^{-1}B \end{bmatrix}$$

and

$$\begin{bmatrix} I_m & 0 \\ -CA^{-1} & I_{k-m} \end{bmatrix}$$

is nonsingular (where $I_j$ is the $j \times j$ identity matrix). Therefore

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ and } \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

have the same rank, but

$$\begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

has rank $m$ if and only if $D - CA^{-1}B = 0$.

We see that $M^m_{k,n}$ actually becomes a manifold of dimension $m(n+k-m)$ by defining

$$h: U \longrightarrow \mathbb{R}^{m^2+(n-m)m+(k-m)m} = \mathbb{R}^{m(n+k-m)},$$

where

$$h(X) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \in \mathbb{R}^{m(n+k-m)} \text{ for } PXQ = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

as above. Note that we can define an inverse for $h$ by

$$h^{-1} \left( \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \right) = P^{-1} \left[ \begin{bmatrix} A & B \\ C & CA^{-1}B \end{bmatrix} \right] Q^{-1}.$$ 

Therefore $h$ is, in fact, bijective and is easily shown to be a homeomorphism. Moreover, if $h_1$ and $h_2$ are given as above,

$$h_2 \circ h_1^{-1} \left( \begin{bmatrix} A_1 & B_1 \\ C_1 & 0 \end{bmatrix} \right) = \begin{bmatrix} A_2 & B_2 \\ C_2 & 0 \end{bmatrix},$$

where

$$P_2 P_1^{-1} \begin{bmatrix} A_1 & B_1 \\ C_1 & C_1A_1^{-1}B_1 \end{bmatrix} Q_1^{-1} Q_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix},$$

and these maps are clearly diffeomorphisms (in fact, real-analytic), and so $M^m_{k,n}(\mathbb{R})$ is a differentiable submanifold of $M_{k,n}(\mathbb{R})$. The same procedure can be used to define complex-analytic structures on $M^m_{k,n}(\mathbb{C})$, $M_{k,n}(\mathbb{C})$, and $M^m_{k,n}(\mathbb{C})$, the corresponding sets of matrices over $\mathbb{C}$. 

Sec. 1

Manifolds

7
Example 1.8 (Grassmannian manifolds): Let $V$ be a finite dimensional $K$-vector space and let $G_k(V) := \{\text{the set of } k\text{-dimensional subspaces of } V\}$, for $k < \dim_K V$. Such a $G_k(V)$ is called a Grassmannian manifold. We shall use two particular Grassmannian manifolds, namely

$$G_{k,n}(\mathbb{R}) := G_k(\mathbb{R}^n) \quad \text{and} \quad G_{k,n}(\mathbb{C}) := G_k(\mathbb{C}^n).$$

The Grassmannian manifolds are clearly generalizations of the projective spaces [in fact, $\mathbb{P}(V) = G_1(V)$; see Example 1.6] and can be given a manifold structure in a fashion analogous to that used for projective spaces.

Consider, for example, $G_{k,n}(\mathbb{R})$. We can define the map

$$\pi: M_{k,n}(\mathbb{R}) \longrightarrow G_{k,n}(\mathbb{R}),$$

where

$$\pi(A) = \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} = \{k\text{-dimensional subspace of } \mathbb{R}^n \text{ spanned by the row vectors } \{a_j\} \text{ of } A\}.$$

We notice that for $g \in GL(k, \mathbb{R})$ (the $k \times k$ nonsingular matrices) we have $\pi(gA) = \pi(A)$ (where $gA$ is matrix multiplication), since the action of $g$ merely changes the basis of $\pi(A)$. This is completely analogous to the projection $\pi: \mathbb{R}^{n+1} - \{0\} \rightarrow \mathbb{P}_n(\mathbb{R})$, and, using the same reasoning, we see that $G_{k,n}(\mathbb{R})$ is a compact Hausdorff space with the quotient topology and that $\pi$ is a surjective, continuous open map.\(^\dagger\)

We can also make $G_{k,n}(\mathbb{R})$ into a differentiable manifold in a way similar to that used for $\mathbb{P}_n(\mathbb{R})$. Consider $A \in M_{k,n}$ and let $\{A_1, \ldots, A_l\}$ be the collection of $k \times k$ minors of $A$ (see Example 1.7). Since $A$ has rank $k$, $A_\alpha$ is nonsingular for some $1 \leq \alpha \leq l$ and there is a permutation matrix $P_\alpha$ such that

$$AP_\alpha = [A_\alpha \tilde{A}_\alpha],$$

where $\tilde{A}_\alpha$ is a $k \times (n-k)$ matrix. Note that if $g \in GL(k, \mathbb{R})$, then $gA_\alpha$ is a nonsingular minor of $gA$ and $gA_\alpha = (gA)_\alpha$. Let $U_\alpha = \{S \in G_{k,n}(\mathbb{R}) : S = \pi(A), \text{ where } A_\alpha \text{ is nonsingular}\}$. This is well defined by the remark above concerning the action of $GL(k, \mathbb{R})$ on $M_{k,n}(\mathbb{R})$. The set $U_\alpha$ is defined by the condition $\det A_\alpha \neq 0$; hence it is an open set in $G_{k,n}(\mathbb{R})$, and $\{U_\alpha\}_{\alpha=1}^l$ covers $G_{k,n}(\mathbb{R})$. We define a map

$$h_\alpha : U_\alpha \longrightarrow \mathbb{R}^{k(n-k)}$$

by setting

$$h_\alpha(\pi(A)) = A_\alpha^{-1} \tilde{A}_\alpha \in \mathbb{R}^{k(n-k)},$$

where $AP_\alpha = [A_\alpha \tilde{A}_\alpha]$. Again this is well defined and we leave it to the reader to show that this does, indeed, define a differentiable structure on $G_{k,n}(\mathbb{R})$.

\(^\dagger\)Note that the compact set $\{A \in M_{k,n}(\mathbb{R}) : A'A = I\}$ is analogous to the unit sphere in the case $k = 1$ and is mapped surjectively onto $G_{k,n}(\mathbb{R})$. 

\(\dagger\)}
Example 1.9 (Algebraic submanifolds): Consider $P_n = \mathbb{P}^n(C)$, and let

$$H = \{ [z_0, \ldots, z_n] \in \mathbb{P}^n: a_0z_0 + \cdots + a_nz_n = 0 \},$$

where $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1} - \{0\}$. Then $H$ is called a projective hyperplane. We shall see that $H$ is a submanifold of $P_n$ of dimension $n - 1$. Let $U_0$ be the coordinate systems for $P_n$ as defined in Example 1.6. Let us consider $U_0 \cap H$, and let $(\zeta_1, \ldots, \zeta_n)$ be coordinates in $C^n$. Suppose that $[z_0, \ldots, z_n] \in H \cap U_0$; then, since $z_0 \neq 0$, we have

$$a_1 \frac{z_1}{z_0} + \cdots + a_n \frac{z_n}{z_0} = -a_0,$$

which implies that if $\zeta = (\zeta_1, \ldots, \zeta_n) = h_0([z_0, \ldots, z_n])$, then $\zeta$ satisfies

$$(1.2) \quad a_1 \zeta_1 + \cdots + a_n \zeta_n = -a_0,$$

which is an affine linear subspace of $C^n$, provided that at least one of $a_1, \ldots, a_n$ is not zero. If, however, $a_0 \neq 0$ and $a_1 = \cdots = a_n = 0$, then it is clear that there is no point $(\zeta_1, \ldots, \zeta_n) \in C^n$ which satisfies (1.2), and hence in this case $U_0 \cap H = \emptyset$ (however, $H$ will then necessarily intersect all the other coordinate systems $U_1, \ldots, U_n$). It now follows easily that $H$ is a submanifold of dimension $n - 1$ of $P_n$ (using equations similar to (1.2) in the other coordinate systems as a representation for $H$). More generally, one can consider

$$V = \{ [z_0, \ldots, z_n] \in \mathbb{P}^n(C): p_1(z_0, \ldots, z_n) = \cdots = p_r(z_0, \ldots, z_n) = 0 \},$$

where $p_1, \ldots, p_r$ are homogeneous polynomials of varying degrees. In local coordinates, one can find equations of the form (for instances, in $U_0$)

$$(1.3) \quad p_1 \left( 1, \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0} \right) = 0,
\quad p_r \left( 1, \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0} \right) = 0,$$

and $V$ will be a submanifold of $P_n$ if the Jacobian matrix of these equations in the various coordinate systems has maximal rank. More generally, $V$ is called a projective algebraic variety, and points where the Jacobian has less than maximal rank are called singular points of the variety.

We say that an $S$-morphism

$$f: (M, S_M) \longrightarrow (N, S_N)$$

of two $S$-manifolds is an $S$-embedding if $f$ is an $S$-isomorphism onto an $S$-submanifold of $(N, S_N)$. Thus, in particular, we have the concept of differentiable, real-analytic, and holomorphic embeddings. Embeddings are most often used (or conceived of as) embeddings of an “abstract” manifold as a submanifold of some more concrete (or more elementary) manifold. Most common is the concept of embedding in Euclidean space and in projective space, which are the simplest geometric models (noncompact and compact, respectively). We shall state some results along this line to give the reader some feeling for the differences among the three categories we have been dealing with. Until now they have behaved very similarly.
Theorem 1.10 (Whitney [1]): Let $M$ be a differentiable $n$-manifold. Then there exists a differentiable embedding $f$ of $M$ into $\mathbb{R}^{2n+1}$. Moreover, the image of $M$, $f(M)$, can be realized as a real-analytic submanifold of $\mathbb{R}^{2n+1}$.

This theorem tells us that all differentiable manifolds (compact and non-compact) can be considered as submanifolds of Euclidean space, such submanifolds having been the motivation for the definition and concept of manifold in general. The second assertion, which is a more difficult result, tells us that on any differentiable manifold $M$ one can find a subfamily of the family $\mathcal{E}$ of differentiable functions on $M$ so that this subfamily gives a real-analytic structure to the manifold $M$; i.e., every differentiable manifold admits a real-analytic structure. It is strictly false that differentiable manifolds admit complex structures in general, since, in particular, complex manifolds must have even topological dimension. We shall discuss this question somewhat more in Sec. 3. We shall not prove Whitney’s theorem since we do not need it later (see, e.g., de Rham [1], Sternberg [1], or Whitney’s original paper for a proof of Whitney’s theorems).

A deeper result is the theorem of Grauert and Morrey (see Grauert [1] and Morrey [1]) that any real-analytic manifold can be embedded, by a real-analytic embedding, into $\mathbb{R}^N$, for some $N$ (again either compact or non-compact). However, when we turn to complex manifolds, things are completely different. First, we have the relatively elementary result.

**Theorem 1.11:** Let $X$ be a connected compact complex manifold and let $f \in \mathcal{O}(X)$. Then $f$ is constant; i.e., global holomorphic functions are necessarily constant.

