# Poisson geometry and holomorphic curves

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# Chapter 1

# Introduction

In this thesis we study global properties of Poisson manifolds using techniques from symplectic topology, especially holomorphic curves. In particular, we study the topology of regular Poisson manifolds (that is, symplectic foliations), log-symplectic manifolds, and scattering-symplectic manifolds. The first two are examined by looking at certain spaces of holomorphic curves, the last by relating it to a composition of symplectic cobordisms between contact manifolds (such relation can also be used to provide a framework for holomorphic curves).

## Poisson manifolds

Poisson manifolds are manifolds endowed with a Poisson bracket, i.e., an antisymmetric binary operation on the space of smooth functions, which is a derivation in both entries and satisfies the Jacobi identity. In formulas, on a manifold $X$, a Poisson structure is a bilinear map

$$\{\cdot,\cdot\} : C^\infty(X) \times C^\infty(X) \longrightarrow C^\infty(X) \tag{1.1}$$

such that

- $\{f,\, g\} = -\{g,\, f\}$
- $\{f,\, gh\} = g\{f,\, h\} + \{f,\, g\}h$
- $\{f,\, \{g,\, h\}\} = \{\{f,\, g\},\, h\} + \{g,\, \{f,\, h\}\}$

As first noted by Weinstein [38], Poisson brackets can be studied geometrically: one can identify them with bivector fields $\pi \in \Gamma(\bigwedge^2 TX)$ (or, equivalently, with antisymmetric linear maps $\pi^\# : T^*X \longrightarrow TX$), via the formula $\{f,\, g\} := \pi(df,\, dg)$. The Jacobi identity is equivalent to the vanishing of the tensor $[\pi,\pi]$, where $[\cdot,\cdot]$ is the Schouten bracket.

A Poisson bivector induces a singular symplectic foliation, such that the tangent spaces to the leaves are pointwise generated by the Hamiltonian vector fields $X_f := \{f, \cdot\}$. Conversely every singular symplectic foliation determines a Poisson structure – in particular, symplectic manifolds, and symplectically foliated manifolds, are examples.

In contrast with symplectic geometry, Poisson manifolds have plenty of local models, and the local theory is very rich. Moreover, every manifold admits many different Poisson structures, hence it is clear that there is no correlation between the existence of a Poisson structure and the topology of the manifold. However, there are *classes* of Poisson structures that do relate to the topology of the manifold, and have an interesting global theory: of course this is the case for the class of non-degenerate Poisson structures (symplectic forms), as well as codimension–1 symplectic foliations, just to name a few. The first step in "Poisson topology" is thus to single out some classes of Poisson structures that do interact non-trivially with the topology.

Many global aspects of Poisson brackets can be studied using Lie algebroids and Lie groupoids, which we now define.

**Definition 1.** A *Lie algebroid* is a vector bundle $A \longrightarrow X$ with a Lie bracket $[\cdot,\cdot]$ on $\Gamma(A)$, and a vector bundle map $a : A \longrightarrow TX$ (the *anchor map*), such that $[v, fw] = f[v,w] + a(v)(f)w$. A *Lie groupoid* is a category such that all arrows are invertible, with set of arrows $G$ and set of objects $X$, such that both $G$ and $X$ are smooth manifolds, the source and target maps $s, t : G \longrightarrow X$ are smooth submersions, and all the structure maps (composition, inversion) are smooth.

Each Poisson bivector gives rise to a Lie algebroid structure $(T^*X, [\cdot,\cdot]_\pi, a_\pi)$ on $T^*X$, by linearization. Each Lie groupoid gives rise to a Lie algebroid (via a *differentiation* process), but not every Lie algebroid comes from a Lie groupoid though. The requirement that the Lie algebroid $(T^*X, [\cdot,\cdot]_\pi, a_\pi)$ be coming from a Lie groupoid (possibly with some extra properties) is a natural example of a non-trivial global Poisson geometric property ([8], [9], [10]).

Another example which is going to play an important role in this thesis is that of *Lie algebroid symplectic structures*, defined in [20] (see also [19]). Let $A \longrightarrow X$ be a Lie algebroid, with anchor map $a : A \longrightarrow TX$. A Lie algebroid symplectic form on $A$ is a 2–form $\omega_A \in \Gamma(\bigwedge^2 A^*)$ which is non-degenerate, and closed with respect to the Lie algebroid differential $d_A$ (an operator defined analogously to the de Rham differential).
Let $\omega_A^{-1} : A^* \longrightarrow A$ be the inverse of a symplectic form on $A$. The following diagram defines a Poisson structure on $X$

$$\begin{array}{ccc} A & \xrightarrow{\omega_A^{-1}} & A^* \\ {\scriptstyle a^*}\uparrow & & \downarrow{\scriptstyle a} \\ T^*X & \xrightarrow{\pi} & TX \end{array} \tag{1.2}$$

## Our setup

In this thesis, we identify certain classes of Poisson manifolds for which it is possible to set up a theory of holomorphic curves, and study their topological properties.
Originally, holomorphic curves were introduced in symplectic geometry by Gromov [14]. On a symplectic manifold $(X, \omega)$ one can always find an almost complex structure $J : TX \longrightarrow TX$ which is suitably compatible with $\omega$, and is unique up to contractible choice. A *J-holomorphic curve* is a map $u : (\Sigma, j) \longrightarrow (X, J)$, with $du \circ j = Jdu$, where $(\Sigma, j)$ is a Riemann surface. Gromov noticed that on a compact symplectic manifold the space of $J$-holomorphic curves is well-behaved, and in particular has a natural compactification. Holomorphic curves have since then been a major tool in the study of symplectic topology.

Trying to apply the theory of holomorphic curves to the leaves of a general Poisson manifold presents some problems, because, for example, the leaves might be non-compact, and if the foliation is singular it is not clear how to sensibly define almost complex structures that vary smoothly from leaf to leaf.
We focus on the following three special cases:

> **Symplectic foliations**, i.e., foliations $\mathcal{F}$ of rank $2k$ together with a leafwise symplectic form $\omega \in \Gamma(\bigwedge^2 T^*\mathcal{F})$, such that $d_{\mathcal{F}}\omega = 0$ ($\omega$ is leafwise closed) and $\omega \wedge \cdots \wedge \omega \neq 0 \in \Gamma(\bigwedge^{2k} T^*\mathcal{F})$
>
> **Log-symplectic manifolds**, i.e., even dimensional Poisson manifolds $(X^{2n}, \pi)$ such that $\pi \wedge \cdots \wedge \pi \pitchfork 0 \in \Gamma(\bigwedge^{2n} TX)$ ([33], [16]).
>
> **Scattering-symplectic manifolds**, i.e., a class of even dimensional Poisson manifolds that are almost everywhere symplectic, except on a hypersurface where the symplectic leaves are points ([20])

All the Poisson structures above are Lie algebroid symplectic structures, in the sense of diagram (1.2), for some Lie algebroid $A$: the case of symplectic foliations can be recovered setting $A = T\mathcal{F}$; log-symplectic manifolds are the same as symplectic structures on the logarithmic tangent bundle $TX(-\log Z)$ (or $b$-tangent bundle ${}^bTX$; see Definition 3.2.6), where $Z$ is the set $\pi\wedge\cdots\wedge\pi = 0$; while scattering symplectic structures are by definition the Poisson structures coming from symplectic structures on the scattering tangent bundle ${}^{sc}TX$ (see Definition 4.2.1).
The Lie algebroid $A$ being symplectic, it admits a contractible set of compatible complex structures, and it makes sense to try and define holomorphic maps as those smooth functions whose tangent map lifts holomorphically to $A$. That is indeed the strategy that we will follow.

## Main results

**Symplectic foliations.** Our main results about symplectic foliations are contained in Chapter 5. We discuss the general theory of closed holomorphic curves in compact symplectically foliated manifolds, with no assumptions on the leaves or on the leafwise symplectic form. The main general result is the following comstruction of a moduli space of *simple* curves, i.e., curves that do not factor through a multiple cover (Corollary 5.4.3).

**Theorem 2.** *Let* $(X^{2k+q}, \mathcal{F}^{2k}, \omega)$ *be a symplectically foliated manifold, let* $A \in H_2(X; \mathbb{Z})$*. There exists a comeager set* $\mathcal{J}^{reg} \subset \mathcal{J}(\mathcal{F})$ *of the set of all compatible complex structures on* $T\mathcal{F}$ *such that* $\forall J \in \mathcal{J}^{reg}$ *the moduli space* $\mathcal{M}_g(A, J)^*$ *of simple* $J$*-holomorphic curves of genus* $g$ *representing the class* $A$ *is a smooth manifold of dimension* $(k-3)(2-2g) + 2c_1(A) + q$

Moreover, we provide a simple argument to prove a foliated version of Gromov's compactness theorem, for a special class of foliations (Theorem 5.5.2).

**Theorem 3.** *Assume* $(X^{2k+q}, \mathcal{F}^{2k}, \omega)$ *is a symplectically foliated manifold with stably complex normal bundle. Then Gromov's compactness holds: a sequence of holomorphic curves with bounded energy converges to a nodal holomorphic curve with values in a leaf.*

In particular the theorem holds for coorientable codimension–1 foliations. Both theorems have versions that take marked points into account.
We use the previous two results to show the following (Theorem 5.6.7, Corollary 5.6.8, Corollary 5.6.9).

**Theorem 4.** *Let* $(X^5, \mathcal{F}^4, \omega)$ *be a symplectically foliated* $5$*–manifold. Assume one of the leaves contains a symplectic sphere with trivial normal bundle. Then*

- *for each leaf* $F$*,* $\pi_2(F) \cdot \omega \neq 0$
- *if all leaves are compact, then they are all diffeomorphic to either* $\mathbb{C}P^2$*, or to a blow-up of a symplectic fibration with fiber* $\mathbb{C}P^1$

This is based on McDuff's work from the early nineties ([26], [25], [24]). In brief, one can always assume that a symplectic sphere with trivial normal bundle is $J$–holomorphic with respect to some $J \in \mathcal{J}^{\text{reg}}$, and, using a standard argument by McDuff ([25]), one can "transport" holomorphic curves throughout the entire manifold, proving the first statement. The second statement follows from the classification of (closed) symplectic rational and ruled surfaces ([24], [40]), in terms of the existence of certain holomorphic spheres. In our case, the existence of such spheres comes from the first part of the statement.

The proof of Theorem 2 mimics the standard proof for symplectic manifolds, as presented in [28]: one realizes the Cauchy-Riemann equation as a section of a Banach vector bundle over a Banach manifold of weakly differentiable maps, and proves that such section can be made transverse by generically perturbing the complex structure. In order to apply this strategy in the foliated case, we construct a manifold whose elements are maps $f : M \longrightarrow X$, with values in the leaves of a foliation (*leafwise maps*). This follows the construction by Eliasson ([12]) of a smooth structure on the space of all maps. More precisely, one fixes a suitable regularity class $\mathfrak{S}$ (e.g., $C^0$, $C^k$, Sobolev...), and considers $\mathfrak{S}$–regular maps, which form a manifold $\mathfrak{S}(M, X)$. We give two equivalent constructions (Theorem 5.2.8, Theorem 5.2.18) of a manifold structure on the space $\mathfrak{S}_0(M, \mathcal{F})$ of all leafwise maps that satisfy a constraint on the holonomy (we call them *leafwise maps with trivial holonomy*). This result holds for a general foliated manifold $(X, \mathcal{F})$ and compact domain $M$. We present it here as we could not find it in the literature.

**Theorem 5.** *The space $\mathfrak{S}_0(M, \mathcal{F})$ of leafwise maps with trivial holonomy is a smooth submanifold of $\mathfrak{S}(M, X)$.*

**Log-symplectic manifolds.** A log-symplectic manifold $(X, \pi)$ is symplectic in the complement of its *singular locus* $Z$, where $\pi$ is not invertible. Using a normal form from [16], we show that $X \setminus Z$ is an example of *symplectic manifold with cylindrical ends* (Section 2.3.1). Restricting one's attention to complex structures on $TX(-\log Z)$ that are compatible with the cylindrical structure on $X \setminus Z$, one can apply the tools of Symplectic Field Theory ([11]) and construct moduli spaces of curves, and study their compactness.
More precisely, it is known ([16]) that the singular locus $Z$ inherits a codimension–1 symplectic foliation. A complex structure on $TX(-\log Z)$ induces both a complex structure on $X \setminus Z$ and a complex structure on the leaves of the symplectic foliation. Given a Riemann surface $\Sigma$, we will consider the following types of maps:

(i) if $\partial\Sigma = \emptyset$, we study $u : \Sigma \longrightarrow X$ such that either $\mathrm{im}(u) \subset X \setminus Z$, or $\mathrm{im}(u)$ is contained in a symplectic leaf in $Z$

(ii) if $\partial\Sigma \neq \emptyset$, we require $u^{-1}(Z) = \partial\Sigma$, and $u|_{\partial\Sigma}$ to be transverse to the symplectic foliation

We prove that the space of curves (i) is always compact up to nodal curves (Definition 2.2.1), while in case (ii) compactness is achieved up to holomorphic buildings (Section 2.3.5), which are the limiting objects of holomorphic curves with cylindrical ends, studied in symplectic field theory. Smoothness of the moduli space also follows from general results in case (ii), while it is problematic in case (i). This is due to the fact that we are imposing the constraint that the complex structure is cylindrical, while smoothness of the moduli space can be achieved if one is allowed to perturb the complex structure generically. The situation can be saved in the case $\Sigma \cong S^2$, under some hypothesis on $Z$, which are always satisfied in dimension 4.
The results on the moduli spaces can be informally phrased as follows (in the case of closed curves we focus on dimension 4 for simplicity, see Section 3.4.4, Theorem 3.4.24, Theorem 3.5.1).

**Theorem 6.** *Let* $(X, \pi)$ *be a* 4*–dimensional log-symplectic manifold. Then, for a generic choice of cylindrical complex structure* $J$ *on* $TX(-\log Z)$*, the moduli space of simply covered* $J$*-holomorphic spheres is a smooth manifold, and it is compact up to nodal curves.*

**Theorem 7.** *Let* $(X, \pi)$ *be any log-symplectic manifold. Then, for a generic choice of cylindrical complex structure* $J$ *on* $TX(-\log Z)$*, the space of simply covered holomorphic curves with non-empty boundary is a smooth manifold, and it is compact up to holomorphic buildings.*

Using Theorem 6, we extend a result of McDuff ([26]) on the classification of rational and ruled symplectic manifolds to the log-symplectic case. It turns out that the nice behaviour of the moduli space of spheres implies that the log-symplectic 4–manifolds whose singular locus has a spherical leaf are very special. In fact, they are disjoint from the log-symplectic manifolds with aspherical leaves, and can be classified up to symplectic deformation equivalence. In the following statement, we will assume that the manifold is minimal, i.e., it is not the blow-up of another log-symplectic manifold at a point in the symplectic locus (Corollary 3.4.30, Proposition 3.4.37).

**Theorem 8.** *Let* $(X, \pi)$ *be a minimal log-symplectic* 4*–manifold. Assume that* $Z$ *contains a copy of* $S^1 \times S^2$*. Then* $X$ *supports an* $S^2$*-fibration over a (not necessarily orientable) closed surface* $B$*, with symplectic fibers, and such that* $Z$ *is a union of fibers. Moreover, the log-symplectic form depends only on the diffeomorphism type, up to deformations.*

Note that $X$ need not be orientable. The proof of Theorem 8 is very similar to that of Theorem 4, in that we use the moduli space to transport holomorphic spheres throughout the manifold. One then needs an extra step to show that such spheres actually foliate the manifold.
It is also possible to obtain a result similar to Theorem 8 using the moduli space of curves with boundary constructed in Theorem 7. Such result will have as input a sphere which intersects $Z$ transversely, and is holomorphic, in the complement of $Z$, with respect to a given cylindrical complex structure $J$. We mention here a simplified version of the statement (see Corollary 3.5.18).

**Theorem 9.** *Let* $(X, \pi)$ *be a minimal log-symplectic* 4*–manifold, such that* $Z \cong \bigsqcup_i S^1 \times \Sigma_{g_i}$*. Fix a cylindrical complex structure* $J$*, and assume there exists a* $J$*–holomorphic sphere* $S \subset X$ *with* $S \pitchfork Z$ *and* $S \cdot S = 0$*. Then*

- $g_i = g_j$ *for all* $i$, $j$
- *there is a continuous fibration* $f : X \longrightarrow \Sigma_{g_i}$ *with fiber* $S^2$*, such that* $Z$ *is a union of sections.*

As before, there is a non-orientable version of the statement, where the assumption of the existence of a holomorphic sphere is replaced by the existence of an $\mathbb{R}P^2$ that is holomorphic in the complement of $Z$. The proof of Theorem 9 is conceptually similar to that of Theorem 8, but technically much harder. The reason is that while we would like to move the holomorphic sphere through the manifold, we can only move its holomorphic pieces, in the complement of $Z$. This is possible if the pieces have genus 0, thanks to the properties of the moduli spaces in such case. Then, we try to glue the pieces back together. The gluing can be performed only if the resulting surface is topologically $S^2$ or $\mathbb{R}P^2$, for purely combinatorial reasons. The fact that the resulting fibration is only continuous is due to the fact that while performing the gluing we lose control over the derivatives at $Z$ of the holomorphic embeddings.

**Scattering-symplectic manifolds.** The scattering tangent bundle is a Lie algebroid ${}^{sc}TX$ associated with a manifold $X$ with a hypersurface $Z$. It is by definition the bundle whose sheaf of sections is generated by $x^2\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial y_i}$, with $Z = \{x = 0\}$, and $y_i$ forming a set of coordinates on $Z$. A scattering symplectic structure is a Poisson structure induced by a symplectic structure on ${}^{sc}TX$, in the sense of the diagram (1.2).
Like in the log-sympectic case, a scattering-symplectic structure $\pi$ is symplectic on $X \setminus Z$. The structure induced on $Z$ is not a symplectic foliation, but a *cooriented contact structure*. If $Z$ is connected, then, each component of $X \setminus Z$ is a symplectic manifold such that the boundary inherits a contact structure – i.e. it is a symplectic filling, which we call a *scattering symplectic filling*. It is natural to ask how does such notion of fillability compare to the other notions that already exist in the literature, and in fact such question was already answered in a special case in [20]. We study the problem in the general case, and show that scattering-fillability is equivalent to weak fillability, in the sense of [23] (see Theorem 4.4.2).

**Theorem 10.** *A cooriented contact manifold has an orientable scattering-filling if and only if it is weakly fillable, in the sense of [23].*

The proof is an explicit manipulation in normal form in a collar neighbourhood of the boundary. In fact, what the proof really shows is that any scattering symplectic manifold is a gluing of weak symplectic cobordisms of contact manifolds (up to deformation). The relation with holomorphic curves is somehow implicit, in that [23] and [31] have shown that there is a sensible theory of holomorphic curves in weak symplectic cobordisms (based once again on symplectic field theory). In the same paper [23] the authors prove (using a holomorphic curve argument) that weak fillability is obstructed by overtwistedness. Hence:

**Corollary 11.** *The singular locus of a scattering symplectic manifold, if connected, is tight (i.e. not overtwisted).*

## Organization of the thesis

In Chapter 2 we provide some background material on holomorphic curves, including a quick review of some aspects of the intersection theory of punctured holomorphic curves that we will use in Chapter 3.

Chapter 3 is devoted to log-symplectic manifolds. We quickly review some general facts, making the relation with symplectic manifolds with cylindrical ends explicit. We introduce a natural notion of holomorphic curves using the language of logarithmic tangent bundles, and prove its equivalence with the usual holomorphic curves in symplectic field theory, and construct the moduli spaces via this equivalence. We first focus on closed curves, and use them to classify ruled surfaces; then we move on to curves with boundary, and prove Theorem 9.
Chapter 4 deals with scattering-symplectic manifolds, contains a review of a few notions of symplectic fillability of contact manifolds, and proves the equivalence between scattering fillings and weak fillings.
Finally Chapter 5 contains the construction of the manifold of leafwise maps (with trivial holonomy), and a construction of a moduli space of closed holomorphic curves. We conclude with a study of codimension–1 foliations on 5–manifolds containing certain symplectic spheres.

# Chapter 2

# Holomorphic curves

This chapter contains the definitions and main properties of holomorphic curves in symplectic manifolds, and the construction of their moduli spaces. It consists of three parts: first we introduce and study curves with closed domain, and their moduli spaces (Section 2.2). The second part deals with Symplectic Field Theory (SFT), i.e. the theory of punctured holomorphic curves. The third part develops the intersection theory of holomorphic curves, with a focus on the punctured case.
The notation we use is the standard one, thus the reader who is familiar with the basics of holomorphic curves, but not with SFT and/or intersection theory, can safely skip the first few pages.
Our treatment is by no means exhaustive, as we only focus on the results that we are going to need in the applications in the later chapters. We refer to [28] for a general treatment of holomorphic curves, with a focus on the compact case; to [42], [11], [3] for punctured holomorphic curves, and to [39], [35] for the intersection theory of punctured holomorphic curves.

## 2.1 General definitions

Pseudo-holomorphic curves were introduced in [14], and studied ever since to address a variety of problems in symplectic geometry.
The setup is the following. Given two almost complex manifolds $(Y, J_Y)$ and $(X, J_X)$, it makes sense to say when a smooth map $f : (Y, J_Y) \longrightarrow (X, J_X)$ is *holomorphic*: one requires that the differential be complex-linear, i.e.

$$df \circ J_Y = J_X \circ df \tag{2.1}$$

Equation (2.1) is the *Cauchy-Riemann equation.* The *Cauchy-Riemann operator* is the non-linear differential operator $\overline{\partial}_{J_Y J_X} f = \frac{1}{2}(df + J_X \circ df \circ J_Y)$. Of course $\overline{\partial}_{J_Y J_X} f = 0$ if and only if Equation (2.1) holds.

In principle, finding solutions of Equation (2.1) in the "almost" setting might be harder than in the integrable case. For example, consider maps $f : (Y, J) \longrightarrow \mathbb{C}$. The integrability of $J_Y$ is the existence of $\dim(Y)$ independent solutions of Equation (2.1). However, whenever $Y$ is 2-dimensional (i.e. it is a Riemann surface) the behaviour of the set of solutions of the Cauchy-Riemann equation does not depend (in general) on the integrability of $J_X$. If $Y$ is 2-dimensional, we normally use the notation $u : (\Sigma, j) \longrightarrow (X, J)$, and we say that $u$ is a *holomorphic curve.*

The most common strategy to employ holomorphic curves consists of constructing good *moduli spaces* of them. First of all, one declares two holomorphic curves to be equivalent is they differ by a reparametrization of the domain. An equivalence class of holomorphic curves modulo reparametrization of the domain is called an *unparametrized holomorphic curve.* Then one considers sets of unparametrized holomorphic curves, normally with the same domain $S$ (at least as smooth surfaces). These sets come naturally endowed with the topology induced from the Fréchet topology on the set of smooth maps $\{u : S \longrightarrow X\}$. The two main problems that one faces in general are:

(i) is the space of holomorphic curves a smooth manifold[1]?

(ii) is the resulting space compact?

In general, the answer to (*ii*) is no, but the failure of compactness is well understood – it generally depends on the topology of $\Sigma$ and $X$. Question (*i*) is also known as the *transversality problem*, as one wants to realize the moduli space as the zero set of a transverse section of a vector bundle.

How is this related to symplectic geometry? First of all any symplectic manifold $(X, \omega)$ is almost complex. Moreover, the symplectic form and the complex structure can be used to define a Riemannian metric, with which one can estimate the area of surfaces, which proves very useful in practise.
More precisely, there exists a non-empty, contractible set of *tame* almost complex structures, i.e. almost complex structures $J$ such that the symmetrization $g(\cdot, \cdot)$ of $\omega(\cdot, J\cdot)$ is a Riemannian metric. One could also consider *compatible* almost complex structures, for which the tensor $\omega(\cdot, J\cdot)$ is already a metric. Compatible almost complex structures also form a non-empty contractible space. What one does is to choose an almost complex structure which is compatible/tame with the given symplectic form $\omega$, and study holomorphic curves with respect to such complex structure. Another natural question is

(iii) (how) does the space of holomorphic curves depend on the choice of almost complex structure?

It turns out that the space of $J$-holomorphic curves does depend on the choice of $J$. Even the answer to question (*i*) above does, and complex structures for

[1] In a way compatible with the setup, for example compatible with the Fréchet manifold structure of the space of smooth maps.

which (a certain class of) holomorphic curves form smooth moduli spaces are called *(Fredholm) regular* (for that particular class). It is a crucial problem to determine which complex structures are regular (and before that, if they exist at all). Given two regular almost complex structures, under some conditions one can prove that the moduli spaces of curves with respect to the two structures are closely related (though not diffeomorphic). More details will be given in the next sections.

## 2.2 Closed holomorphic curves in closed manifolds

We restrict now our attention to the case of closed Riemann surfaces $(\Sigma, j)$ mapped to closed symplectic manifolds $(X, \omega)$. The standard reference for this is [28]. Endow $X$ with a tame almost complex structure $J$.

### 2.2.1 Energy

To each smooth map $u : (\Sigma, j) \longrightarrow (X, J)$ one can associate a non-negative real number, called *energy* of the map, defined as

$$E(u) := \int_{\Sigma} |du|^2 d\text{vol}_{\Sigma} \tag{2.2}$$

Here the norm of $du$ is defined using the Riemannian metrics on $\Sigma$ and $X$ constructed using the complex structures and the symplectic forms $d\text{vol}_{\Sigma}$ and $\omega$. It is a key fact that $E(u)$ is a topological quantity, whenever $u$ is holomorphic and $J$ is tame. Indeed, if $u$ is holomorphic, one has

$$E(u) = \int_{\Sigma} u^* \omega \tag{2.3}$$

This equality is known as the *energy identity*. It implies that the energy of a holomorphic curve only depends on its homology class $[u] := u_*([\Sigma]) \in H_2(X; \mathbb{Z})$ (in particular it does not depend on the choice of tame almost complex structure). A first consequence of the energy identity is that a homologically trivial holomorphic curve needs to be constant.

### 2.2.2 Compactness

A sequence of (unparametrized) holomorphic curves might not converge to a holomorphic curve, in the $C^\infty$ topology. However, there exist more general objects, called *nodal holomorphic curves*, that can appear as limits of sequences of holomorphic curves, with respect to a certain notion of limit known as "Gromov convergence". It is a well-known fact, due to Gromov, that any sequence of holomorphic curves with bounded energy admits a subsequence converging to a nodal curve - which implies that the set of nodal curves is compact. Here we give the definition of nodal curve, after which we state the results.

**Definition 2.2.1** (Nodal curves)**.** A *nodal holomorphic curve* consists of the following data:

- a Riemann surface $(S, j) = \bigsqcup_a (S_a, j_a)$
- a holomorphic map $u = (u_a)_a : \bigsqcup_a S_a \longrightarrow X$, where none of the $u_a$'s is constant
- a set $N$ of points in $\sqcup S_a$, called *nodes*, with a fixed-point-free involution $\sigma : N \longrightarrow N$, such that $u(n) = u(\sigma(n))\ \forall n \in N$, and such that the singular surface obtained by glueing $n$ with $\sigma(n)$ is connected

We define the homology class of a nodal curve as $\sum_a [u_a]$. Taking connected sums of the components $S_a$ at corresponding nodes, one obtains a smooth surface $S$. The *genus* of a nodal curve $(S, u, N)$ is by definition the genus of $S$.

**Theorem 2.2.2** (Gromov compactness, [28], Theorem 4.6.1)**.** *Let $X$ be a compact manifold, and let $J_k$ be a convergent sequence of almost complex structures on $X$, and let $J_\infty$ be the limit. Let $u_k$ be a sequence of $J_k$-holomorphic curves satisfying a uniform energy bound $E(u_k) \leq C$. Then there exists a subsequence converging to a $J_\infty$-holomorphic nodal curve $u_\infty$. If all the curves have the same genus, then so does the limit. If $[u_k] = A$, then $[u_\infty] = A$.*

**Remark 2.2.3.** If $(X, \omega)$ is symplectic, and $J_k$ is a sequence of tame almost complex structures, the energy bound can be provided by the requirement that all curves represent the same homology class $[u_k] = A$ (Equation (2.3)).

Note that if the homology class $A$ cannot be split as a sum $A = A_1 + \cdots + A_N$, with $A_i$ representable by a holomorphic map (e.g. if $A = A_1 + \cdots + A_N$ (non-trivially) then $A_i \cdot \omega \leq 0$), then Theorem 2.2.2 implies that the set of curves $u : \Sigma \longrightarrow X$ with $[u] = A$ is compact.

### 2.2.3 Moduli spaces

One can collect holomorphic curves of the same genus in moduli spaces, i.e. one can give a geometric structure to the set of holomorphic maps from a surface with a fixed genus. More precisely, one would like to consider *unparametrized holomorphic curves*, in the following sense. We say that two holomorphic curves $u : (\Sigma, j) \longrightarrow (X, J)$, $u' : (\Sigma', j') \longrightarrow (X, J)$ are equivalent if there is a biholomorphism $\phi : (\Sigma, j) \longrightarrow (\Sigma', j')$ such that $u' \circ \phi = u$ (they differ by a reparametrization). The set of self equivalences of a holomorphic curve is called automorphism group.
We consider

$$\mathcal{M}_g(J) := \{(j, u) : u : \Sigma_g \longrightarrow X \text{ smooth such that } du \circ j = J du\}/\sim$$

The topology of this set is the quotient topology induced by the (Fréchet) topology on the set of all smooth maps. To give a smooth structure to this space, one views the operator $\overline{\partial}_J$, defined as

$$\overline{\partial}_J u = \frac{1}{2}(du + J du \circ j) \tag{2.4}$$

as a section of an infinite dimensional vector bundle, in the following way. Enlarge the set of smooth maps considering a set $\mathcal{B} = \mathcal{B}(\Sigma, X)$ of continuous Sobolev maps $u : \Sigma \longrightarrow X$. Given $u \in \mathcal{B}$ smooth, take the Banach space $\mathcal{E}_u \supset \Omega^{0,1}(\Sigma; u^*TX)$, (a suitable Sobolev completion of the space) of complex antilinear 1-forms on $\Sigma$ with coefficients in $u^*TX$. The union of those Banach spaces forms a Banach bundle $\pi : \mathcal{E} \longrightarrow \mathcal{B}$, of which $\overline{\partial}_J$ is a section. This section is Fredholm. If it is transverse (i.e. its vertical component at solutions is surjective) then the zero set is smooth, as a consequence of the implicit function theorem, and finite dimensional.

**Definition 2.2.4.** A holomorphic curve $u$ is *Fredholm regular* if the linearization at $u$ of the operator $\overline{\partial}_J$ is surjective.

