Exercise sheet 2

From McDuff and Salamon: 6.20, 7.30, 7.31.

Resolution of singularities

Exercise 1.

- Describe the blow-up procedure which separates the following curves in \( \mathbb{C}^2 \):
  
  \[ c_1 = [y = 0] \quad c_2 = [y^3 = x^{10}] \]

- Let \( \mathbb{Z}_n \) act on \( \mathbb{C}^2 \) by multiplication by a \( n^{th} \) root of 1. Describe the resolution of the singularity at 0 of \( \mathbb{C}^2/\mathbb{Z}_n \).

Lefschetz pencils

Let \( L \) be the line bundle over \( \mathbb{P}^2 \) whose Chern class corresponds to the Poincaré dual of a complex line \( \mathbb{P}^1 \subset \mathbb{P}^2 \). If we cover \( \mathbb{P}^2 \) by 3 affine coordinate charts \( (x_i, y_i) \in U_i = \mathbb{C}^2, i = 1, 2, 3 \), related to the homogeneous coordinates \( [Z_0, Z_1, Z_2] \) by

\[
\begin{align*}
x_1 &= \frac{Z_1}{Z_0} \quad y_1 = \frac{Z_2}{Z_0} \\
x_2 &= \frac{Z_2}{Z_1} \quad y_2 = \frac{Z_0}{Z_1} \\
x_3 &= \frac{Z_0}{Z_2} \quad y_3 = \frac{Z_1}{Z_2}
\end{align*}
\]

Then \( L \) is trivial over each of these charts a has non vanishing sections \( s_i \) over \( U_i \) which satisfy

\[ s_1 = y_2 s_2 = x_3 s_3. \]

In this exercise, we consider the line bundles \( L^n \), which have trivializations \( s_i \) over \( U_i \) which are related by the transition functions

\[ s_1 = y_2^n s_2 = x_3^n s_3. \]

Exercise 2.

- Using Taylor series show that the holomorphic sections of \( L \) (i.e., sections whose coefficients are given by holomorphic functions in local coordinates) correspond to linear polynomials in \( \mathbb{C}^2 \). What do sections of \( L^2 \) correspond to? What about sections of \( L^n \)?

- For \( L^2 \), consider the sections \( s_1 \) and \( s_2 \) determined over \( U_1 \) by the polynomials
  
  \[ p_1(x_1, y_1) = x^2 + y^2 + 1 \quad p_2(x_1, y_1) = xy. \]

Find the base locus of the pencil determined by these two sections. Find the singular values of the induced pencil. By considering the projection onto the first coordinate, or otherwise, determine the genus of the fibers of the fibration associated to this pencil.

- For \( L^3 \), consider two sections \( s_1 \) and \( s_2 \) determined over \( U_1 \) by two generic cubic polynomials. How many points are there in the base locus? How many singular fibers are there? By considering the projection onto the first coordinate, or otherwise, determine the genus of the fibers of the fibration associated to this pencil.

- Can you guess what the genus of the fibers of a Lefschetz fibration associated to two generic sections of \( L^n \) is?