**Proof:** Suppose that $f \in \mathcal{O}(X)$. Then, since $f$ is a continuous function on a compact space, $|f|$ assumes its maximum at some point $x_0 \in X$ and $S = \{x: f(x) = f(x_0)\}$ is closed. Let $z = (z_1, \ldots, z_n)$ be local coordinates at $x \in S$, with $z = 0$ corresponding to the point $x$. Consider a small ball $B$ about $z = 0$ and let $z \in B$. Then the function $g(\lambda) = f(\lambda z)$ is a function of one complex variable $\lambda$ which assumes its maximum absolute value at $\lambda = 0$ and is hence constant by the maximum principle. Therefore, $g(1) = g(0)$ and hence $f(z) = f(0)$, for all $z \in B$. By connectedness, $S = X$, and $f$ is constant.

Q.E.D.

**Remark:** The maximum principle for holomorphic functions in domains in $\mathbb{C}^n$ is also valid and could have been applied (see Gunning and Rossi [1]).

**Corollary 1.12:** There are no compact complex submanifolds of $\mathbb{C}^n$ of positive dimension.

**Proof:** Otherwise at least one of the coordinate functions $z_1, \ldots, z_n$ would be a nonconstant function when restricted to such a submanifold.

Q.E.D.
Therefore, we see that not all complex manifolds admit an embedding into Euclidean space in contrast to the differentiable and real-analytic situations, and of course, there are many examples of such complex manifolds [e.g., \( P_n(\mathbb{C}) \)]. One can characterize the (necessarily noncompact) complex manifolds which admit embeddings into \( \mathbb{C}^n \), and these are called Stein manifolds, which have an abstract definition and have been the subject of much study during the past 20 years or so (see Gunning and Rossi [1] and Hörmander [2] for an exposition of the theory of Stein manifolds). In this book we want to develop the material necessary to provide a characterization of the compact complex manifolds which admit an embedding into projective space. This was first accomplished by Kodaira in 1954 (see Kodaira [2]) and the material in the next several chapters is developed partly with this characterization in mind. We give a formal definition.

**Definition 1.13:** A compact complex manifold \( X \) which admits an embedding into \( P_n(\mathbb{C}) \) (for some \( n \)) is called a projective algebraic manifold.

**Remark:** By a theorem of Chow (see, e.g., Gunning and Rossi [1]), every complex submanifold \( V \) of \( P_n(\mathbb{C}) \) is actually an algebraic submanifold (hence the name projective algebraic manifold), which means in this context that \( V \) can be expressed as the zeros of homogeneous polynomials in homogeneous coordinates. Thus, such manifolds can be studied from the point of view of algebra (and hence algebraic geometry). We will not need this result since the methods we shall be developing in this book will be analytical and not algebraic. As an example, we have the following proposition.

**Proposition 1.14:** The Grassmannian manifolds \( G_{k,n}(\mathbb{C}) \) are projective algebraic manifolds.

**Proof:** Consider the following map:

\[
\tilde{F} : M_{k,n}(\mathbb{C}) \longrightarrow \wedge^k \mathbb{C}^n
\]

defined by

\[
\tilde{F}(A) = \tilde{F} \left( \begin{array}{c} a_1 \\ \vdots \\ a_k \end{array} \right) = a_1 \wedge \cdots \wedge a_k.
\]

The image of this map is actually contained in \( \wedge^k \mathbb{C}^n \setminus \{0\} \) since \( \{a_j\} \) is an independent set. We can obtain the desired embedding by completing the following diagram by \( F \):

\[
\begin{array}{ccc}
M_{k,n}(\mathbb{C}) & \xrightarrow{\tilde{F}} & \wedge^k \mathbb{C}^n \setminus \{0\} \\
\downarrow{\pi_G} & & \downarrow{\pi_p} \\
G_{k,n}(\mathbb{C}) & \xrightarrow{F} & P(\wedge^k \mathbb{C}^n),
\end{array}
\]
where $\pi_G, \pi_P$ are the previously defined projections. We must show that $F$ is well defined; i.e.,

$$\pi_G(A) = \pi_G(B) \implies \pi_P \circ \tilde{F}(A) = \pi_P \circ \tilde{F}(B).$$

But $\pi_G(A) = \pi_G(B)$ implies that $A = gB$ for $g \in GL(k, \mathbb{C})$, and so

$$a_1 \wedge \cdots \wedge a_k = \det g(b_1 \wedge \cdots \wedge b_k),$$

where

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix},$$

but

$$\pi_P(a_1 \wedge \cdots \wedge a_k) = \pi_P(\det g(b_1 \wedge \cdots \wedge b_k)) = \pi_P(b_1 \wedge \cdots \wedge b_k),$$

and so the map $F$ is well defined. We leave it to the reader to show that $F$ is also an embedding.

Q.E.D.

2. Vector Bundles

The study of vector bundles on manifolds has been motivated primarily by the desire to linearize nonlinear problems in geometry, and their use has had a profound effect on various modern fields of mathematics. In this section we want to introduce the concept of a vector bundle and give various examples. We shall also discuss some of the now classical results in differential topology (the classification of vector bundles, for instance) which form a motivation for some of our constructions later in the context of holomorphic vector bundles.

We shall use the same notation as in Sec. 1. In particular $S$ will denote one of the three structures on manifolds $(E, A, \emptyset)$ studied there, and $K = \mathbb{R}$ or $\mathbb{C}$.

**Definition 2.1:** A continuous map $\pi: E \to X$ of one Hausdorff space, $E$, onto another, $X$, is called a $K$-vector bundle of rank $r$ if the following conditions are satisfied:

(a) $E_p := \pi^{-1}(p)$, for $p \in X$, is a $K$-vector space of dimension $r$ ($E_p$ is called the fibre over $p$).

(b) For every $p \in X$ there is a neighborhood $U$ of $p$ and a homeomorphism

$$h: \pi^{-1}(U) \longrightarrow U \times K^r$$

such that $h(E_p) \subset \{p\} \times K^r$, and $h^p$, defined by the composition

$$h^p: E_p \xrightarrow{h} \{p\} \times K^r \xrightarrow{\text{proj}} K^r,$$

is a $K$-vector space isomorphism [the pair $(U, h)$ is called a local trivialization].

For a $K$-vector bundle $\pi: E \to X$, $E$ is called the total space and $X$ is called
the base space, and we often say that \( E \) is a vector bundle over \( X \). Notice that for two local trivializations \((U_α, h_α)\) and \((U_β, h_β)\) the map 
\[
h_α \circ h_β^{-1} : (U_α \cap U_β) \times K^r \longrightarrow (U_α \cap U_β) \times K^r
\]
induces a map
\[
(2.1) \quad g_{αβ} : U_α \cap U_β \longrightarrow GL(r, K),
\]
where
\[
g_{αβ}(p) = h_α^p \circ (h_β^p)^{-1} : K^r \longrightarrow K^r.
\]
The functions \( g_{αβ} \) are called the transition functions of the \( K \)-vector bundle \( π : E \rightarrow X \) (with respect to the two local trivializations above).†

The transition functions \( g_{αβ} \) satisfy the following compatibility conditions:

(2.2a) \[
g_{αβ} \circ g_{βγ} \circ g_{γα} = I_r \text{ on } U_α \cap U_β \cap U_γ,
\]
and

(2.2b) \[
g_{αα} = I_r \text{ on } U_α,
\]
where the product is a matrix product and \( I_r \) is the identity matrix of rank \( r \). This follows immediately from the definition of the transition functions.

**Definition 2.2:** A \( K \)-vector bundle of rank \( r, π : E \rightarrow X \), is said to be an \( S \)-bundle if \( E \) and \( X \) are \( S \)-manifolds, \( π \) is an \( S \)-morphism, and the local trivializations are \( S \)-isomorphisms.

Note that the fact that the local trivializations are \( S \)-isomorphisms is equivalent to the fact that the transition functions are \( S \)-morphisms. In particular, then, we have differentiable vector bundles, real-analytic vector bundles, and holomorphic vector bundles (\( K \) must equal \( C \)).

**Remark:** Suppose that on an \( S \)-manifold we are given an open covering \( \mathfrak{A} = \{U_α\} \) and that to each ordered nonempty intersection \( U_α \cap U_β \) we have assigned an \( S \)-function
\[
g_{αβ} : U_α \cap U_β \longrightarrow GL(r, K)
\]
satisfying the compatibility conditions (2.2). Then one can construct a vector bundle \( E \xrightarrow{π} X \) having these transition functions. An outline of the construction is as follows: Let
\[
\tilde{E} = \bigcup_a U_α \times K^r \quad \text{(disjoint union)}
\]
equipped with the natural product topology and \( S \)-structure. Define an equivalence relation in \( \tilde{E} \) by setting
\[
(x, υ) \sim (y, w), \text{ for } (x, υ) \in U_β \times K^r, (y, w) \in U_α \times K^r
\]
if and only if
\[
y = x \quad \text{and} \quad w = g_{αβ}(x)υ.
\]

†Note that the transition function \( g_{αβ}(p) \) is a linear mapping from the \( U_β \) trivialization to the \( U_α \) trivialization. The order is significant.
The fact that this is a well-defined equivalence relation is a consequence of the compatibility conditions (2.2). Let $E = \tilde{E}/\sim$ (the set of equivalence classes), equipped with the quotient topology, and let $\pi: E \to X$ be the mapping which sends a representative $(x, v)$ of a point $p \in E$ into the first coordinate. One then shows that an $E$ so constructed carries on $S$-structure and is an $S$-vector bundle. In the examples discussed below we shall see more details of such a construction.

Example 2.3 (Trivial bundle): Let $M$ be an $S$-manifold. Then

$$\pi: M \times K^n \longrightarrow M,$$

where $\pi$ is the natural projection, is an $S$-bundle called a trivial bundle.

Example 2.4 (Tangent bundle): Let $M$ be a differentiable manifold. Then we want to construct a vector bundle over $M$ whose fibre at each point is the linearization of the manifold $M$, to be called the tangent bundle to $M$. Let $p \in M$. Then we let

$$\mathcal{E}_{M,p} := \lim_{\substack{p \in U \subset M \text{ open}}} \mathcal{E}_M(U)$$

be the algebra (over $\mathbb{R}$) of germs of differentiable functions at the point $p \in M$, where the inductive limit† is taken with respect to the partial ordering on open neighborhoods of $p$ given by inclusion. Expressed differently, we can say that if $f$ and $g$ are defined and $C^\infty$ near $p$ and they coincide on some neighborhood of $p$, then they are equivalent. The set of equivalence classes is easily seen to form an algebra over $\mathbb{R}$ and is the same as the inductive limit algebra above; an equivalence class (element of $\mathcal{E}_{M,p}$) is called a germ of a $C^\infty$ function at $p$. A derivation of the algebra $\mathcal{E}_{M,p}$ is a vector space homomorphism $D: \mathcal{E}_{M,p} \to \mathbb{R}$ with the property that $D(fg) = D(f) \cdot g(p) + f(p) \cdot D(g)$, where $g(p)$ and $f(p)$ denote evaluation of a germ at a point $p$ (which clearly makes sense). The tangent space to $M$ at $p$ is the vector space of all derivations of the algebra $\mathcal{E}_{M,p}$, which we denote by $T_p(M)$. Since $M$ is a differentiable manifold, we can find a diffeomorphism $h$ defined in a neighborhood $U$ of $p$ where

$$h: U \longrightarrow U' \subset \mathbb{R}^n$$

and where, letting $h^* f(x) = f \circ h(x)$, $h$ has the property that, for $V \subset U'$,

$$h^*: \mathcal{E}_{\mathbb{R}^n}(V) \longrightarrow \mathcal{E}_M(h^{-1}(V))$$

is an algebra isomorphism. It follows that $h^*$ induces an algebra isomorphism on germs, i.e., (using the same notation),

$$h^*: \mathcal{E}_{\mathbb{R}^n,h(p)} \overset{\simeq}{\longrightarrow} \mathcal{E}_{M,p},$$