Given $u : \Sigma \longrightarrow X$, denote with $[u] := u_*([\Sigma]) \in H_2(X; \mathbb{Z})$ the *homology class of the map.* Letting $A \in H_2(X; \mathbb{Z})$ be a singular homology class, define

$$\mathcal{M}_g(A, J) := \{[(j, u)] \in \mathcal{M}_g(J) \,:\, [u] = A\}$$

The dimension of the component of this set containing a solution $u$ is computed using the Fredholm index of the operator $\overline{\partial}_J$ at $u$ (which is also called the *index of* $u$). This only depends on the homology class $[u] = A$: one has $\operatorname{ind}(u) = n\chi(\Sigma) + 2c_1(A)$. The dimension of the moduli space is obtained from the index adding the dimension of the space of complex structures on $\Sigma$, and subtracting the dimension of the automorphism groups. The resulting integer is called the *virtual dimension* of $\mathcal{M}_g(A, J)$. One can compute that for all $\Sigma$ and $A$ the virtual dimension is $\operatorname{vdim}\mathcal{M}_g(A, J) = (n-3)\chi(\Sigma) + 2c_1(A)$.

**Theorem 2.2.5** ([41], Theorem 4.43)**.** *The set $\mathcal{M}_g(A, J)^{reg} \subset \mathcal{M}_g(A, J)$ of Fredholm regular curves is a smooth orbifold of dimension $vdim\mathcal{M}_g(A, J)$.*

In order to study the regularity of holomorphic curves it is important to make a distinction between simple and multiply covered curves:

**Definition 2.2.6.** A holomorphic map $u : \Sigma \longrightarrow X$ is *multiply covered* if there exists a (non-trivial) branched cover $c : \Sigma \longrightarrow \Sigma'$ and a holomorphic map $v : \Sigma' \longrightarrow X$ such that $u = v \circ c$. A curve which is not multiply covered is called *simple.*

In general, multiply covered curves pose more problems to the transversality. For closed simply covered curves one has the following result. Denote with $\mathcal{J}(\omega)$ a set of $\omega$-compatible or $\omega$-tame almost complex structures on $X$.

**Theorem 2.2.7** ([41], Theorem 4.48)**.** *There exists a comeager[2] subset of complex structures $\mathcal{J}^{reg} = \mathcal{J}^{reg}(\omega) \subset \mathcal{J}(\omega)$ such that for all $J \in \mathcal{J}^{reg}$, every simple $u \in \mathcal{M}_g(J)$ is Fredholm regular.*

[2] A countable intersection of open dense subsets.

Let $A \in H_2(X;\mathbb{Z})$. Denote with $\mathcal{M}^*_g(A,J)$ the set of genus $g$ simple $J$-holomorphic curves in the homology class $A$.

**Corollary 2.2.8.** *There exists a comeager subset of complex structures $\mathcal{J}^{reg} = \mathcal{J}^{reg}(\omega) \subset \mathcal{J}(\omega)$ such that for all $J \in \mathcal{J}^{reg}$ the space $\mathcal{M}^*_g(A,J)$ is a smooth manifold of dimension $vdim \mathcal{M}_g(A,J) = (n-3)\chi(\Sigma) + 2c_1(A)$*

The proof of Corollary 2.2.8 goes more or less as follows. One considers the manifold $\mathcal{B}(\Sigma,\, X)$ of suitably regular maps $u : \Sigma \longrightarrow X$, and the space $\mathcal{J}(X,\, \omega)$ of suitably regular tame or compatible almost complex structures. Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus $g$. The key point of the proof is to show that the subset

$$\tilde{\mathcal{M}}_g(X) \subset \mathcal{B}(\Sigma,\, X) \times \mathcal{M}_g \times \mathcal{J}(X,\, \omega)$$

defined as

$$\tilde{\mathcal{M}}_g(X) = \{(u,\, j,\, J) \,:\, \overline{\partial}_{j,J} u = 0\}$$

is a smooth submanifold. This can be proven showing that the vertical differential of the map

$$\overline{\partial} : \mathcal{B} \times \mathcal{M}_g \times \mathcal{J} \longrightarrow \mathcal{E} \tag{2.5}$$

is surjective. Now, denoting by $p_{\mathcal{J}} : \tilde{\mathcal{M}}_g(X) \longrightarrow \mathcal{J}$ the restriction of the projection onto the second factor, one has by definition that $\mathcal{M}_g(J) = p_{\mathcal{J}}^{-1}(J)/\sim$ (as sets). One can check (see proof of 3.1.6 (II) in [28]) that the differential of $p_{\mathcal{J}}$ and the vertical differential of $\overline{\partial}_J$ have isomorphic kernel and cokernel, in particular one is surjective if and only if the other is. Hence, for regular values $J$ of $p_{\mathcal{J}}$, $J$ is Fredholm regular. In particular, by Sard's theorem, the set of regular almost complex structures is comeager, and in particular dense.

The discussion above is useful to prove that "regular holomorphic curves are stable under deformations", in the sense of the following lemma.

**Lemma 2.2.9** ([28], Remark 3.2.8)**.** *Let $J$ be an almost complex structure, and let $u$ be a Fredholm regular $J$-holomorphic curve. Then there exists a regular almost complex structure $\tilde{J}$, close to $J$, and a $\tilde{J}$-holomorphic curve $\tilde{u}$ close to $u$.*

*Proof.* Since $p_{\mathcal{J}}$ is a submersion at $(u,\, J)$, it is a submersion in a neighbourhood $U$ of $(u,\, J)$. By density of the set of regular complex structures, there is a regular $\tilde{J}$ in a neighbourhood of $J$ contained in $p_{\mathcal{J}}(U)$. The pair $(\tilde{u},\, \tilde{J})$ can be found in $U \cap p_{\mathcal{J}}^{-1}(\tilde{J})$. □

We conclude the section mentioning some criteria to ensure that a holomorphic curve is Fredholm regular.

**Proposition 2.2.10** ([28], Lemma 3.3.1)**.** *Let $(X,\, J)$ be an integrable complex manifold, and $u : \mathbb{C}P^1 \longrightarrow X$ holomorphic. Write $u^*TX = L_1 \oplus \cdots \oplus L_n$, $L_i$ holomorphic line bundle. $u$ is Fredholm regular if and only if $c_1(L_i) \geq -1\ \forall\, i$'s.*

The proof of this is a simple cohomology computation using the Riemann-Roch formula and Serre duality. A variation of Proposition 2.2.10 can be used to prove the following:

**Proposition 2.2.11** ([28], Corollary 3.3.5). *Let $(M, \omega)$ be a symplectically aspherical symplectic manifold ($\pi_2(M) \cdot \omega = 0$). Let $X = S^2 \times M$, and $A = [S^2 \times \{*\}]$. Consider a product complex structure $J = j \times J_M$ on $X$, where $J_M$ is an $\omega$-tame almost complex structure. Then any holomorphic $u$ in the homology class $A$ is Fredholm regular.*

A different more difficult automatic transversality result, that only holds in dimension 4, is the following.

**Theorem 2.2.12** ([17], Theorem 1). *Let $(M, J)$ be an almost complex 4-manifold, and $u : \Sigma \longrightarrow X$ an immersed holomorphic curve, such that $ind(u) > 2g - 2$. Then $u$ is Fredholm regular. In particular if $u$ is an immersed sphere of non-negative index it is Fredholm regular.*

## 2.3 Punctured holomorphic curves

We describe here some general aspects of the theory of punctured holomorphic curves in open symplectic manifolds, under some assumptions on the geometry at infinity. In particular, we will study open manifolds which can be constructed from a compact manifold with boundary by attaching a cylindrical end (i.e. a copy of the boundary times a half line). Moreover the boundary needs to be a special type of manifold, i.e. it needs to carry a so called *stable Hamiltonian structure*.

The study of punctured holomorphic curves in manifolds with cylindrical ends is part of a theory known as Symplectic Field Theory (SFT), which was introduced in 2000 in [11].

A piece of notation: we will denote with $\Sigma$ a closed Riemann surface, and given a finite set of punctures $\Gamma \subset \Sigma$, we will denote the corresponding punctured surface $\Sigma \setminus \Gamma$ with $\dot{\Sigma}$.

### 2.3.1 Stable Hamiltonian structures and cylindrical ends.

**Definition 2.3.1.** A *stable Hamiltonian structure* on a $(2n-1)$-dimensional manifold $Z$ is a pair $(\alpha, \beta)$ consisting of a 1-form $\alpha$ and a 2-form $\beta$ such that

- $d\beta = 0$
- $\alpha \wedge \beta^{n-1}$ is a volume form
- $\ker \beta \subset \ker d\alpha$

The *Reeb vector field* of a stable Hamiltonian structure is the unique vector field $R$ such that $\iota_R \beta = 0$ and $\alpha(R) = 1$.

**Remark 2.3.2.** The form $\omega = d(s\alpha) + \beta$ is symplectic on $(-\varepsilon, \varepsilon) \times Z$.

**Example 2.3.3.** If $\alpha$ is a contact form, then $(\alpha, d\alpha)$ is stable Hamiltonian.

**Example 2.3.4.** If $\alpha$ and $\beta$ are both closed, with $\alpha \wedge \beta^{n-1} \neq 0$, then $(\alpha, \beta)$ is stable Hamiltonian. In this case $d(s\alpha) + \beta = ds \wedge \alpha + \beta$ is symplectic on $\mathbb{R} \times Z$. These stable Hamiltonian structures are called *cosymplectic structures*.

**Definition 2.3.5.** A symplectic manifold $(W, \omega)$ has *stable Hamiltonian boundary* $\partial W = Z^+ \sqcup Z^-$ if there exists a vector field $V$ (defined in a neighbourhood of the boundary), which points outwards at $Z^+$ and inwards at $Z^-$, such that $((\iota_V \omega)|_{T\partial W}, \omega|_{T\partial W})$ is a stable Hamiltonian structure on $\partial W$, inducing the boundary orientation on $Z^+$, and the opposite orientation on $Z^-$.

Let us assume for simplicity that $Z^- = \emptyset$. A collar neighbourhood of a stable Hamiltonian boundary is symplectomorphic to $(-\varepsilon, 0] \times \partial W$ with symplectic form $\omega = d(s\alpha) + \beta$ (Moser argument). The symplectomorphism is obtained by realizing the vector field $V$ as $V = \frac{\partial}{\partial s}$. This naturally endowes $\widehat{W} := W \cup [0, \infty) \times \partial W$ with a smooth structure. Moreover, the form $\omega$ extends smoothly as a closed form $\widehat{\omega}_\varphi$ on $\widehat{W}$, with $\varphi : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ a diffeomorphism equal to the identity near 0. The extension is defined by the formula $\widehat{\omega}_\varphi|_{[0, \infty) \times \partial W} := d(s\alpha) + \beta$. One calls the manifold $(\widehat{W}, \widehat{\omega}_\varphi)$ the *completion* of $(W, \omega)$.

### 2.3.2 Cylindrical complex structures and holomorphic maps

We shall restrict our attention to a special class of complex structures.

**Definition 2.3.6.** A complex structure $J$ on $\widehat{W}$ is *cylindrical* if it is $\widehat{\omega}$-tame, and on the cylindrical end it satisfies

- $J$ is $s$-invariant
- $J\partial_s = R$
- $J(\ker \alpha) = \ker \alpha$
- $J$ is $\beta$-compatible on $\ker \alpha$

As in the closed case, there is a useful notion of energy of a holomorphic curve. Let $u : \dot{\Sigma} \longrightarrow \widehat{W}$ be a holomorphic map.

**Definition 2.3.7.** The *energy* of a punctured holomorphic curve is the quantity $E(u) := \sup_{\varphi} \int_\Sigma \hat{\omega}_\varphi$. The supremum is taken over all diffeomorphisms $\varphi : [0, +\infty) \longrightarrow [0, \varepsilon)$ equal to the identity near 0.

Finite energy curves enjoy special geometric properties, under the following non-degeneracy assumption on the stable Hamiltonian boundary.

**Definition 2.3.8.** A stable Hamiltonian structure $(\alpha, \beta)$ on $Z$ is *Morse-Bott* if for all $T > 0$

- the set $Z_T \subset Z$ of points belonging to a $T$-periodic Reeb orbit is a closed submanifold
- rank $d\alpha|_{M_T}$ is locally constant
- $T_p Z_T = \ker(d_p \phi^T - 1)$ for all $p \in Z_T$, where $\phi^T$ is the time–$T$ map of the Reeb flow.

**Example 2.3.9.** Take a closed disc $(D^2, \omega_{\mathrm{std}})$ and a symplectic manifold $(M, \eta)$. The product $(D^2 \times M, \omega_{\mathrm{std}} + \eta)$ is a symplectic manifold whose boundary is $S^1 \times M$. The boundary inherits a stable Hamiltonian structure of cosymplectic type, namely $(dt, \eta)$ ($t$ is the coordinate on the circle $\mathbb{R}/\mathbb{Z}$). Such structure is Morse-Bott. Indeed, the Reeb vector field is $\partial_t$, thus the Reeb orbits have integer period, and are iterations of the standard cover of the circle, with image $S^1 \times p$. In the notation of Definition 2.3.8 $Z_N = S^1 \times M$ for all $N \in \mathbb{N}$, hence it is a closed manifold. Also the second condition in Definition 2.3.8 is satisfied, as $\alpha = dt$ is closed. Also the Reeb flow is the identity map, so the third condition is trivially satisfied. This example is going to play an important role in Chapter 3.

The following theorem is proved in [2].

**Theorem 2.3.10** ([2], Proposition 3.4)**.** *A finite energy holomorphic curve with values in the completion of a symplectic manifold with Morse-Bott stable boundary is proper, and is asymptotic to a closed Reeb orbit. This means that for each puncture, there are holomorphic coordinates $re^{i\theta}$ centered at the puncture, and a closed Reeb orbit $\gamma$, such that $u(re^{i\theta}) \to \gamma(e^{i\theta})$ for $r \to 0$.*

If $\dot{\Sigma} = \Sigma \setminus \Gamma$, we say that a puncture $z \in \Gamma$ is *positive* is it is asymptotic to a Reeb orbit in $Z^+$, *negative* otherwise. We split the set of punctures accordingly as $\Gamma = \Gamma^+ \cup \Gamma^-$.

### 2.3.3 Asymptotic operators

In order to study punctured holomorphic curves, it is useful to have a detailed understanding of the behaviour of the Cauchy-Riemann operator near the punctures. In this section we describe some properties of (non-degenerate) *asymptotic operators*, which are exactly the operators that arise as certain limits of Cauchy-Riemann operators at the punctures. We refer to [42] for a complete account.

**Definition 2.3.11.** Let $(E, J, \omega) \longrightarrow S^1$ be a Hermitian vector bundle over a punctured Riemann surface. An *asymptotic operator* $A : \Gamma(E) \longrightarrow \Gamma(E)$ is an operator that can be written, in a unitary trivialization $\tau : E \cong S^1 \times \mathbb{R}^{2n}$, as $A^\tau : C^\infty(S^1, \mathbb{R}^{2n}) \longrightarrow C^\infty(S^1, \mathbb{R}^{2n})$, $A^\tau = -J_0 \frac{d}{dt} - S(t)$, where $J_0$ is the standard complex structure and $S(t)$ is a loop of symmetric matrices. We say that $A$ is *non-degenerate* if $\ker(A) = 0$.

It is crucial for the index theory of Cauchy-Riemann operators and the intersection theory of holomorphic curves to understand the Conley-Zehnder index and the spectrum of asymptotic operators.

### Conley-Zehnder index

The Conley-Zehnder index is an integer $\mu_{CZ}(\Psi)$ which is associated to a path of symplectic matrices $\Psi(t)$, such that $\Psi(0) = \mathbb{I}$, and $\det(\Psi(1) - \mathbb{I}) \neq 0$. We will not present a construction of $\mu_{CZ}$, but we will define it as the unique map satisfying certain properties (see [34]).

**Definition 2.3.12.**

- $\mu_{CZ}$ is invariant under homotopy with fixed end points;
- let $\Phi$ be a loop of symplectic matrices, with $\Phi(0) = \Phi(1) = \mathbb{I}$. Then $\mu_{CZ}(\Phi\Psi) = \mu_{CZ}(\Psi) = 2\mu(\Psi)$, where $\mu$ denotes the Maslov index[3];
- if $S$ is a symmetric matrix with norm $||S|| < 2\pi$, and $\Psi(t) = \exp(J_0 St)$, then $\mu_{CZ}(\Psi) = \frac{1}{2}\text{sign}(S)$, where $\text{sign}(S)$ is the signature of $S$.

The last property is the most important to us, as we will only need to compute the Conley-zehnder index explicitly for matrices of the form $\exp(J_0 St)$ (see Chapter 3).

One can define $\mu_{CZ}(A)$ for a non-degenerate trivialized asymptotic operator of the form $A = -J_0 \frac{d}{dt} - S(t)$ as $\mu_{CZ}(\Psi(t))$, where $\Psi(t)$ is the unique solution to

$$\begin{cases} (-J_0 \dfrac{d}{dt} - S(t))\Psi = 0 \\ \Psi(0) = \mathbb{I} \end{cases} \tag{2.6}$$

The solution to this system is unique and symplectic, and by non-degeneracy of $A$ one has $\det(\Psi(1) - \mathbb{I}) \neq 0$, thus the definition makes sense.

**Definition 2.3.13.** Let $A$ be an asymptotic operator on smooth sections of $(E, J, \omega) \longrightarrow S^1$. Let $\tau : E \cong S^1 \times \mathbb{R}^{2n}$ be a unitary trivialization. The *Conley-Zehnder index (relative to $\tau$) of* $A$ is by definition the Conley-Zehnder index of the trivialized operator $A^\tau$. That is,

$$\mu^\tau_{CZ}(A) := \mu_{CZ}(A^\tau)$$

### Spectrum of asymptotic operators

We collect here some facts about the spectrum of asymptotic operators, as well as the relation between eigenvalues and the Conley-Zehnder index. Again, in applications we will only deal with very simple explicit cases, where we can compute spectrum and eigenfunctions explicitely. Nonetheless, the general treatment will be useful later, especially when discussing intersection theory of punctured holomorphic curves (see Section 2.4.3).

[3]The Maslov index is a map realizing the isomorphism $\pi_1(Sp(n)) \cong \mathbb{Z}$. Roughly, one first retracts the symplectic group onto the unitary group, then one takes the complex determinant. The Maslov index $\mu$ is the induced map in $\pi_1$. See [27]

First of all, asymptotic operators are Fredholm and self-adjoint, as operators between Sobolev spaces $H^1(S^1, \mathbb{R}^{2n}) \longrightarrow L^2(S^1, \mathbb{R}^{2n})$ ([42]), hence the spectrum is real and discrete. Moreover, they have infinitely many positive and negative eigenvalues (see Proposition 3.27 in [42]).
In low dimension ($E \cong \mathbb{R}^2$) one can also get a precise description of the eigenfunctions, due to [18] (see Theorem 3.35 in [42]).

**Theorem 2.3.14.** *Let $A : H^1(S^1, \mathbb{R}^{2n}) \longrightarrow L^2(S^1, \mathbb{R}^{2n})$ be an asymptotic operator, of the form $A = -J_0 \frac{d}{dt} - S(t)$. Denote by $\sigma(A) \subset \mathbb{R}$ the spectrum of $A$, and for each $\lambda \in \sigma(A)$ let $E_\lambda \subset H^1(S^1, \mathbb{R}^2)$ be the corresponding eigenspace. Then*

- *every non-zero $e_\lambda \in E_\lambda$ is nowhere $0$. In particular, the winding number $wind(e_\lambda)$ of $e_\lambda$ is well defined;*
- *every two non-zero elements in $E_\lambda$ have the same winding number, which we denote by $wind(\lambda)$;*
- *if $\lambda, \lambda' \in \sigma(A)$ with $\lambda < \lambda'$, then $wind(\lambda) \leq wind(\lambda')$;*
- *for each $k \in \mathbb{Z}$, there exist exactly two eigenvalues (counted with multiplicity), such that their winding number is $k$.*

Given this result, it makes sense to define the *extremal winding numbers* of an asymptotic operator $A$ defined on sections of a two-dimensional vector bundle (relative to a trivialization). Given a trivialization $\tau$, these numbers are, by definition

$$\begin{aligned} \alpha_+^\tau(A) &:= \min\{\text{wind}(\lambda) :\ \lambda \in \sigma(A^\tau) \cap (0, +\infty)\} \\ \alpha_-^\tau(A) &:= \max\{\text{wind}(\lambda) :\ \lambda \in \sigma(A^\tau) \cap (-\infty, 0)\} \end{aligned} \tag{2.7}$$

One defines the *parity* of $A$ as

$$p(A) := \alpha_+^\tau(A) - \alpha_-^\tau(A) \tag{2.8}$$

The number $p(A)$ does not depend on the trivialization $\tau$, and if $A$ is non-degenerate, is either 0 or 1. These numbers will appear in the intersection formulas of punctured holomorphic curves. They are also related to the Conley-Zehnder index, in the following way.

**Theorem 2.3.15** ([42], Theorem 3.36)**.** *If $A$ is a non-degenerate asymptotic operator on a complex line bundle, one has*

$$\mu_{CZ}^\tau(A) = 2\alpha_-^\tau(A) + p(A) = 2\alpha_+^\tau(A) - p(A)$$

### Asymptotic operators and Reeb orbits

Let $(Z, \alpha, \beta)$ be stable Hamiltonian, with $\xi := \ker \alpha$, and let $\gamma : S^1 \longrightarrow Z$ be a Reeb orbit of period $T$. Let $J$ be a $\beta$-compatible complex structure on $\xi$.

**Lemma 2.3.16** ([42], Exercise 3.27). *Let $\nabla$ be a symmetric connection on $TZ$. Let $A_\gamma : \Gamma(\gamma^*\xi) \longrightarrow \Gamma(\gamma^*\xi)$ be defined as*

$$A_\gamma \eta = -J(\nabla_t - T\nabla_\eta R) \tag{2.9}$$

*Then $A_\gamma$ is an asymptotic operator in the sense of Definition 2.3.11*

**Definition 2.3.17.** $A_\gamma$ is the *asymptotic operator associated to the Reeb orbit* $\gamma$.

**Remark 2.3.18.** Given a punctured holomorphic curve $u : \dot{\Sigma} \longrightarrow \mathbb{R} \times Z$, with a puncture $z \in \Sigma$ asymptotic (at $+\infty$) to $\gamma$, one can realize $A_\gamma$ as a limit at $+\infty$ of the linearized Cauchy-Riemann operator on $u^*T(\mathbb{R} \times Z)$. The details are provided in [42].

### 2.3.4 Moduli spaces

Consider a punctured surface $\dot{\Sigma} = \Sigma \setminus \Gamma$, and let $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_k)$ be a $k$-tuple of connected sets of Reeb orbits.

**Definition 2.3.19.** We say that a set of asymptotic constraints $\mathfrak{c}$ is the datum of

- a partition $\Gamma = \Gamma^\pm_C \sqcup \Gamma^\pm_U$ into (positive or negative) constrained and unconstrained punctures;
- for each $z \in \Gamma_C$, a Reeb orbit $\gamma_z$;
- for each $z \in \Gamma_U$, a connected manifold $\mathcal{S}_z \subset \partial W$ of points belonging to a family $\mathcal{R}_z$ of Reeb orbits.

We say that a punctured holomorphic curve $u : \dot{\Sigma} \longrightarrow \widehat{W}$ satisfies the constraints $\mathfrak{c}$ if each puncture $z \in \Gamma_C$ is asymptotic to $\gamma_z$, and each puncture $z \in \Gamma_U$ is asymptotic to a Reeb orbit belonging to $\mathcal{R}_z$.

A curve with $k$ punctures, subject to a constraint $\mathfrak{c}$, determines a relative homology class $A \in H_2(W, \bigcup_{z\in\Gamma_C} \gamma_z \cup \bigcup_{z\in\Gamma_U} \mathcal{S}_z)$. Define

$$\mathcal{M}_g(A, J)^{\mathfrak{c}}$$

as the set of equivalence classes of punctured genus $g$ $J$-holomorphic curves, subject to the constraint $\mathfrak{c}$, in the homology class $A$. As in the closed case, there is a generic transversality result, and a dimension formula for the moduli space.

In order to give the dimension formula, we need to define the *relative first Chern number* of a vector bundle over a surface with boundary, and the *total Conley-Zehnder index.*
Let $E \longrightarrow \Sigma$ be a complex vector bundle, let $\tau$ be a trivialization of $E|_{\partial\Sigma}$. If

$\mathrm{rk} = \mathbb{C}(E) = 1$, one defines $c_1^\tau(E)$ as the number of zeros of a transverse section of $E$, which is constant on $\partial\Sigma$ with respect to the trivialization $\tau$. For higher rank bundles, we extend the definition additively.
Let $u : \Sigma \longrightarrow \widehat{W}$ be holomorphic. Fix a trivialization $\tau$ of $u^*TW$ in a neighbourhood of the punctures, we can define

$$c_1^\tau(u) := c_1(u^*T\widehat{W})$$

Define the *total Conley-Zehnder index* as

$$\mu^\tau(\mathfrak{c}) := \sum_{z\in\Gamma^+} \mu_{CZ}^\tau(\gamma_z + \delta_z) - \sum_{z\in\Gamma^-} \mu_{CZ}^\tau(\gamma_z - \delta_z) \tag{2.10}$$

where

$$\delta_z = \begin{cases} \varepsilon & \text{if } z \in \Gamma_C \\ -\varepsilon & \text{if } z \in \Gamma_U \end{cases} \tag{2.11}$$

for $\varepsilon > 0$ suitably small.
Define

$$\mathrm{vdim}\mathcal{M}_g(A, J)^{\mathfrak{c}} := (n-3)\chi(\dot{\Sigma}) + 2c_1^\tau(A) + \mu^\tau(\mathfrak{c}) \tag{2.12}$$

(if $u \in \mathcal{M}_g(A, J, S)$, this is also called the *index of* $u$, and denoted by $\mathrm{ind}(u)$).

**Theorem 2.3.20** ([42], Theorem 7.1)**.** *Let $\widehat{W}$ be the completion of a manifold with Morse-Bott stable Hamiltonian boundary. Let $V$ be an open subset. Fix a complex structure $J_{fix}$ on $\widehat{W} \setminus V$, and consider the subset $\mathcal{M}_g^*(A, J; V)^{\mathfrak{c}} \subset \mathcal{M}_g(A, J; V)^{\mathfrak{c}}$ of simple curves with an injective point mapped to $V$. Let $\mathcal{J}(\omega, J_{fix})$ be the set of compatible almost complex structures that coincide with $J_{fix}$ on $\widehat{W} \setminus V$. Then there exists a comeager subset $\mathcal{J}(\omega, J_{fix})^{reg} \subset \mathcal{J}(\omega, J_{fix})$ such that for all $J \in \mathcal{J}(\omega, J_{fix})^{reg}$ the space $\mathcal{M}_g^*(A, J; V)^{\mathfrak{c}}$ is smooth of dimension $vdim\mathcal{M}_g(A, J; V)^{\mathfrak{c}}$.*

Analogously to the closed case presented in Theorem 2.2.12, an automatic transversality result for punctured holomorphic curves holds in dimesion 4. In order to state it, we introduce the *normal Chern number*:

$$c_1^N(u) = c_1^\tau(u) - \chi(\dot{\Sigma}) + \frac{1}{2}\mu(\mathfrak{c}) + \sum_{z\in\Gamma^+} \alpha_-^\tau(\gamma_z + \delta_z) - \sum_{z\in\Gamma^-} \alpha_+^\tau(\gamma_z - \delta_z) \tag{2.13}$$

**Theorem 2.3.21** ([39], Theorem 1)**.** *Let $u$ be an immersed punctured holomorphic curve. If $ind(u) > c_1^N(u)$, then $u$ is Fredholm regular.*

### 2.3.5 Compactness

The compactness theory of punctured holomorphic curves is more complicated than its closed analogue. As in the closed case, bubbling of spheres might occur; moreover, another phenomenon, called *breaking*, might occur. The objects that describe the bubbling behaviour and the breaking are called *holomorphic buildings*. We will not explain the definition of buildings in full generality, but indicate the main ideas. For a complete definition see ([42], Section 9.4.2).

Assume that $\widehat{W} = \mathbb{R} \times Z$, and consider spheres with two punctures (i.e. cylinders), with one puncture asymptotic to $\{-\infty\} \times \gamma_-$ (negative puncture), and the other asymptotic to $\{+\infty\} \times \gamma_+$ (positive puncture). Assuming that no bubbling can occur, the limiting objects in the moduli space of cylinders are so-called *broken cylinders*: finite sets $u = (u_1, \ldots, u_N)$ of cylinders with values in $\mathbb{R} \times Z$, such that: $u_1(-\infty, t) = \{-\infty\} \times \gamma_-$, $u_N(+\infty, t) = \{+\infty\} \times \gamma_+$, and $u_j(+\infty, t) = u_{j+1}(-\infty, t)$. The integer indexing the cylinders is called the *level*. In the case of a completion $\widehat{W}$, the behaviour is the same, except that $u_1$ will be a map with values in $\widehat{W}$, while $u_j$ takes values in $\mathbb{R} \times Z$ for all $j > 1$. Another important point is that if the complex structure is $\mathbb{R}$ invariant on $\mathbb{R} \times Z$, the compactness can only be achieved if we mod out by the group of translations.

A building is, roughly, a set of curves as above, that additionally allows for the curves $u_j$ to be nodal curves. When one considers curves with a bigger number of punctures and positive genus, the notation becomes more elaborate, but the behaviour of sequences is similar. See [3], Section 7.2 and 8.1, and [42] Section 9.4, for details.

**Theorem 2.3.22** ([3], Theorem 10.2)**.** *A sequence $u_k$ of punctured holomorphic curves with Morse-Bott asymptotics and uniformly bounded energy admits a subsequence converging to a holomorphic building.*

One denotes by $\overline{\mathcal{M}} = \overline{\mathcal{M}}_g(A, J; V)^{\mathfrak{c}}$ the set of holomorphic buildings. The Theorem 2.3.22 implies that $\overline{\mathcal{M}}$ is compact – we will refer to it as the compactification of $\mathcal{M}$.

## 2.4 Intersection theory and adjunction formula

We recall here some useful results on the intersection theory of holomorphic curves, as stated in [41] (see also [28]).

### 2.4.1 Local theory

Consider two functions $f_1 : M_1 \longrightarrow X$, $f_2 : M_2 \longrightarrow X$, with $\dim(M_1) + \dim(M_2) = \dim(X)$. We say that an intersection point $(p_1, p_2) \in M_1 \times M_2$, $f_1(p_1) = f_2(p_2)$ is *isolated* if there are neighbourhoods $p_i \in U_i \subset M_i$ such that $f_1(U_1 \setminus p_1) \cap f_2(U_2 \setminus p_2) = \emptyset$. Recall that an isolated intersection has an *intersection index*, defined as $\pm 1$ if the intersection is transverse (depending on the orientation), while if the intersection is not transverse, one takes a small perturbation of $f_i$ on $U_i$, to make them transverse, and counts the (signed) number of intersections between the perturbed functions on the neighbourhoods $U_i$.

The first theorem, so-called "positivity of intersections", states that the local intersection index of a holomorphic curve is positive.

**Theorem 2.4.1** (Positivity of intersections, [41], Theorem 2.88)**.** *Let $(X, J)$ be an almost complex manifold of dimension $2n > 2$, let $H \subset X$ be a complex hypersurface, and $u : \mathbb{D} \longrightarrow X$ a holomorphic disc, with $u(0) \in H$. Then either $u(\mathbb{D}) \subset H$, or $u(0)$ is an isolated intersection. Moreover, the local intersection index at $0$ is bigger or equal to $1$, with equality if and only if the intersection is transverse at $0$.*

### 2.4.2 Intersections of closed curves

In dimension 4 it makes sense to compute the (global) intersection number between two closed curves. The adjunction formula relates the topology of the curve with its self-intersection number, and gives a tool to detect whether a curve is embedded.

**Theorem 2.4.2** (Adjunction formula, [40] Theorem 2.51)**.** *Let $u : \Sigma \longrightarrow X$ be a simple holomorphic curve with values in a 4-manifold $X$. Then there exist a non-negative integer $\delta(u)$, called the singularity index, such that $\delta(u) = 0 \Leftrightarrow u$ is embedded. Moreover, letting $A = u_*([\Sigma]) \in H_2(X; \mathbb{Z})$, the following equality holds, known as the adjunction formula:*

$$A \cdot A = 2\delta(u) + c_1(A) - \chi(\Sigma) = 2\delta(u) + \frac{1}{2}(ind(u) - \chi(\Sigma)) \tag{2.14}$$

In particular, one can read the fact that a curve is embedded off its homology class.

### 2.4.3 Intersection theory of punctured holomorphic curves

In the punctured case things become more intricate, as there is no notion of homotopically invariant topological intersection number between two punctured surfaces. However, there is a well defined notion of number of intersections of two punctured holomorphic curves in the context of symplectic field theory, which is invariant under homotopies of punctured holomorphic curves. This is developed in [35] in the non-degenerate case, and [39] and [36] in the Morse-Bott case. Moreover, an adjunction formula generalizing Theorem 2.4.2 holds.

### 2.4.4 Intersection number

Let $(W, \partial W, \omega)$ be a 4-dimensional symplectic manifold, and assume $\partial W$ inherits a Morse-Bott stable Hamiltonian structure. Let $u$ be a punctured holomorphic curve, with set of punctures $\Gamma$.
Let us assume for simplicity that all Reeb orbits are simply covered.

**Theorem 2.4.3** ([39], Section 4.1)**.** *Given two distinct holomorphic curves $u$, $u'$, with asymptotic constraints $\mathfrak{c}$, $\mathfrak{c}'$, there exists a number $i_\infty(u, u')$ such* *that the pairing*

$$u \star u' := u \cdot u' + i_\infty(u, u') \in \mathbb{Z} \tag{2.15}$$

*(where $u \cdot u'$ is the number of intersection points, with multiplicity) depends only on the components of the moduli spaces $\mathcal{M}^{\mathfrak{c}}$, $\mathcal{M}^{\mathfrak{c}'}$ containing respectively $u$ and $u'$.*

The number $u \star u'$ is the *intersection number* between the two punctured holomorphic curves. The dependence only on the corresponding components of the moduli spaces can be stated as the invariance of the intersection number under homotopies of punctured holomorphic curves with asymptotics $\mathfrak{c}$, $\mathfrak{c}'$. The intersection number does depend on the choice of asymptotic constraint (i.e., whether we consider a punctured to be constrained or unconstrained), so it really should be seen as a pairing between pairs of curves together with constraints.

A way to compute the intersection number is by relating it to the spectrum of the asymptotic operators. Let us introduce some notation. Let $\tau$ denote a choice of asymptotic trivialization of the hyperplane distribution for each puncture. Consider the asymptotic operator associated to a Reeb orbit $\gamma$, which in the trivialization $\tau$ can be written as $A^\tau_\gamma = -J_0 \frac{d}{dt} - S(t)$ (Lemma 2.3.16).
For all $|\delta|$ small enough, the asymptotic operator $A^\tau_\gamma + \delta := -J_0 \frac{d}{dt} - S(t) + \delta \mathbb{I}$ is non-degenerate. Moreover, the extremal winding numbers $\alpha_\pm(A^\tau_\gamma + \delta)$ (see Equation (2.7)) are independent on $\delta$ if $|\delta|$ is small enough. We define the extremal winding numbers of a Reeb orbit as

$$\alpha^\tau_\pm(\gamma + \delta) := \alpha_\pm(A^\tau_\gamma + \delta) \tag{2.16}$$

In order to write a computable formula for the $\star$ pairing, we introduce the following two numbers:

- $u \bullet_\tau u'$ is the intersection number of $u$ and a perturbation of $u'$ "in the direction of $\tau$". By definition, the perturbation is supported in a neighbourhood of infinity, and is of the following type. Assume $u'$ is asymptotic to $\gamma$ at $z$. We deform $u'$ so that it is asymptotic to $\exp_{\gamma(t)} \eta(t)$, for $\eta \in \Gamma(\gamma^* \xi)$ small, with $\mathrm{wind}(\eta) = 0$. One can ensure that $u$ and $u'$ intersect in a finite number of points.

- Let $z$, $z'$ be punctures of the domain of $u$, $u'$ respectively, asymptotic to $\gamma$, $\gamma'$. Define

$$\Omega^\tau_\pm(\gamma + \delta, \gamma' + \delta) = \begin{cases} 0 & \text{if } \gamma \neq \gamma' \\ \mp \alpha^\tau_\mp(\gamma + \delta) & \text{if } \gamma = \gamma' \end{cases} \tag{2.17}$$

Now, given a puncture $z$, and $\varepsilon > 0$ suitably small, define

$$\delta_z = \begin{cases} \varepsilon & \text{if } z \in \Gamma_C \\ -\varepsilon & \text{if } z \in \Gamma_U \end{cases} \tag{2.18}$$

And we define the *perturbed asymptotic operator* as $\{A_z \pm \delta_z\}_{z \in \Gamma^\pm}$

**Theorem 2.4.4** ([39], Section 4.1)**.**

$$u \star u' = u \bullet_\tau u' - \sum_{(z,\, z') \in \Gamma^\pm \times \Gamma'^\pm} \Omega^\tau_\pm(\gamma_z \pm \delta_z,\, \gamma_{z'} \pm \delta_{z'}) \tag{2.19}$$

Note that this allows to define the self-intersection number $u \star u$.

### 2.4.5 Adjunction formula

Let us introduce some notation in order to state the adjunction formula. Define the *normal Chern number* of a punctured holomorphic curve $u$ as

$$c_N(u) = c_1^\tau(u) - \chi(\dot{\Sigma}) + \sum_{z \in \Gamma^+} \alpha^\tau_-(\gamma_z + \delta_z) - \sum_{z \in \Gamma^-} \alpha^\tau_+(\gamma_z - \delta_z) \tag{2.20}$$

Since $A_{\gamma_z} \pm \delta_z$ is non-degenerate, Equation (2.8) holds. We can write $\Gamma = \Gamma_0 \sqcup \Gamma_1$, with $z \in \Gamma_i \cap \Gamma^\pm$ if and only if $p(A_{\gamma_z} \pm \delta_z) = i$ ($\Gamma_0$ and $\Gamma_1$ are said the sets of *even/odd punctures*). Using Equation (2.12), one easily checks that

$$2c_N(u) = \mathrm{ind}(u) - \chi(\dot{\Sigma}) - \#\Gamma_1 \tag{2.21}$$

**Theorem 2.4.5** (Adjunction formula, [39], Section 4.1)**.** *For a holomorphic curve with constraint $\mathfrak{c}$, with value in a 4-manifold with cylindrical ends, there exist a homotopy invariant number $sing(u;\, \mathfrak{c}) \geq 0$, such that $u$ is embedded whenever $sing(u;\, \mathfrak{c}) = 0$, and*

$$u \star u = 2sing(u;\, \mathfrak{c}) + c_N(u) = 2sing(u;\, \mathfrak{c}) + \frac{1}{2}(ind(u) - \chi(\dot{\Sigma}) - \#\Gamma_1) \tag{2.22}$$

# Chapter 3

# Log-symplectic manifolds

## 3.1 Introduction

A log-symplectic structure on a manifold $X^{2n}$ is a Poisson tensor $\pi \in \Gamma(\bigwedge^2 TX)$ for which $\pi^{\wedge n} \in \Gamma(\bigwedge^{2n} TX)$ vanishes transversely. Morally this means that a log-symplectic structure is very close to being a symplectic structure: the definition implies that $\pi$ is invertible in the complement of the codimension–1 submanifold $Z = (\pi^{\wedge n})^{-1}(0)$. Hence $\omega = \pi^{-1}$ is a symplectic form on $X \setminus Z$. The singular behaviour in a neighbourhood of $Z$ is also very much constrained by the transverse vanishing requirement: there exist local coordinates such that $\omega = \frac{dx}{x} \wedge dy_1 + dy_2 \wedge dy_3 + \cdots + dy_{2n-2} \wedge dy_{2n-1}$.
Log-symplectic structures were introduced by Radko on orientable surfaces in [33], and a general definition first appeared in [16]. A crucial observation of [16] is that one can view a log-symplectic structure as a symplectic form on a Lie algebroid, called the "log-tangent bundle". That is a vector bundle, denoted by $TX(-\log Z)$, with a Lie bracket on the space of its sections, and a compatible infinitesimal action on $X$ (this bundle is often also denoted by ${}^bTX$, and called the "$b$-tangent bundle"). The Lie algebroid picture is very useful as it allows us to apply some techniques of symplectic geometry to log-symplectic structures: this is the case for Moser's argument and its consequences like the Darboux theorem, the existence of some cohomological constraints to the existence ([21], [4]), and also for more sophisticated results like Gompf's construction ([13]) of symplectic forms on Lefschetz fibrations ([6]).

In this chapter we go further in this direction, extending the use of pseudoholomorphic curves to log-symplectic structures. We study what we call "log-holomorphic maps", meaning maps from a surface with boundary $\Sigma$ to a log-symplectic manifold $X$ with singular locus $Z$, satisfying a holomorphicity condition. To be more precise, the log-tangent bundle $T\Sigma(-\log \partial\Sigma)$ of a surface relative to the boundary, as well as the log-tangent bundle $TX(-\log Z)$ of a log-symplectic manifold, both carry almost complex structures. We study moduli spaces of maps $u : \Sigma \longrightarrow X$ such that there is an induced complex linear map $du : T\Sigma(-\log \partial\Sigma) \longrightarrow TX(-\log Z)$. These moduli spaces turn out to behave well enough. As in symplectic geometry there are natural compactifications, and we give criteria for them to be smooth. In particular, we construct moduli spaces of spheres, and of Riemann surfaces with non-empty boundary of arbitrary genus. In order to prove compactness and smoothness we translate everything back to the symplectic world, and borrow from the general theory (especially Symplectic Field Theory).

As applications we deduce several results on the (symplectic) topology of log-symplectic manifolds, especially in dimension 4, in which case the theory is more powerful. To start with, we use an argument of McDuff ([25]) to prove a list of non-existence results for log-symplectic manifolds (possibly with boundary). The key point is that McDuff's argument relies on a maximum principle for holomorphic curves near a contact hypersurface, which also holds in a neighbourhood of the singular locus in a log-symplectic manifold. In dimension 4 the results can be phrased as follows.

**Theorem 3.1.1.** *Let $(X, Z, \omega)$ be a closed 4–dimensional log-symplectic manifold. If one component of $Z$ contains a symplectic sphere, then all components are diffeomorphic to $S^1 \times S^2$. If $(X, Z, \omega)$ has non-empty boundary of contact type. Then none of the components of $Z$ contains a symplectic sphere. Moreover, the boundary cannot be contactomorphic to the standard sphere.*

A finer analysis of the moduli spaces of spheres allows to prove a classification theorem up to diffeomorphism for a certain class of log-symplectic 4–manifolds. To be more precise, we prove that if the singular locus of a log-symplectic manifold contains a symplectic sphere, then the moduli space of holomorphic spheres can be compactified adding nodal curves with $(-1)$-spheres as irreducible components. In particular the moduli space is compact whenever the log-symplectic manifold is *minimal*, i.e. its symplectic locus is not a blow-up of another symplectic manifold. In that case we say that the manifold is a *ruled surface*. This is the log-symplectic analogue of a famous result of McDuff ([26]).

**Theorem 3.1.2.** *Let $(X, Z, \omega)$ be a minimal log-symplectic 4–manifold. Assume that $Z$ contains a copy of $S^1 \times S^2$. Then $X$ supports an $S^2$-fibration over a (not necessarily orientable) closed surface $B$, with symplectic fibers, and such that $Z$ is a union of fibers. Moreover, the log-symplectic form depends only on the diffeomorphism type, up to deformations.*

Moreover, the non-minimal case can be characterized in terms of Lefschetz fibrations. It might also be worth pointing out that in the Theorems A and B above, as well as in the general construction of the moduli spaces of curves, $X$ is not assumed to be orientable.

The results above only use maps of spheres. As an application of our study of surfaces with boundary, we can prove a result analogous to Theorem 3.1.2, constructing a "log-symplectic fibration", instead of a symplectic one. We state here a simplified version of the result (for a precise statement we refer to Corollary 3.5.18).

**Theorem 3.1.3.** *Let $(X, Z, \omega)$ be log-symplectic, such that $Z \cong \bigsqcup_i S^1 \times \Sigma_{g_i}$. Assume there exists a holomorphic sphere $S \subset X$ with $S \pitchfork Z$ and $S \cdot S = 0$. Then*

- $g_i = g_j$ *for all* $i$, $j$
- *there is a continuous fibration* $f : X \longrightarrow \Sigma_{g_i}$ *with fiber* $S^2$*, such that* $Z$ *is a union of sections.*

In fact, one can show that $f$ is smooth on $X \setminus Z$, and the fibers of $f|_{X \setminus Z}$ are symplectic. There is also a "non-orientable" version of this statement, producing a fibration with fiber $\mathbb{R}P^2$, out of a single holomorphic $\mathbb{R}P^2$ (holomorphic in the logarithmic sense). Theorem 3.1.3 is proven using the intersection theory of punctured holomorphic curves ([35], [36]).

## 3.2 Log-symplectic manifolds

In this section we give the definition of log-symplectic manifold and list some well-known properties. The properties are formulated in a way which is useful for the purpose of studying holomorphic curves.

### 3.2.1 Definition and first properties.

**Definition 3.2.1.** A *log-symplectic manifold* is a manifold $X$ together with a Poisson bivector $\pi$, such that

- $\pi^n \pitchfork 0$
- $(\pi^n)^{-1}(0) \neq \emptyset$

**Remark 3.2.2.** A Poisson bivector satisfying the first but not the second condition in Definition 3.2.1 is the inverse of a symplectic form. In the literature people mostly omit the second condition in the definition of a log-symplectic structure. In [4] and [6] the authors call a Poisson bivector satisfying our Definition 3.2.1 a *bona fide log-symplectic structure.* Since we will mostly consider log-symplectic manifolds which are not symplectic, we preferred to avoid the use of additional terminology, and put this requirement in the definition.

**Remark 3.2.3.** Log-symplectic manifolds are necessarily even dimensional. We will always denote $\dim X = 2n$.

A log-symplectic manifold contains a distinguished submanifold, namely the locus where the Poisson structure does not have maximal rank.

**Definition 3.2.4.** The *singular locus* of a log-symplectic structure $\pi$ on a manifold $X$ is the codimension–1 submanifold $Z := (\pi^n)^{-1}(0)$.

We will always denote the singular locus of a log-symplectic manifold $X$ as $Z_X$, or just $Z$.

**Remark 3.2.5.** Since the singular locus $Z$ is the zero set of a generic section of the bundle $\bigwedge^{\mathrm{top}} TX$, it is coorientable if $X$ is orientable. For the same reason if $X$ is orientable, then $Z$ is trivial in homology.

**Local forms.**

As a consequence of the Weinstein splitting theorem, for each point in $Z$ there exists a coordinate neighbourhood $V$ with coordinate functions $(x, y_1, \dots, y_{2n-1})$ such that

- $V \cap Z = \{x = 0\}$
- $\pi = x\partial_x \wedge \partial_{y_1} + \partial_{y_2} \wedge \partial_{y_3} + \cdots + \partial_{y_{2n-2}} \wedge \partial_{y_{2n-1}}$

One could also look at the inverse of the Poisson structure, obtaining a non-smooth symplectic form

$$\frac{dx}{x} \wedge dy_1 + dy_2 \wedge dy_3 + \cdots + dy_{2n-2} \wedge dy_{2n-1} \tag{3.1}$$

(the existence of such local form is referred to as *Darboux theorem*, and was originally proven in [16]). This can actually be viewed as a smooth form in a Lie algebroid called the *logarithmic tangent bundle*.

**Definition 3.2.6.** Let $(X, Z)$ be a manifold with a codimension–1 submanifold. The *logarithmic tangent bundle* $TX(-\log Z)$ is the rank-$2n$ vector bundle whose sheaf of section is the sheaf $\mathfrak{X}_Z(X)$ of vector fields on $X$ tangent to $Z$. We call its dual *logarithmic cotangent bundle*, and we denote it with $T^*X(\log Z)$.

This bundle exists and is unique up to isomorphism by the Serre-Swan theorem. The inclusion $\mathfrak{X}_Z \hookrightarrow \mathfrak{X}$ induces a map $\rho : TX(-\log Z) \longrightarrow TX$, called the *anchor map*, which is an isomorphism almost everywhere (namely, on $X \setminus Z$). $TX(-\log Z)$ is in fact a Lie algebroid.

**Definition 3.2.7.** A *logarithmic differential form* is a section of $\bigwedge^\bullet(T^*X(\log Z))$. We say that a logarithmic differential form $\omega$ is *closed* if $d\omega = 0$, where $d$ denotes the Lie algebroid exterior differential.

**Remark 3.2.8.** $d\omega = 0$ is equivalent to $d((\rho|_{X\setminus Z}^{-1})^*\omega) = 0$, as an ordinary differential form.

The local form Equation (3.1) implies that one can view a log-symplectic structure as a logarithmic 2-form. This implies in particular that the vector bundle $TX(-\log Z)$ admits a complex structure. To be more precise, it admits

a contractible space of compatible complex structures, and a contractible space of tame complex structures ([27]).
The correspondence between log-symplectic structures and logarithmic symplectic forms is in fact one to one ([16]). Based on this, from now on we will treat a log-symplectic manifold as a triple $(X, Z, \omega)$ consisting of:

- a smooth manifold $X$
- a codimension–1 submanifold $Z$
- a logarithmic symplectic form $\omega$ on $TX(-\log Z)$

### 3.2.2 Maps of log-symplectic manifolds.

**Definition 3.2.9** ([6])**.** Let $(X, Z_X)$, $(Y, Z_Y)$ be pairs of manifold with a codimension–1 submanifold. A *(smooth) map of pairs* $f : (Y, Z_Y) \longrightarrow (X, Z_X)$ is a smooth map $f : Y \longrightarrow X$ such that $f \pitchfork Z_X$, and $f^{-1}(Z_X) = Z_Y$.

**Example 3.2.10.** $f : (Y, \emptyset) \longrightarrow (X, Z)$ is a smooth map of pairs if and only if $f$ is an ordinary smooth map $f : Y \longrightarrow X \setminus Z$.

The differential of a smooth map of pairs lifts uniquely to the logarithmic tangent bundles, meaning that there is a commutative diagram

$$\begin{array}{ccc} TY(-\log Z_Y) & \xrightarrow{\overline{df}} & TX(-\log Z_X) \\ \downarrow & & \downarrow \\ TY & \xrightarrow{df} & TX \end{array} \tag{3.2}$$

The restricted bundle $TX(-\log Z)|_Z$ carries a "canonical transverse section" $\xi_X = \xi$, constructed as follows. Choose a local coordinate system $\phi = (x, y_1, \ldots, y_{2n-1}))$ on an open set $U$ with $x$ a defining function for $Z \cap U$ and $y_i$ a coordinate chart for $Z$. Define $\xi|_U := \phi_*(x\partial_x)$. Given another chart $\phi' = (x', y'_1, \ldots, y'_{2n-1}))$ as above, it is easy to compute that $\phi_*(x\partial_x)|_Z = \phi'_*(x'\partial_{x'})|_Z$, which ensures that $\xi$ is well defined on $Z$. Moreover, this canonical transverse section is preserved under maps of pairs:

**Proposition 3.2.11.** *Let $f : (Y, Z_Y) \longrightarrow (X, Z_X)$ be a smooth map of pairs. Then $f_*(\xi_{Z_Y}) = \xi_{Z_X}$.*

*Proof.* Consider a local defining function $x$ for $Z_X$. By transversality, its pullback $y = f^*x$ via $f$ is a local defining function for $Z_Y$. By definition we can compute $\xi_{Z_X}$ and $\xi_{Z_Y}$ by looking at $x\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$. $f_*(y\frac{\partial}{\partial y}) = x\frac{\partial}{\partial x} + xV$, where $V$ is tangent to $Z_X$. Restricting to $Z_X$ gives the desired equality. $\square$

The following lemma due to Cavalcanti and Klaasse [6] is useful in order to define embeddings of pairs, and in particular log-symplectic submanifolds. Below we provide a short proof in local coordinates, different from the original one.

**Lemma 3.2.12** ([6], Proposition 2.14)**.** *Let $f : (Y, Z_Y) \longrightarrow (X, Z_X)$ be a smooth map of pairs, such that $f^{-1}(Z_X) = Z_Y$. The anchor map $\rho : TY(-log\, Z_Y) \longrightarrow TY$ induces an isomorphism $\rho : \ker \overline{df} \longrightarrow \ker df$ at all points, where $\overline{df}$ is the lift of $df$, in the sense of (3.2).*

*Proof.* The statement is local. Pick coordinates $(x, y^i)$ on $Y$, such that $Z_Y = \{x = 0\}$, and $(x', y'^j)$ on $X$ such that $Z_X = \{x' = 0\}$. The lift $\overline{df}$, at points in $Z$, acts as

$$\overline{df}(x\partial_x) = x'\partial_{x'} \tag{3.3}$$

$$\overline{df}(\partial_{y^i}) = df(\partial_{y^i}) \in \operatorname{span}(\partial_{y^i}) \tag{3.4}$$

where the first line is a consequence of Proposition 3.2.11. Since $f$ is a map of pairs, $\partial_x \notin \ker df$. Then $0 = \overline{df}(v) = \overline{df}(a\xi + b^i\partial_{y^i}) = a\xi + b^i\overline{df}(\partial_{y^i})$. Now since $\overline{df}(\partial_{y^i}) \in \operatorname{span}(\partial_{y^i})$, one has $a = 0$, and $b^i\overline{df}(\partial_{y^i}) = b^i df(\partial_{y^i}) = 0$, proving the lemma. □

This means that an embedding induces an inclusion at the level of the logarithmic tangent bundles. Hence the following definition makes sense.

**Definition 3.2.13.** A *log-symplectic submanifold* of $(X, Z_X, \omega)$ is a pair of manifolds $(Y, Z_Y)$ with a smooth embedding $f : (Y, Z_Y) \longrightarrow (X, Z_X)$ such that $\omega$ restricts to a symplectic form (i.e. nondegenerately) to $f_*(TY(-\log Z_Y))$.

The same meaning can be given to the expression *complex submanifold* of a manifold with a logarithmic complex structure. The following fact is obvious:

**Proposition 3.2.14.** *Let $J$ be an $\omega$-tame complex structure on $TX(-log\, Z)$. Any complex submanifold is symplectic. Moreover, for each log-symplectic submanifold $(Y, Z_Y)$ there exists a compatible complex structure on $TX(-log\, Z)$ for which $(Y, Z_Y)$ is a complex submanifold.*

### 3.2.3 A normal form around the boundary, and the stable Hamiltonian geometry of the singular locus.

Fix a Riemannian metric on $X$ and consider the function "distance from $Z$", denoted with $\lambda$. We can write the log-symplectic form in a neighbourhood of $Z$ as

$$\omega = \frac{d\lambda}{\lambda} \wedge a + b$$

for $a$ and $b$ respectively a 1- and a 2-form (in the ordinary sense) on a neighbourhood of $Z$. It follows easily from $\omega$ being symplectic that $\alpha := a|_{TZ}$ and $\beta := b|_{TZ}$ determine a *cosymplectic structure* on $Z$ (that is by definition a pair $(\alpha, \beta)$ such that $d\alpha = d\beta = 0$, $\alpha \wedge \beta^{n-1} \neq 0$). The cosymplectic structure on $Z$ completely determines the log-symplectic structure in a neighbourhood of $Z$.

**Proposition 3.2.15** ([16], [21])**.** *Let $(X, Z, \omega)$ be log-symplectic, with $Z$ compact. There exists a cosymplectic structure $(\alpha, \beta)$ on $Z$, and a metric $g$ on $NZ$*

*such that a tubular neighbourhood of $Z$ is isomorphic to the normal bundle $NZ$ with the log-symplectic form*

$$\frac{d\lambda}{\lambda} \wedge \alpha + \beta \tag{3.5}$$

*and $\lambda$ is the $g$-distance from the zero section of $NZ$.*

As a consequence, there exists a vector field $V$ on $X$ – defined by $\alpha(V) = \beta(V) = 0$, $\frac{d\lambda}{\lambda}(V) = 1$ – such that $L_V\omega = 0$, and $V$ is transverse to the level sets of $\lambda$, for values of $\lambda$ small enough. Let $U$ be a tubular neighbourhood of $Z$, with smooth boundary, so that $V \pitchfork \partial U$. The condition $L_V\omega = 0$ implies that $W := X \setminus U$ is a symplectic manifold with stable Hamiltonian boundary $-\partial U$. The induced stable Hamiltonian structure is pulled back from $(-\alpha, \beta)$ as above, via the normal bundle projection. Via the change of coordinates $s := -\log\lambda$, one can write $\omega = ds \wedge (-\alpha) + \beta$, as in Example 2.3.4. Hence one has the following proposition, which is crucial to set up a theory of holomorphic curves.

**Proposition 3.2.16.** *The complement $X \setminus Z$ of the singular locus is symplectomorphic to the completion of the manifold $W := X \setminus U$, with stable Hamiltonian boundary of cosymplectic type. $\partial W$ is a double cover of $Z$ (disconnected if and only if $X$ is orientable) and the cosymplectic structure on the boundary coincides (up to isomorphism) with the pullback via the covering map of the cosymplectic structure on $Z$.*

**An alternative normal form around the singular locus.**

One could write a more concrete model around $Z$, just by writing explicitely what the normal bundle looks like. To this end, note that every real line bundle over $Z$ can be written as a fiber product $\mathbb{R} \times_{\mathbb{Z}_2} \tilde{Z}$, where $c : \tilde{Z} \longrightarrow Z$ is a double cover of $Z$. More precisely, a double cover $\tilde{Z}$ of $Z$ is acted on by an involution $\sigma$, and we define an action of $\mathbb{Z}_2 = \{\pm 1\}$ on $\mathbb{R} \times \tilde{Z}$ as

$$-1 \cdot (x, z) := (-x, \sigma(z)) \tag{3.6}$$

and define

$$\mathbb{R} \times_{\mathbb{Z}_2} \tilde{Z} := (\mathbb{R} \times \tilde{Z})/\mathbb{Z}_2 \tag{3.7}$$

Given a real line bundle over $Z$, one recovers the manifold $\tilde{Z}$ as the set of vectors of length 1 with respect to some fiber metric, and the involution $\sigma$ is the multiplication by $-1$.
For a cosymplectic $Z$, $\tilde{Z}$ inherits a cosymplectic structure (via pullback along $c$) which is invariant under $\sigma$. Hence the log-symplectic structure $\omega := \frac{dx}{x} \wedge c^*\alpha + c^*\beta$ on $\mathbb{R} \times \tilde{Z}$ is invariant under the action (3.6). Thus $\omega$ descends to the quotient. The resulting log-symplectic structure on $\mathbb{R} \times_{\mathbb{Z}_2} \tilde{Z}$ has the form

$$\frac{dx}{x} \wedge \alpha + \beta \tag{3.8}$$

and is symplectomorphic to the normal form (3.5). This implies:

**Corollary 3.2.17.** *Let* $(X, Z, \omega)$ *be compact log-symplectic. There exists a double cover* $c : \tilde{Z} \longrightarrow Z$ *such that a neighbourhood of the singular locus can be written (up to symplectomorphism) as*

$$((-\varepsilon, \varepsilon) \times_{\mathbb{Z}_2} \tilde{Z}, \frac{dx}{x} \wedge c^*\alpha + c^*\beta)$$

*where the fiber product is taken using the deck transformation.*

**Notation 3.2.18.** We will most of the time omit the pullback sign from the formula for the symplectic form.

Note that if $\tilde{Z}$ is the trivial double cover, then the action of $\mathbb{Z}_2$ just exchanges corresponding connected components, and the resulting quotient is the trivial bundle over $Z$. Since this happens if and only if $Z$ is coorientable, and $NZ \cong \bigwedge^{\mathrm{top}} TX|_Z$ (see Remark 3.2.5), we obtain the following obvious specialization.

**Corollary 3.2.19.** *If* $X$ *is orientable, a neighbourhood of the singular locus can be written (up to symplectomorphism) as*

$$((-\varepsilon, \varepsilon) \times Z, \frac{dx}{x} \wedge \alpha + \beta)$$

**Simple codimension–1 foliations.**

The cosymplectic structure on the singular locus induces a codimension–1 symplectic foliation. Assume now that $Z$ is compact and connected. There is a map

$$p : Z \longrightarrow \mathbb{R}/\alpha(H_1(Z, \mathbb{Z})) \tag{3.9}$$

defined as $p(z) := \int_{z_0}^{z} \alpha$. Whenever $\alpha$ is a multiple of a rational form, then $\mathbb{R}/\alpha(H_1(Z, \mathbb{Z})) = \mathbb{R}/c\mathbb{Z}$ for some positive number $c$. In this case, one can prove that the map is a fibration, such that the connected components of the fibers are leaves. Moreover, one can this map to realize the leaves of the foliation $\ker \alpha$ as the fibers of a fibration.

**Proposition 3.2.20** ([37])**.** *Let* $(Z, \alpha, \beta)$ *be a cosymplectic manifold such that* $\alpha$ *is a real multiple of a rational form. Then* $Z$ *is a symplectic mapping torus. More precisely, there exists a closed symplectic manifold* $(F, \beta_F)$*, a symplectomorphism* $\phi : (F, \beta_F) \longrightarrow (F, \beta_F)$*, and a constant* $c$*, such that* $(\mathbb{R} \times F/\mathbb{Z}, dt, \beta_F) \cong (Z, \alpha, \beta)$*, where the* $\mathbb{Z}$ *action is generated by* $1 \cdot (t, z) = (t + c, \phi(z))$*.*

If $Z$ is disconnected, each of its components $Z_i$ inherits a fibration over $\mathbb{R}/c_i\mathbb{Z}$. The number $c_i$ is a Poisson geometric invariant, called the *period of the modular vector field* ([33], [16]). Log-symplectic manifolds whose singular locus is a fibration are called *proper* ([4]). It is simple to observe that for a given log-symplectic form $\omega$ there exists a nearby log-symplectic form $\omega'$ such that the resulting log-symplectic manifold is proper (Theorem 3.6 in [4]).

## 3.3 Holomorphic curves in log-symplectic manifolds

In this section we study some general aspects of holomorphic curves in log-symplectic manifolds. We start by introducing the class of almost complex structures that we will consider (namely the cylindrical ones). Then we will introduce log-holomorphic curves, and we prove that log-holomorphic curves coincide with the punctured curves in SFT. We will also express the notion of energy that is used in standard Gromov-Witten theory and SFT in terms of log-symplectic data. Finally, we prove a maximum principle which is going to be crucial for our construction of the moduli spaces. The construction of the moduli spaces is carried out in the later sections, separately for closed and open curves.