†We denote by $\lim \to$ the inductive (or direct) limit and by $\lim \leftarrow$ the projective (or inverse) limit of a partially ordered system.
and hence induces an isomorphism on derivations:

\[ h_*: T_p(M) \xrightarrow{\cong} T_{h(p)}(R^n). \]

It is easy to verify that

(a) \( \partial/\partial x_j \) are derivations of \( \mathcal{E}_{R^n,h(p)}, \ j = 1, \ldots, n \), and that

(b) \{\partial/\partial x_1, \ldots, \partial/\partial x_n\} is a basis for \( T_{h(p)}(R^n) \),

and thus that \( T_p(M) \) is an \( n \)-dimensional vector space over \( R \), for each point \( p \in M \) [the derivations are, of course, simply the classical directional derivatives evaluated at the point \( h(p) \)]. Suppose that \( f: M \to N \) is a differentiable mapping of differentiable manifolds. Then there is a natural map

\[ df_p: T_p(M) \longrightarrow T_{f(p)}(N) \]

defined by the following diagram:

\[
\begin{array}{ccc}
\mathcal{E}_{M,p} & \xrightarrow{f^*} & \mathcal{E}_{N,f(p)} \\
D_p & \downarrow & D_p \circ f^* \\
\mathbb{R} & \xrightarrow{df_p} & \mathbb{R} \\
\end{array}
\]

for \( D_p \in T_p(M) \). The mapping \( df_p \) is a linear mapping and can be expressed as a matrix of first derivatives with respect to local coordinates. The coefficients of such a matrix representation will be \( C^\infty \) functions of the local coordinates. Classically, the mapping \( df_p \) (the derivative mapping, differential mapping, or tangent mapping) is called the Jacobian of the differentiable map \( f \). The tangent map represents a first-order linear approximation (at \( p \)) to the differentiable map \( f \). We are now in a position to construct the tangent bundle to \( M \). Let

\[ T(M) = \bigcup_{p \in M} T_p(M) \quad (\text{disjoint union}) \]

and define

\[ \pi: T(M) \longrightarrow M \]

by

\[ \pi(\nu) = p \quad \text{if} \quad \nu \in T_p(M). \]

We can now make \( T(M) \) into a vector bundle. Let \{\( U_\alpha, h_\alpha \)\} be an atlas for \( M \), and let \( T(U_\alpha) = \pi^{-1}(U_\alpha) \) and

\[ \psi_\alpha: T(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n \]

be defined as follows: Suppose that \( \nu \in T_p(M) \subset T(U_\alpha) \). Then \( dh_{a,p}(\nu) \in T_{h_\alpha(p)}(\mathbb{R}^n) \). Thus

\[ dh_{a,p}(\nu) = \sum_{j=1}^n \xi_j(p) \frac{\partial}{\partial x_j} \bigg|_{h_\alpha(p)}, \]

where \( \xi_j \in \mathcal{E}_M(U_\alpha) \) (the fact that the coefficients are \( C^\infty \) follows easily from the proof that \{\( \partial/\partial x_1, \ldots, \partial/\partial x_n \)\} is a basis for the tangent vectors at a point in \( \mathbb{R}^n \)). Now let

\[ \psi_\alpha(\nu) = (p, \xi_1(p), \ldots, \xi_n(p)) \in U_\alpha \times \mathbb{R}^n. \]
It is easy to verify that $\psi_\alpha$ is bijective and fibre-preserving and moreover that

$$\psi_\alpha^p: T_p(M) \xrightarrow{\psi_\alpha} \{p\} \times \mathbb{R}^n \xrightarrow{\text{proj.}} \mathbb{R}^n$$

is an $\mathbb{R}$-linear isomorphism. We can define transition functions $g_{\alpha\beta}: U_\beta \cap U_\alpha \rightarrow GL(n, \mathbb{R})$

by setting

$$g_{\alpha\beta}(p) = \psi_\alpha^p \circ (\psi_\beta^p)^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n.$$ 

Moreover, it is easy to check that the coefficients of the matrices $\{g_{\alpha\beta}\}$ are $C^\infty$ functions in $U_\alpha \cap U_\beta$, since $g_{\alpha\beta}$ is a matrix representation for the composition $dh_\alpha \circ dh_\beta^{-1}$ with respect to the basis $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$ at $T_{h_\beta(p)}(\mathbb{R}^n)$ and $T_{h_\alpha(p)}(\mathbb{R}^n)$, and that the tangent maps are differentiable functions of local coordinates. Thus the $\{(U_\alpha, \psi_\alpha)\}$ become the desired trivializations. We have only to put the right topology on $T(M)$ so that $T(M)$ becomes a differentiable manifold. We simply require that $U \subset T(M)$ be open if and only if $\psi_\alpha(U \cap T(U_\alpha))$ is open in $U_\alpha \times \mathbb{R}^n$ for every $\alpha$. This is well defined since

$$\psi_\alpha \circ \psi_\beta^{-1}: (U_\alpha \cap U_\beta) \times \mathbb{R}^n \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is a diffeomorphism for any $\alpha$ and $\beta$ such that $U_\alpha \cap U_\beta \neq \emptyset$ (since $\psi_\alpha \circ \psi_\beta^{-1} = id \times g_{\alpha\beta}$, where $id$ is the identity mapping). Because the transition functions are diffeomorphisms, this defines a differentiable structure on $T(M)$ so that the projection $\pi$ and the local trivializations $\psi_\alpha$ are differentiable maps.

**Example 2.5 (Tangent bundle to a complex manifold):** Let $X = (X, \mathcal{O}_X)$ be a complex manifold of complex dimension $n$, let

$$\mathcal{O}_{X,x} := \lim_{x \in U \subseteq X \text{ open}} \mathcal{O}(U)$$

be the $\mathbb{C}$-algebra of germs of holomorphic functions at $x \in X$, and let $T_x(X)$ be the derivations of this $\mathbb{C}$-algebra (defined exactly as in Example 2.4). Then $T_x(X)$ is the holomorphic (or complex) tangent space to $X$ at $x$. In local coordinates, we see that $T_x(X) \cong T_x(\mathbb{C}^n)$ (abusing notation) and that the complex partial derivatives $\{\partial/\partial z_1, \ldots, \partial/\partial z_n\}$ form a basis over $\mathbb{C}$ for the vector space $T_x(\mathbb{C}^n)$ (see also Sec. 3). In the same manner as in Example 2.4 we can make the union of these tangent spaces into a holomorphic vector bundle over $X$, i.e., $T(X) \rightarrow X$, where the fibres are all isomorphic to $\mathbb{C}^n$.

**Remark:** The same technique used to construct the tangent bundles in the above examples can be used to construct other vector bundles. For instance, suppose that we have $\pi: E \rightarrow X$, where $X$ is an $S$-manifold and $\pi$ is a surjective map, so that

(a) $E_p$ is a $K$-vector space,
(b) For each $p \in X$ there is a neighborhood $U$ of $p$ and a bijective map:

$$h: \pi^{-1}(U) \rightarrow U \times K^r$$

such that $h(E_p) \subset \{p\} \times K^r$. 

(c) $h_p: E_p \to \{p\} \times K^r \xrightarrow{\text{proj}} K^r$ is a $K$-vector space isomorphism.

Then, if for every $(U_\alpha, h_\alpha), (U_\beta, h_\beta)$ as in (b) $h_\alpha \circ h_\beta^{-1}$ is an $S$-isomorphism, we can make $E$ into an $S$-bundle over $X$ by giving it the topology that makes $h_\alpha$ a homeomorphism for every $\alpha$.

**Example 2.6 (Universal bundle):** Let $U_{r,n}$ be the disjoint union of the $r$-planes ($r$-dimensional $K$-linear subspaces) in $K^n$. Then there is a natural projection

$$\pi: U_{r,n} \to G_{r,n},$$

where $G_{r,n} = G_{r,n}(K)$, given by $\pi(v) = S$, if $v$ is a vector in the $r$-plane $S$, and $S$ is considered as a point in the Grassmannian manifold $G_{r,n}$. Thus the inverse image under $\pi$ of a point $p$ in the Grassmannian is the subspace of $K^n$ which is the point $p$, and we may regard $U_{r,n}$ as a subset of $G_{r,n} \times K^n$.

We can make $U_{r,n}$ into an $S$-bundle by using the coordinate systems of $G_{r,n}$ to define transition functions, as was done with the tangent bundle in Example 2.4, and by then applying the remark following Example 2.5. To simplify things somewhat consider $U_{1,n} \to G_{1,n} = \mathbb{P}_{n-1}$. First we note that any point $v \in U_{1,n}$ can be represented (not in a unique manner) in the form

$$v = (tx_0, \ldots, tx_{n-1}) = t(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n,$$

where $(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n - \{0\}$, and $t \in \mathbb{R}$. Moreover, the projection $\pi: U_{1,n} \to \mathbb{P}_{n-1}$ is given by

$$\pi(t(x_0, \ldots, x_{n-1})) = \pi(x_0, \ldots, x_{n-1}) = [x_0, \ldots, x_{n-1}] \in \mathbb{P}_{n-1}.$$

Letting $U_\alpha = \{[x_0, \ldots, x_{n-1}] \in \mathbb{P}_{n-1}: x_\alpha \neq 0\}$, (cf. Example 1.6), we see that

$$\pi^{-1}(U_\alpha) = \{v = t(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n: t \in \mathbb{R}, x_\alpha \neq 0\}.$$

Now if $v = t(x_0, \ldots, x_{n-1}) \in \pi^{-1}(U_\alpha)$, then we can write $v$ in the form

$$v = t_\alpha \left( x_0 \frac{x_0}{x_\alpha}, \ldots, \frac{x_{n-1}}{x_\alpha} \right),$$

and $t_\alpha = tx_\alpha \in \mathbb{R}$ is uniquely determined by $v$. Then we can define the mapping

$$h_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}$$

by setting $h_\alpha(v) = h_\alpha(t(x_0, \ldots, x_{n-1})) = ([x_0, \ldots, x_{n-1}], t_\alpha).$

The mapping $h_\alpha$ is bijective and is $\mathbb{R}$-linear from the fibres of $\pi^{-1}(U_\alpha)$ to the fibres of $U_\alpha \times \mathbb{R}$. Suppose now that $v = t(x_0, \ldots, x_{n-1}) \in \pi^{-1}(U_\alpha \cap U_\beta)$, then we have two different representations for $v$ and we want to compute the relationship. Namely,

$$h_\alpha(v) = ([x_0, \ldots, x_{n-1}], t_\alpha)$$

$$h_\beta(v) = ([x_0, \ldots, x_{n-1}], t_\beta)$$
and then \( t_\alpha = tx_\alpha, t_\beta = tx_\beta \), Therefore
\[
I = \frac{t_\alpha}{x_\alpha} = \frac{t_\beta}{x_\beta},
\]
which implies that
\[
I_\alpha = \frac{x_\alpha}{x_\beta} t_\beta.
\]
Thus if we let \( g_{\alpha\beta} = x_\alpha/x_\beta \), then it follows that \( g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1 \), and thus by the remark following Example 2.5, we see that \( U_{1,n} \) can be given the structure of a vector bundle by means of the functions \( \{h_\alpha\} \), (the trivializations), and the transition functions of \( U_{1,n} \)
\[
g_{\alpha\beta}(\{x_0, \ldots, x_{n-1}\}) = \frac{x_\alpha}{x_\beta},
\]
are mappings of \( U_\alpha \cap U_\beta \rightarrow GL(1,R) = R - \{0\} \). These are the standard transition functions for the universal bundle over \( P_{n-1} \). Exactly the same relation holds for \( U_{1,n}(C) \rightarrow P_{n-1}(C) \), which we meet again in later chapters.