### 3.3.1 Cylindrical complex structures on log-symplectic manifolds

Let us recall the notation we introduced previously: $U$ denotes a tubular neighbourhood of $Z$, with a fixed isomorphisms to the unit disc bundle of the normal bundle $NZ$ of $Z$. Let $\lambda$ denote the length of the fiber coordinate (with respect to some fiberwise Riemannian metric). We know from Proposition 3.2.15 and Proposition 3.2.16 that there exist closed forms $\alpha$, $\beta$ on $Z$, respectively a 1- and a 2-form, such that the log-symplectic form is

$$\omega = d\log\lambda \wedge \alpha + \beta \tag{3.10}$$

Viewing $X \setminus Z$ as $\widehat{W} := \widehat{X \setminus U}$, it is natural to consider $\omega$-compatible almost complex structures on $T(X \setminus Z)$ which are cylindrical, as in Definition 2.3.6. For reasons that will become apparent, we actually use a slightly modified definition of cylindrical complex structure. Recall that each connected component $Z_i$ of $Z$ has a well defined period $c_i \in (0, +\infty)$.

**Definition 3.3.1.** Let $(X, Z, \omega)$ be log-symplectic. A complex structure $J$ on $X \setminus Z$ is *cylindrical* if it is $\widehat{\omega}$-tame, and on each component $U_i$ of the cylindrical end $U$ it satisfies

- $J$ is $s$-invariant
- $J\partial_s = c_i R$
- $J(\ker\alpha) = \ker\alpha$
- $J$ is $\beta$-compatible on $\ker\alpha$

**Notation 3.3.2.** For simplicity, we will denote with $cR$ the vector field on $Z$ defined as $c_i R$ on each component $Z_i$. The above definition hence requires $J\partial_s = cR$.

We need also to impose a further compatibility condition with $Z$. Recall that $U$ is isomorphic to $\mathbb{R} \times_{\mathbb{Z}_2} \tilde{Z}$, where $\mathbb{Z}_2$ acts by multiplication by $-1$ on $\mathbb{R}$, and by an involution $\sigma$ on $\tilde{Z}$ (Corollary 3.2.17). Hence a cylindrical complex structure is an $\mathbb{R}$ invariant complex structure on $(\mathbb{R} \times \tilde{Z} \setminus \{0\} \times \tilde{Z})/\mathbb{Z}_2$. In particular it induces an almost complex structure on the each fiber $\tilde{F}$ of the codimension–1 foliation on $\tilde{Z}$.

**Definition 3.3.3.** A cylindrical complex structure is *adapted* to the log-symplectic structure if its restriction to $\tilde{F}$ is $\sigma$-invariant.

**Example 3.3.4.** If $X$ is orientable, then $\tilde{Z}$ is a union of two disjoint copies of $Z$. The definition is asking for the complex structures on two corresponding copies of $F$ to be equal.

The definition is designed so that the complex structure extends over $Z$ as a complex structure on $TX(-\log Z)$.

**Proposition 3.3.5.** *An adapted cylindrical almost complex structure induces a smooth complex structure on $TX(-log Z)$.*

Moreover, an adapted complex structure $J$ on $TX(-\log Z)$ induces a complex structure on $\ker\alpha \subset TZ$, compatibly with the leafwise symplectic form $\beta$ if $J$ is $\omega$-compatible. Indeed, consider the canonical section $\xi \in \Gamma(TX(-\log Z)|_Z)$. Consider the logarithmic 1-form $\rho^*\alpha \in \Gamma(T^*X(\log Z))$: $\xi \in \ker(\rho^*\alpha)$, and $\rho : \ker(\rho^*\alpha)|_Z \longrightarrow \ker(\alpha)|_Z \subset TZ$ is well-defined and surjective. Moreover, $J(\xi) \pitchfork \ker(\rho^*\alpha)$. Further, the symplectic orthogonal to the subspace generated by $\xi$ and $J(\xi)$ is contained in $\ker(\rho^*\alpha)$, and projects isomorphically to $\ker(\alpha)$. Finally, $\mathrm{span}(\xi,\, J(\xi))^{\perp\omega}$ is closed under $J$, hence $\rho$ induces an almost complex structure on $\ker(\alpha)$. This complex structure is the same that is induced by the quotient map $\tilde{Z} \longrightarrow \tilde{Z}/\mathbb{Z}_2 = Z$ (this follows from the normal form).
Moreover, since $\rho(J(\lambda\partial_\lambda)) = c\tilde{R}$ the Reeb vector field for all $\lambda \neq 0$ by definition, then $\rho(J(\xi)) = cR$ on $Z$.

**Notation 3.3.6.** We will normally use adapted compatible complex structures, and leave the word "adapted" implicit. We will use the expression "compatible complex structure" for both the almost complex structure on $X \setminus Z$ and its extension to $TX(-\log Z)$.

## 3.3.2 Riemann surfaces and log-holomorphic structures

In this section we will specialize the definition of cylindrical complex structure from the previous section to 2-dimensional log-symplectic manifolds $(\Sigma, \partial\Sigma)$, for which the singular locus coincides with the boundary. We explain how these "logarithmic Riemann surfaces" correspond to punctured Riemann surfaces, and that their automorphisms correspond to automorphisms of punctured surfaces.

### Logarithmic holomorphic structures

Let $\Sigma$ be a compact surface, possibly with boundary. Consider the log-tangent bundle of the pair $(\Sigma, \partial\Sigma)$, denoted by $T\Sigma(-\log\partial\Sigma)$.

**Definition 3.3.7.** A *(logarithmic) holomorphic structure* on $(\Sigma, \partial\Sigma)$ is a complex structure $j$ on $T\Sigma(-\log\partial\Sigma)$, such that there exist a positive defining function $r$ for $\partial\Sigma$, and a global parametrization $\theta$ of the boundary, such that $j(r\partial_r) = \partial_\theta$ in a collar neighbourhood of $\partial\Sigma$

**Remark 3.3.8.** Normally a holomorphic structure on a surface with boundary is defined as a conformal structure on its double, commuting with the canonical involution. This notion is different fromDefinition 3.3.7, as a logarithmic holomorphic structure does not extend to a holomorphic structure on the double surface (rather, it extends to a degenerate one). Nonetheless, as we will only deal with logarithmic holomorphic structures, we will often omit the word "logarithmic".

**Remark 3.3.9.** A logarithmic holomorphic structure on $(\Sigma, \partial\Sigma)$ is cylindrical, and adapted to a log-symplectic structure which is of the form $\frac{dr}{r} \wedge d\theta$ in a neighbourhood of $\partial\Sigma$.

### Punctured surfaces

Definition 3.3.7 implies that a collar neighbourhood of a connected component of the boundary is biholomorphic to $[0, \epsilon) \times S^1$, $j(r\partial_r) = \partial_\theta$. There is a map

$$\pi : [0, \epsilon) \times S^1 \longrightarrow D_\varepsilon \tag{3.11}$$

where $D_\varepsilon$ is the disc of radius epsilon, defined as

$$\pi(r, \theta) := re^{i\theta}$$

A simple computation shows that $\pi$ is a biholomorphism when restricted to the interior. Globally, using (3.11) one can construct a closed Riemann surface $\hat{\Sigma}$, with a map

$$\pi : \Sigma \longrightarrow \hat{\Sigma}.$$

The closed surface $\hat{\Sigma}$ comes with a finite set of distinguished points $p_1, \ldots, p_k$, namely $\pi(\partial\Sigma)$. The restriction $\pi : \Sigma \setminus \partial\Sigma \longrightarrow \hat{\Sigma} \setminus \{p_1, \ldots, p_k\} := \dot{\Sigma}$ is a biholomorphism.

Conversely, given a closed Riemann surface $\hat{\Sigma}$ with a finite set of punctures $p_1, \ldots, p_k$, one can view local charts centered at the punctures as a choice of cylindrical ends for $\dot{\Sigma} := \hat{\Sigma} \setminus \{p_1, \ldots, p_k\}$. One can compactify the punctured surface $\dot{\Sigma}$ adding circles at infinity – the resulting surface with boundary $\Sigma$ has a logarithmic holomorphic structure.

### Maps of Riemann surfaces

Surfaces with a holomorphic structure are related by the following natural notion of map.

**Definition 3.3.10.** A map of pairs $(\Sigma, \partial\Sigma) \longrightarrow (\Sigma', \partial\Sigma')$ between surfaces with logarithmic holomorphic structures is *holomorphic* if the induced map on the log-tangent bundles is complex linear.

**Lemma 3.3.11.** *(1) given $\Sigma$, the corresponding closed surface $\hat{\Sigma}$ is the unique closed surface with the property that there is a map $\pi : \Sigma \longrightarrow \hat{\Sigma}$ which is an isomorphism when restricted: $\pi : \Sigma \setminus \partial\Sigma \longrightarrow \hat{\Sigma} \setminus \{p_1, \ldots, p_k\}$. Unique means that given two surfaces with these properties there exists a unique isomorphism commuting with the projections.*

*(2) given a holomorphic map $\Sigma \longrightarrow \Sigma'$ there is a unique holomorphic map $\hat{\Sigma} \longrightarrow \hat{\Sigma}'$ making the following diagram commute*

$$\begin{array}{ccc} \Sigma & \longrightarrow & \Sigma' \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \hat{\Sigma} & \longrightarrow & \hat{\Sigma}' \end{array} \tag{3.12}$$

*(3) given a holomorphic map $\hat{\Sigma} \longrightarrow \hat{\Sigma}'$ there is a unique holomorphic map $\Sigma \longrightarrow \Sigma'$ making the following diagram commute*

$$\begin{array}{ccc} \Sigma & \longrightarrow & \Sigma' \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi} \\ \hat{\Sigma} & \longrightarrow & \hat{\Sigma}' \end{array} \tag{3.13}$$

*Proof.* (1) If there are two surfaces with a map $\pi : \Sigma \longrightarrow \hat{\Sigma}$ and $\pi' : \Sigma \longrightarrow \hat{\Sigma}'$, then there is an isomorphism in the complement of a finite number of points (namely $\pi' \circ \pi^{-1}|_{\Sigma\setminus\{p_1, \ldots, p_k\}}$). Bounded neighbourhoods of the punctures isomorphic to punctured discs are mapped to bounded neighbourhoods of the punctures. These maps extend as holomorphic automorphisms of the disc, so we obtain an isomorphism $\phi : \Sigma \longrightarrow \Sigma'$ such that $\pi\phi = \pi'$. □

*Proof.* (2), (3) Similar to the above, using also the fact that the only automorphisms of the punctured disc are rotations. □

### 3.3.3 Holomorphic curves

We are ready to introduce our notion of holomorphic curve in the log-setting. Fix a log-symplectic manifold $(X, Z, \omega)$, a tubular neighbourhood $U$ of the singular locus $Z$ in $X$, and pick a compatible cylindrical complex structure. Let $(\Sigma, \partial\Sigma)$ be a logarithmic Riemann surface; recall that a map of pairs $u : (\Sigma, \partial\Sigma) \longrightarrow (X, Z)$ induces by definition a diagram

$$\begin{array}{ccc} T\Sigma(-\log \partial\Sigma) & \xrightarrow{\overline{du}} & TX(-\log Z) \\ \big\downarrow & & \big\downarrow \\ T\Sigma & \xrightarrow{du} & TX \end{array}$$

**Definition 3.3.12.** A map of pairs $u : (\Sigma, \partial\Sigma) \longrightarrow (X, Z)$ is *log-holomorphic* (or simply *holomorphic*) if the lift of the differential $\overline{du} : T\Sigma(-\log \partial\Sigma) \longrightarrow TX(-\log Z)$ is complex linear.

The choice of using cylindrical complex structures implies that the image of the boundary of a holomorphic curve is constrained: it needs to coincide with a Reeb orbit.

**Proposition 3.3.13.** *Let $u : (\Sigma, \partial\Sigma) \longrightarrow (X, Z)$ be holomorphic. Then the boundary is mapped to a simple periodic orbit of the Reeb vector field.*

*Proof.* This follows from Proposition 3.2.11, and the fact that $\rho(J(-\lambda\partial_\lambda)) = \rho(J(\partial_s)) = cR$. □

We can also easily compute the period of the Reeb orbit: write $Z = \sqcup_i Z_i$, with $Z_i$ connected. Let $c_i > 0$ such that $\alpha(H_1(Z_i)) = c_i\mathbb{Z}$ (the period of the component $Z_i$). Then the period of the Reeb orbit to which a component of the boundary is mapped only depends on the connected components of $Z$ that contains it, and it coincides with $c_i$. In particular the orbit has minimal period, hence it is simply covered. Notice that a Reeb orbit $\gamma$ appearing as boundary condition of a holomorphic curve actually lifts to a closed loop $\tilde{\gamma}$ in $\tilde{Z}$. The cosymplectic structure on $\tilde{Z}$ is the pull-back of the one on $Z$, $(p^*\alpha, p^*\beta)$. Hence the period of $\tilde{\gamma}$ is also $c_i$, and is also the minimal period.

**Proposition 3.3.14.** *A log-holomorphic curve with non-empty boundary is simply covered.*

*Proof.* Our remarks above imply that a log-holomorphic is simply covered in a neighbourhood of the singular locus. It is a general fact that a holomorphic curve is simply covered if and only if it is so in an open neighbourhood ([28]) □

**Remark 3.3.15.** Consider the punctured surface $\dot{\Sigma}$ associated to $\Sigma$. A holomorphic map $u : (\Sigma, \partial\Sigma) \longrightarrow (X, Z)$ induces (by restriction) a map $u : \dot{\Sigma} \longrightarrow X \setminus Z$ which is holomorphic in the usual sense. The converse holds for simple maps if we assume that the holomorphic curve has finite energy (see [2]).

### 3.3.4 Energy

We give a definition of energy of a holomorphic curve in a log-symplectic manifold, and prove that it coincides with the SFT energy of the curve in the symplectic locus. We keep the same notation as in the previous section.

**Definition 3.3.16** ([22])**.** Consider a function $\mu : X \longrightarrow \mathbb{R}$, smooth on $X \setminus Z$, that coincides with $\lambda$ in a neighbourhood of $Z$, vanishing only at $\lambda = 0$, and such that $\mu - 1$ is compactly supported in $U$. Define the form $\beta^\mu$ so that the following equality holds:

$$\omega = d\log\mu \wedge \alpha + \beta^\mu$$

**Remark 3.3.17.** The cohomology class of $\beta^\mu$ is well defined and non-zero ([22]).

**Definition 3.3.18** (Energy)**.** Let $u : \Sigma \longrightarrow X$ be a smooth map. Define the *energy* of $u$ as

$$E(u) := \int_\Sigma u^*\beta^\mu + \int_{\partial\Sigma} u^*\alpha$$

**Proposition 3.3.19.** *$E(u)$ is well defined.*

*Proof.* Choose two functions $\mu$, $\mu'$ as in Definition 3.3.16. One sees immediately that

$$\beta^{\mu} - \beta^{\mu'} = d(\log(\frac{\mu'}{\mu})\alpha)$$

By Stokes' formula

$$\int_{\Sigma} u^* \beta^{\mu} - \int_{\Sigma} u^* \beta^{\mu'} = \int_{\partial\Sigma} u^* \log(\frac{\mu'}{\mu})\alpha = 0$$

where the last equality holds as $\mu = \mu'$ in a neighbourhood of $Z$. □

**Remark 3.3.20.** Since $\beta^{\mu}$ is closed, the energy of $u$ depends only on $u(\partial\Sigma)$ and on the homology class of $u$ rel $u(\partial\Sigma)$.

The energy of a closed holomorphic curve $\Sigma$ mapped to the symplectic manifold $(X \setminus Z, \omega|_{X\setminus Z})$ coincides with the usual notion from symplectic geometry.

**Proposition 3.3.21.** *If $\Sigma$ is closed and $u : \Sigma \longrightarrow X$ is holomorphic, with image contained in $X \setminus Z$, then*

$$E(u) = \int_{\Sigma} u^* \omega$$

*Proof.* The image of $u$ is contained in the complement of some open set $V$ containing $Z$. We can choose $\mu$ to be equal to 1 on $X \setminus V$. □

We prove here that Definition 2.3.7 is equivalent to Definition 3.3.18. Notice that for log-symplectic manifolds it is actually unnecessary to take the supremum in Definition 2.3.7, since $\alpha$ is closed.

**Proposition 3.3.22.** *$E_{SFT}(u) = \varepsilon E(u)$. In particular, the two notions are equivalent, and coincide if we choose $\varepsilon = 1$.*

*Proof.* Note first that $\omega|_{X\setminus U} = \omega_{\varphi}|_{X\setminus U}$, hence we only need to check the equality in the cylindrical end $U$. To this aim, we assume without loss of generality that $u^{-1}(U)$ is a Riemann surface with smooth boundary. On $U$ we can write $\omega = d\log\lambda \wedge \alpha + \beta$. Choosing a function $\mu$ as in Definition 3.3.16, one has $\omega = d\log(\mu) \wedge \alpha + \beta^{\mu} = \omega = d\text{log}\lambda \wedge \alpha + \beta$, hence

$$E(u) := \int_{\partial\Sigma} u^* \alpha + \int_{u^{-1}(U)} u^* \beta^{\mu}$$

$$= \int_{\partial\Sigma} u^* \alpha + \int_{u^{-1}(U)} u^* d(\log\lambda - \log\mu) \wedge u^* \alpha + \int_{u^{-1}(U)} u^* \beta$$

$$= \int_{\partial\Sigma} u^* \alpha + \int_{u^{-1}(U)} u^* d(\log(\frac{\lambda}{\mu})) \wedge u^* \alpha + \int_{u^{-1}(U)} u^* \beta$$

Noting that $\partial u^{-1}(U) = -u^{-1}(\partial\overline{U}) \cup \partial\Sigma$, Stokes' theorem gives

$$E(u) = \int_{\partial\Sigma} u^*\alpha + \int_{u^{-1}(U)} u^*\beta - \int_{u^{-1}(\partial\overline{U})} u^* \log(\frac{\lambda}{\mu}) \wedge u^*\alpha + \int_{\partial\Sigma} u^* \log(\frac{\lambda}{\mu}) \wedge u^*\alpha$$

Now, $\lambda \equiv \mu$ near $Z$, which implies $\int_{\partial\Sigma} u^* \log(\frac{-\lambda}{\mu}) \wedge u^*\alpha = 0$. As $\mu \equiv 1$ near $\partial\overline{U}$, one has that $\int_{u^{-1}(\partial\overline{U})} u^* \log(\frac{\lambda}{\mu}) \wedge u^*\alpha = \int_{u^{-1}(\partial\overline{U})} u^* \log\lambda \wedge u^*\alpha$. Hence

$$E(u) = \int_{\partial\Sigma} u^*\alpha + \int_{u^{-1}(U)} u^*\beta - \int_{u^{-1}(\partial\overline{U})} u^* \log\lambda \wedge u^*\alpha \tag{3.14}$$

Let us turn to the SFT energy. Writing out explicitly what the SFT energy is, one gets

$$E_{SFT}(u) = \int_{u^{-1}(U)} u^* d(\varphi(-\log(\lambda))) \wedge u^*\alpha + u^*\beta$$

Comparing with Equation (3.14), one sees that $E_{SFT}(u) = \varepsilon E(u)$ if and only if

$$\begin{aligned} &-\int_{u^{-1}(\partial\overline{U})} u^*\varphi(-\log\lambda) \wedge u^*\alpha + \int_{\partial\Sigma} u^*\varphi(-\log\lambda) \wedge u^*\alpha \\ &\qquad = \varepsilon\Big[\int_{\partial\Sigma} u^*\alpha + \int_{u^{-1}(\partial\overline{U})} u^* \log\lambda \wedge u^*\alpha\Big] \end{aligned}$$

To conclude, we only need to observe that $\varphi = \mathit{identity}$ near $\partial\overline{U}$, and $\varphi(-log\lambda) \to \varepsilon$ as $\lambda \to 0$ (i.e. on $\partial\Sigma$). □

### 3.3.5 The maximum and minimum principle

We prove in this section a maximum principle for holomorphic maps. This is crucial in the proof of all compactness theorems for families of holomorphic curves.

Realize $X \setminus Z$ as $\widehat{W}$, as in Proposition 3.2.16. Assume $X$ is endowed with a cylindrical complex structure. Restrict to the cylindrical end $[0, +\infty) \times \partial W$, and consider the first projection $p_1 : [0, +\infty) \times \partial W \longrightarrow [0, +\infty)$. Denote by $D^2_r$ the open disc of radius $r$.

**Proposition 3.3.23.** *Let $u : D^2_r \longrightarrow [0, +\infty) \times \partial W$ be holomorphic. Then $p_1 \circ u$ is harmonic.*

*Proof.* $\Delta(p_1 \circ u) = -dd^c(p_1 \circ u) = d(d(p_1 \circ u) \circ J) = d(dp_1 \circ J \circ du) = d(\alpha \circ du) = u^*d\alpha = 0$. □

**Corollary 3.3.24.** *Let $\Sigma$ be a closed Riemann surface. Let $u : \Sigma \longrightarrow X$ be holomorphic and such that $u(\Sigma) \cap U \neq \emptyset$. Then $\Sigma$ is mapped to a symplectic leaf in $\{\lambda\} \times \partial W$, $\lambda \neq 0$.*

**Corollary 3.3.25.** *Let $\Sigma$ be compact with non-empty connected boundary, $u : \Sigma \longrightarrow X$ holomorphic with $u^{-1}(Z) = \partial\Sigma$. Then $u(\Sigma) \cap (X \setminus U) \neq \emptyset$.*

*Proof.* If $X \setminus U \cap u(\Sigma) = \emptyset$, then $p_1 \circ u$ has a minimum, in contradiction with Proposition 3.3.23. □

## 3.4 Closed curves

In this section we study the moduli space of closed curves in a log-symplectic manifold, and prove several results on the topology of log-symplectic manifolds, including Theorem 3.1.1 and Theorem 3.1.2.

### 3.4.1 A simpler case: genus $0$, aspherical $Z$

Let us begin with a discussion of a simpler setup: we look at holomorphic spheres in manifolds where there can be no holomorphic spheres in a neighbourhood of the singular locus $Z$.

**Definition 3.4.1.** We say $(Z, \alpha, \beta)$ is *symplectically aspherical* if $A \cdot \beta = 0$ for all homotopy classes $A \in \pi_2(Z, z_0)$ for all $z_0 \in Z$.

Manifolds for which $\pi_2(Z_i) = 0$ for all components $Z_i$ of $Z$ are symplectically aspherical. In the proper case, since $\pi_2(Z) \cong \pi_2(F)$ for a symplectic fiber $F$, the definition coincides with the usual asphericity condition for the symplectic manifold $(F, \beta)$.
If $(Z, \alpha, \beta)$ is aspherical, then all double covers $p : \tilde{Z} \longrightarrow Z$ with the pulled-back cosymplectic structure are. By the energy identity and Proposition 3.3.22, there exists no holomorphic sphere $u : S^2 \longrightarrow \tilde{Z}$. The maximum and minimum principles (Proposition 3.3.23) imply the following:

**Corollary 3.4.2.** *Let $(X, Z, \omega)$ be log-symplectic, with $Z$ symplectically aspherical. Then all non-constant holomorphic spheres are mapped into $X \setminus U$.*

This specializes, for instance, to:

**Corollary 3.4.3.** *Assume $X$ has dimension $4$, and $Z$ has no component diffeomorphic to $S^1 \times S^2$. Then all non-constant holomorphic spheres are mapped into $X \setminus U$.*

When $Z$ is aspherical, then, all holomorphic spheres live in a compact subset of the symplectic locus of $X$. Moreover, they all intersect (and are, in fact, contained in) the subset of the symplectic locus where the almost complex structure is allowed to vary arbitrarily. Thus, the transversality for the moduli space can be achieved for a generic choice of complex structure (Corollary 2.2.8). Further, since all the curves take value in a compact subset, Gromov's compactness theorem also applies (Theorem 2.2.2).

**Proposition 3.4.4.** *Let $(X, Z, \omega)$ be log-symplectic, with $Z$ symplectically aspherical. Let $A \in H_2(X)$ be a spherical homology class. Then there exist a comeager set of cylindrical complex structures $\mathcal{J}_{reg}$ such that for all $J \in \mathcal{J}_{reg}$ the moduli space $\mathcal{M}^*_{0,m}(J, A)$ is smooth of dimension $2n - 6 + 2c_1(A) + 2m$, and it is compact modulo bubbling.*

In particular, if bubbling can be prevented (for example by considering classes $A$ such that each non-trivial partition $A = A_1 + \cdots + A_k$ has $A_i \cdot \omega \leq 0$)

then $\mathcal{M}_0^*(A)$ is a compact smooth manifold of dimension vdim$\mathcal{M}(A)$. In general, bubbling can be controlled if $X \setminus Z$ is low-dimensional (dim$X = 4, 6$), or is, more generally, a *semipositive manifold* ([28]).

**Definition 3.4.5.** A symplectic manifold $(M, \omega)$ is *semipositive* if for all $A \in \pi_2(M)$, $3 - n \leq c_1(A) < 0$ implies $\omega \cdot A \leq 0$.

This is automatically satisfied in dimension $\leq 6$. The usefulness of this definition lies in the fact that in semipositive manifolds, and for generic complex structures, bubbling can only happen in codimension two. This means that fixing a homology class $A$, the set of nodal curves representing $A$ has codimension at least 2 in the space of all curves representing $A$.
For convenience we will say that log-symplectic manifold $(X, Z, \omega)$ is *semipositive* $(X \setminus Z, \omega|_{X \setminus Z})$ is semipositive.

### 3.4.2 Obstructing certain log-symplectic manifolds

A standard argument by McDuff ([25]) can be used to obstuct the existence of certain classes of log-symplectic manifolds, of which Theorem 3.1.1 mentioned in the introduction to this chapter is a special case. Let us summarize McDuff's argument here. We keep on assuming that $Z$ is aspherical, and that $X \setminus Z$ is semipositive.
Consider $A \in H_2(X \setminus Z)$ with $c_1(A) = 2$, so that vdim$\mathcal{M}_{0,1}^*(A, J) = 2n$, and pick a generic $J$. Consider the evaluation map $\mathrm{ev} : \mathcal{M}_{0,1}^*(A, J) \longrightarrow X \setminus Z$. Assuming for the moment that the evaluation ev is a proper map, we can compute its degree. Since $Z$ is aspherical there is no holomorphic sphere in a neighbourhood of $Z$, thus $\deg(ev) = 0$.
However, in some explicit example one might be able to prove that there is a point $x \in X \setminus Z$ such that there exists a unique Fredholm regular holomorphic sphere passing through that point, leading to a contradiction with the asphericity of $Z$.
For instance, one can prove that such holomorphic spheres exist in the following situations (see [25] for details and a more general discussion):

- if there exist symplectic submanifolds with trivial normal bundle, symplectomorphic to $S^2 \times V$, with $V$ a Kähler manifold with $\pi_2(V) \cdot \omega = 0$. Here extend the product complex structure on $S^2 \times V$ to the whole manifold, take $A = [S^2 \times \{p\}]$ and apply [28], Chapter 3.3, to show that the obvious holomorphic sphere if Fredholm regular.

- there is a boundary component of contact type, contactomorphic to $S^{2n-1}$ with the standard contact structure. Here realize the sphere as the boundary of a Darboux ball in $S^2 \times \cdots \times S^2$, and take $A = [S^2 \times \{p\}]$

- if dim$X \leq 6$, a codimension-2 symplectic submanifold $S$ symplectomorphic to $\mathbb{C}P^{n-1}$, with $c_1(NS) \geq 0$. Blow-up a copy of $\mathbb{C}P^{n-2}$ until the normal bundle to $S$ becomes trivial, and take the proper transform of a $\mathbb{C}P^1 \subset \mathbb{C}P^{2n-1}$ transverse to $\mathbb{C}P^{n-2}$.

If ev is not proper, but $X \setminus Z$ is semipositive, one can argue as follows. Pick a point $x$ in a neighbourhood of $Z$, and a point $x' \in X \setminus Z$ (in the same component as $x$) such that there is an element in $\mathcal{M}^*_{0,1}(A, J)$ passing through $x$. Let $N := ev(\overline{\mathcal{M}}_0(A, J) \setminus ev(\mathcal{M}^*_0(A, J))$ be the set of points in the image of a nodal curve. This set has codimension at least 2 by semipositivity ([25], Corollary 4.9), hence there is a path $\gamma$ with values in $(X \setminus Z) \setminus N$ joining $x$ and $x'$. There is a neighbourhood of $\gamma$ which does not intersect $N$, and the evaluation map is proper over this neighbourhood (no nodal curves are mapped to this neighbourhood). Now we can argue as in the compact case above.
As examples of applications, we point out the following corollaries.

**Corollary 3.4.6.** *Let $(V, \sigma_V)$ be a symplectic manifold such that $\pi_2(V)\cdot\sigma_V = 0$, and such that $V_k$ supports a $\sigma_V$-compatible integrable complex structure. Let $\sigma_{S^2}$ be any symplectic form on $S^2$. Let $(X, Z, \omega)$ be a closed semipositive log-symplectic manifold, such that a component of $Z$ is symplectomorphic to a mapping torus over $(S^2 \times V, \sigma_{S^2} \times \sigma_V)$. Then none of the components of $Z$ are symplectically aspherical.*

Given that any symplectic submanifold $W$ with trivial normal bundle can be used to produce a singular component diffeomorphic to $S^1 \times W$ ([4]), the conclusion of the Corollary 3.4.6 holds if one assumes that $X$ contains a symplectic $(S^2 \times V, \sigma_{S^2} \times \sigma_V)$ with trivial normal bundle.
A log-symplectic manifold with boundary is a manifold $X$ with boundary $\partial X$ with a codimension–1 submanifold $Z$ such that $\partial X \cap Z = \emptyset$, together with a symplectic form $\omega$ on $TX(-\log Z)$.

**Corollary 3.4.7.** *Assume that $(X, Z, \omega)$ is compact, semipositive, and has non-empty boundary of contact type, contactomorphic to the standard $(2n-1)$-dimensional sphere. Then $Z$ cannot be symplectically aspherical.*

The following is also an easy corollary of McDuff's argument, applied to log-symplectic manifolds with non-aspherical singular locus.

**Corollary 3.4.8.** *Let $(V_k, \sigma_{V_k})$ be symplectic manifolds such that $\pi_2(V_k)\cdot\sigma_{V_k} = 0$, and such that $V_k$ supports a $\sigma_{V_k}$-compatible integrable complex structure. Let $\sigma_{S^2}$ be any symplectic form on $S^2$. Let $Z$ be a union of symplectic mapping tori over $(S^2 \times V_k, \sigma_{S^2} \times \sigma_{V_k})$. Then there exists no log-symplectic manifold $(X, Z, \omega)$ with non-empty boundary of contact type.*

*Proof.* Let $U$ be a neighbourhood of the singular locus, with smooth boundary. The boundary of $U$ is also a mapping torus over $S^2 \times \tilde{V}_k$, for some double cover $\tilde{V}_k$. We can use the $S^2$ factor to produce a holomorphic curve with Chern number 2. By the properties of the moduli space of spheres there must then be a holomorphic curve with values in a neighbourhood of the boundary, which is a contradiction with the contact type hypothesis. More precisely, due to a maximum principle analogous to Proposition 3.3.23 any sphere with a point mapped close to the boundary must be entirely contained in a neighbourhood of the boundary. Moreover, one can ensure that there exists a sphere mapped in a neighbourhood of the boundary where the symplectic form is exact. By the energy identity this implies that the holomorphic sphere has 0 energy, and is therefore constant, which contradicts the fact that it has Chern number 2. □

Specializing the above discussion to the 4–dimensional case, one obtains the following statements, which provide a strengthening of Theorem 3.1.1. Recall that every 4–manifold is semipositive.

**Corollary 3.4.9.** *If a log-symplectic 4–manifold contains a symplectic sphere with 0 self-intersection, then the singular locus is a union of copies of $S^1 \times S^2$.*

**Corollary 3.4.10.** *If one component of the singular locus of a closed 4–dimensional log-symplectic manifold is diffeomorphic to $S^1 \times S^2$, then all components are diffeomorphic to $S^1 \times S^2$.*

**Corollary 3.4.11.** *In a symplectic 4–manifold, symplectic spheres with non-negative normal Chern number, and symplectic surfaces of positive genus with trivial normal bundle, must intersect (in particular they cannot be homologous).*

*Proof.* This is because around a symplectic surface $\Sigma$ with trivial normal bundle the symplectic structure can be modified into a log-symplectic form, with singular locus $S^1 \times \Sigma$ ([4], Theorem 5.1). □

**Corollary 3.4.12.** *Let $(X, Z, \omega)$ be compact log-symplectic 4–manifold with non-empty boundary of contact type. Then $Z$ is a union of $S^1 \times S^2$.*

**Corollary 3.4.13.** *Let $(X, Z, \omega)$ be compact log-symplectic 4–manifold with non-empty boundary of contact type. Then $\partial X$ is not contactomorphic to the standard contact 3-sphere.*

### 3.4.3 Compactness of the moduli space

Let us now consider an arbitrary **proper** log-symplectic manifold. In order to compactify the moduli space of curves one needs to take care not only of bubbling, but also of the non-compactness caused by the presence of the singular locus, as shown in the following simple example.

**Example 3.4.14.** Consider $(X, Z) = (\mathbb{R} \times S^1 \times \Sigma, \{0\} \times S^1 \times \Sigma)$, with coordinates $(x, t, z)$, and complex structure $J(x\partial_x) = \partial_t$, $J|_{T\Sigma} = j_\Sigma$ on $TX(-\log Z)$. The inclusion $u_{(x,t)}$ of $\Sigma$ in $X$ as $\{(x, t)\} \times \Sigma$ is a log-holomorphic map for all $x \neq 0$. Letting $x \to 0$, one finds the map $u_{(0,t)}$, the inclusion of $\Sigma$ in $X$ as $\{(0, t)\} \times \Sigma$. This is not a map of pairs, as in Definition 3.2.9.

This example suggests that we might want to add the following set to the moduli space:

$$\mathcal{M}_g(A, J; Z) := \left\{ [(u, j)] \, : \, \begin{array}{l} u : \Sigma_g \longrightarrow Z \text{ has values in a symplectic leaf } F \\ u \text{ is } (j, J)\text{-holomorphic as an } F\text{-valued map} \end{array} \right\} / \sim \tag{3.15}$$

The notion of holomorphicity as an $F$-valued map is well-defined, thanks to the discussion following Proposition 3.3.5 – a cylindrical complex structure on $TX(-\log Z)$ induces a compatible leafwise complex structure on $\ker\alpha \subset TZ$. In order to show that one can compactify the moduli space by adding (possibly nodal) holomorphic curves in $Z$, and nodal curves in $X \setminus Z$, one applies a very simple idea: one just observes that a sequence of holomorphic curves has a subsequence contained in the compact symplectic manifold $X \setminus U$, or has a subsequence contained in a compact neighbourhood of $Z$. In both cases Gromov's theorem gives us a convergent subsequence. In the remainder of this section we make this idea precise.

**Lemma 3.4.15.** *$\mathcal{M}_g(A, J;\, \tilde{Z})$ is compact modulo nodal curves.*

*Proof.* In the proper case, this is an immediate consequence of Theorem 2.2.2, applied to a (compact) fiber $\tilde{F}$ with varying complex structure $\tilde{F}_t$. □

**Remark 3.4.16.** Of course one does not need to fix the homology class $A$: it is enough to uniformly bound the energy.

**Theorem 3.4.17.** *Let $(X,\, Z,\, \omega)$ be a closed log-symplectic manifold. $\mathcal{M}_g(A, J) \sqcup \mathcal{M}_g(A, J; Z)$ is compact modulo nodal curves*

*Proof.* Take a sequence $u_k$ of elements in $\mathcal{M}_g(A, J) \sqcup \mathcal{M}_g(A, J; Z)$. If there is a subsequence with values in $\mathcal{M}_g(A, J; Z)$, the statement is a consequence of Lemma 3.4.15, hence we can assume that $u_k \in \mathcal{M}_g(A, J)$. Let $\tilde{U} \cong (-\varepsilon,\, \varepsilon) \times \tilde{Z}$. Let $U := \tilde{U}/\mathbb{Z}_2$. A sequence of holomorphic curves $u_k$ satisfies one of the following two:

- it has a subsequence contained in $X \setminus U$
- it has a subsequence contained in $\overline{U} \setminus Z$

In the first case, there is a subsequence converging to a nodal curve in $X \setminus U$, by the standard compactness result. In the second case, there are non-zero real numbers $\lambda_k$, and holomorphic maps $v_k : \Sigma \longrightarrow \tilde{Z}$, such that $u_k = (\lambda_k,\, v_k)$. Up to a subsequence, this converges in $\tilde{U} = (-\varepsilon,\, \varepsilon) \times \tilde{Z}$ to $u_\infty = (\lambda_\infty,\, v_\infty)$, with $\lambda_\infty$ a possibly zero real number, and $v_\infty$ a nodal holomorphic curve in $\tilde{Z}$. Denoting the projection to the quotient with $q : \tilde{U} \longrightarrow U$, we find that $q \circ u_\infty$ is the limit (in $U \subset X$) of $u_k = q \circ u_k$ (up to subsequence). □

The proof shows that we do not need to add the whole of $\mathcal{M}_g(A, J;\, Z)$ in order to compactify $\mathcal{M}_g(A, J)$. It is enough to take the union of $\mathcal{M}_g(A, J)$ the following set:

$$\mathcal{N}(A, J; U) := \{q \circ u \,:\, u : \Sigma \longrightarrow (-\varepsilon,\, \varepsilon) \times \tilde{Z}\}/\sim \tag{3.16}$$

It is readily seen, using Corollary 3.3.24, that

$$\mathcal{N}(A, J; U) = (-\varepsilon,\, \varepsilon) \times_{\mathbb{Z}_2} \mathcal{M}_g(A, J;\, \tilde{Z}) \tag{3.17}$$

where the action of $\mathbb{Z}_2$ on $\tilde{Z}$ is by post-composition with the involution $\sigma$ (recall that we chose a $\sigma$-equivariant $J$).

**Definition 3.4.18.** Denote with $\mathcal{N}_g(A, J)$ the compactification of the moduli space of curves of genus $g$, obtained by adding to $\mathcal{M}_g(A, J)$ the maps of the form $q \circ u : \Sigma_g \longrightarrow Z$, with $u \in \mathcal{M}_g(A, J; \tilde{Z})$. The space $\mathcal{N}(A, J; U)$ is the set of all elements in $\mathcal{N}_g(A, J)$ with values in $U$.

**Remark 3.4.19.** Using the results from Chapter 5 (Theorem 5.5.2 and Corollary 5.4.3) one can apply the argument above in the non-proper case. The properness assumption simplifies the proof of compactness, allowing to reduce it to a compactness result for sequences of complex structures on the same manifold.

### 3.4.4 Transversality for closed holomorphic curves

We would like to prove that the moduli space is a smooth manifold. We consider here the compactification $\mathcal{N}_g(A, J)$, as in Definition 3.4.18, and prove that it is smooth, under some assumption. In particular, we prove smoothness for (the compactification of) the moduli space of genus 0 maps in 4–dimensional log-symplectic manifolds.
By the maximum principle (Corollary 3.3.24), one has

$$\mathcal{N}_g(A, J) = \mathcal{N}_g(A, J; U) \sqcup \mathcal{M}_g(A, J; X \setminus U) \tag{3.18}$$

Since regularity can be expected for somewhere injective curves, let us introduce a notation. We denote by

$$\mathcal{N}_g^*(A, J) := \mathcal{N}_g^*(A, J; U) \sqcup \mathcal{M}_g^*(A, J; X \setminus U)$$

where $\mathcal{M}_g^*(A, J; X \setminus U)$ is the subset of somewhere injective curves, and $\mathcal{N}_g^*(A, J; U)$ is the set of curves of the form $q \circ u$ with $u$ somewhere injective.
We know from the general theory that the curves in $\mathcal{M}_g^*(A, J; X \setminus U)$ are Fredholm regular for a generic choice of almost complex structure (Corollary 2.2.8). The regularity of curves in a neighbourhood of the singular locus is more delicate: we observe in the following example that we cannot hope for Fredholm regularity for positive-genus holomorphic curves.

**Example 3.4.20.** Consider $\mathbb{R} \times S^1 \times \Sigma_g$, and the simple map $u : \Sigma_g \longrightarrow \mathbb{R} \times S^1 \times \Sigma_g$. This is holomorphic for appropriate choices of cylindrical complex structures. The pull-back of the tangent bundle is $\underline{\mathbb{C}} \times T\Sigma_g$. The $\overline{\partial}$ operator splits, because of integrability of the complex structure. This operator has always a non-trivial cokernel by the Riemann-Roch formula, unless $g = 0$.

So let us content ourselves to consider the genus 0 case. The first step is to show that in some cases one can deduce the Fredholm regularity from the symplectic leaves. Assume we are in a neighbourhood $U$ of the singular locus $Z$, and $u : S^2 \longrightarrow U$ is of the form $u = (\lambda_0, [t_0, v])$ with $v : S^2 \longrightarrow \tilde{F}_{t_0}$, $\lambda_0 \neq 0$. Then $u^*TX = \underline{\mathbb{C}} \oplus v^*T\tilde{F}$.

**Corollary 3.4.21.** *If the complex structure on $\tilde{F}_{t_0}$ is integrable, $u$ is regular if and only if $v$ is regular.*

*Proof.* This follows immediately from Proposition 2.2.10. □

This is enough to prove that all simple holomorphic spheres are generically regular in dimension 4. Indeed, if $Z$ has a component $Z_0$ for which the symplectic fibers are not spheres, then we know that there are no holomorphic spheres with values in the correponding component $U_0$ of $U$. Hence we can restrict to the case where $Z$ is a collection of $S^1 \times S^2$'s. By the formula (3.17) we only need to prove that

- $\mathcal{M}^*(A, J; \tilde{Z})$ is smooth
- the action of $\mathbb{Z}_2$ on $(-\varepsilon, \varepsilon) \times \mathcal{M}_g(A, J; \tilde{Z})$ is free

The first item follows from generic transversality, and Corollary 3.4.21. Actually, one could also show this explicitly. In fact, the only somewhere injective maps of a sphere into a sphere are the biholomorphisms, which all represent the same homology class $A$. For that homology class, $\mathcal{M}^*(A, J; \tilde{Z}) = S^1$. Let us turn the second item into a lemma.

**Lemma 3.4.22.** *$\mathbb{Z}_2$ acts freely on $(-\varepsilon, \varepsilon) \times \mathcal{M}_g(A, J; \tilde{Z})$.*

*Proof.* Assume $(-1) \cdot (\lambda, v) = (\lambda, v)$. Then there exists a biholomorphism $\varphi : S^2 \longrightarrow S^2$ such that

$$(\lambda, v) = (-\lambda, \sigma \circ v \circ \varphi)$$

or equivalently

$$\sigma \circ v = v \circ \varphi$$

It is enough to show that $\sigma$ induces a fixed-point free map on the leaf space ($S^1$) of $\tilde{Z} = S^1 \times S^2$. That has to be the case because if there exists a $t \in S^1$ such that $\sigma(t, z) = (t, \tau(z))$, for a symplectomorphism $\tau$. The map $\tau$ is Hamiltonian, hence has fixed points. Since $\sigma$ needs to be free, that's a contradiction. □

**Corollary 3.4.23.** *$\mathcal{N}^*(A, J; U)$ is smooth and diffeomorphic to $(-\varepsilon, \varepsilon) \times_{\mathbb{Z}_2} \mathcal{M}_g(A, J; \tilde{Z})$*

The proof actually shows that $\mathcal{N}^*(A, J; U)$ is a cylinder when $X$ is orientable and a Möbius band when $X$ is not orientable.

**Theorem 3.4.24.** *Let $(X, Z, \omega)$ be a log-symplectic manifold of dimension 4. There is a comeager set of cylindrical complex structures $\mathcal{J}_{reg}$ such that for all $J$ in $\mathcal{J}_{reg}$ all simple holomorphic maps $u : S^2 \longrightarrow X$ are Fredholm-regular. In particular the subset $\mathcal{N}_0^*(A, J) \subset \mathcal{N}_0(A, J)$ consisting of simple curves is a smooth manifold of dimension $2c_1(A) - 2$.*

*Proof.* Consider a tubular neighbourhood $U \cong (-\varepsilon, \varepsilon) \times_{\mathbb{Z}_2} Z$. We already know that for a generic choice of $J$ all the spheres with values in $X \setminus U$ are Fredholm regular, so we only need to look at $U$. Let $F$ be the (2-dimensional) fiber. If $F$ has genus $g > 0$, then the theorem is just a consequence of Proposition 3.4.4.

If $F$ has genus $g = 0$, we need to study the regularity of simple curves with values in $U \setminus Z$. Since all complex structures on $F$ are integrable, we can apply the regularity theorem Corollary 3.4.21. The only simple holomorphic map has Chern number 2, which is bigger than $-1$. Hence it is Fredholm regular. To conclude, as a consequence of Lemma 3.4.22 we obtain that $\mathcal{N}^*(A, J; U)$ is smooth also at points corresponding to $Z$-valued maps. □

### 3.4.5 Ruled surfaces

We noted in Corollary 3.4.10 that for a log-symplectic 4–manifold the presence of a singular component diffeomorphic to $S^1 \times S^2$ forces all the singular locus to be a union of $S^1 \times S^2$. This condition turns out to be even more restrictive, as it forces the entire manifold to be a blow-up of a *ruled surface*, i.e. a blow-up a 4–manifold supporting a fibration $X \longrightarrow B$ over a surface, with symplectic fiber $S^2$. This result is part of what we called Theorem 3.1.2 in the introduction, and is analogous to what McDuff proves in [26]. There it is proved that in a symplectic 4–manifold the existence of a symplectically embedded sphere with trivial normal bundle is equivalent to the manifold being ruled (up to blow-up). The key fact is the following refined compactness theorem from [40].

**Theorem 3.4.25** ([40], Theorem 4.6 )**.** *Let $(M, \omega)$ be a closed symplectic 4–manifold. Let $A$ be a homology class with $A \cdot A = 0$. Let $J$ be a generic complex structure, and consider a sequence of embedded $J$-holomorphic spheres representing the class $A$. Then a convergent subsequence converges to either an embedded sphere (in the same class $A$), or to a nodal curve with two irreducible components, in classes $A_1$ and $A_2$, with self-intersection $-1$, and intersecting each other positively at exactly one point. Moreover the moduli space of such nodal curves is compact and 0-dimensional.*

The proof of this result is based on an analysis of the virtual dimension, combined with the adjunction formula (in particular the fact that the index determines whether the curve is embedded). The genericity of $J$ is needed in order to prevent negative-index holomorphic curves to appear. The exact same proof applies to sequences of holomorphic curves in $X \setminus Z$ converging to a nodal curve in $X \setminus Z$. Hence, we can prove a log-symplectic version of Theorem 3.4.25 as soon as we can prevent bubbling in a neighbourhood of $Z$.
Let us give a name to the 4–manifolds with singular locus containing $S^1 \times S^2$.

**Definition 3.4.26.** A log-symplectic 4–manifold $(X, Z, \omega)$ is called *genus–0 log-symplectic* if one component of the singular locus is diffeomorphic to $S^1 \times S^2$.

We already know (Corollary 3.4.10) that if $(X, Z, \omega)$ is genus–0 then all of the components of $Z$ are $S^1 \times S^2$. Moreover if $(X, Z, \omega)$ contains a symplectic sphere with trivial normal bundle then $X$ is genus–0 (Corollary 3.4.9), and conversely if $X$ is genus 0 then the $S^2$-factor in the singular locus is an embedded symplectic sphere with trivial normal bundle. We first show the spheres in the singular locus are in fact all homologous (modulo double covers). In particular they all appear in the same moduli space.

**Proposition 3.4.27.** *Let $(X, Z, \omega)$ be a closed log-symplectic 4–manifold. Assume there exists an embedded symplectic sphere representing a homology class $A$ with zero self-intersection. Then all components of the singular locus are diffeomorphic to $S^1 \times S^2$. The double cover $\tilde{Z}$ is also diffeomorphic to $S^1 \times S^2$, and for each component $[\{p\} \times S^2 \subset \tilde{Z}] = A$ of $\tilde{Z}$.*

*Proof.* The proof goes as in Section 3.4.2. We already noted that all the components of the singular locus are $S^1 \times S^2$'s, and the same is true for the components of $\tilde{Z}$. Take neighbourhood of $Z$ such that $U \cong (-\varepsilon, \varepsilon) \times_{\mathbb{Z}_2} (S^1 \times S^2)$; the complement of $\{0\} \times_{\mathbb{Z}_2} (S^1 \times S^2)$ is just $(-\varepsilon, 0) \times S^1 \times S^2$. Pick a cylindrical complex structure that makes the embedded sphere $S$ into a $J$-holomorphic sphere, and such that $\{(\lambda, t)\} \times S^2 \subset (-\varepsilon, 0) \times S^1 \times S^2$ is holomorphic. By automatic transversality we can assume that $J$ is generic. The moduli space of simple spheres in the class $A$, together with the spheres in the singular locus, is then a smooth manifold (Theorem 3.4.24). It has dimension 2 by the adjunction formula. Let $N$ be the set of points belonging to a reducible nodal curve in the class $A$. The evaluation map

$$\mathrm{ev} : \mathcal{M}_{0,1}(A, J) \cap \mathrm{ev}^{-1}((X \setminus Z) \setminus N) \longrightarrow (X \setminus Z) \setminus N$$

is a proper map. By positivity of intersection and the fact that $A \cdot A = 0$, there is at most one $A$ curve through each point. Thus the degree of the evaluation is 1, hence the evalutation is a bijection (it is in fact a diffeomorphism). In particular there exists an $A$-curve through each point of $U \setminus Z$. By the maximum principle it has to be contained in $\{(\lambda, t)\} \times S^2 \subset \tilde{Z}$, and hence coincide with $\{(\lambda, t)\} \times S^2$. □

**Theorem 3.4.28.** *Let $(X, Z, \omega)$ be a closed log-symplectic 4–manifold. Let $A$ be a homology class with $A \cdot A = 0$. Let $J$ be a generic complex structure, and consider a sequence of embedded $J$-holomorphic spheres representing the class $A$. Then a convergent subsequence converges to either an embedded sphere (in the same class $A$), or to a nodal curve with two irreducible components, in classes $A_1$ and $A_2$, with self-intersection $-1$, and intersecting each other positively at exactly one point. Moreover the moduli space of such nodal curves is compact and 0-dimensional, and the nodal curves only appear in the symplectic locus.*

*Proof.* We know from the above Proposition 3.4.27 that the singular locus consists of $S^1 \times S^2$'s in the same class. Hence, by Corollary 3.4.23, the moduli space of curves with values in $U$ looks like $(-\varepsilon, \varepsilon) \times_{\mathbb{Z}_2} \mathcal{M}_0(A, J; \tilde{Z}) \cong (-\varepsilon, \varepsilon) \times_{\mathbb{Z}_2} S^1$. Hence no bubbling can occur in $U$. The statement now follows from Theorem 3.4.25 □

This result, together with the fact that the nodal singularities can be seen as Lefschetz singularities ([40]) implies the following characterization.

**Theorem 3.4.29.** *Let $(X, Z, \omega)$ be a closed genus–0 log-symplectic 4–manifold. Then $X$ supports a Lefschetz fibration $X \longrightarrow B$ with fibers of genus 0, so that $Z$ is a union of regular fibers. Moreover, the singular fibers have exactly one* *singular point, and their irreducible components are $(-1)$-curves. In particular if $X \setminus Z$ is minimal, then $X \longrightarrow B$ has no critical points.*

*Proof.* (see also [40]) The statement follows from the evaluation map

$$\mathrm{ev} : \overline{\mathcal{N}}_{0,1}(A, J) \longrightarrow X$$

being a diffeomorphism. The base of the Lefschetz fibration is $\overline{\mathcal{N}}_0(A, J)$. □

**Corollary 3.4.30.** *A genus–0 log-symplectic 4–manifold $X$ is diffeomorphic to a blow-up of one of the manifolds in the following list:*

- $S^2 \times \Sigma_g$
- $S^2 \times (\#^k \mathbb{R}P^2)$
- $S^2 \widehat{\times} \Sigma_g$
- $S^2 \widehat{\times} (\#^k \mathbb{R}P^2)$

*where by $S^2 \widehat{\times} B$ we mean the unique (up to diffeomorphism) non-trivial oriented sphere bundle over $B$.*

### 3.4.6 Uniqueness of Lie algebroid symplectic structures

The goal of this section is to show that the classification form Corollary 3.4.30 holds up to deformation equivalence, thus concluding the proof of Theorem 3.1.2 (see Corollary 3.4.38). More precisely, this means that any two log-symplectic structures on one of the manifolds in Corollary 3.4.30 can be joined by a path of log-symplectic forms. The method used to prove Corollary 3.4.38 actually hold for general symplectic structures on Lie algebroids ([20]), hence we state the results at that level of generality. In this section we introduce the relevant terminology, and prove the deformation equivalence of Lie algebroid symplectic structures supported by a Lie algebroid Lefschetz fibration. This applies in particular to symplectic, log-symplectic, and stable generalized complex structures ([6], [7]).

Let us recall what is the statement in the case of symplectic forms. Recall that a symplectic form $\omega$ on $M$ is said to be *supported by a Lefschetz fibration* if there is a Lefschetz fibration $f : M \longrightarrow B$ such that

- $\omega$ restricts symplectically on the smooth part of the fibers
- in complex coordinates $(z_1, z_2)$ around the critical points, such that $f(z_1, z_2) = z_1^2 + z_2^2$, the complex structure $i$ is tamed by $\omega$.

**Theorem 3.4.31** ([40], Theorem 3.33)**.** *Let $M$ be an oriented 4–manifold, and let $f : M \longrightarrow B$ be a Lefschetz fibration. Let $\omega_0$, $\omega_1$ be symplectic structures supported by $f$. Then there is a family $\omega_s$, $s \in [0, 1]$, of symplectic structures supported by $f$, joining $\omega_0$ and $\omega_1$.*

Let us now introduce the language of symplectic structures and Lefschetz fibrations on Lie algebroids.

**Definition 3.4.32.** Let $A \longrightarrow M$ be a Lie algebroid of rank $2k$. A *Lie algebroid symplectic structure* on $A$ is a closed algebroid 2-form $\omega \in \Omega^2_A$ such that $\omega^{\wedge k} \neq 0$.

Obviously, symplectic and log-symplectic structures are examples of Lie algebroid symplectic structures, $A = TM$, $TM(-\log Z)$ respectively.

**Definition 3.4.33.** Let $A_M \longrightarrow M$, $A_B \longrightarrow B$ be Lie algebroid, with anchor maps $\rho_M$ and $\rho_B$ respectively. A *Lie algebroid morphism* is a vector bundle map

$$\begin{array}{ccc} A_M & \xrightarrow{F} & A_B \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & B \end{array} \tag{3.19}$$

such that

$$\begin{array}{ccc} A_M & \xrightarrow{F} & A_B \\ {\scriptstyle \rho_M}\downarrow & & \downarrow{\scriptstyle \rho_B} \\ TM & \xrightarrow{df} & TB \end{array} \tag{3.20}$$

commutes, and $dF^* = F^* d$, where $d$ denotes the suitable Lie algebroid differential.

Given a Lie algebroid $A_M \longrightarrow M$, let $M^{\text{iso}} \subset M$ be the maximal open set such that $\rho_M|_{M^{\text{iso}}}$ is an isomorphism.

**Definition 3.4.34.** A *Lie algebroid Lefschetz fibration* is a Lie algebroid morphism

$$\begin{array}{ccc} A_M & \xrightarrow{F} & A_B \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & B \end{array} \tag{3.21}$$

between oriented Lie algebroids, with a (possibly empty) set of critical points $M^{\text{crit}} \subset M^{\text{iso}}$ such that

- $f(M^{\text{crit}}) \subset B^{\text{iso}}$
- in a neighbourhood of $M^{\text{crit}}$, $f$ is a Lefschetz fibration with critical points $M^{\text{crit}}$
- $F$ is fiberwise surjective on each fiber over $M \setminus M^{\text{crit}}$

**Definition 3.4.35.** Let $F : A^4_M \longrightarrow A^2_B$ be a Lie algebroid Lefschetz fibration. A *fiberwise symplectic form* is a closed Lie algebroid 2-form $\eta$ on $A_M$ such that

- $\eta$ is symplectic on the fibers of $F|_{M \setminus M^{\text{crit}}}$

- for any almost complex structure $J$ in a neighbourhood of $M^{\text{crit}}$, that restricts to a positive complex structure on the fibers, $\eta$ tames $J$ at $M^{\text{crit}}$

A Lie algebroid symplectic form which is fiberwise symplectic is said to be *supported by the fibration.*

As was already mentioned, Lie algebroid Lefschetz fibrations provide a framework to study Lie algebroid symplectic forms, and in particular to construct examples.

**Proposition 3.4.36** ([6],[7]). *Let $F : A^4_M \longrightarrow A^2_B$ be a Lie algebroid Lefschetz fibration, with $M$ and $B$ compact, and $A^2_B$ orientable. Assume there is a fiberwise symplectic form $\eta$. Then $A_M$ admits a symplectic structure, of the form*

$$\omega = \eta + KF^*\sigma \tag{3.22}$$

*for $K >> 0$, $\sigma$ a volume form on $A_B$.*

The resulting symplectic form is apparently supported by the fibration.

We can now prove the Lie algebroid version of Theorem 3.4.31. The proof is almost word by word the one explained in [40] – the proof being essentially linear algebra, it works for any map of symplectic vector bundles. We provide a proof for completeness.

**Proposition 3.4.37.** *Let $F : A^4_M \longrightarrow A^2_B$ be a Lie algebroid Lefschetz fibration, and $\omega_0$, $\omega_1$ be symplectic forms supported by the fibration and inducing the same orientation on $A_M$. Then there is a path $\omega_s$ of symplectic forms joining $\omega_0$ and $\omega_1$.*

*Proof.* Consider $\omega'_s := (1 - s)\omega_0 + s\omega_1$. This is a fiberwise symplectic form, and non-degenerate for $s \in [0, \varepsilon) \cup (1 - \varepsilon, 1]$, and at $p \in M^{\text{crit}}$, $\forall s$. Let $V := \ker F$, $H_s = \{a \in A_M : \omega'_s|_V(a, \cdot) = 0\}$. One has $A_M = V \oplus H_s$, and since $\omega'_s(V, H_s) = 0$, $\omega'_s$ is non-degenerate if and only if $\omega'_s|_{H_s}$ is.

Since $\omega_0$ and $\omega_1$ induce the same orientation, there exists a volume form $\sigma$ on $A_B$ such that $F^*\sigma|_{H_s}$ and $\omega'_s|_{H_s}$ induce the same orientation, for all $s \in [0, \varepsilon) \cup (1 - \varepsilon, 1]$. Thus, for all positive numbers $K$, $\omega'_s + KF^*\sigma$ is non-degenerate for all $s \in [0, \varepsilon) \cup (1 - \varepsilon, 1]$. Moreover, for $K$ big enough, $\omega'_s + KF^*\sigma$ is symplectic for all $s$, by compactness of $B$.

Take a function $\rho$ which is equal to 0 in a neighbourhood of 0 and 1, and equal to 1 on $[\varepsilon, 1 - \varepsilon]$. By what we said above, the forms

$$\omega_s := \omega'_s + K\rho(s)F^*\sigma \tag{3.23}$$

are all symplectic, and join $\omega_0$ to $\omega_1$. □

**Corollary 3.4.38.** *A 4–dimensional genus–0 log-symplectic manifold $(X, Z, \omega)$ is diffeomorphic to a blow-up of one of the list in Corollary 3.4.30, and any two diffeomorphic ones have deformation equivalent log-symplectic structures.*

## 3.5 Curves with boundary

The aim of this section is to study logarithmic holomorphic curves with boundary on the singular locus, as well as closed logarithmic holomorphic curves with non-empty singular locus. We will construct moduli spaces of the latter in a particular case, and apply our construction to the study of ruled surfaces.

### 3.5.1 Moduli spaces of curves with boundary

We consider here (moduli spaces of) holomorphic maps of compact logarithmic Riemann surfaces with boundary. We know from Proposition 3.3.13 and Remark 3.3.15 that such curves correspond to finite energy punctured holomorphic curves, as studied in SFT, and their boundary components are mapped to simple closed Reeb orbits.
We construct the moduli spaces using this correspondence. To be more precise, fix a tubular neighbourhood of $Z$ so that the resulting stable Hamiltonian structure is Morse-Bott, and pick a cylindrical complex structure. As in Section 2.3.4, one can fix a set $\mathfrak{c}$ of asymptotic constraints, and consider the space $\mathcal{M}_g(A, J)^{\mathfrak{c}}$. Let $\tau$ be a trivialization of $TZ$ on the Reeb orbits to which the punctures are asymptotic. Applying Theorem 2.3.20 one obtains immediately the following generic transversality theorem:

**Theorem 3.5.1.** *Let $(X, Z, \omega)$ be log-symplectic. Fix a tubular neighbourhood $U$ of $Z$ such that the induced stable Hamiltonian structure is Morse-Bott, and pick a cylindrical complex structure $J_{\mathit{fix}}$ on $U$. Consider the set $\mathcal{J}_{\mathit{fix}}$ of compatible complex structures that coincide with $J_{\mathit{fix}}$ on $U$. Let $\mathfrak{c}$ be a set of asymptotic constraints. Then for a generic choice of $J \in \mathcal{J}_{\mathit{fix}}$ all curves in $\mathcal{M}_g(A, J)^{\mathfrak{c}}$ are Fredholm regular, so that the moduli space is a smooth manifold of dimension $(n-3)(2-2g-\#\pi_0(\partial\Sigma)) + 2c_1^\tau(A) + \mu^\tau(\mathfrak{c})$.*

*Proof.* All curves are embedded near the boundary, hence they are somewhere injective (or, equivalently, simply covered). Hence Theorem 2.3.20 applies, yielding the result. □

In the same way the SFT compactness theorem applies.