Namely, for complex homogeneous coordinates \( [z_0, \ldots, z_{n-1}] \) we have the transition functions for the universal bundle over \( P_{n-1}(C) \):
\[
g_{\alpha\beta}(\{z_0, \ldots, z_{n-1}\}) = \frac{z_\alpha}{z_\beta}.
\]
The more general case of \( U_{r,n} \rightarrow G_{r,n} \) can be treated in a similar manner, using the coordinate systems developed in Sec. 1. We note that \( U_{r,n}(R) \rightarrow G_{r,n}(R) \) is a real-analytic (and hence also differentiable) \( R \)-vector bundle and that \( U_{r,n}(C) \rightarrow G_{r,n}(C) \) is a holomorphic vector bundle. The reason for the name “universal bundle” will be made more apparent later in this section.

**Definition 2.7:** Let \( \pi: E \rightarrow X \) be an \( \mathcal{S} \)-bundle and \( U \) an open subset of \( X \). Then the restriction of \( E \) to \( U \), denoted by \( E|_U \) is the \( \mathcal{S} \)-bundle
\[
\pi|_{\pi^{-1}(U)}: \pi^{-1}(U) \rightarrow U.
\]

**Definition 2.8:** Let \( E \) and \( F \) be \( \mathcal{S} \)-bundles over \( X \); i.e., \( \pi_E: E \rightarrow X \) and \( \pi_F: F \rightarrow X \). Then a **homomorphism of \( \mathcal{S} \)-bundles**, \( f: E \rightarrow F, \) is an \( \mathcal{S} \)-morphism of the total spaces which preserves fibres and is \( K \)-linear on each fibre; i.e., \( f \) commutes with the projections and is a \( K \)-linear mapping when restricted to fibres. An **\( \mathcal{S} \)-bundle isomorphism** is an \( \mathcal{S} \)-bundle homomorphism which is an \( \mathcal{S} \)-isomorphism on the total spaces and a \( K \)-vector space isomorphism on the fibres. Two \( \mathcal{S} \)-bundles are **equivalent** if there is some \( \mathcal{S} \)-bundle isomorphism between them. This clearly defines an equivalence relation on the \( \mathcal{S} \)-bundles over an \( \mathcal{S} \)-manifold, \( X \).

The statement that a bundle is locally trivial now becomes the following:
For every \( p \in X \) there is an open neighborhood \( U \) of \( p \) and a bundle isomorphism

\[
h: E|_U \sim U \times K^r.
\]

Suppose that we are given two \( K \)-vector spaces \( A \) and \( B \). Then from them we can form new \( K \)-vector spaces, for example,

(a) \( A \oplus B \), the direct sum.
(b) \( A \otimes B \), the tensor product.
(c) \( \text{Hom}(A, B) \), the linear maps from \( A \) to \( B \).
(d) \( A^* \), the linear maps from \( A \) to \( K \).
(e) \( \wedge^k A \), the antisymmetric tensor products of degree \( k \) (exterior algebra of \( A \)).
(f) \( S^k(A) \), the symmetric tensor products of degree \( k \) (symmetric algebra of \( A \)).

Using the remark following Example 2.5, we can extend all the above algebraic constructions to vector bundles. For example, suppose that we have two vector bundles \( \pi_E: E \to X \) and \( \pi_F: F \to X \).

Then define

\[
E \oplus F = \bigcup_{p \in X} E_p \oplus F_p.
\]

We then have the natural projection

\[
\pi: E \oplus F \to X
\]
given by

\[
\pi^{-1}(p) = E_p \oplus F_p.
\]

Now for any \( p \in X \) we can find a neighborhood \( U \) of \( p \) and local trivializations

\[
h_E: E|_U \sim U \times K^n
\]

\[
h_F: F|_U \sim U \times K^m,
\]

and we define

\[
h_{E \oplus F}: E \oplus F|_U \to U \times (K^n \oplus K^m)
\]

by \( h_{E \oplus F}(v + w) = (p, h^v_E(v) + h^w_F(w)) \) for \( v \in E_p \) and \( w \in F_p \). Then this map is bijective and \( K \)-linear on fibres, and for intersections of local trivializations we obtain the transition functions

\[
g_{E \oplus F}^{E \oplus F}(p) = \begin{bmatrix} g^E_{\alpha \beta}(p) & 0 \\ 0 & g^F_{\alpha \beta}(p) \end{bmatrix}.
\]

So by the remark and the fact that \( g^E_{\alpha \beta} \) and \( g^F_{\alpha \beta} \) are bundle transition functions, \( \pi: E \oplus F \to X \) is a vector bundle. Note that if \( E \) and \( F \) were \( S \)-bundles over an \( S \)-manifold \( X \), then \( g^E_{\alpha \beta} \) and \( g^F_{\alpha \beta} \) would be \( S \)-isomorphisms, and so
$E \oplus F$ would then be an $S$-bundle over $X$. The same is true for all the other possible constructions induced by the vector space constructions listed above. Transition functions for the algebraically derived bundles are easily determined by knowing the transition functions for the given bundle.

The above examples lead naturally to the following definition.

**Definition 2.9:** Let $E \xrightarrow{\pi} X$ be an $S$-bundle. An $S$-submanifold $F \subset E$ is said to be an $S$-subbundle of $E$ if

(a) $F \cap E_x$ is a vector subspace of $E_x$.

(b) $\pi|_F: F \rightarrow X$ has the structure of an $S$-bundle induced by the $S$-bundle structure of $E$, i.e., there exist local trivializations for $E$ and $F$ which are compatible as in the following diagram:

\[
\begin{array}{ccc}
E|_U & \xrightarrow{\sim} & U \times K' \\
\uparrow i & & \uparrow id \times j \\
F|_U & \xrightarrow{\sim} & U \times K^s, \quad s \leq r,
\end{array}
\]

where the map $j$ is the natural inclusion mapping of $K^s$ as a subspace of $K'$ and $i$ is the inclusion of $F$ in $E$.

We shall frequently use the language of linear algebra in discussing homomorphisms of vector bundles. As an example, suppose that $E \xrightarrow{f} F$ is a vector bundle homomorphism of $K$-vector bundles over a space $X$. We define

\[
\operatorname{Ker} f = \bigcup_{x \in X} \operatorname{Ker} f_x
\]
\[
\operatorname{Im} f = \bigcup_{x \in X} \operatorname{Im} f_x,
\]

where $f_x = f|_{E_x}$. Moreover, we say that $f$ has constant rank on $X$ if rank $f_x$ (as a $K$-linear mapping) is constant for $x \in X$.

**Proposition 2.10:** Let $E \xrightarrow{f} F$ be an $S$-homomorphism of $S$-bundles over $X$. If $f$ has constant rank on $X$, then $\operatorname{Ker} f$ and $\operatorname{Im} f$ are $S$-subbundles of $E$ and $F$, respectively. In particular, $f$ has constant rank if $f$ is injective or surjective.

We leave the proof of this simple proposition to the reader.

Suppose now that we have a sequence of vector bundle homomorphisms over a space $X$,

\[
\cdots \rightarrow E \xrightarrow{f} F \xrightarrow{g} G \rightarrow \cdots,
\]

then the sequence is said to be exact at $F$ if $\operatorname{Ker} g = \operatorname{Im} f$. A short exact sequence of vector bundles is a sequence of vector bundles (and vector bundle homomorphisms) of the following form,

\[
0 \rightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \rightarrow 0,
\]

which is exact at $E'$, $E$, and $E''$. In particular, $f$ is injective and $g$ is surjective,
and $\text{Im } f = \text{Ker } g$ is a subbundle of $E$. We shall see examples of short exact sequences and their utility in the next two chapters.

As we have stated before, vector bundles represent the geometry of the underlying base space. However, to get some understanding via analysis of vector bundles, it is necessary to introduce a generalized notion of function (reflecting the geometry of the vector bundle) to which we can apply the tools of analysis.

**Definition 2.11:** An $\mathcal{S}$-section of an $\mathcal{S}$-bundle $E \xrightarrow{\pi} X$ is an $\mathcal{S}$-morphism $s: X \rightarrow E$ such that

$$\pi \circ s = 1_X,$$

where $1_X$ is the identity on $X$; i.e., $s$ maps a point in the base space into the fibre over that point. $\mathcal{S}(X, E)$ will denote the $\mathcal{S}$-sections of $E$ over $X$. $\mathcal{S}(U, E)$ will denote the $\mathcal{S}$-sections of $E|_U$ over $U \subset X$; i.e., $\mathcal{S}(U, E) = \mathcal{S}(U, E|_U)$ [we shall also occasionally use the common notation $\Gamma(X, E)$ for sections, provided that there is no confusion as to which category we are dealing with].

**Example 2.12:** Consider the trivial bundle $M \times \mathbb{R}$ over a differentiable manifold $M$. Then $\mathcal{E}(M, M \times \mathbb{R})$ can be identified in a natural way with $\mathcal{E}(M)$, the global real-valued functions on $M$. Similarly, $\mathcal{E}(M, M \times \mathbb{R}^n)$ can be identified with global differentiable mappings of $M$ into $\mathbb{R}^n$ (i.e., vector-valued functions). Since vector bundles are locally of the form $U \times \mathbb{R}^n$, we see that sections of a vector bundle can be viewed as vector-valued functions (locally), where two different local representations are related by the transition functions for the bundle. Therefore sections can be thought of as “twisted” vector-valued functions.

**Remarks:**

(a) A section $s$ is often identified with its image $s(X) \subset E$; for example, the term *zero section* is used to refer to the section $0: X \rightarrow E$ given by $0(x) = 0 \in E_x$ and is often identified with its image, which is, in fact, $\mathcal{S}$-isomorphic with the base space $X$.

(b) For $\mathcal{S}$-bundles $E \xrightarrow{\pi} X$ and $E' \xrightarrow{\pi'} X$ we can identify the set of $\mathcal{S}$-bundle homomorphisms of $E$ into $E'$, with $\mathcal{S}(X, \text{Hom}(E, E'))$. A section $s \in \mathcal{S}(X, \text{Hom}(E, E'))$ picks out for each point $x \in X$ a $K$-linear map $s(x): E_x \rightarrow E'_x$, and $s$ is identified with $f_s: E \rightarrow E'$ which is defined by

$$f_s|_{E\pi(e)} = s(\pi(e)) \quad \text{for } e \in E.$$

(c) If $E \rightarrow X$ is an $\mathcal{S}$-bundle of rank $r$ with transition functions $\{g_{ab}\}$ associated with a trivializing cover $\{U_a\}$, then let $f_a: U_a \rightarrow K'$ be $\mathcal{S}$-morphisms satisfying the compatibility conditions

$$f_a = g_{ab} f_\beta \quad \text{on } U_a \cap U_\beta \neq \emptyset.$$

Here we are using matrix multiplication, considering $f_a$ and $f_\beta$ as column vectors. Then the collection $\{f_a\}$ defines an $\mathcal{S}$-section $f$ of $E$, since each $f_a$ gives a section of $U_a \times K'$, and this pulls back by the trivialization to a section of $E|_{U_a}$. These sections of $E|_{U_a}$ agree on the overlap regions $U_a \cap U_\beta$. 


by the compatibility conditions imposed on \( \{ f_a \} \), and thus define a global section. Conversely, any \( S \)-section of \( E \) has this type of representation. We call each \( f_a \) a trivialization of the section \( f \).

**Example 2.13:** We use remark (c) above to compute the global sections of the holomorphic line bundles \( E^k \rightarrow \mathbf{P}_1(\mathbf{C}) \), which we define as follows, using the transition function \( g_{01} \) for the universal bundle \( U_{1,2}(\mathbf{C}) \rightarrow \mathbf{P}_1(\mathbf{C}) \) of Example 2.6. Let the \( \mathbf{P}_1(\mathbf{C}) \) coordinate maps (Example 1.6) \( \tilde{\varphi} : U_a \rightarrow \mathbf{C} \) be denoted by \( \varphi_0([z_0, z_1]) = z_1/z_0 = z \) and \( \varphi_1([z_0, z_1]) = z_0/z_1 = w \) so that \( z = \varphi_0 \circ \varphi_1^{-1}(w) = 1/w \) for \( w \neq 0 \). For a fixed integer \( k \) define the line bundle \( E^k \rightarrow \mathbf{P}_1(\mathbf{C}) \) by the transition function \( g_{01}^k : U_0 \cap U_1 \rightarrow GL(1, \mathbf{C}) \) where \( g_{01}^k([z_0, z_1]) = (z_0/z_1)^k \). \( E^k \) is the \( k \)th tensor power of \( U_{1,2}(\mathbf{C}) \) for \( k > 0 \), the \( k \)th tensor power of the dual bundle \( U_{1,2}(\mathbf{C})^* \) for \( k < 0 \), and trivial for \( k = 0 \). If \( f \in \mathcal{O}(\mathbf{P}_1(\mathbf{C}), E^k) \), then each trivialization of \( f, f_a \) is in \( \mathcal{O}(U_a, U_a \times \mathbf{C}) = \mathcal{O}(U_a) \) and the \( f_a \circ \varphi_a^{-1} \) are entire functions, say \( f_0 \circ \varphi_0^{-1}(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( f_1 \circ \varphi_1^{-1}(w) = \sum_{n=0}^{\infty} b_n w^n \). If \( z = \varphi_0(p) \) and \( w = \varphi_1(p) \), then by remark (c), \( f_0(p) = g_{01}^k(p) f_1(p) \), for \( p \in U_0 \cap U_1 \) in the \( w \)-coordinate plane, and this becomes

\[
\sum_{n=0}^{\infty} a_n (1/w)^n = f_0 \circ \varphi_0^{-1}(\varphi_0 \circ \varphi_1^{-1}(w)) = g_{01}^k(\varphi_1^{-1}(w)) f_1 \circ \varphi_1^{-1}(w) = w^k \sum_{n=0}^{\infty} b_n w^n.
\]

Hence,

\[
\mathcal{O}(\mathbf{P}_1(\mathbf{C}), E^k) = \begin{cases} 
0 & \text{for } k > 0, \\
\mathbf{C} & \text{for } k = 0 \text{ (Theorem 1.11),} \\
\text{homogeneous polynomials} & \text{for } k < 0.
\end{cases}
\]

When dealing with certain categories of \( S \)-manifolds, it is possible to define algebraic structures on \( S(X, E) \). First, \( S(X, E) \) can be made into a \( K \)-vector space under the following operations:

(a) For \( s, t \in S(X, E) \),
\[
(s + t)(x) := s(x) + t(x) \quad \text{for all } x \in X.
\]

(b) For \( s \in S(X, E) \) and \( \alpha \in K \), \((\alpha s)(x) := \alpha(s(x)) \) for all \( s \in X \). Moreover, \( S(X, E) \) can be given the structure of an \( S_X(X) \) module [where the \( S_X(X) \) are the globally defined \( K \)-valued \( S \)-functions on \( X \)] by defining

(c) For \( s \in S(X, E) \) and \( f \in S_X(X) \),
\[
f s(x) := f(x)s(x) \quad \text{for all } x \in X.
\]

To ensure that the above maps actually are \( S \)-morphisms and thus \( S \)-sections, it is necessary that the vector space operations on \( K^n \) be \( S \)-morphisms in the \( S \)-structure on \( K^n \). But this is clearly the case for the three categories with which we are dealing.

Let \( M \) be a differentiable manifold and let \( T(M) \rightarrow M \) be its tangent bundle. Using the techniques outlined above, we would like to consider new differentiable vector bundles over \( M \), derived from \( T(M) \). We have
(a) The cotangent bundle, \( T^*(M) \), whose fibre at \( x \in M, T^*_x(M) \), is the \( \mathbb{R} \)-linear dual to \( T_x(M) \).

(b) The exterior algebra bundles, \( \wedge^p T(M), \wedge^p T^*(M) \), whose fibre at \( x \in M \) is the antisymmetric tensor product (of degree \( p \)) of the vector spaces \( T_x(M) \) and \( T^*_x(M) \), respectively, and

\[
\wedge T(M) = \bigoplus_{p=0}^{n} \wedge^p T(M)
\]

\[
\wedge T^*(M) = \bigoplus_{p=0}^{n} \wedge^p T^*(M).
\]

(c) The symmetric algebra bundles, \( S^k(T(M)), S^k(T^*(M)) \), whose fibres are the symmetric tensor products (of degree \( k \)) of \( T_x(M) \) and \( T^*_x(M) \), respectively.

We define

\[ E^p(U) = \mathcal{E}(U, \wedge^p T^*(M)), \]

the \( C^\infty \) differential forms of degree \( p \) on the open set \( U \subset M \). As usual, we can define the exterior derivative

\[ d: E^p(U) \rightarrow E^{p+1}(U). \]

We recall how this is done. First, consider \( U \subset \mathbb{R}^n \) and recall that the derivations \( \{\partial/\partial x_1, \ldots, \partial/\partial x_n\} \) form a basis for \( T_x(\mathbb{R}^n) \) at \( x \in U \). Let \( \{dx_1, \ldots, dx_n\} \) be a dual basis for \( T^*_x(\mathbb{R}^n) \). Then the maps

\[ dx_j: U \rightarrow T^*(\mathbb{R}^n)|_x \]

given by

\[ dx_j(x) = dx_j|_x \]

form a basis for the \( \mathcal{E}(U)(= \mathcal{E}_{\mathbb{R}^n}(U)) \)-module \( \mathcal{E}(U, T^*(\mathbb{R}^n)) = \mathcal{E}^1(U) \). Moreover, \( \{dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}\} \), where \( I = (i_1, \ldots, i_p) \) and \( 1 \leq i_1 < i_2 < \cdots < i_p \leq n \), form a basis for the \( \mathcal{E}(U) \)-module \( \mathcal{E}^p(U) \). We defined \( d: \mathcal{E}^p(U) \rightarrow \mathcal{E}^{p+1}(U) \) as follows:

**Case 1 \((p = 0)\):** Suppose that \( f \in \mathcal{E}^0(U) = \mathcal{E}(U) \). Then let

\[ df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \in \mathcal{E}^1(U). \]

**Case 2 \((p > 0)\):** Suppose that \( f \in \mathcal{E}^p(U) \). Then

\[ f = \sum_{|I|=p} f_I dx_I, \]

where \( f_I \in \mathcal{E}(U), I = (i_1, \ldots, i_p) \), \(|I| = \) the number of indices, and \( \sum' \) signifies that the sum is taken over strictly increasing indices. Then

\[ df = \sum_{|I|=p} dx_I \wedge df_I = \sum_{|I|=p} \sum_{j=1}^{n} \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_I. \]
Suppose now that $(U, h)$ is a coordinate system on a differentiable manifold $M$. Then we have that $T(M)|_U \sim T(\mathbb{R}^n)|_{h(U)}$; hence $\mathcal{E}^p(U) \sim \mathcal{E}^p(h(U))$, and the mapping
defined above induces a mapping (also denoted by $d$)

d: \mathcal{E}^p(U) \to \mathcal{E}^{p+1}(U).

This defines the exterior derivative $d$ locally on $M$, and it is not difficult to show, using the chain rule, that the definition is independent of the choice of local coordinates. It follows that the exterior derivative is well defined globally on the manifold $M$.

We have previously defined a bundle homomorphism of two bundles over the same base space (Definition 2.8). We now would like to define a mapping between bundles over different base spaces.

**Definition 2.14:** An $S$-bundle morphism between two $S$-bundles $\pi_E: E \to X$ and $\pi_F: F \to Y$ is an $S$-morphism $f: E \to F$ which takes fibres of $E$ isomorphically (as vector spaces) onto fibres in $F$. An $S$-bundle morphism $f: E \to F$ induces an $S$-morphism $\tilde{f}(\pi_E(e)) = \pi_F(f(e))$; in other words, the following diagram commutes:

$$
\begin{array}{ccc}
E & \to & F \\
\pi_E \downarrow & & \downarrow \pi_F \\
X & \to & Y.
\end{array}
$$

If $X$ is identified with $0(X)$, the zero section, then $\tilde{f}$ may be identified with $\tilde{f} = f|_0: X \to Y = 0(Y)$ since $f$ is a homomorphism on fibres and maps the zero section of $X$ into the zero section of $Y$, which can likewise be identified with $Y$. If $E$ and $F$ are bundles over the same space $X$ and $\tilde{f}$ is the identity, then $E$ and $F$ are said to be equivalent (which implies that the two vector bundles are $S$-isomorphic and hence equivalent in the sense of Definition 2.8).

**Proposition 2.15:** Given an $S$-morphism $f: X \to Y$ and an $S$-bundle $\pi: E \to Y$, then there exists an $S$-bundle $\pi': E' \to X$ and an $S$-bundle morphism $g$ such that the following diagram commutes:

$$
\begin{array}{ccc}
E' & \to & E \\
\pi' \downarrow & & \downarrow \pi \\
X & \to & Y.
\end{array}
$$

Moreover, $E'$ is unique up to equivalence. We call $E'$ the pullback of $E$ by $f$ and denote it by $f^*E$.

**Proof:** Let
(2.3) $E' = \{(x, e) \in X \times E: f(x) = \pi(e)\}$. 
We have the natural projections

\[ g: E' \rightarrow E \quad \text{and} \quad \pi': E' \rightarrow X \]

\[(x, e) \rightarrow e \quad (x, e) \rightarrow x.\]

Giving \(E'_x = \{x\} \times E_{f(x)}\) the structure of a \(K\)-vector space induced by \(E_{f(x)}\), \(E'\) becomes a fibered family of vector spaces over \(X\).