**Theorem 3.5.2.** *With the same notation as above, the moduli space $\mathcal{M}_g(A, J)^{\mathfrak{c}}$ is compactified by the space $\overline{\mathcal{M}}_g(A, J)^{\mathfrak{c}}$ of stable holomorphic buildings.*

In the special case when no two boundary components are mapped to the same component of the singular locus, the geometric setup allows to rule out certain buildings.

**Proposition 3.5.3.** *Let $u_k$ be a sequence of holomorphic curves with asymptotic constraint $\mathfrak{c}$ so that no two boundary components are mapped to the same component of the singular locus. Let $u_\infty$ be a limiting holomorphic building. Then all the levels other than the main one consist of disjoint unions of nodal curves with exactly two punctures (one for each component of the singular locus), asymptotic to simple Reeb orbits. The main level must be a nodal curve* *with as many punctures as the $u_k$'s, with at most one puncture asymptotic to each component of $Z$.*

*Proof.* For each component of the singular locus, the top level is a nodal curve with a single positive puncture asymptotic to a single simple curve. The sum of the negative punctures must be homotopic (in $Z$) to a simple curve. In particular, their projection to $S^1$ need to be homotopic. By holomorphicity, all the negative punctures are multiples of the positive generator of $\pi_1(S^1)$, which implies there must be a single simple negative puncture. Iterating the argument one sees that it has to hold for all lower levels (main exluded), and that the punctures of the main level are in one to one correspondence with the punctures of $u_k$. □

### 3.5.2 Punctured spheres with 0 self-intersection

In this section we investigate the consequences of the existence of an embedded holomorphic punctured sphere with 0 self-intersection in certain log-symplectic 4–manifolds. Let $(X, Z, \omega)$ be an oriented log-symplectic 4–manifold. Let $X_0$ be a component of $X \setminus Z$. We know (Proposition 3.2.16) that we can realize $X_0$ (non-uniquely) as a completion $\widehat{W}$ of a manifold $(W, \partial W)$ with stable Hamiltonian boundary, such that the 1-form $\alpha$ is closed, and all components of $\partial W$ are positive. We will study punctured spheres with 0 self-intersection in the case when the stable Hamiltonian structure induces the trivial fibration $S^1 \times \Sigma_{g_i}$ on each component of $\partial W$. The goal is again to apply the results to the study of ruled surfaces (Section 3.5.3).

The precise statement is the following.

**Theorem 3.5.4.** *Let $(W, \partial W, \omega)$ be a symplectic 4–manifold with an induced positive stable Hamiltonian structure on $\partial W$, of the form $(\alpha, \beta)$, with $d\alpha = 0$. Assume that $(\alpha, \beta)$ induces a trivial symplectic fibration, so that $\partial W = \sqcup_i S^1 \times \Sigma_{g_i}$.*
*Let $J$ be a "generic" cylindrical complex structure, and let $u : S^2 \backslash \{p_1, \ldots, p_k\} \longrightarrow \widehat{W}$ be a $k$-punctured $J$-holomorphic sphere such that*

- *different punctures are asymptotic to Reeb orbits in different components of $\partial W$*
- *$u \star u = 0$ as a punctured holomorphic curve with unconstrained asymptotics*

*Let $\mathcal{M}$ denote the component containing $u$ of the moduli space of $J$-holomorphic maps. Then $\mathcal{M}$ is a smooth 2-dimensional manifold, all elements of which are embedded curves. Moreover $\mathcal{M}$ can be compactified to a smooth closed 2-dimensional manifold $\overline{\mathcal{M}}$, by adding nodal curves consisting exactly of 2 components: an embedded $k$-punctured sphere of self-intersection $-1$, and an embedded sphere of self-intersection $-1$. The two components intersect exactly at one point.*

The proof will consist of an index computation, and an application of the adjunction formula. We will start with an arbitrary holomorphic building, and compare the index of each level with the index of $u$. The genericity of $J$ will be used to rule out all such buildings containing levels with negative index. Using the adjunction formula, we can relate the indices of the remaining buildings with their self-intersection, and their embeddedness. As a result, we will prove that the only holomorphic buildings that can appear as limits of curves homotopic to $u$ are those mentioned in the statement.

A useful tool in the proof is the following lemma, which allows to tell whether a cylinder is trivial.

**Lemma 3.5.5.** *Let $Z$ be a symplectic mapping torus, isomorphic to $(S^1 \times \Sigma_g,\, cdt,\, vol_{\Sigma_g})$, and let $J$ be a compatible almost complex structure on $\mathbb{R} \times Z$. Let $u : \mathbb{C}^* \longrightarrow \mathbb{R} \times Z$ be a finite energy holomorphic cylinder. Then, if $g > 0$, $u$ is a trivial cylinder. If $g = 0$, then the relative Chern class $c_1(u)$, defined with respect to the isomorphism $Z \cong S^1 \times S^2$, is a non-negative even number, and $c_1(u) = 0 \Leftrightarrow u$ is a trivial cylinder.*

*Proof.* Fix an isomorphism $Z = S^1 \times \Sigma_g$ as symplectic mapping tori, and write $u = (u_1,\, u_2)$. The map $u_2 : \mathbb{C}^* \longrightarrow \Sigma_g$ is holomorphic and has finite energy, thus extends to $u_2 : S^2 \longrightarrow \Sigma_g$. If $g > 0$, this means that $u_2$ is a constant.

If $g = 0$, one has $u_2 : S^2 \longrightarrow S^2$, and $c_1(u_2) = 2\deg(u_2) \geq 0$ because $u_2$ is orientation preserving. In particular $u_2$ is constant if and only if $\deg(u_2) = 0$ (by holomorphicity). The fact that $c_1(u) = c_1(u_2)$ implies the statement. □

*Proof of Theorem 3.5.4.* First of all, using the adjunction formula one computes that $\mathrm{ind}(u) = 2$ (Corollary 3.6.2), and that all elements in $\mathcal{M}$ are embedded. Since $J$ is a generic complex structure, there are no $J$-holomorphic curves of negative index.
Let us assume now that $u : S^2 \setminus \{p_1,\, \ldots,\, p_k\} \longrightarrow \widehat{W}$ is $J$-holomorphic; let $\gamma_i$ be the orbit to which $p_i$ is asymptotic. Consider the moduli space $\mathcal{M} := \mathcal{M}_0(A, J)^{\mathfrak{c}}$ of $J$-holomorphic punctured spheres, with punctures $p_1,\, \ldots,\, p_k$ asymptotic to simple orbits in the same components as $\gamma_1,\, \ldots,\, \gamma_k$, respectively. The transversality theorem 3.5.1 can be applied to each element of $\mathcal{M}$, because by the maximum principle each curve has a point mapped to $X \setminus U$, and such point is injective because $u$ is an embedding. Thus $\mathcal{M}$ is a smooth 2–dimensional manifold.

By Proposition 3.5.3, we can compactify $\mathcal{M}$ by adding holomorphic buildings $\mathbf{u}_\infty = (\mathbf{u}_0, \mathbf{u}_1^{(a)}, \ldots, \mathbf{u}_L^{(a)})$, where $\mathbf{u}_0$ is a nodal punctured sphere in $\widehat{W}$, and $\mathbf{u}_i^{(a)}$, $a = 1, \ldots, a_i$ is a nodal cylinder in $\mathbb{R} \times \partial W$. In particular we can write each $\mathbf{u}_i^{(a)}$ as a collection of curves: $\mathbf{u}_i^{(a)} = (u_{i,0}^{(a)},\, u_{i,1}^{(a)},\, \ldots,\, u_{i,m_i}^{(a)})$, where $u_{i,j}^{(a)}$ are spheres for $j > 0$, and $u_{i,0}^{(a)}$ is either a $k$-punctured sphere (when $i = 0$) or a cylinder (when $i > 0$). We need to show that:

- $m_i = 0$ for all $i > 0$
- $u_{i,0}^{(a)}$ is the trivial cylinder for all $i > 0$, and all $a$
- $m_0 = 1$
- $\mathbf{u}_0 = (u_0, v_1)$, is such that $u_0 \star u_0 = -1 = v_1 \cdot v_1$, and $u_0 \cdot v_1 = 1$

We will do this by index counting, assuming that there are no holomorphic curves of negative index (which is true due to our assumption that $J$ is generic).

Denote by $N_{i,j}^{(a)}$ the number of nodes of the curve $u_{i,j}^{(a)}$, and let $S_{i,j}^{(a)}$ be its domain. Let $S_i^{(a)}$ be the (topological) connected sum of the $S_{i,j}^{(a)}$'s at the nodes. Then

$$\chi(S_i^{(a)}) = \sum_{j\geq 0}(\chi(S_{i,j}^{(a)}) - N_{i,j}^{(a)}) \tag{3.24}$$

and

$$2 - k = \chi(S^2 \setminus \{p_1, \dots, p_k\}) = \sum_{i,j,a} \chi(S_i^{(a)}) = \chi(S_0) \tag{3.25}$$

(the last equality follows from $S_i^{(a)}$ being a cylinder for $i > 0$). Denote by $A_i^{(a)}$ the relative homology class of $u_{i,0}^{(a)}$, and $B_{i,j}^{(a)}$ the homology class of $u_{i,j}^{(a)}$.

Define the following numbers:

- $\text{ind}(\mathbf{u}_\infty) := (n-3)\chi(\dot{\Sigma}) + 2c_1^\tau(A) + \mu^\tau(\mathfrak{c}) = -\sum_{i,j}(\chi(S_{i,j}) - N_{i,j}) + \sum_i 2c_1^\tau(A_i) + \sum_{i,j} 2c_1(B_{i,j}) + \mu^\tau(\mathfrak{c})$
- $\text{ind}(\mathbf{u}_0) := -\sum_j(\chi(S_{0,j}) - N_{0,j}) + 2c_1^\tau(A_0) + \sum_j 2c_1(B_{0,j}) + \mu^\tau(\mathfrak{c}')$
- $\text{ind}(\mathbf{u}_i^{(a)}) := -\sum_j(\chi(S_{i,j}^{(a)}) - N_{i,j}^{(a)}) + 2c_1^\tau(A_i^{(a)}) + \sum_j 2c_1(B_{i,j}^{(a)}) + \mu^\tau(\mathfrak{c}')$, for $i > 0$

One has

$$\begin{aligned} 2 = \text{ind}(\mathbf{u}_\infty) &= \text{ind}(\mathbf{u}_0) + \sum_{i\geq 1,\, a}(\text{ind}(\mathbf{u}_i^{(a)}) - \mu^\tau(\mathfrak{c}')) \\ &= \text{ind}(\mathbf{u}_0) + \sum_{i\geq 1,\, a}(\text{ind}(\mathbf{u}_i^{(a)}) - 2) \end{aligned} \tag{3.26}$$

We would like to prove that $\text{ind}(\mathbf{u}_0)$ and $\text{ind}(\mathbf{u}_i^{(a)}) - 2$ are all non-negative.

It is immediate to check that

$$\text{ind}(\mathbf{u}_0) = \text{ind}(u_0) + N_{0,0} + \sum(\text{ind}(u_{0,j}) + N_{0,j}) \tag{3.27}$$

and all the summands are non-negative. Similarly, for $i > 0$,

$$\begin{aligned}\operatorname{ind}(\mathbf{u}_i^{(a)}) = (\operatorname{ind}(u_{i,0}^{(a)}) + N_{i,0}^{(a)} - 2) + \sum_{j\geq 1}(\operatorname{ind}(u_{i,j}^{(a)} + N_{i,j}^{(a)})) \\ \geq \sum_{j\geq 0}(N_{i,j}^{(a)}) - 2 \quad (3.28)\end{aligned}$$

We claim that $\operatorname{ind}(\mathbf{u}_i^{(a)}) - 2 \geq 0$.
First of all, Lemma 3.5.5 ensures that $c_1(A_i^{(a)}) \geq 0$ and is even whenever $i > 0$. Moreover, also $c_1(B_{i,j}^{(a)}) \geq 0$, because by the maximum principle $u_{i,j}^{(a)} : S^2 \longrightarrow \{\star\} \times S^2$. Finally, $c_1(A_i^{(a)}) = 0$ if and only if $A_i^{(a)}$ represents a trivial cylinder (in which case $N_{i,0}^{(a)} > 0$, by stability), while $c_1(B_{i,j}^{(a)}) = 0$ if and only if $B_{i,j}^{(a)}$ represents a constant bubble (in which case $N_{i,j}^{(a)} > 2$, again by stability).

Now,

$$\begin{aligned}\operatorname{ind}(\mathbf{u}_i^{(a)}) - 2 = 2c_1^{\tau}(A_i^{(a)}) + \sum_j 2c_1(B_{i,j}^{(a)}) - \sum_j(\chi(S_{i,j}^{(a)}) - N_{i,j}^{(a)}) \\ \geq 0 + \operatorname{ind}(u_{i,j}^{(a)}) + \sum_j N_{i,j}^{(a)} \geq 0 \quad (3.29)\end{aligned}$$

which proves the claim.

In fact, we can prove that $\operatorname{ind}(\mathbf{u}_i^{(a)}) - 2 \geq 3$, which contradicts the estimate $2 \geq \operatorname{ind}(\mathbf{u}_i^{(a)}) - 2$. Indeed,

$$\operatorname{ind}(\mathbf{u}_i^{(a)}) - 2 = 2c_1^{\tau}(A_i^{(a)}) - (0 - N_{i,0}^{(a)}) + \sum_j 2c_1(B_{i,j}^{(a)}) - (2 - N_{i,j}^{(a)}) \quad (3.30)$$

It was already observed that the stability condition implies $2c_1^{\tau}(A_i^{(a)}) + N_{i,0}^{(a)} \geq 1$. If there are no bubbles, then $2c_1^{\tau}(A_i^{(a)}) \geq 4$. If there is a non-constant bubble, then $2c_1(B_{i,j}^{(a)}) - (2 - N_{i,j}^{(a)}) \geq 4 - 2 + 1 = 3$ (a bubble has at least one node). If all bubbles are constant, then there must be at least 6 nodes (by stability, and the fact that nodes come in pairs), hence $\operatorname{ind}(\mathbf{u}_i^{(a)}) - 2 \geq 6 - 2 = 4$. All cases contradict $2 \geq \operatorname{ind}(\mathbf{u}_i^{(a)}) - 2$, implying that there cannot be any levels other than the main one.

We are left to prove that the main level is either an embedded curve with 0 self-intersection, or it consists of two components, one of which is a punctured sphere and the other a sphere, intersecting each other in a point, and with self-intersection $-1$. We know that $2 = \operatorname{ind}(\mathbf{u}_\infty) = \operatorname{ind}(\mathbf{u}_{0,0}) = \operatorname{ind}(u_0) + N_{0,0} + \sum_j \operatorname{ind}(u_{0,j}) + N_{0,j}$.
Since the orbits to which $u_0$ is asymptotic are simple and of minimal period, $u_0$

is a simply covered curve, hence the adjunction formula applies. Moreover, the index of $u_0$ is an even number. We can distinguish two cases:

- $\mathrm{ind}(u_0) = 2$. Then there are no nodes, and no bubbles. Since $u_0$ is a simple curve, limit of curves with 0 self-intersection, it has 0 self-intersection. The adjunction formula then implies that it is embedded.
- $\mathrm{ind}(u_0) = 0$. Then the only possibility is that there is a single bubble, say $v = u_{0,1}$, such that $\mathrm{ind}(v) = 0$, and $N_{0,0} = N_{0,1} = 1$. Moreover, we claim that $v$ is simply covered. Indeed, assume it is a $k$-fold cover of a simple curve $v'$, then $v'$ has to be a sphere. Further, $0 = \mathrm{ind}(v) = 2c_1(kB') - 2 = \mathrm{ind}(v') + 2(k-1)c_1(B')$, where $[v] = B$, and $[v'] = B'$. Since $\mathrm{ind}(v') \geq 0$, $c_1(B') \geq 1$, so that one needs to have $k = 1$.

Now, we would like to prove that $v \cdot v = u_0 \star u_0 = -1$, $v \cdot u_0 = 1$. We know (Theorem 2.4.4) that $u \star u = u \bullet_\tau u + i^\tau_\infty(u)$, where the first term is topological, and consists of the number of intersections of $u$ with a slight perturbation of $u$ along the direction of the trivialization $\tau$, and $i^\tau_\infty(u)$ (the number of "hidden intersections at infinity"), only depends on the asymptotic trivialization and on the Reeb orbits. We know that the pair $(u_0, v)$ is a limit of curves with 0 self-intersection, and that $v$ is closed. Hence, if $u$ is a curve in the moduli space, one has $u \star u = 0 = (u_0+v)\bullet_\tau(u_0+v)+i^\tau_\infty(u) = u_0\bullet_\tau u_0+i^\tau_\infty(u_0)+v\cdot v+2u_0\cdot v = u_0\star u_0+v\cdot v+2$. Denote by $\mathrm{sing}(w)$ the contribution of the singularity of a curve $w$ to the self intersection number. By the adjunction formula, $w\star w = \mathrm{sing}(w)+\frac{1}{2}\mathrm{ind}(w)-1$. Moreover $u_0\cdot v \geq 1$. Thus $u_0\star u_0+v\cdot v+2 \geq \mathrm{sing}(u_0)+\mathrm{sing}(v) \geq 0$. This implies that both curves are embedded, and their self intersection is $-1$. As a consequence, $u_0\cdot v = 1$ (the single point of intersection is the node, and is transverse).

This automatically implies that the moduli space is a smooth closed surface if the manifold is not a blow-up. If the manifold is a blow-up, the compactified moduli space in the Gromov topology is a topological surface, with a smooth structure coming from the blow-down map.

This concludes the proof of the theorem. □

A version of Theorem 3.5.4 for holomorphic curves with a marked point can be stated as follows.

**Theorem 3.5.6.** *Same hypotheses and notation as in Theorem 3.5.4. Let $\mathcal{M}_1$ be the component containing $u$ of the moduli space of $J$-holomorphic curves with one marked point. Then $\mathcal{M}_1$ is a smooth 4–dimensional manifold, fibering over $\mathcal{M}$. Moreover $\mathcal{M}_1$ can be compactified to a manifold with boundary $\overline{\mathcal{M}}_1$, by adding:*

- *punctured nodal curves, with two irreducible components of self-intersection $-1$, intersecting in one point*
- *holomorphic buildings of height 2, where the main level is a curve of $\mathcal{M}$, and the second level is a trivial cylinder with a marked point*

*The boundary of $\mathcal{M}_1$ consists of the latter elements. The evaluation map extends continuously to $\overline{\mathcal{M}}_1$, with values in the compactification of $\widehat{W}$ in $X$.*

*Proof.* The proof follows from Theorem 3.5.4 and the SFT compactness theorem. Indeed, we can compactify $\mathcal{M}_1$ using the space of stable holomorphic buildings with one marked point. The buildings of $\overline{\mathcal{M}}$ together with a marked point are examples of elements of the SFT compactification. They are not all, as there might be building with levels consisting entirely of trivial cylinders, as long as they have enough marked points to be stable. In the case of the present theorem, it is possible for a level to contain exactly one trivial cylinder, if the marked point belongs to it. From this consideration the first part of the statement is proved. It is easily computed that the dimension of the set of height 2 buildings is 3, i.e. it has codimension 1. The evaluation map extends continuously, by the very definition of the topology on the space of buildings. □

As an application, we mention the following consequence of the analysis above.

**Proposition 3.5.7.** *Let $W$ be symplectic with stable Hamiltonian boundary isomorphic to a union of $S^1 \times \Sigma_{g_i}$. Let $J$ be a cylindrical complex structure, and assume there is an embedded punctured $J$-holomorphic sphere $S$, with at most one puncture for each boundary component, and 0 self-intersection (with unconstrained asymptotics). Then:*

*(i) the number of punctures is equal to the number of components of $\partial W$*

*(ii) there is a diffeomorphism between the moduli space of curves homotopic to $S$, and $\Sigma_{g_i}$. In particular $g_i = g_j$ for all $i$, $j$*

*Proof. Part* $(i)$. Assume first that $W$ be minimal. View the domain of $S$ as a compact surface with boundary $(\Sigma, \partial\Sigma)$, with a logarithmic holomorphic structure. Consider the space $\mathcal{M}_1$ of holomorphic curves homotopic to $S$, with one marked point in $\Sigma$. By Theorem 3.5.6 and the minimality assumption this is a compact smooth manifold with boundary. One can naturally compactify $\widehat{W}$ to a log-symplectic manifold $Y$ with boundary, such that $\partial Y$ is the singular locus. Since $\pi_0(\partial\mathcal{M}_1) = \pi_0(\partial\overline{S})$, by assumption $\mathrm{ev}_* : \pi_0(\partial\mathcal{M}_1) \longrightarrow \pi_0(\partial Y)$ is injective. Hence, if it is surjective the first part of the statement is true.

Consider the moduli space $\mathcal{M}_1$ of punctured $J$-holomorphic spheres homotopic to $S$, and its compactification $\overline{\mathcal{M}}_1$. By Theorem 3.5.6, $\overline{\mathcal{M}}_1$ is a compact manifold with boundary, and the evaluation map $\mathrm{ev} : \mathcal{M}_1 \longrightarrow Y$ is continuous, sends the boundary to the boundary, and is smooth in the interior. If ev is surjective, then the number of boundary components of $S$ needs to be equal to the number of boundary components of $Y$.

Consider the doubles of $\mathcal{M}_1$ and $Y$, and extend the evaluation map in the obvious way. Its degree is well defined, and needs to be non-zero: for each point in the image of the evaluation, there is a unique curve in $\mathcal{M}$ mapping to it.

In the nonminimal case one can run a similar argument noting that the limiting buildings form a codimension-2 space. Thus one can proceed as in Section 3.4.2.

*Part (ii).* For each component of the boundary, one can define a map $r_i : \mathcal{M} \longrightarrow \mathcal{R}_i$, from the moduli space of curves homotopic to $S$ to the set $\mathcal{R}_i \cong \Sigma_{g_i}$ of simple unparametrized Reeb orbits in the $i$-th component of $\partial X$. The map assigns to each curve the Reeb orbit to which its $i$-th puncture is asymptotic. We will show that for all $i$ the map $r_i$ is a homeomorphism.

The map $r_i$ is injective. Indeed, assume that there are two curves in $\mathcal{M}$ with a puncture asymptotic to the same Reeb orbit $\gamma$. Then their intersection, as curves with constrained asymptotic orbit $\gamma$ and the other punctures unconstrained, is equal to $-1$ (this follows from Lemma 3.6.4). Hence the curves must coincide, by positivity of intersection (Theorem 2.4.1).

The map $r_i$ is a local diffeomorphism. The tangent space to $\mathcal{M}$ at embedded curves consists of holomorphic sections of the normal bundle; the image of the differential is the restriction of the sections to the boundary. A non-zero section that vanishes at the boundary produces a new holomorphic curve with the same asymptotic orbit, which contradicts the injectivity. If the manifold is minimal, this is enough to conclude that $r_i$ is a diffeomorphism. Since $\mathcal{R}_i \cong \Sigma_{g_i}$, and all boundary components are reached by holomorphic curves, we get $g_i = g_j$ for all $i$, $j$.
In the non-minimal case, it suffices to recall that the blow-down map induces a diffeomorphism at the level of compactified moduli spaces, as in Theorem 3.5.4. □

### 3.5.3 Logarithmic ruled surfaces

We use the results from the previous section to show that in certain log-symplectic 4–manifolds, the presence of a holomorphic sphere with 0 self-intersection implies that the manifold is (topologically) ruled, up to blow-up, proving Theorem 3.1.3 (Corollary 3.5.18). The main ingredient is the construction of a smooth moduli space of log-holomorphic spheres.

Let us start with some observations on the combinatorics of the singular locus of a log-symplectic sphere. First of all, to any pair $(X, Z)$ consisting of a manifold and a codimension–1 submanifold (in particular to the pair determined by a log-symplectic structure), one can attach a finite unoriented planar graph $\Gamma(X, Z)$. The vertices are the components of $X \setminus Z$. Each component of $Z$ determines an edge, joining the vertices corresponding to the components of $X \setminus Z$ that it bounds.

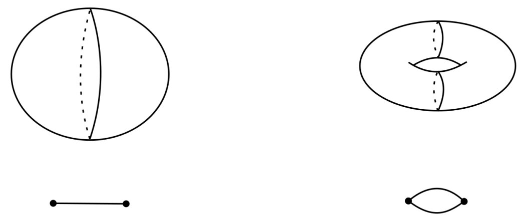

Figure 3.1: Manifolds with hypersurface, and corresponding graph

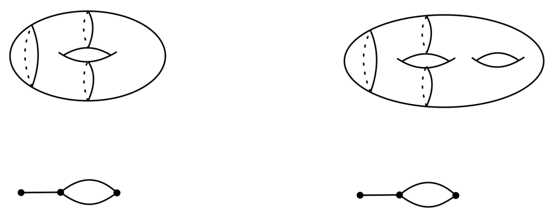

Figure 3.2: Different manifolds may give rise to the same graph

Clearly $\pi_0(\Gamma(X, Z)) = \pi_0(X)$, in particular the graph is connected if and only if $X$ is. Let us introduce some terminology that is going to be useful.

**Definition 3.5.8.** Let $\Gamma$ be a finite connected graph. We call a *circle* an edge with both endpoints on the same vertex. A *loop* is a connected subgraph, with no circles, such that each vertex is connected to exactly two edges. We say that a vertex is an *extreme* if all the edges connected to it, except at most one, are circles. We say that a graph is 1-*connected* if it contains no loops, and *orientable* if it contains no circles.

**Lemma 3.5.9.** *Let $(X, Z, \omega)$ be log-symplectic. Then $X$ is orientable if and only if $\Gamma(X, Z)$ is orientable.*

*Proof.* If $X$ is orientable, then $Z = \{f = 0\}$ for some globally defined function vanishing linearly. $X \setminus Z$ splits as $\{f > 0\} \cup \{f < 0\}$, and each component of $Z$, being coorientable, bounds both a component of $\{f > 0\}$ and a component of $\{f < 0\}$, which are necessarily different.
If $X$ is not orientable, then at least one component $Z_i$ of $Z$ is not coorientable. Then there exists a neighbourhood $U_i$ of $Z_i$ such that $u_i \setminus Z_i$ is connected. Hence the edge of $\Gamma(X, Z)$ corresponding to $Z_i$ has both extremes on the same vertex. □

**Lemma 3.5.10.** *A* 1*-connected graph with at least* 2 *vertices has at least* 2 *extremes.*

*Proof.* The statement and proof are insensitive of circles, we might assume there are none. Pick any vertex $v_0$. Construct a sequence $v_k$ simply by selecting at

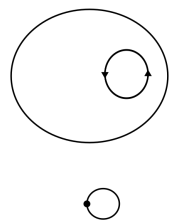

Figure 3.3: A circle with a single vertex, representing $\mathbb{R}P^2$ with the nontrivial loop as singular locus.

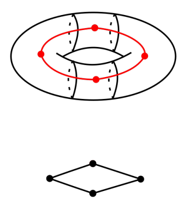

Figure 3.4: A loop in the graph producing an infinite order element in $\pi_1$

each step a vertex (different from $v_k$) which is connected to $v_{k-1}$ by an edge. There are two possibilities. Case one, the process must stop, as we get to a vertex connected by only one edge. In this case such vertex is an extreme. Case two, the process can continue indefinitely. But then since the graph is finite, there must be a loop, so the graph is not 1-connected.
In order to find the second extreme, one can repeat the process above, taking as $v_0$ the extreme found with the procedure above. The process must end to a different extreme than $v_0$. □

For surfaces, we can reconstruct $(\Sigma, \sigma)$ given $\Gamma(\Sigma, \sigma)$ and the genus of the component associated to each vertex. In particular we have the following lemma.

**Lemma 3.5.11.** *Let $(\Sigma, \sigma)$ be a surface with a codimension–1 submanifold. If $\Sigma$ is a sphere, then $\Gamma(\Sigma, \sigma)$ is orientable and 1-connected. If $\Sigma$ is $\mathbb{R}P^2$, then $\Gamma(\Sigma, \sigma)$ has exactly one circle, and is 1-connected.*

*Proof.* If $\Sigma$ is a sphere then every component of $\Sigma\backslash\sigma$ is a punctured sphere, while if $\Sigma \cong \mathbb{R}P^2$, then $\Sigma \setminus \sigma$ consists of punctured spheres and punctured Möbius bands. Either way, a loop in $\Gamma(\Sigma, \sigma)$ would determine a(t least one) element of infinite order in $\pi_1(\Sigma)$ (see Figure 3.4), which cannot happen. Finally, adding a circle to the graph corresponds with taking connected sum with $\mathbb{R}P^2$, and this can be done exactly once by the classification of surfaces. □

**Definition 3.5.12.** Let $(X, Z, \omega)$ be a log-symplectic 4–manifold. A closed embedded surface $S \subset X$ is *nice* if

- $S \pitchfork Z$
- the map $\pi_0(S \cap Z) \longrightarrow \pi_0(Z)$ induced from the inclusion is injective
- the graph $\Gamma(S, S \cap Z)$ is 1-connected, and has at least one edge

**Lemma 3.5.13.** *Embedded spheres and embedded real projective planes, intersecting $Z$ transversely, and intersecting each component of $Z$ in a connected set, are nice.*

*Proof.* This is a direct consequence of Lemma 3.5.11. □

**Theorem 3.5.14.** *Let $(X, Z, \omega)$ be a log-symplectic 4–manifold. Assume that there is a neighbourhood of $Z$ realizing $Z$ as a union of $S^1 \times \Sigma_{g_i}$, with the trivial stable Hamiltonian structure. Let $J$ be a cylindrical complex structure with respect to such trivialization. Then the compactified moduli space $\overline{\mathcal{M}}$ of nice $J$-holomorphic spheres with 0 self-intersection is a smooth closed 2-dimensional manifold.*

*Proof.* A log $J$-holomorphic curve $u : (\Sigma, \sigma) \longrightarrow (X, Z)$ determines several punctured holomorphic curves $u_i : \dot{\Sigma}_i \longrightarrow X_i$, $i = 1, \ldots, N$, where $\dot{\Sigma}_i$ and $X_i$ are components respectively of $\Sigma \setminus \sigma$ and $X \setminus Z$. In the case at hand, each $\dot{\Sigma}_i$ is a punctured sphere. By automatic transversality (Theorem 2.3.21), we may assume that $J$ is regular (Lemma 2.2.9). Then, each $u_i$ satisfies the hypotheses of Proposition 3.5.7. Thus there are moduli spaces $\mathcal{M}_i$, one for each component, and they are all diffeomorphic to each other.

More precisely, recall that for each component $X_i$ of $X \setminus Z$, and each component $Z_{i,j}$ bounding $X_i$, one can define a map $r_{i,j} : \mathcal{M}_i \longrightarrow \mathcal{R}_{i,j}$, where $\mathcal{R}_{i,j}$ is the set of simple unparametrized Reeb orbits in $Z_{i,j}$. The map is defined so that $r_{i,j}(v)$ is the Reeb orbit to which the $j$-th puncture of $v$ in $X_i$ is asymptotic; we showed in Proposition 3.5.7 that $r_{i,j}$ is a diffeomorphism.