If \((U, h)\) is a local trivialization for \(E\), i.e.,

\[ h: E|_U \rightarrow U \times K^n, \]

then it is easy to show that \(E'|_{f^{-1}(U)} \sim f^{-1}(U) \times K^n\) is a local trivialization of \(E'\); hence \(E'\) is the necessary bundle.

Suppose that we have another bundle \(\tilde{\pi}: \tilde{E} \rightarrow X\) and a bundle morphism \(\tilde{g}\) such that

\[
\begin{array}{ccc}
\tilde{E} & \xrightarrow{\tilde{g}} & E \\
\downarrow \tilde{\pi} & & \downarrow \pi \\
X & \xrightarrow{f} & Y
\end{array}
\]

commutes. Then define the bundle homomorphism \(h: \tilde{E} \rightarrow E'\) by

\[ h(\tilde{e}) = (\tilde{\pi}(\tilde{e}), \tilde{g}(\tilde{e})) \in \{\pi(\tilde{e})\} \times E. \]

Note that \(h(\tilde{e}) \in E'\) since the commutativity of the above diagram yields \(f(\tilde{\pi}(\tilde{e})) = \pi(\tilde{g}(\tilde{e}))\); hence this is a bundle homomorphism. Moreover, it is a vector space isomorphism on fibres and hence an \(S\)-bundle morphism inducing the identity \(1_X: X \rightarrow X\), i.e., an equivalence.

Q.E.D.

Remark: In the diagram in Proposition 2.15, the vector bundle \(E'\) and the maps \(\pi'\) and \(g\) depend on \(f\) and \(\pi\), and we shall sometimes denote this relation by

\[
\begin{array}{ccc}
f^* E & \xrightarrow{f_*} & E \\
\downarrow \pi_f & & \downarrow \pi \\
X & \xrightarrow{f} & Y
\end{array}
\]

to indicate the dependence on the map \(f\) of the pullback. For convenience, we assume from now on that \(f^* E\) is given by (2.3) and that the maps \(\pi_f\) and \(f_*\) are the natural projections.

The concepts of \(S\)-bundle homomorphism and \(S\)-bundle morphism are related by the following proposition.

**Proposition 2.16:** Let \(E \xrightarrow{\pi} X\) and \(E' \xrightarrow{\pi'} Y\) be \(S\)-bundles. If \(f: E \rightarrow E'\) is an \(S\)-morphism of the total spaces which maps fibres to fibres and which is a vector space homomorphism on each fibre, then \(f\) can be expressed as the composition of an \(S\)-bundle homomorphism and an \(S\)-bundle morphism.
Proof: Let $\bar{f}$ be the map on base spaces $\bar{f}: X \rightarrow Y$ induced by $f$. Let $\bar{f}^*E'$ be the pullback of $E'$ by $\bar{f}$, and consider the following diagram,

$$
\begin{array}{ccc}
E & \xrightarrow{h} & \bar{f}^*E' & \xrightarrow{f_*} & E' \\
\pi & \downarrow & \pi'_{\bar{f}} & \downarrow & \pi' \\
X & \xrightarrow{\bar{f}} & Y,
\end{array}
$$

where $h$ is defined by $h(e) = (\pi(e), f(e))$ [see (2.3)]. It is clear that $f = \bar{f}_* \circ h$. Moreover, $\bar{f}_*$ is an $S$-bundle morphism, and $h$ is an $S$-bundle homomorphism.

Q.E.D.

There are two basic problems concerning vector bundles on a given space: first, to determine, up to equivalence, how many different vector bundles there are on a given space, and second, to decide how “twisted” or how far from being trivial a given vector bundle is. The second question is the motivation for the theory of characteristic classes, which will be studied in Chap. III. The first question has different “answers,” depending on the category. A special important case is the following theorem. Let $U = U_{r,n}$ denote the universal bundle over $G_{r,n}$ (see Example 2.6).

**Theorem 2.17:** Let $X$ be a differentiable manifold and let $E \rightarrow X$ be a differentiable vector bundle of rank $r$. Then there exists an $N > 0$ (depending only on $X$) and a differentiable mapping $f: X \rightarrow G_{r,N}(\mathbb{R})$, so that $f^*U \cong E$. Moreover, any mapping $\tilde{f}$ which is homotopic to $f$ has the property that $\tilde{f}^*U \cong E$.

We recall that $f$ and $\tilde{f}$ are homotopic if there is a one-parameter family of mappings $F: [0, 1] \times X \rightarrow G_{r,N}$ so that $F|_{[0] \times X} = f$ and $F|_{[1] \times X} = \tilde{f}$. The content of the theorem is that the different isomorphism classes of differentiable vector bundles over $X$ are classified by homotopy classes of maps into the Grassmannian $G_{r,N}$. For certain spaces, these are computable (e.g., if $X$ is a sphere, see Steenrod [1]). If one assumes that $X$ is compact, one can actually require that the mapping $f$ in Theorem 2.17 be an embedding of $X$ into $G_{r,N}$ (by letting $N$ be somewhat larger). One could have phrased the above result in another way: Theorem 2.17 is valid in the category of continuous vector bundles, and there is a one-to-one correspondence between isomorphism classes of continuous and differentiable (and also real-analytic) vector bundles. However, such a result is not true in the case of holomorphic vector bundles over a compact complex manifold unless additional assumptions (positivity) are made. This is studied in Chap. VI. In fact, the problem of finding a projective algebraic embedding of a given compact complex manifold (mentioned in Sec. 1) is reduced to finding a class of holomorphic bundles over $X$ so that Theorem 2.17 holds for these bundles and the mapping $f$ gives an embedding into $G_{r,n}(\mathbb{C})$, which by Proposition 1.14 is itself projective algebraic. We shall not need the classification given by Theorem 2.17 in our later chapters and we refer the reader to the classical reference Steenrod.
[1] (also see Proposition III.4.2). A thorough and very accessible discussion of the topics in this section can be found in Milnor [2].

The set of all vector bundles on a space $X$ (in a given category) can be made into a ring by considering the free abelian group generated by the set of all vector bundles and introducing the equivalence relation that $E - (E' + E'')$ is equivalent to zero if there is a short exact sequence of the form $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$. The set of equivalence classes form a ring $K(X)$ (using tensor product as multiplication), which was first introduced by Grothendieck in the context of algebraic geometry (Borel and Serre [2]) and generalized by Atiyah and Hirzebruch [1]. For an introduction to this area, as well as a good introduction to vector bundles which is more extensive than our brief summary, see the text by Atiyah [1]. The subject of $K$-theory plays an important role in the Atiyah-Singer theorem (Atiyah and Singer [1]) and in modern differential topology. We shall not develop this in our book, as we shall concentrate more on the analytical side of the subject.

3. Almost Complex Manifolds and the $\bar{\partial}$-Operator

In this section we want to introduce certain first-order differential operators which act on differential forms on a complex manifold and which intrinsically reflect the complex structure. The most natural context in which to discuss these operators is from the viewpoint of almost complex manifolds, a generalization of a complex manifold which has the first-order structure of a complex manifold (i.e., at the tangent space level). We shall first discuss the concept of a $C$-linear structure on an $R$-linear vector space and will apply the (linear algebra) results obtained to the real tangent bundle of a differentiable manifold.

Let $V$ be a real vector space and suppose that $J$ is an $R$-linear isomorphism $J: V \rightarrow V$ such that $J^2 = -I$ (where $I$ = identity). Then $J$ is called a complex structure on $V$. Suppose that $V$ and a complex structure $J$ are given. Then we can equip $V$ with the structure of a complex vector space in the following manner:

$$(\alpha + i\beta)v := \alpha v + \beta Jv, \quad \alpha, \beta \in R, \quad i = \sqrt{-1}.$$ 

Thus scalar multiplication on $V$ by complex numbers is defined, and it is easy to check that $V$ becomes a complex vector space. Conversely, if $V$ is a complex vector space, then it can also be considered as a vector space over $R$, and the operation of multiplication by $i$ is an $R$-linear endomorphism of $V$ onto itself, which we can call $J$, and is a complex structure. Moreover, if $\{v_1, \ldots, v_n\}$ is a basis for $V$ over $C$, then $\{v_1, \ldots, v_n, Jv_1, \ldots, Jv_n\}$ will be a basis for $V$ over $R$.

Example 3.1: Let $C^n$ be the usual Euclidean space of $n$-tuples of complex numbers, $\{z_1, \ldots, z_n\}$, and let $z_j = x_j + iy_j, j = 1, \ldots, n$, be the real and imaginary parts. Then $C^n$ can be identified with $R^{2n} = \{x_1, y_1, \ldots, x_n, y_n\}$,
Scalar multiplication by \(i\) in \(\mathbb{C}^n\) induces a mapping \(J: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}\) given by

\[
J(x_1, y_1, \ldots, x_n, y_n) = (-y_1, x_1, \ldots, -y_n, x_n),
\]

and, moreover, \(J^2 = -1\). This is the \textit{standard complex structure} on \(\mathbb{R}^{2n}\).

The coset space \(GL(2n, \mathbb{R})/GL(n, \mathbb{C})\) determines all complex structures on \(\mathbb{R}^{2n}\) by the mapping \([A] \rightarrow A^{-1}JA\), where \([A]\) is the equivalence class of \(A \in GL(2n, \mathbb{R})\).

**Example 3.2:** Let \(X\) be a complex manifold and let \(T_x(X)\) be the (complex) tangent space to \(X\) at \(x\). Let \(X_0\) be the underlying differentiable manifold of \(X\) (i.e., \(X\) induces a differentiable structure on the underlying topological manifold of \(X\)) and let \(T_x(X_0)\) be the (real) tangent space to \(X_0\) at \(x\). Then we claim that \(T_x(X_0)\) is canonically isomorphic with the underlying real vector space of \(T_x(X)\) and that, in particular, \(T_x(X)\) induces a complex structure \(J_x\) on the real tangent space \(T_x(X_0)\). To see this, we let \((h, U)\) be a holomorphic coordinate system near \(x\). Then \(h: U \rightarrow U' \subset \mathbb{C}^n\), and hence, by taking real and imaginary parts of the vector-valued function \(h\), we obtain

\[
\tilde{h}: U \rightarrow \mathbb{R}^{2n}
\]

given by

\[
\tilde{h}(x) = (\text{Re } h_1(x), \text{Im } h_1(x), \ldots, \text{Re } h_n(x), \text{Im } h_n(x)),
\]

which is a real-analytic (and, in particular, differentiable) coordinate system for \(X_0\) near \(x\). Then it suffices to consider the claim above for the vector spaces \(T_0(\mathbb{C}^n)\) and \(T_0(\mathbb{R}^{2n})\) at \(0 \in \mathbb{C}^n\), where \(\mathbb{R}^{2n}\) has the standard complex structure. Let \(\{\partial/\partial z_1, \ldots, \partial/\partial z_n\}\) be a basis for \(T_0(\mathbb{C}^n)\) and let \(\{\partial/\partial x_1, \partial/\partial y_1, \ldots, \partial/\partial x_n, \partial/\partial y_n\}\) be a basis for \(T_0(\mathbb{R}^{2n})\). Then we have the diagram

\[
\begin{array}{ccc}
T_0(\mathbb{C}^n) & \cong & \mathbb{C}^n \\
\| & \|_R & \|_R \\
T_0(\mathbb{R}^{2n}) & \cong & \mathbb{R}^{2n}
\end{array}
\]

where \(\alpha\) is the \(\mathbb{R}\)-linear isomorphism between \(T_0(\mathbb{R}^{2n})\) and \(T_0(\mathbb{C}^n)\) induced by the other maps, and thus the complex structure of \(T_0(\mathbb{C}^n)\) induces a complex structure on \(T_0(\mathbb{R}^{2n})\), just as in Example 3.1. We claim that the complex structure \(J_x\) induced on \(T_x(X_0)\) in this manner is independent of the choice of local holomorphic coordinates. To check that this is the case, consider a biholomorphism \(f\) defined on a neighborhood \(N\) of the origin in \(\mathbb{C}^n\), \(f: N \rightarrow N\), where \(f(0) = 0\). Then, letting \(\zeta = f(z)\) and writing in terms of real and imaginary coordinates, we have the corresponding diffeomorphism expressed in real coordinates:

\[
\xi = u(x, y) \\
\eta = v(x, y),
\]

where \(\xi, \eta, x, y \in \mathbb{R}^n\) and \(\xi + i\eta = \zeta \in \mathbb{C}^n\), \(x + iy = z \in \mathbb{C}^n\). The map \(f(z)\) corresponds to a holomorphic change of coordinates on the complex
manifold \( X \); the pair of mappings \( u, v \) corresponds to the change of coordinates for the underlying differentiable manifold. The Jacobian matrix (differential) of these mappings corresponds to the transition functions for the corresponding trivializations for \( T(X) \) and \( T(X_0) \), respectively. Let \( J \) denote the standard complex structure in \( \mathbb{C}^n \), and we shall show that \( J \) commutes with the Jacobian of the real mapping. The real Jacobian of (3.1) has the form of an \( n \times n \) matrix of \( 2 \times 2 \) blocks,

\[
M = \begin{bmatrix}
  \frac{\partial u_\alpha}{\partial x_\beta} & \frac{\partial u_\alpha}{\partial y_\beta} \\
  \frac{\partial v_\alpha}{\partial x_\beta} & \frac{\partial v_\alpha}{\partial y_\beta}
\end{bmatrix}, \quad \alpha, \beta = 1, \ldots, n,
\]

which, by the Cauchy-Riemann equations (since \( f \) is a holomorphic mapping), is the same as

\[
\begin{bmatrix}
  \frac{\partial v_\alpha}{\partial y_\beta} & \frac{\partial u_\alpha}{\partial y_\beta} \\
  \frac{\partial u_\alpha}{\partial y_\beta} & \frac{\partial v_\alpha}{\partial y_\beta}
\end{bmatrix}, \quad \alpha, \beta = 1, \ldots, n.
\]

Thus the Jacobian is an \( n \times n \) matrix consisting of \( 2 \times 2 \) blocks of the form

\[
\begin{bmatrix}
  a & b \\
  -b & a
\end{bmatrix}.
\]

Moreover, \( J \) can be expressed in matrix form as an \( n \times n \) matrix of \( 2 \times 2 \) blocks with matrices of the form

\[
\begin{bmatrix}
  0 & 1 \\
  -1 & 0
\end{bmatrix}
\]

along the diagonal and zero elsewhere. It is now easy to check that \( MJ = JM \). It follows then that \( J \) induces the same complex structure on \( T_x(X_0) \) for each choice of local holomorphic coordinates at \( x \).

Let \( V \) be a real vector space with a complex structure \( J \), and consider \( V \otimes_{\mathbb{R}} \mathbb{C} \), the complexification of \( V \). The \( \mathbb{R} \)-linear mapping \( J \) extends to a \( \mathbb{C} \)-linear mapping on \( V \otimes_{\mathbb{R}} \mathbb{C} \) by setting \( J(v \otimes \alpha) = J(v) \otimes \alpha \) for \( v \in V, \alpha \in \mathbb{C} \). Moreover, the extension still has the property that \( J^2 = -I \), and it follows that \( J \) has two eigenvalues \( \{i, -i\} \). Let \( V^{1,0} \) be the eigenspace corresponding to the eigenvalue \( i \) and let \( V^{0,1} \) be the eigenspace corresponding to \( -i \). Then we have

\[
V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}.
\]

Moreover, conjugation on \( V \otimes_{\mathbb{R}} \mathbb{C} \) is defined by \( \overline{v \otimes \alpha} = v \otimes \bar{\alpha} \) for \( v \in V \) and \( \alpha \in \mathbb{C} \). Thus \( V^{1,0} \cong_{\mathbb{R}} V^{0,1} \) (conjugation is a conjugate-linear mapping). It is easy to see that the complex vector space obtained from \( V \) by means of the complex structure \( J \), denoted by \( V_J \), is \( \mathbb{C} \)-linearly isomorphic to \( V^{1,0} \), and we shall identify \( V_J \) with \( V^{1,0} \) from now on.
We now want to consider the exterior algebras of these complex vector spaces. Namely, denote $V \otimes_{\mathbb{R}} \mathbb{C}$ by $V_c$ and consider the exterior algebras $\wedge V_c, \wedge V^{1,0},$ and $\wedge V^{0,1}$.

Then we have natural injections

\[
\wedge V^{1,0} \hookrightarrow \wedge V_c, \\
\wedge V^{0,1} \hookrightarrow \wedge V_c,
\]

and we let $\wedge^{p,q} V$ be the subspace of $\wedge V_c$ generated by elements of the form $u \wedge w$, where $u \in \wedge^p V^{1,0}$ and $w \in \wedge^q V^{0,1}$. Thus we have the direct sum (letting $n = \dim_c V^{1,0}$)

\[
\wedge V_c = \sum_{r=0}^{2n} \sum_{p+q=r} \wedge^{p,q} V.
\]

We now want to carry out the above algebraic construction on the tangent bundle to a manifold. First, we have a definition.

**Definition 3.3:** Let $X$ be a differentiable manifold of dimension $2n$. Suppose that $J$ is a differentiable vector bundle isomorphism $J: T(X) \rightarrow T(X)$ such that $J_x: T_x(X) \rightarrow T_x(X)$ is a complex structure for $T_x(X)$; i.e., $J^2 = -I$, where $I$ is the identity vector bundle isomorphism acting on $T(X)$. Then $J$ is called an **almost complex structure** for the differentiable manifold $X$. If $X$ is equipped with an almost complex structure $J$, then $(X, J)$ is called an **almost complex manifold**.

We see that a differentiable manifold having an almost complex structure is equivalent to prescribing a $\mathbb{C}$-vector bundle structure on the $\mathbb{R}$-linear tangent bundle.

**Proposition 3.4:** A complex manifold $X$ induces an almost complex structure on its underlying differentiable manifold.

**Proof:** As we saw in Example 3.2, for each point $x \in X$ there is a complex structure induced on $T_x(X_0)$, where $X_0$ is the underlying differentiable manifold. What remains to check is that the mapping

\[
J_x: T_x(X_0) \rightarrow T_x(X_0), \quad x \in X_0,
\]

is, in fact, a $C^\infty$ mapping with respect to the parameter $x$. To see that $J$ is a $C^\infty$ vector bundle mapping, choose local holomorphic coordinates $(h, U)$ and obtain a trivialization for $T(X_0)$ over $U$, i.e.,

\[
T(X_0)|_U \cong h(U) \times \mathbb{R}^{2n},
\]

where we let $z_j = x_j + iy_j$ be the coordinates in $h(U)$ and $(\xi_1, \eta_1, \ldots, \xi_n, \eta_n)$
be the coordinates in $\mathbb{R}^{2n}$. Then the mapping $J|_U$ is defined by (with respect to this trivialization)

$$id \times J: h(U) \times \mathbb{R}^{2n} \longrightarrow h(U) \times \mathbb{R}^{2n},$$

where

$$J(\xi_1, \eta_1, \ldots, \xi_n, \eta_n) = (-\eta_1, \xi_1, \ldots, -\eta_n, \xi_n),$$

as before. That is, in this trivialization $J$ is a constant mapping, and hence $C^\infty$. Since differentiability is a local property, it follows that $J$ is a differentiable bundle mapping.

Q.E.D.

**Remark:** There are various examples of almost complex structures which do not arise from complex structures. The 2-sphere $S^2$ carries a complex structure $\cong \mathbb{P}_1(\mathbb{C})$, and the 6-sphere $S^6$ carries an almost complex structure induced on it by the unit Cayley numbers in $S^7$ (see Steenrod [1]). However, this almost complex structure does not come from a complex structure (it is not integrable; see the discussion below). Moreover, it is unknown whether $S^6$ carries a complex structure. A theorem of Borel and Serre [1] asserts that only $S^2$ and $S^6$ admit almost complex structures among the even dimensional real spheres. For more information about almost complex structures on manifolds, consult, e.g., Kobayashi and Nomizu [1] or Helgason [1].

Let $X$ be a differentiable $m$-manifold, let $T(X) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the tangent bundle, and let $T^*(X) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the cotangent bundle. We can form the exterior algebra bundle $\wedge T^*(X) \otimes_{\mathbb{R}} \mathbb{C}$, and we let $E^r(X) = E(X, \wedge^r T^*(X) \otimes_{\mathbb{R}} \mathbb{C})$. These are the complex-valued differential forms of total degree $r$ on $X$. We shall usually drop the subscript $c$ and denote them simply by $\mathcal{E}^r(X)$ when there is no chance of confusion with the real-valued forms discussed in Sec. 2. In local coordinates we have $\varphi \in \mathcal{E}^r(X)$ if and only if $\varphi$ can be expressed in a coordinate neighborhood by

$$\varphi(x) = \sum_{|I|=r} \varphi_I(x) dx_I,$$

where we use the multiindex notation of Sec. 2. and $\varphi_I(x)$ is a $C^\infty$ complex-valued function on the neighborhood. The exterior derivative $d$ is extended by complex linearity to act on complex-valued differential forms, and we have the sequence

$$\mathcal{E}^0(X) \xrightarrow{d} \mathcal{E}^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^m(X) \longrightarrow 0,$$

where $d^2 = 0$.