In order to glue the moduli spaces $\mathcal{M}_i$, we exploit our understanding of the graph $\Gamma(\Sigma, \sigma)$. First of all we know that to each vertex there corresponds a punctured holomorphic sphere, and a corresponding moduli space. Denote $\mathcal{M}_x$ the moduli space corresponding to the vertex $x$. Since there are no loops, each pair of vertices determines at most one edge. To the edge $xy$ between vertices $x$ and $y$ one can associate the manifold $R_{xy}$ of Reeb orbits on the corresponding component of $Z$. Moreover, to each edge one can associate two diffeomorphisms $r_{xy} : \mathcal{M}_x \longrightarrow R_{xy}$ and $r_{yx} : \mathcal{M}_y \longrightarrow R_{xy}$.

By Lemma 3.5.10 we know that there exists one extreme. We can fix one of the extremes, say $x_0$, and partition the set of vertices by their distance from $x_0$, i.e. the number of edges that it takes to reach $x_0$. This is well defined as there are no loops. In particular, for each vertex there is exactly one set of edges connecting it to $x_0$.

For each $x$, let $x_0x_1$, $x_1x_2$, …, $x_ix$ be the sequence of edges joining $x$ with $x_0$. Define the map $r_x = r_{x_0x_1}^{-1} \circ r_{x_1x_0} \circ r_{x_1x_2}^{-1} \circ r_{x_2x_1} \circ \cdots \circ r_{xx_i} : \mathcal{M}_x \longrightarrow \mathcal{M}_{x_0}$ ($r_{x_0} = \mathrm{id}$). It is a diffeomorphisms for each vertex $x$.

Finally, denoting with $V(\Sigma, \sigma)$ the set of vertices of $\Gamma(\Sigma, \sigma)$, we can define the moduli space as $\mathcal{M} = \{(v_x)_{x \in V(\Sigma,\sigma)} : v_x \in \mathcal{M}_x, r_x(v_x) = v_{x_0}\}$. The map $(v_x)_x \mapsto v_x$ induces a diffeomorphism $\mathcal{M} \cong \mathcal{M}_x$ for all $x \in V(\Sigma, \sigma)$. □

In fact, in the proof we only used the properties of the graph $\Gamma(\Sigma, \sigma)$, and the fact that $\Sigma \setminus \sigma$ is a collection of punctured spheres. This suggests that if $\Sigma \cong \mathbb{R}P^2$, a similar statement holds.

Let $(\mathbb{R}P^2, \sigma)$ be a real projective plane admitting a log-symplectic structure with singular locus $\sigma$. Let $j$ be a complex structure on $T\mathbb{R}P^2(-\log(\sigma))$, and let $u : (\mathbb{R}P^2, \sigma, j) \longrightarrow (X, Z, J)$ be holomorphic.

**Definition 3.5.15.** We say that a holomorphic map $u : (\mathbb{R}P^2, \sigma, j) \longrightarrow (X, Z, J)$ has 0 self intersection if each component of $u|_{\mathbb{R}P^2 \setminus \sigma}$ has 0 self intersection as a punctured holomorphic curve.

**Theorem 3.5.16.** *Let $(X, Z, \omega)$ be a log-symplectic 4–manifold. Assume that there is a neighbourhood of $Z$ realizing $Z$ as a union of $S^1 \times \Sigma_{g_i}$, with the trivial stable Hamiltonian structure. Let $J$ be a cylindrical complex structure with respect to such trivialization. Then the moduli space $\mathcal{M}$ of nice $J$-holomorphic projective planes with 0 self-intersection is a smooth closed 2-dimensional manifold.*

*Proof.* The proof is almost exactly the same as Theorem 3.5.14. We can split a holomorphic projective plane as a union of punctured holomorphic spheres, and apply automatic transversality to make sure $J$ is generic. One then constructs a moduli space for each components, and glues along the singular locus. The only difference with the above is that the graph $\Gamma(\Sigma, \sigma)$ contains a circle. However, the circle does not prescribe any gluing condition between moduli spaces (moduli spaces are associated to vertices, and two vertices are glued using the edges joining them). □

One can construct a moduli space of curves with one marked point using a similar gluing construction. A delicate point is that the compactification of the spaces of curves with a marked point in an open symplectic manifold is a manifold with boundary, and we can only construct a $C^0$ structure at the boundary points. As a result, the moduli space of maps with values in $X$ inherits only a $C^0$ structure. We don't know in general how to control the infinite jet of the punctured spheres along their asymptotic orbits; hence we can only glue their images as topologically embedded surfaces.

**Theorem 3.5.17.** *Same assumptions as Theorem 3.5.14 or Theorem 3.5.16. There exist moduli spaces $\mathcal{M}_1$ of log-holomorphic spheres and projective planes with one marked point, which is a closed topological 4–dimensional manifold. It* *comes with a canonical smooth structure on the complement of a codimension–1 submanifold $\mathcal{M}_Z$. Moreover there is a well defined evaluation map $ev : \mathcal{M}_1 \longrightarrow X$, such that $ev^{-1}(Z) = \mathcal{M}_Z$, which is continuous, and smooth on $\mathcal{M}_1 \setminus \mathcal{M}_Z$.*

*Proof.* For each $x \in V(\Sigma, \sigma)$, define $\overline{\mathcal{M}}_{1\,x}$ as the compactification of the space of holomorphic curves (in the corresponding component of $X \setminus Z$) with one marked point. This is a manifold whose boundary can be identified with a union of copies of $\overline{\mathcal{M}}_x \times S^1$, one for each edge having $x$ as a vertex. Using $r_{yx}^{-1} \circ r_{xy}$ to identify $\overline{\mathcal{M}}_x \times S^1$ with $\overline{\mathcal{M}}_y \times S^1$, define $\overline{\mathcal{M}}_1$ as the space obtained by gluing the manifolds $\overline{\mathcal{M}}_{1\,x}$ along their boundaries, using the identifications just explained, as dictated by the graph $\Gamma(\Sigma, \sigma)$, in the same fashion as in the proof of Theorem 3.5.14. The resulting space is a topological manifold, with the properties claimed in the statement. ☐

The following corollary is what was described in the Introduction as Theorem 3.1.3.

**Corollary 3.5.18.** *Let $(X, Z, \omega)$ be a log-symplectic 4–manifold satisfying the assumptions from Theorem 3.5.14. Assume further that $X$ is minimal, and that there exists a cylindrical complex structure $J$ (with respect to the trivialization in the assumption) and a $J$ holomorphic sphere with 0 self-intersection, or a $J$-holomorphic projective plane with 0 self intersection. Then there is a continuous map $f : X \longrightarrow B$, with $B$ a smooth closed orientable surface, such that*

- *$f|_{X\setminus Z}$ is a smooth submersion;*
- *$Z$ is a union of copies of $S^1 \times B$, and $f|_Z$ is the projection to $B$;*
- *the fibers of $f$ are spheres if $X$ is orientable, projective planes if $X$ is not orientable;*
- *the fibers are symplectic on $X \setminus Z$, and the closures in $X$ of the components of $f^{-1}(p) \cap X \setminus Z$ are log-symplectic surfaces with boundary, of which the boundary is the singular locus.*

*Proof.* There is an obvious map $g : \mathcal{M}_1 \longrightarrow \mathcal{M}$, which can be described as follows. Let $(u, z) \in \mathcal{M}_1 \setminus \mathcal{M}_Z$, and let $x \in V(\Sigma, \sigma)$ correspond to the component that contains the point $z$. Let $u_x$ be the restriction of $u$ to that component. This defines a unique element $(u_x)_x \in \mathcal{M}$ (via the canonical isomorphism $\mathcal{M}_x \cong \mathcal{M}$). This map extends continuously to $\mathcal{M}_Z$, by the way we chose the gluing maps.

In order to prove the theorem, now, it suffices to show that the evaluation $\mathrm{ev} : \mathcal{M}_1 \longrightarrow X$ is a homeomorphism, which is a diffeomorphism on $\mathcal{M}_1 \setminus Z$. This follows from the same arguments as the proof of Proposition 3.4.27. ☐

**Remark 3.5.19.** In Theorem 3.4.29 we were able to produce a moduli space of holomorphic curves starting from a symplectic submanifold. In the setting of Corollary 3.5.18 instead we need to start from a holomorphic curve. This is because we know how to construct good moduli spaces of $J$-holomorphic curves only when $J$ is a cylindrical complex structure of a specific type. If one starts from, say, a log-symplectic submanifold, it is guaranteed that one will find a log-complex structure $J$ making it holomorphic, but $J$ won't be cylindrical in general.

## 3.6 Computation of intersection numbers

In this section we compute the intersection numbers used in the proof of Theorem 3.5.4. Until the end of the section, we will be considering a symplectic manifold $W$ with boundary $\partial W$, inheriting a stable Hamiltonian structure $(\alpha, \beta)$ such that $(\partial W, \alpha, \beta) \cong (S^1 \times \Sigma_g, cdt, \mathrm{vol}_{\Sigma_g})$. In particular, the Reeb vector field is $\frac{1}{c}\frac{\partial}{\partial t}$, and the simple Reeb orbits are $\gamma(t) = (t, x_0)$. The pullback bundle $\gamma^*\xi := \gamma^* \ker\alpha \cong S^1 \times T_{x_0}\Sigma_g$, hence a trivialization $\tau$ can be given by a Hermitian isomorphism $T_{x_0}\Sigma_g \cong \mathbb{C}$. We will always assume that this choice of trivialization is fixed, unless otherwise stated.
Finally, we will denote the completion of $W$ (Section 2.3.1) with $\widehat{W}$.

Let us compute the asymptotic operator and its spectrum, in order to compute intersection numbers using Theorem 2.4.4. One sees immediately from Equation (2.9) that $A_\gamma = -J_0\frac{d}{dt}$. As expected, the kernel is 2-dimensional (i.e. equal to the dimension of the space of Reeb orbits), consisting of all constant complex functions. All eigenvalues are $2\pi k$, $k \in \mathbb{Z}$ with eigenfunctions $C\exp(2\pi k J_0 t)$, $C \in \mathbb{C}$.

Taking $\varepsilon$ small ($\varepsilon < 2\pi$), consider $A_\gamma \pm \varepsilon$. This operator is nondegenerate with eigenvalues $\lambda_k^{\pm} = 2\pi k \pm \varepsilon$, and eigenfunctions $f = C\exp(2\pi k t J_0) = C\exp((\lambda_k^{\pm} \mp \varepsilon) t J_0)$. The corresponding extremal winding numbers are

$$\begin{aligned}
\alpha_+(A+\varepsilon) &= \min\{\mathrm{wind}(\lambda_k^+) : \lambda_k > 0\} = \min\{\mathrm{wind}(\exp(2\pi k t J_0)) = k : k \geq 0\} = 0\\
\alpha_-(A+\varepsilon) &= \max\{\mathrm{wind}(\lambda_k^+) : \lambda_k < 0\} = \max\{\mathrm{wind}(\exp(2\pi k t J_0)) = k : k < 0\} = -1\\
\alpha_+(A-\varepsilon) &= \min\{\mathrm{wind}(\lambda_k^-) : \lambda_k^- > 0\} = \min\{\mathrm{wind}(\exp(2\pi k t J_0)) = k : k \geq 1\} = 1\\
\alpha_-(A-\varepsilon) &= \max\{\mathrm{wind}(\lambda_k^-) : \lambda_k^- < 0\} = \max\{\mathrm{wind}(\exp(2\pi k t J_0)) = k : k \leq 0\} = 0
\end{aligned} \tag{3.31}$$

In particular this implies

**Lemma 3.6.1.** *All punctured holomorphic curves $u : \dot{\Sigma} = \Sigma \setminus \Gamma \longrightarrow W$ with simple asymptotics have only odd punctures.*

*Proof.* For notational simplicity, we prove this for $z \in \Gamma^+$ a positive puncture (the computation is identical for negative punctures). Recall that the parity of a Morse-Bott puncture asymptotic to an orbit $\gamma$ is defined as $p(A_z) = \alpha_+(A_z \pm \varepsilon) - \alpha_-(A_z \pm \varepsilon)$, where the sign depends on whether we see the puncture as constrained or unconstrained. Either way, Equation (3.31) gives $p(A_z) = 1$. □

**Corollary 3.6.2.** *Let $u : \dot{\Sigma} \longrightarrow \widehat{W}$ be an embedded punctured holomorphic sphere. Then*

$$ind(u) = 2 + 2(u \star u) \tag{3.32}$$

*Proof.* The equation follows immediately from the adjunction formula Theorem 2.4.5 and Lemma 3.6.1. □

It is also easy to compute the Conley-Zehnder indices. Indeed, from the axiomatic definition 2.3.12 one sees immediately that

$$\mu_{CZ}(-J_0\frac{d}{dt} \pm \varepsilon\mathbb{I}) = \frac{1}{2}(\text{sign}(\mp\varepsilon\mathbb{I})) = \mp 1 \tag{3.33}$$

This can also be seen using formula 2.3.15 and (3.31). As a result we can compute the total Conley-Zehnder index of a punctured holomorphic curve.

**Lemma 3.6.3.** *Let $(W, \partial W, \omega)$ be symplectic with stable Hamiltonian boundary $(\bigsqcup_i S^1 \times \Sigma_{g_i},\, c_i dt,\, vol_{\Sigma_{g_i}})$. Let $\mathfrak{c}$ be a set of constraints, consisting of simple orbits. Let $\Gamma$ be a set of punctures, split $\Gamma = \Gamma_C^{\pm} \cup \Gamma_U^{\pm}$. Then*

$$\mu(\mathfrak{c}) = \#\Gamma_U - \#\Gamma_C \tag{3.34}$$

*Proof.* Recall that

$$\mu(\mathfrak{c}) := \sum_{z\in\Gamma^+} \mu_{CZ}(\gamma_z + \delta_z) - \sum_{z\in\Gamma^-} \mu_{CZ}(\gamma_z - \delta_z) \tag{3.35}$$

(see Equation (2.10)), where

$$\delta_z = \begin{cases} \varepsilon & \text{if } z \in \Gamma_C^{\pm} \\ -\varepsilon & \text{if } z \in \Gamma_U^{\pm} \end{cases} \tag{3.36}$$

Hence

$$\mu(\mathfrak{c}) = \sum_{z\in\Gamma_U^+} \mu_{CZ}(\gamma_z - \varepsilon) + \sum_{z\in\Gamma_C^+} \mu_{CZ}(\gamma_z + \varepsilon) - \sum_{z\in\Gamma_U^-} \mu_{CZ}(\gamma_z + \varepsilon) - \sum_{z\in\Gamma_C^-} \mu_{CZ}(\gamma_z - \varepsilon) =$$

$$\sum_{z\in\Gamma_U^+} 1 + \sum_{z\in\Gamma_C^+} (-1) - \sum_{z\in\Gamma_U^-} (-1) - \sum_{z\in\Gamma_C^-} 1 = \#\Gamma_U - \#\Gamma_C$$

□

Finally, we conclude by computing the difference between self intersection numbers in the constrained and unconstrained case. Denote by $u \star_U u$ the self intersection number of a curve $u$ where all the asymptotic orbits are considered unconstrained. Analogously, $u \star_C u$ is the intersection number where all the orbits are considered constrained.

**Lemma 3.6.4.**

$$u \star_U u = u \star_C u + \#\Gamma \tag{3.37}$$

*Proof.* We use Theorem 2.4.4. Then

$$u\star_U u - u\star_C u = -\sum_{z\in\Gamma^\pm} \mp\alpha_\mp(\gamma_z \mp\varepsilon) + \sum_{z\in\Gamma^\pm} \mp\alpha_\mp(\gamma_z \pm\varepsilon) = -\sum_{z\in\Gamma^\pm} 0 + \sum_{z\in\Gamma^\pm} 1 = \#\Gamma$$

□

# Chapter 4

# Scattering-symplectic manifolds

## 4.1 Introduction

A scattering symplectic structure on a manifold $X$ is a Poisson bivector $\pi$ which is almost everywhere symplectic, and vanishes on a codimension–1 submanifold $Z$, in a prescribed way. The prescription can be described in terms of the *scattering Lie algebroid*, which is a vector bundle ${}^{\mathrm{sc}}TX$ such that $\Gamma({}^{\mathrm{sc}}TX) = \langle x^2\partial_x,\, x\partial_{y_1},\, \ldots,\, x\partial_{y_k}\rangle$, where $y_1, \ldots, y_k$ is a coordinate system on $Z$, and locally $Z = \{x = 0\}$. The bundle ${}^{\mathrm{sc}}TX$ comes with a map $\rho : {}^{\mathrm{sc}}TX \longrightarrow TX$, induced from the inclusion on sections, which is an isomorphism on $X \setminus Z$. We say that $\pi$ is scattering symplectic if there exists a symplectic form $\omega : {}^{\mathrm{sc}}TX \longrightarrow {}^{\mathrm{sc}}T^*X$ such that the diagram

$$\begin{array}{ccc} {}^{\mathrm{sc}}T^*X & \xrightarrow{\omega^{-1}} & {}^{\mathrm{sc}}TX \\ \rho\downarrow & & \downarrow\rho \\ T^*X & \xrightarrow{\pi} & TX \end{array} \tag{4.1}$$

commutes. Scattering symplectic structures were introduced in [20], the idea being to view $\omega$ (or rather its inverse $\omega^{-1}$) as a desingularization of $\pi$, and that much of the Poisson geometry of $\pi$ is actually encoded in the symplectic geometry of $\omega$. Notably, one is able to prove normal form theorems using Moser-trick type of arguments, and to compute Poisson cohomology. Scattering symplectic structures are examples of Poisson structures that can be "desingularized by a Lie algebroid", i.e. for which there exists a Lie algebroid with a symplectic structure that can be fit in a diagram like (4.1) (see also [19]). Log-symplectic structures ([16]) and elliptic symplectic structures ([5]) are other examples of this.

A scattering symplectic structure on $(X,\, Z)$ induces a canonical cooriented contact structure on $Z$. As in the log-symplectic case, then, $X$ is orientable if and only if $Z$ is coorientable in $X$. Moreover, there is a normal form theorem around the singularity: every scattering symplectic structure looks like

$$\omega = \frac{d\lambda}{\lambda^3} \wedge (-\alpha + x^2\beta_1) + \frac{1}{2x^2} d\alpha + \beta_2 \tag{4.2}$$

Here $\lambda$ is a distance function from the zero section of the normal bundle $NZ$, $\alpha$ is a contact form on $Z$, and $\beta_1$ and $\beta_2$ are closed differential forms on $Z$. The cohomology classes of the forms $\beta_i$ are invariants of the Poisson structure.
When $\beta_1 = \beta_2 = 0$, and $X$ is oriented, this is the normal form of a strong symplectic filling of $(Z, \ker\alpha)$; conversely given a filling $W$ of $(Z, \ker\alpha)$ one can glue two copies of $W$ with opposite orientations and get a scattering symplectic manifold with singularity on $Z$, and $\beta_1 = \beta_2 = 0$. This can be phrased as follows:

**Fact** ([20])**.** $(Z, \xi = \ker\alpha)$ is the singular locus of an oriented scattering symplectic manifold with $\beta_1 = \beta_2 = 0$ if and only if it is strongly fillable.

Motivated by the above result, we give the following definition:

**Definition 4.1.1.** A cooriented contact manifold $(Z, \xi)$ has an *(orientable) scattering-filling* if it is the singular locus of a closed (orientable) scattering symplectic manifold.

The question that we want to address here is:

**Question.** How does scattering-fillability compare to the other notions of fillability existing in the contact topology literature?

The main result of this paper is the following.

**Theorem 4.1.2.** *The cooriented contact manifold* $(Z, \ker\alpha)$ *has an orientable scattering-filling if and only if it is weakly fillable, in the sense of [23].*

Also non-orientable scattering manifolds can be described in terms of symplectic fillings.

**Theorem 4.1.3.** *Assume that* $(Z, \ker\alpha)$ *is the singular locus of a non-orientable scattering manifold. Then there exists a connected double cover* $p : \tilde{Z} \longrightarrow Z$ *such that* $(\tilde{Z}, p^*\alpha)$ *is weakly fillable, in the sense of [23].*

## 4.2 Scattering symplectic manifolds

### 4.2.1 The scattering tangent bundle

Let $(X, Z)$ be a pair consisting of a manifold $X$ with a codimension–1 closed submanifold $Z$. Let ${}^{\mathrm{sc}}\mathfrak{X}_Z$ be the sheaf of vector fields vanishing on $Z$, quadratically in the transverse direction, linearly in the tangent direction. Explicitely, for each point $p \in Z$ choose a coordinate neighbourhood $U$, with coordinates $(x, y_1, \ldots, y_N)$, such that $Z \cap U = \{x = 0\}$, and $(y_1, \ldots, y_N)$ is a coordinate system for $Z$ on $Z \cap U$. We define ${}^{\mathrm{sc}}\mathfrak{X}_Z(U) := \langle x^2\partial_x, x\partial_{y_1}, \ldots, x\partial_{y_N}\rangle$. This defines a locally free sheaf ${}^{\mathrm{sc}}\mathfrak{X}_Z$ of rank $\dim X$.

**Definition 4.2.1.** The *scattering tangent bundle* ${}^{sc}TX = {}^{sc}TX_Z$ is the unique vector bundle such that $\Gamma({}^{sc}TX) = {}^{sc}\mathfrak{X}_Z$. Its dual is denoted with ${}^{sc}T^*X$ and called the *scattering cotangent bundle*

Since ${}^{sc}\mathfrak{X}_Z$ is closed under the Lie bracket, ${}^{sc}TX$ is a Lie algebroid, with anchor map induced by the inclusion ${}^{sc}\mathfrak{X} \hookrightarrow \mathfrak{X}$. Moreover, the anchor induces an isomorphism between ${}^{sc}TX_Z|_{X\setminus Z}$ and $TX|_{X\setminus Z}$.

### 4.2.2 Scattering symplectic structures

We collect here the definition and some known results on scattering symplectic manifolds. All the results mentioned here are due to [20].

**Definition 4.2.2.** A *scattering symplectic structure* is a symplectic structure on ${}^{sc}TX$. That is, it is a closed non-degenerate section of $\Lambda^2({}^{sc}T^*X)$. Here "closed" means closed in the Lie algebroid sense, or equivalently as an ordinary differential form on $X \setminus Z$.

We will denote a scattering symplectic manifold with a triple $(X, Z, \omega)$. A scattering symplectic structure induces an ordinary symplectic structure on $X \setminus Z$. In particular $\dim X = 2n$, and $X \setminus Z$ is oriented.

**Remark 4.2.3.** In [20], the attention is restricted to pairs of oriented manifolds $(X, Z)$. However, none of the results that we will mention depends on the orientability of $X$.

The following propositions are proven in [20], in the orientable case. The same proofs given in [20] hold in the non-orientable case.

**Proposition 4.2.4.** *A scattering symplectic structure $\omega$ on $(X, Z)$ induces a canonical cooriented contact structure $\xi$ on $Z$. In particular, $Z$ is oriented.*

A contact form can be recovered as follows: pick a fiber metric on the normal bundle, and denote with $\lambda$ the distance from $Z$ function. Write

$$\omega = \alpha_\lambda \wedge \frac{d\lambda}{\lambda^3} + \frac{\beta_\lambda}{\lambda^2}$$

It follows from closure and nondegeneracy of $\omega$ that $\alpha = \alpha_\lambda|_{TZ}$ is a contact form. The form $\alpha$ does not depend on the choice of the metric up to a positive factor. Hence there is a well-defined induced cooriented contact structure. Moreover, a scattering symplectomorphism preserves the contact form up to a positive scaling factor, hence scattering symplectomorphisms preserve the cooriented contact structure on the singular locus (i.e. induce orientation preserving contactomorphisms on the singular locus).

### 4.2.3 A normal form theorem around the singular locus

In order to state the normal form theorem around the singular locus, we need to spend a word on tubular neighbourhoods of hypersurfaces (compare with

Corollary 3.2.17 and the related discussion). Identify a neighbourhood of $Z$ with an open neighbourhood of the zero section in the normal bundle $NZ$. Let $\lambda$ be a distance function from $Z$, induced by a fiber metric on $NZ$. Define $\tilde{Z} := \{\lambda = 1\}$. Fiberwise multiplication by $-1$ induces a diffeomorphism $\sigma : \tilde{Z} \longrightarrow \tilde{Z}$. This induces a free $\mathbb{Z}_2$ action, and $\tilde{Z}/\mathbb{Z}_2 = Z$. Moreover, one has $NZ \cong \mathbb{R} \times_{\mathbb{Z}_2} \tilde{Z}$, where $\mathbb{R} \times_{\mathbb{Z}_2} \tilde{Z} := (\mathbb{R} \times \tilde{Z})/\mathbb{Z}_2$, with action $(-1) \cdot (t, x) = (-t, \sigma(x))$.
When $NZ$ is trivial, then $\tilde{Z} = Z \sqcup Z$, and one gets back $NZ \cong \mathbb{R} \times Z$. Moreover, one can always assume that a tubular neighbourhood with smooth boundary is of the form $(-\varepsilon, \varepsilon) \times_{Z_2} \tilde{Z}$.

**Theorem 4.2.5.** *Let $(X, Z, \omega)$ be scattering symplectic. Take a fiber metric on the normal bundle of $Z$, denote the distance function from $0$ with $\lambda$. Let $\tilde{Z} = \{\lambda = 1\}$, and denote the projection with $p : \tilde{Z} \longrightarrow Z$. There exist*

- *a contact form $\alpha$ on $Z$*
- *a closed 1-form $\beta_1$ on $Z$*
- *a closed 2-form $\beta_2$ on $Z$*

*such that, in a tubular neighbourhood of the form $(-\varepsilon, \varepsilon) \times_{\mathbb{Z}_2} \tilde{Z}$,*

$$\omega \cong \frac{d\lambda}{\lambda^3} \wedge (-p^*\alpha + \lambda^2 p^*\beta_1) + \frac{1}{2\lambda^2} dp^*\alpha + p^*\beta_2 \tag{4.3}$$

*Here the expression $\omega \cong \eta$ means: there exists a diffeomorphism $\phi$ such that $\phi|_Z = id$, and such that $\phi^*\omega = \eta$ on a tubular neighbourhood of $Z$.*

## 4.3 Symplectic fillings of contact manifolds

In this section we recall several different notions of symplectic filling in contact geometry. In general, a symplectic filling of a cooriented contact manifold $(Z, \xi)$ is a connected symplectic manifold $(W, \omega)$ with boundary $\partial W = Z$ as oriented manifolds (i.e., the boundary orientation matches the contact orientation) such that the symplectic form is "compatible" with the contact distribution.
Different ways of making the word "compatible" precise give different flavours of symplectic fillings. We will always consider notions of compatibility that only involve a neighbourhood of the boundary. Let us start with the strongest notion appearing in this paper.

**Definition 4.3.1.** $(W, \omega)$ is a *strong symplectic filling* of $(Z, \xi)$ if there is a collar neighbourhood $U = (-\varepsilon, 0] \times Z$ of $\partial W$, with $\partial W = \{0\} \times Z$, and transverse coordinate $s$, such that $\omega = d(s\alpha)$ on $U$.

In particular $\omega$ pulls-back to $\xi$ on $Z$ as a symplectic form, belonging to the canonical conformal symplectic structure of $\xi$.
Denote the conformal symplectic structure on $\xi$, viewed as a set of symplectic forms, with $CS_\xi$.

**Definition 4.3.2** ([23]). $(W, \omega)$ is a *weak symplectic filling* of $(Z, \xi)$ if $\omega|_\xi$ is symplectic, and for all representatives $\eta$ of $CS_\xi$, the form $\omega|_\xi + \eta$ is symplectic on $\xi$.

Of course strong implies weak. In dimension 4 the contact distribution is 2-dimensional, and this definition of weak filling is equivalent to the requirement that $\omega|_\xi$ is a positive form, i.e. it induces the same orientation as the contact orientation of $\xi$. This is equivalent, still in dimension 4, to $\omega|_\xi$ belonging to $CS_\xi$. In higher dimension, only requiring $\omega|_\xi \in CS_\xi$ turns out to be equivalent to the existence of a strong filling ([25]). Instead, in all dimensions (including 4) the notions of strong and weak fillability are different ([23]). The notion of weak fillability can be understood in terms of almost complex structures.

**Theorem 4.3.3** ([23], Theorem D). *A symplectic manifold $(W, \omega)$ with boundary $\partial W = Z$ is a weak filling if and only if there exists an almost complex structure $J$ on $W$ such that*

- *$J$ tames $\omega$*
- *$\xi = TZ \cap JTZ$*
- *for any contact form $\alpha$ and $v \in \xi$, $d\alpha(v, Jv) > 0$ (with respect to the boundary orientation)*

A possible intermediate notion, motivated by dynamics and the theory of holomorphic curves is that of a *stable filling*, where the boundary is required to inherit a *stable Hamiltonian structure* (Definition 2.3.1), in a strong sense (meaning, in a way that induces a normal form in a collar neighbourhod of the boundary, see Definition 2.3.5). As explained in Chapter 2, this is the setup in which one is able to study moduli spaces of (punctured) holomorphic curves, in the framework of symplectic field theory ([11]). A stable filling is what one expects.

**Definition 4.3.4.** $(W, \omega)$ is a *stable (Hamiltonian) filling* if there exists an outward pointing vector field $X$ aroud the boundary such that $((\iota_X\omega)|_{T\partial W}, \omega|_{T\partial W})$ is a stable Hamiltonian structure.

## 4.4 Scattering fillings versus weak fillings

### 4.4.1 Statement of the result

Scattering symplectic geometry suggests yet another notion of filling of a contact manifold.

**Definition 4.4.1.** A *(orientable) scattering filling* of a cooriented contact manifold $(Z, \ker\alpha)$ is a (orientable) scattering symplectic manifold $(X, Z, \omega)$, such that $Z$ with the induced contact structure is contactomorphic to $(Z, \ker\alpha)$. A scattering filling is *strong* if the forms $\beta_1$, $\beta_2$ are exact.

In [20] it is shown that

strong orientable scattering fillability $\Leftrightarrow$ strong fillability

In this section we prove the main results of the present paper.

**Theorem 4.4.2.**
- *$(Z, \ker\alpha)$ has an orientable scattering filling $\Leftrightarrow$ $(Z, \ker\alpha)$ is weakly fillable.*
- *$(Z, \ker\alpha)$ has a non-orientable scattering filling $\Leftrightarrow$ there is a connected double cover $p : \tilde{Z} \longrightarrow Z$ such that $(\tilde{Z}, p^* \ker\alpha)$ is weakly fillable.*

### 4.4.2 Manifolds with scattering boundary

When $X$ is a manifold with boundary, one can define the scattering tangent bundle ${}^{sc}TX$ exactly as in Definition 4.2.1, with $Z = \partial X$. A scattering symplectic form on $X$ is defined accordingly as a symplectic form on ${}^{sc}TX$. When we mention *symplectic manifolds with scattering boundary* we will always mean we are in the situation just described.
On a closed orientable scattering symplectic manifold $(X, Z, \omega)$, one can fix an orientation, and compare it with the symplectic orientation induced by $\omega$ on $X \setminus Z$. These orientations are going to coincide on some components of $X \setminus Z$, and differ on other components. This induces a splitting $X = X^+ \cup X^-$, where $X^\pm$ are compact submanifolds with boundary $\partial X^\pm = Z$. Both $X^+$ and $X^-$ are symplectic manifolds with scattering boundary.
Conversely, any two symplectic manifolds with (connected) scattering boundary, and matching normal form around the boundary, can be glued along the boundary to form a (closed) scattering symplectic manifold. Moreover, given a symplectic manifold with scattering boundary $W$, it's easy to show that its double inherits a scattering symplectic structure with singular locus $\partial W$.
Hence, as far as the contact manifold is concerned, being the boundary of a symplectic manifold with scattering boundary or being the singular locus of a scattering symplectic manifold are equivalent notions. Let us formulate the result just explained in the following proposition.