Suppose now that $(X, J)$ is an almost complex manifold. Then we can apply the linear algebra above to $T(X) \otimes_{\mathbb{R}} \mathbb{C}$. Namely, $J$ extends to a $\mathbb{C}$-linear bundle isomorphism on $T(X) \otimes_{\mathbb{R}} \mathbb{C}$ and has (fibrewise) eigenvalues $\pm i$. Let $T(X)^{1,0}$ be the bundle of $(+i)$-eigenspaces for $J$ and let $T(X)^{0,1}$ be the bundle of
(−i)-eigenspaces for J [note that these are differentiable subbundles of T(X)J]. We can define a conjugation on T(X)J,

\[ Q: T(X)J \rightarrow T(X)J, \]

by fibrewise conjugation, and, as before,

\[ Q: T(X)^{1,0} \rightarrow T(X)^{0,1} \]
is a conjugate-linear isomorphism. Moreover, there is a C-linear isomorphism

\[ T(X)^J \cong T(X)^{1,0}, \]

where T(X)J is the C-linear bundle constructed from T(X) by means of J. Let T*(X)^{1,0}, T*(X)^{0,1} denote the C-dual bundles of T(X)^{1,0} and T(X)^{0,1}, respectively. Consider the exterior algebra bundles \( \wedge T^*(X)_c, \wedge T^*(X)^{1,0} \), and \( \wedge T^*(X)^{0,1} \), and, as in the case of vector spaces, we have

\[ T^*(X)_c = T^*(X)^{1,0} \oplus T^*(X)^{0,1} \]

and natural bundle injections

\[ \wedge T^*(X)^{1,0} \rightarrow \wedge T^*(X)_c, \]
\[ \wedge T^*(X)^{0,1} \rightarrow \wedge T^*(X)_c, \]

and we let \( \wedge^{p,q} T^*(X) \) be the bundle whose fibre is \( \wedge^{p,q} T^*(X) \). This bundle is the one we are interested in, since its sections are the complex-valued differential forms of type \( (p, q) \) on X, which we denote by

\[ E^{p,q}(X) = E(X, \wedge^{p,q} T^*(X)). \]

Moreover, we have that

\[ E^r(X) = \sum_{p+q=r} E^{p,q}(X). \]

Note that the differential forms of degree r do not reflect the almost complex structure J, whereas its decomposition into subspaces of type \((p, q)\) does.

We want to obtain local representations for differential forms of type \((p, q)\). To do this, we make the following general definition.

**Definition 3.5:** Let \( E \rightarrow X \) be an S-bundle of rank \( r \) and let \( U \) be an open subset of X. A frame for \( E \) over \( U \) is a set of \( r \) S-sections \( \{s_1, \ldots, s_r\} \), \( s_j \in S(U, E) \), such that \( \{s_1(x), \ldots, s_r(x)\} \) is a basis for \( E_x \) for any \( x \in U \).

Any S-bundle \( E \) admits a frame in some neighborhood of any given point in the base space. Namely, let \( U \) be a trivializing neighborhood for \( E \) so that

\[ h: E|_U \rightarrow U \times K', \]

and thus we have an isomorphism

\[ h_*: S(U, E|_U) \rightarrow S(U, U \times K'). \]

Consider the vector-valued functions

\[ e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), \ldots, e_r = (0, \ldots, 0, 1), \]
which clearly form a (constant) frame for $U \times K^n$, and thus \{(h_a)^{-1}(e_1), \ldots, (h_a)^{-1}(e_r)\} forms a frame for $E|_U$, since the bundle mapping $h$ is an isomorphism on fibres, carrying a basis to a basis. Therefore we see that having a frame is equivalent to having a trivialization and that the existence of a global frame (defined over $X$) is equivalent to the bundle being trivial.

Let now $(X, J)$ be an almost complex manifold as before and let \{\$w_1, \ldots, w_n\$\} be a local frame (defined over some open set $U$) for $T^*(X)^{1,0}$. It follows that \{\$\bar{w}_1, \ldots, \bar{w}_n$\} is a local frame for $T^*(X)^{0,1}$. Then a local frame for $\wedge^{p,q}T^*(X)$ is given by (using the multiindex notation of Sec. 2)

\[
\{w^I \wedge \bar{w}^J\}, \quad |I| = p, \quad |J| = q, \quad (I, J \text{ strictly increasing}).
\]

Therefore any section $s \in \mathcal{E}^{p,q}(X)$ can be written (in $U$) as

\[
s = \sum_{|I| = p, |J| = q} a_{IJ} w^I \wedge \bar{w}^J, \quad a_{IJ} \in \mathcal{E}^0(U).
\]

Note that

\[
ds = \sum_{|I| = p, |J| = q} d a_{IJ} \wedge w^I \wedge \bar{w}^J + a_{IJ} d(w^I \wedge \bar{w}^J),
\]

where the second term is not necessarily zero, since $w_i(x)$ is not necessarily a constant function of the local coordinates in the base space (which will, however, be the case for a complex manifold and certain canonical frames defined with respect to local holomorphic coordinates, as will be seen below).

We now have, based on the almost complex structure, a direct sum decomposition of $\mathcal{E}^r(X)$ into subspaces $\{\mathcal{E}^{p,q}(X)\}$. Let $\pi_{p,q}$ denote the natural projection operators

\[
\pi_{p,q}: \mathcal{E}^r(X) \rightarrow \mathcal{E}^{p,q}(X), \quad p + q = r.
\]

We have in general

\[
d: \mathcal{E}^{p,q}(X) \rightarrow \mathcal{E}^{p+q+1}(X) = \sum_{r+s = p+q+1} \mathcal{E}^{r,s}(X)
\]

by restricting $d$ to $\mathcal{E}^{p,q}$. We define

\[
\partial: \mathcal{E}^{p,q}(X) \rightarrow \mathcal{E}^{p+1,q}(X)
\]

by setting

\[
\partial = \pi_{p+1,q} \circ d
\]

\[
\bar{\partial}: \mathcal{E}^{p,q}(X) \rightarrow \mathcal{E}^{p,q+1}(X)
\]

by setting

\[
\bar{\partial} = \pi_{p,q+1} \circ d.
\]

We then extend $\partial$ and $\bar{\partial}$ to all

\[
\mathcal{E}^*(X) = \sum_{r=0}^{\dim X} \mathcal{E}^r(X)
\]

by complex linearity.

Recalling that $Q$ denotes complex conjugation, we have the following elementary results.

**Proposition 3.6:**

\[
Q\bar{\partial}(Qf) = \partial f, \quad \text{for } f \in \mathcal{E}^*(X).
\]

†We shall use both $Q$ and overbars to denote the conjugation, depending on the context.
Proof: One has to verify that if \( f \in \mathcal{E}'(X) \) and \( p+q = r \), then \( Q\pi_{p,q}.f = \pi_{q,p}.Qf \) and \( Q(df) = dQf \), which are simple, and we shall omit the details. Q.E.D.

In general, we know that \( d^2 = 0 \), but it is not necessarily the case that \( \bar{\partial}^2 = 0 \). However, it follows from Proposition 3.6 that \( \bar{\partial}^2 = 0 \) if and only if \( \partial^2 = 0 \).

In general \( d: \mathcal{E}^{p,q}(X) \rightarrow \mathcal{E}^{p+q+1}(X) \) can be decomposed as

\[
d = \sum_{r+s=p+q+1} \pi_{r,s} \circ d = \partial + \bar{\partial} + \cdots.
\]

If, however, \( d = \partial + \bar{\partial} \), then

\[
d^2 = \partial^2 + \bar{\partial} \partial + \bar{\partial} \partial + \bar{\partial}^2,
\]

and since each operator projects to a different summand of \( \mathcal{E}^{p+q+2}(X) \) (in which case the operators are said to be of different type), we obtain

\[
\partial^2 = \bar{\partial} \partial + \bar{\partial} \partial = \bar{\partial}^2 = 0.
\]

If \( d = \partial + \bar{\partial} \) then we say that the almost complex structure is integrable.

**Theorem 3.7:** The induced almost complex structure on a complex manifold is integrable.

Proof: Let \( X \) be a complex manifold and let \( (X_0, J) \) be the underlying differentiable manifold with the induced almost complex structure \( J \). Since \( T(X) \) is \( \mathbb{C} \)-linear isomorphic to \( T(X_0) \) equipped with the \( \mathbb{C} \)-bundle structure induced by \( J \), it follows that, as \( \mathbb{C} \)-bundles,

\[
T(X) \cong T(X_0)^{1,0}.
\]

and similarly for the dual bundles,

\[
T^*(X) \cong T^*(X_0)^{1,0}.
\]

But \( \{dz_1, \ldots, dz_n\} \) is a local frame for \( T^*(X) \) if \( (z_1, \ldots, z_n) \) are local coordinates (recall that \( \{dz_1, \ldots, dz_n\} \) are dual to \( \{\partial/\partial z_1, \ldots, \partial/\partial z_n\} \)). We set

\[
\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad j = 1, \ldots, n
\]

and

\[
\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right), \quad j = 1, \ldots, n,
\]

where \( \{\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial y_1, \ldots, \partial/\partial y_n\} \) is a local frame for \( T(X_0)_c \) and \( \{\partial/\partial z_1, \ldots, \partial/\partial z_n\} \) is a local frame for \( T(X) \) (cf. Examples 2.4 and 2.5).

We observe that \( \partial/\partial z_j \) so defined is the complex (partial) derivative of a holomorphic function, and thus the assertion that these derivatives form a local frame for \( T(X) \) is valid. From the above relationships, it follows that

\[
dz_j = dx_j + idy_j\]

\[
d\bar{z}_j = dx_j - idy_j, \quad j = 1, \ldots, n,
\]
which gives
\[ dx_j = \frac{1}{2} (dz_j + d\bar{z}_j) \]
\[ dy_j = \frac{1}{2i} (dz_j - d\bar{z}_j), \quad j = 1, \ldots, n. \]
This in turn implies that for \( s \in \mathcal{E}^{p,q}(X) \)
\[ s = \sum_{I,J} a_{IJ} dz^I \wedge d\bar{z}^J. \]
We have
\[ ds = \sum_{j=1}^n \sum_{I,J} \left( \frac{\partial a_{IJ}}{\partial x_j} dx_j + \frac{\partial a_{IJ}}{\partial y_j} dy_j \right) \wedge dz^I \wedge d\bar{z}^J \]
\[ = \sum_{j=1}^n \sum_{I,J} \frac{\partial a_{IJ}}{\partial z_j} dz_j \wedge dz^I \wedge d\bar{z}^J \]
\[ + \sum_{j=1}^n \sum_{I,J} \frac{\partial a_{IJ}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz^I \wedge d\bar{z}^J. \]
The first term is of type \((p + 1, q)_s\) and so
\[ \partial = \sum_{j=1}^n \frac{\partial}{\partial z_j} dz_j, \]
and similarly
\[ \bar{\partial} = \sum_{j=1}^n \frac{\partial}{\partial \bar{z}_j} d\bar{z}_j, \]
and hence \( d = \partial + \bar{\partial} \). Thus the almost complex structure induced by the complex structure of \( X \) is integrable.

Q.E.D.

The converse of this theorem is a deep result due to Newlander and Nirenberg [1], whose proof has been simplified in recent years (see, e.g., Kohn [1], Hörmander [2]).

**Theorem 3.8 (Newlander-Nirenberg):** Let \((X, J)\) be an integrable almost complex manifold. Then there exists a unique complex structure \( \mathcal{O}_X \) on \( X \) which induces the almost complex structure \( J \).

We shall not prove this theorem, and instead refer the reader to Hörmander [2]. We shall mention, however, that it can easily be reduced to a local problem—and, indeed, to solving particular partial differential equations (namely the inhomogeneous Cauchy-Riemann equations) with estimates. In the case where \((X, J)\) is a real-analytic almost complex manifold, there are simpler proofs (see e.g., Kobayashi-Nomizu, Vol. II [1]). We shall not need this theorem, but we shall mention that it plays an important role in the study of deformations of complex structures on a fixed differentiable manifold, a topic we shall discuss in Chap. V.