**Proposition 4.4.3.** *Let $(Z, \ker\alpha)$ be a cooriented contact manifold. Then it is oriented scattering fillable if and only if it can be realized as the boundary of a symplectic manifold with scattering boundary.*

### 4.4.3 Reduction to the orientable case

In this section we relate scattering fillability by a non-orientable manifold to orientable fillability of a double cover. The result is the following.

**Proposition 4.4.4.** *Let $(Z, \ker\alpha)$ be the singular locus of a non-orientable scattering symplectic manifold. Then there exists a double cover $p : \tilde{Z} \longrightarrow Z$, and symplectic manifold with scattering boundary that coincides with $(\tilde{Z}, \ker p^*\alpha)$.*

*Proof.* This proposition is proven by constructing the real oriented blow-up along the singular locus $Z$.

**Lemma 4.4.5.** *Given a non-orientable scattering symplectic manifold* $(X,\, Z,\, \omega)$*, there exists a symplectic manifold with scattering boundary* $(\tilde{X},\, \tilde{Z},\, \tilde{\omega})$*, with a map* $b : (\tilde{X},\, \tilde{Z}) \longrightarrow (X,\, Z)$ *which restricts to a symplectomorphism* $(\tilde{X} \setminus \tilde{Z},\, \tilde{\omega}) \cong (X \setminus Z,\, \omega)$ *on the regular part, and to a double cover on the singular part. This manifold coincides with the real oriented blow-up of* $(X,\, Z)$ *along* $Z$*.*

*Proof of the lemma.* It is immediate to see that $((-\varepsilon,\, \varepsilon) \times_{\mathbb{Z}_2} \tilde{Z}) \setminus (\{0\} \times_{\mathbb{Z}_2} \tilde{Z}) = ((-\varepsilon,\, \varepsilon) \times_{\mathbb{Z}_2} \tilde{Z}) \setminus (\{0\} \times Z) \cong (-\varepsilon,\, 0) \times \tilde{Z}$. Moreover there is a smooth map $b : (-\varepsilon,\, 0] \times \tilde{Z} \longrightarrow (-\varepsilon,\, \varepsilon) \times_{\mathbb{Z}_2} \tilde{Z}$, which on $\{0\} \times \tilde{Z}$ is the projection $p : \tilde{Z} \longrightarrow \tilde{Z}/\mathbb{Z}_2 = Z$. The normal form Theorem 4.2.5 implies that the manifold with boundary obtained by removing $(-\varepsilon,\, \varepsilon) \times_{\mathbb{Z}_2} \tilde{Z}$ and gluing $(-\varepsilon,\, 0] \times \tilde{Z}$ back in has a natural scattering symplectic structure. □

□

Proposition 4.4.4 together with Proposition 4.4.3 imply that the first item in Theorem 4.4.2 implies the second. We will prove the first item in Theorem 4.4.2 in the remaining subsections. In what follows we will always assume orientability of $X$, unless otherwise specified.

### 4.4.4 An alternative normal form theorem

The following alternative normal form result is going to be useful in the proof of the main theorem. It is stated for orientable scattering symplectic manifolds for simplicity, as that is the only case that we will need. Nonetheless, it holds, with the obvious modifications, for arbitrary scattering symplectic manifolds (compare with Theorem 4.2.5).

**Proposition 4.4.6.** *Let* $(X,\, Z,\, \omega)$ *be an orientable scattering symplectic manifold. There exists* $\varepsilon > 0$*, and a symplectic manifold* $(Y,\, \omega)$ *with boundary* $\partial Y = Z \sqcup Z$*, such that* $\omega$ *extends to a symplectic form* $\widehat{\omega}$ *on* $\widehat{Y} = Y \sqcup [\frac{1}{2\varepsilon^2},\, +\infty) \times \partial Y$*. The restriction of* $\widehat{\omega}$ *to* $[\frac{1}{2\varepsilon^2},\, +\infty) \times \partial Y$ *is*

$$d(s\alpha) - \frac{1}{2}\frac{ds}{s} \wedge \beta_1 + \beta_2 \tag{4.4}$$

*and* $(\widehat{Y},\, \widehat{\omega})$ *is symplectomorphic to* $X \setminus Z$*.*

*Proof.* This just follows from Theorem 4.2.5. Consider the change of coordinates $s = \frac{1}{2\lambda^2}$. Since $\log(s) = -2\log(x) + \log(2)$, $\frac{dx}{x} = d\log(x) = -\frac{1}{2}d\log(s)$, then the form $\omega$ becomes of the desired form

$$\omega = d(s\alpha) - \frac{1}{2}d\log(s) \wedge \beta_1 + \beta_2 = d(s\alpha) - \frac{1}{2}\frac{ds}{s} \wedge \beta_1 + \beta_2 \tag{4.5}$$

□

In particular if $\beta_1 = \beta_2 = 0$ that is exactly the normal form at the boundary of a strong filling of $(Z, \ker\alpha)$. To be more precise:

**Corollary 4.4.7.** *Write a connected component* $\widehat{Y}$ *of* $X \setminus Z$ *as in Proposition 4.4.6; write a neighbourhood of infinity as* $[C, +\infty) \times Z$ *. If* $\beta_1 = \beta_2 = 0$*, for all* $C' > C$*, the manifold* $W := \widehat{Y} \cap \{s \leq C'\}$ *is a strong filling of* $(Z, \ker\alpha)$*.*

### 4.4.5 Scattering ⇒ weak

The first step to relate scattering to weak filling is to get closer to stable Hamiltonian fillings, by getting rid of the form $\beta_1$.

**Proposition 4.4.8.** *The symplectic form* $\omega$ *on* $[C, +\infty) \times Z$ *in Equation* (4.4) *can be modified to a symplectic form* $\tilde{\omega}$ *such that*

- $\omega = \tilde{\omega}$ *on* $[C, C'] \times Z$
- $\tilde{\omega} = d(s\alpha) + \beta_2$ *on* $[C'', +\infty) \times Z$

*Proof.* Pick a smooth function $f = f(s) : [C, +\infty) \times Z \longrightarrow \mathbb{R}$ such that

- $f(s) = \frac{1}{s}$ on $[C, C']$
- $f(s) = 0$ for $s \geq C'' > C'$
- $f'(s) \leq 0$

and define

$$\tilde{\omega} = d(s\alpha) - \frac{1}{2} f(s) ds \wedge \beta_1 + \beta_2$$

$\tilde{\omega}$ remains closed and nondegenerate, at least if one chooses $C'$ big enough. □

Note that $(\alpha, \beta_2)$ is not a stable Hamiltonian structure, as we imposed no condition on $\alpha \wedge \beta^{n-1}$.
Note the following consequence.

**Corollary 4.4.9.** *If* $(Z, \ker\alpha)$ *admits a scattering filling, then it admits a filling such that the symplectic form in a collar neighbourhood is of the form* $d(s\alpha) + \beta$*.*

*Proof.* Just realize $Z$ as $\{c\} \times Z$, with $c \geq C''$, in the normal form of Proposition 4.4.8. Since we are not changing the symplectic form in a neighbourhood of $\{C\} \times Z$, it extends to the rest of the filling. □

One can reformulate the previous corollary as a statement about scattering symplectic manifold.

**Corollary 4.4.10.** *If a pair* $(X, Z)$ *supports a scattering symplectic structure with cohomology decomposition* $(a, [\beta_1], [\beta_2])$*, then it supports one with cohomology decomposition* $(a, 0, [\beta_2])$*.*

We now apply Proposition 4.4.8 to show that scattering fillability implies weak fillability.

**Corollary 4.4.11.** *Let $(X, Z, \omega)$ be an orientable scattering filling of $(Z, \ker\alpha)$, with $Z$ connected. Let $U$ be a tubular neighbourhood of $Z$, and $W$ be a component of $X \setminus U$. Then there exists a symplectic form $\tilde{\omega}$ so that $(W, \tilde{\omega})$ is a weak filling of $(Z, \ker\alpha)$.*

*Proof.* By Proposition 4.4.8, we can attach a cylindrical end to $W$ and get a manifold $\widehat{W}$ with a symplectic form $\tilde{\omega}$, such that $\tilde{\omega} = d(s\alpha) + \beta$ in a neighbourhood $V$ of infinity. Define an almost complex structure $J$ on $TV$ such that

- $J$ turns $\ker\alpha$ into a complex vector bundle, and $J\partial_s = R_\alpha$, the Reeb vector field of $\alpha$.
- $d\alpha$ tames $J$ on $\ker\alpha$

In particular $J$ tames $d(s\alpha)$ on $V$. If $s$ is big enough, then $J$ also tames $\tilde{\omega}$. This means that there exists a smaller neighbourhood $V' \subset V$ (of the form $(c, +\infty) \times Z$) on which $J$ tames $\omega$ and satisfies the properties above. By contractibility of the space of tame complex structures we can extend $J|_V'$ to a tame complex structure on the whole $\widehat{W}$. Choosing $c' > c$, and identifying $Z$ with $\{c'\} \times Z$, we see that Theorem 4.3.3 implies $\widehat{W} \cap \{s \leq c'\}$ is a weak filling of $(Z, \ker\alpha)$. □

### 4.4.6 Weak $\Rightarrow$ scattering

This is the easier implication. It follows from the following well-known consequence of Moser's argument:

**Proposition 4.4.12.** *Let $(W, \omega)$ be a symplectic manifold whose boundary $Z$ has a positive cooriented contact structure $\xi = \ker\alpha$. Assume that $\omega|_\xi$ is symplectic. Denoting $\omega_Z := \omega|_{TZ}$, there is a collar neighbourhood $V \cong (-\varepsilon, 0] \times Z$ such that $\omega \cong d(s\alpha) + \omega_Z$.*

In particular we can assume that the symplectic form on a collar neighbourhood of the boundary in a weak filling can be written as $d(s\alpha) + \omega_Z$. As a consequence one has

**Proposition 4.4.13.** *If $(W, \omega)$ is a weak filling of $(Z, \ker\alpha)$, then there is a symplectic form $\tilde{\omega}$ such that $(W, \tilde{\omega})$ is a symplectic manifold with scattering boundary $(Z, \ker\alpha)$.*

*Proof.* Assume that $\omega = d(s\alpha) + \omega_Z$ in a neighbourhood of the boundary. Using Proposition 4.4.12, attach a copy of $(-\varepsilon, +\infty) \times Z$ along a collar neighbourhood of $\partial W$, with symplectic form $d(s\alpha) + \omega_Z$. Denote the resulting symplectic manifold $(\widehat{W}, \widehat{\omega})$. The inverse of the change of coordinates in Proposition 4.4.8 turns $(\widehat{W}, \widehat{\omega})$ into the open part of a symplectic manifold with scattering boundary, with singular locus $(Z, \ker\alpha)$, with $\beta_1 = 0$, and $\beta_2 = \omega_Z$. □

The main theorem follows as a consequence of Proposition 4.4.13 and Proposition 4.4.3.

## 4.5 Implications for scattering symplectic manifolds

Consider $q : \tilde{Z} \longrightarrow Z$ the double cover map. If $Z$ is endowed with an overtwisted contact structure (in the sense of [1]), then the pullback contact structure on $\tilde{Z}$ is also overtwisted. More generally, if $Z$ has a *bordered Legendrian open book (bLob)* ([23]), also $\tilde{Z}$ does. Since a bLob is an obstruction to weak fillability, we obtain the following result.

**Theorem 4.5.1.** *A connected contact manifold containing a bLob cannot appear as the singular locus of a (not necessarily orientable) scattering symplectic manifold. In particular, the singular locus of a scattering symplectic manifold, if connected, is tight (i.e. not overtwisted).*

Moreover, since every weak filling can be deformed to a stable Hamiltonian filling ([23]), one obtains a well behaved theory of possibly punctured holomorphic curves with values in a scattering symplectic manifold, the punctures being asymptotic to closed Reeb orbits in the singular locus (with respect to the deformed structure).

# Chapter 5

# Symplectic foliations

The goal of this chapter is to construct moduli spaces of holomorphic curves in symplectically foliated manifolds. The main results are the constructions of moduli spaces of somewhere injective spheres with values in the leaves of any foliation, and of somewhere injective positive genus surfaces subject to a holonomy restriction. The construction inclues a proof of a "generic transversality theorem", analogue to the one in [28] (see also Theorem 2.2.7), and a discussion of the compactness. In particular we present a simple proof of the same result in a particular case, which includes all cooriented codimension-1 foliations. As an application, we obtain some results on the topology of the leaves of a foliation, which we summarize below.

**Theorem 5.0.1.** *Let $(X^5, \mathcal{F}^4, \omega)$ be a 5–manifold with a symplectic codimension–1 foliation. Assume that one leaf is ruled (i.e. it contains a symplectically embedded sphere with trivial normal bundle). Then*

- *each leaf $F$ contains a homologically non-trivial sphere, pairing positively with $\omega|_F$*
- *if all leaves are compact, then they are all rational or blown-up ruled surfaces*

The strategy to construct the moduli space is, as in the standard symplectic case, to realize the leafwise Cauchy-Riemann operator as a Fredholm section of a Banach bundle. The base of the bundle is not the space of all (suitably differentiable) maps, but a subset of "leafwise maps", which are those with image contained in a leaf. The first and main step is to construct a meaningful Banach manifold structure on such space of maps. The rest of the work is a quite straightforward application, if not a repetition, of the standard theory, as discussed in [28].

The proof of Theorem 5.0.1 uses the arguments from [25], also used in Chapter 3, and a characterizazion of ruled symplectic 4–manifold in terms of immersed symplectic spheres, originally appeared in [24] (see also [40]).

Most of the work in this chapter arose from discussions with Klaus Niederkrüger.

## 5.1 Manifolds of maps

Let us recall the construction of a smooth Banach manifold $\mathfrak{S}(M, X)$ of "$\mathfrak{S}$-regular maps" between a compact smooth manifold $M$, and a smooth manifold $X$, following Eliasson [12] (see also [32] for an alternative approach).
We will use the following notation. We denote with $\mathbb{V}(M)$ the category of smooth vector bundles over $M$, and with $\mathbb{B}$ the category of Banach(able) spaces. Given a vector bundle $E \longrightarrow M$, we denote $C^k(E)$, $0 \leq k \leq \infty$, the topological vector space of $C^k$-regular sections of $E$ (viewed as a Banach space whenever $k \neq \infty$). Given two vector bundles $E$ and $F$ over $M$, we let $\mathrm{Hom}(E, F)$ be the vector bundle of smooth vector bundle maps $E \longrightarrow F$ lifting the identity. There is a natural continuous map $-_* : C^\infty(\mathrm{Hom}(E, F)) \longrightarrow \mathrm{Hom}(C^j(E), C^j(F))$, for all $j \leq \infty$, defined by postcomposition.

In the notation of Eliasson [12], a *section functor* is a functor $\mathfrak{S} : \mathbb{V}(M) \longrightarrow \mathbb{B}$ such that the map $-_*$ defines a continuous inclusion $-_* : C^\infty(\mathrm{Hom}(E, F)) \longrightarrow \mathrm{Hom}(\mathfrak{S}(E), \mathfrak{S}(F))$. The following definition is also Eliasson's.

**Definition 5.1.1.** A *manifold model* is a section functor $\mathfrak{S} : \mathbb{V}(M) \longrightarrow \mathbb{B}$ such that:

- there is a continuous inclusion $\mathfrak{S}(E) \subset C^0(E)$;
- the map $-_*$ induces a continuous inclusion $-_* : \mathfrak{S}(\mathrm{Hom}(E, F)) \longrightarrow \mathrm{Hom}(\mathfrak{S}(E), \mathfrak{S}(F))$;
- let $\mathcal{D} \subset E$ be an open neighbourhood of the zero-section, let $f : \mathcal{D} \longrightarrow F$ be a fiber-preserving $C^\infty$ map. Then the map $f_* : \mathfrak{S}(\mathcal{D}) \longrightarrow \mathfrak{S}(F)$ is well-defined and continuous.

For example, the functors $C^k$ of $k$-times differentiable sections ($k \neq \infty$) satisfy these hypothesis, as well as the functors $W^{k,p}$ of Sobolev sections, provided $kp > \dim(M)$. In the latter case, the requirement $kp > \dim(M)$ is needed in order to ensure all the three conditions in Definition 5.1.1. The first condition follows from the Sobolev embedding theorem, while for the second and the third we refer respectively to Theorem B.1.19 and B.1.20 in [28].

The main tool in the construction of a manifold of maps is the following Lemma, due to Eliasson ([12]).

**Lemma 5.1.2** ([12], Lemma 4.1)**.** *Let $E$, $F$ be vector bundles over $M$, let $\mathcal{O} \subset E$ be a neighbourhood of the zero section. If*

$$f : \mathcal{O} \longrightarrow F$$

*is a fiber-preserving smooth function, then*

$$f_* : \mathfrak{S}(\mathcal{O}) \longrightarrow \mathfrak{S}(F)$$

*is smooth, and* $D^i(f_*) = \mathfrak{S}(D^i f)$.

We define the space $\mathfrak{S}(M, X)$ of $\mathfrak{S}$-regular maps as follows.

**Definition 5.1.3** ([12])**.** Let $\nabla$ be a symmetric connection on $X$. We say that a continuous function $f : M \longrightarrow X$ is an element of $\mathfrak{S}(M, X)$ if there exists a smooth map $h : M \longrightarrow X$, and a section $F \in \mathfrak{S}(h^*TX)$, such that $f = \exp_h^\nabla(F)$ (here $\exp_h^\nabla(F)(p) := \exp_{h(p)}^\nabla(F_p)$, and $\exp^\nabla$ is the exponential map with respect to the connection $\nabla$).

**Remark 5.1.4.** The set $\mathfrak{S}(M, X)$ does not depend on the choice of the connection. Indeed, let $\nabla_1$, $\nabla_2$ be symmetric connections, and let $h : M \longrightarrow X$ be a smooth function. Let $F_1 \in \mathfrak{S}(h^*TX)$, and consider the function $f = \exp_f^{\nabla_1}(F_1)$. Since $f$ is continuous, there exists a section $F_2 \in C^0(h^*TX)$ such that $f = \exp_f^{\nabla_2}(F_2)$; in particular $F_2 = (\exp_f^{\nabla_2})^{-1}(\exp_f^{\nabla_1})(F_1) =: \phi_*(F_1)$. Now, $\phi_*$ is an operator coming from a fiber preserving smooth function $\phi = (\exp_f^{\nabla_2})^{-1}(\exp_f^{\nabla_1}) : \mathcal{D} \subset h^*TX \longrightarrow h^*TX$. By the definition of a manifold model, $\phi_*(\mathfrak{S}(\mathcal{D})) \subset \mathfrak{S}(h^*TX)$. The smoothness of $\phi_*$ follows from Lemma 5.1.2.

The following theorem is proven in [12], section 5.

**Theorem 5.1.5.** *Given a symmetric connection $\nabla$ on $X$, for each smooth function $h : M \longrightarrow X$ the maps $exp_h^\nabla : \mathfrak{S}(h^*TX) \longrightarrow \mathfrak{S}(M, X)$ form a system of smoothly compatible parametrizations, defining a topology and a smooth Banach manifold structure on $\mathfrak{S}(M, X)$. The smooth structure does not depend on the choice of $\nabla$.*

This manifold can be viewed (in certain cases) as a "completion" of the manifold of differentiable maps. In fact, assuming that there is a continuous inclusion

$$C^\infty(E) \hookrightarrow \mathfrak{S}(E)$$

for all differentiable vector bundles $E$, and some $k$, then there is an immersion

$$C^\infty(M, X) \hookrightarrow \mathfrak{S}(M, X)$$

This means that the inclusion is a continuous map, which induces an inclusion at the level of tangent spaces. In fact, since the topology is given by definition using the charts (Theorem 5.1.5), it is enough to show that for each smooth $h \in C^\infty(M, X)$, the inclusion $C^\infty(h^*TX) \hookrightarrow \mathfrak{S}(h^*TX)$ is continuous, which is true by assumption.

Moreover, if we assume that $C^\infty(E) \subset \mathfrak{S}(E)$ is dense for all vector bundles $E$, then $C^\infty(M, X)$ is dense in $\mathfrak{S}(M, X)$. A special case which is most important for us is when $\mathfrak{S} = W^{k,p}$ $(p \geq 1)$, the space of Sobolev maps.

## 5.2 Manifolds of leafwise maps

Throughout this section we assume that $(X, \mathcal{F})$ is a manifold with a smooth codimension–$q$ foliation. We keep on denoting by $M$ a compact manifold, possibly with boundary. We want to construct a manifold $\mathfrak{S}(M, F)$ of "$\mathfrak{S}$-regular" maps with image contained in the leaves of $\mathcal{F}$. To simplify the discussion, let us give a name to such maps.

**Definition 5.2.1.** A *leafwise map* $h : M \longrightarrow X$ is a continuous map such that there exists a leaf $F_h$ of $\mathcal{F}$ such that $\mathrm{im}(h) \subset F_h$.

We will define the space of $\mathfrak{S}$-regular leafwise maps by "completing" the set of leafwise $C^\infty$ maps, in the spirit of the comments at the end of Section 5.1.

### 5.2.1 Maps with trivial holonomy

We will not be able to put a smooth structure on the space of all leafwise maps, but we will restrict to a subset of maps with "trivial holonomy". Morally, those are maps around the image of which the foliation is trivial (i.e., given by the factors of a product).

First of all, we recall what the holonomy group of a leaf is (for more details see, e.g., [29]). Pick a point $x \in X$, and let $F_x$ be the leaf through $x$. Given a transversal $\mathcal{T}$ through $x$, the *holonomy map* is a group homomorphism $\mathrm{Hol}_x^{\mathcal{T}} : \pi_1(F_x, x) \longrightarrow \mathrm{Diff}_x(\mathcal{T}, x)$ to the group of germs of diffeomorphisms of $\mathcal{T}$ fixing $x$. It is defined by covering the image of a path with suitable foliation charts, and parallel transporting along the plaques of the chart.
Fixing an isomorphism $\mathrm{Diff}_x(\mathcal{T}, x) \cong \mathrm{Diff}_0(\mathbb{R}^q, 0)$ (coming from a diffeomorphism of a neighbourhood of $x \in \mathcal{T}$ with a neighbourhood of $0 \in \mathbb{R}^q$), one gets a group homomorphism $\mathrm{Hol}_x : \pi_1(F_x, x) \longrightarrow \mathrm{Diff}_0(\mathbb{R}^q, 0)$, well defined and independent on the choice of $\mathcal{T}$ up to conjugation in the target. We denote with $H_x$ the image of $\mathrm{Hol}_x$, and with $K_x$ the kernel. The holonomy map, and all the groups involved, do not depend on the choice of $x$ up to isomorphism.

**Definition 5.2.2.** We say that a smooth map $h : M \longrightarrow X$, tangent to $\mathcal{F}$, has *trivial holonomy* if the induced map $h_* : \pi_1(M, p) \longrightarrow \pi_1(F_{h(p)}, h(p))$ factors through $K_{h(p)}$. Denote by $C_0^k(M, \mathcal{F})$ the set of $C^k$ maps with trivial holonomy.

### 5.2.2 Local classification of maps with trivial holonomy

Fix a splitting $TX = T\mathcal{F} \oplus N\mathcal{F}$, where $N\mathcal{F}$ is the normal bundle to the foliation, and fix a diagonal symmetric connection $\nabla = \begin{pmatrix} \nabla^{\mathcal{F}} & \\ & \nabla^N \end{pmatrix}$. Given a leaf $F$, with normal bundle $NF = N\mathcal{F}|_F$, the exponential map in the normal direction $\exp^{\nabla^{\mathcal{F}}} : V \subset N\mathcal{F} \longrightarrow U_F \subset X$ defines a tubular neighbourhood of $F$. This induces a foliation on $V$, in the total space of $NF$, such that the zero section is a leaf, and the leaves are transverse to the fibers of $NF \longrightarrow F$.

Given a smooth map $h : M \longrightarrow X$, with image contained in a leaf $F_h$, one can consider the pullback vector bundle $h^*N\mathcal{F}$. Since the foliation on the total space of $N\mathcal{F}|_{F_h}$ is transverse to the fibers of $NF_h \longrightarrow F_h$, one can pull it back to a foliation $h^*\mathcal{F}$ on the total space of $h^*N\mathcal{F}$. The foliation $h^*\mathcal{F}$ is still transverse to the fibers of the bundle projection, and the zero section is a leaf.

The local Reeb stability theorem ([29], Theorem 2.9) readily implies that, if $h$ has trivial holonomy, then $h^*\mathcal{F}$ is the product foliation. More precisely, there is an open neighbourhood $\mathcal{D} \subset h^*N\mathcal{F}$ of the zero section, with $\mathcal{D} \cong M \times \mathcal{T}$, where $\mathcal{T}$ is a disc in a normal fiber, and $h^*\mathcal{F}$ coincides with $M \times \{\tau\}$. One can choose such transversal to be the pullback of a small transversal to a leaf of $\mathcal{F}$ in $X$, intersecting $h(M)$ in exactly one point (this is possible as $M$ is compact). This construction allows us to classify all smooth leafwise maps near $h$ in terms of vector fields, locally homeomorphically in the $C^k$ topology.

**Proposition 5.2.3.** *There is a convex subset $C^{const}(h^*N\mathcal{F}) \subset C^k(h^*N\mathcal{F})$, such that $C^{const}(h^*N\mathcal{F}) \cong \mathcal{T} \cong \mathbb{D}^q$, and such that the map*

$$E_h^{\nabla} : C^{const}(h^*N\mathcal{F}) \times C^k(h^*T\mathcal{F}) \longrightarrow C_0^k(M, \mathcal{F}) \tag{5.1}$$

*defined as*

$$E_h^{\nabla}(v, w) := \exp^{\nabla}_{\exp_h^{\nabla}(v)}(d(\exp_h^{\nabla})_v(w)) \tag{5.2}$$

*maps a neighbourhood of 0 onto a neighbourhood of $h$, homeomorphically.*

*Proof.* Take $\mathcal{D} \subset h^*N\mathcal{F}$ as above, with a diffeomorphism $\mathcal{D} \cong M \times \mathcal{T}$, such that the inuced foliation is the trivial one. Define $C^{\text{const}}(h^*N\mathcal{F})$ as the subset of constant sections of $M \times \mathcal{T}$. By construction, exponentiating such sections sends leaves to leaves. Moreover, parallel transport by a diagonal connection preserves the splitting $TX = T\mathcal{F} \oplus N\mathcal{F}$. Hence $d(\exp_h^{\nabla})_v(w)$ is still a vector tangent to $\mathcal{F}$. The exponential of such vector field preserves leaves, as the connection is diagonal. The map is easily seen to be one-to-one near $h$.
It is a (local) homeomorphism between the space of $C^k$ sections with the $C^k$ topology and the space of maps with the $C^k$ topology. This is because the norms of the derivatives of $E_h^{\nabla}(v, w)$ can be estimated in terms of the derivatives of $v$ and $w$, and conversely, as the exponential map is smooth. □

### 5.2.3 Manifold structure

**Theorem 5.2.4.** *The maps $E_h^{\nabla}$ form a smooth atlas for $C_0^k(M, \mathcal{F})$, giving it the structure of a smooth (Banach, if $k < \infty$) manifold. The inclusion map*

$$C_0^k(M, \mathcal{F}) \hookrightarrow C^k(M, X)$$

*is an embedding. Moreover, for each leaf $F \subset X$ the inclusion*

$$C^k(M, F) \hookrightarrow C^k(M, \mathcal{F})$$

*is an embedding.*

*Proof.* We saw in Proposition 5.2.3 that the maps $E_h^\nabla$ form a topological atlas. Consider the map

$$\overline{E}_h^\nabla : h^*TX = h^*N\mathcal{F} \oplus T\mathcal{F} \longrightarrow M \times X$$

defined by the formula 5.2. It is a local diffeomorphism around the 0 section. In particular, for any two maps $\overline{E}_{h_1}^\nabla$, $\overline{E}_{h_2}^\nabla$, the composition $\overline{E}_{h_1}^\nabla \circ \overline{E}_{h_2}^{\nabla\,-1}$ is smooth where defined, hence restricts to a smooth map at the level of sections (Lemma 5.1.2). Since $C^{\text{const}}(h^*N\mathcal{F}) \times C^k(h^*T\mathcal{F})$ is a submanifold of $C^k(h^*TX)$, it follows that $E_{h_1}^\nabla \circ E_{h_2}^{\nabla\,-1}$ is also smooth.
The fact that

$$C_0^k(M, \mathcal{F}) \hookrightarrow C^k(M, X)$$

is an embedding is a consequence of the fact that $\overline{E}_{h_1}^\nabla$ is smooth with respect to the atlas on $C^k(M, X)$ defined in Section 5.1. This follows again from Lemma 5.1.2.
Finally, fix a leaf $F$, and consider the inclusion map

$$C^k(M, F) \hookrightarrow C_0^k(M, \mathcal{F})$$

Around each $h \in C^k(M, F)$, the map is given by the inclusion

$$C^k(h^*T\mathcal{F}) \cong \{0\} \times C^k(h^*T\mathcal{F}) \subset C^{\text{const}}(h^*N\mathcal{F}) \times C^k(h^*T\mathcal{F})$$

hence it is an embedding. □

### 5.2.4 Manifold of $\mathfrak{S}$-regular maps

Let us generalize the construction above to an arbitrary manifold model $\mathfrak{S}$. First of all, let us define the set of leafwise $\mathfrak{S}$-regular maps with trivial holonomy. Let $\nabla^{\mathcal{F}}$ be a connection on $T\mathcal{F}$.

**Definition 5.2.5.** The space of *$\mathfrak{S}$-regular leafwise maps with trivial holonomy* is the topological subspace $\mathfrak{S}_0(M, \mathcal{F}) \subset \mathfrak{S}(M, X)$ of continuous leafwise maps $h \in C_0^0(M, \mathcal{F})$ such that there exists $h' \in C^\infty(M, \mathcal{F})$ and $v \in \mathfrak{S}(h^*T\mathcal{F})$ small such that $h = \exp_{h'}^{\nabla^{\mathcal{F}}}(v)$.

**Remark 5.2.6.** As in Remark 5.1.4, the definition does not depend on the choice of a connection on $T\mathcal{F}$.

**Lemma 5.2.7.** *Given a smooth map $h \in C_0^\infty(M, \mathcal{F})$, the set of $\mathfrak{S}$-regular leafwise maps is locally parametrized around $h$ by $C^{const}(h^*N\mathcal{F}) \times \mathfrak{S}(h^*T\mathcal{F})$, via the map $E_h^\nabla$ defined as in (5.2).*

*Proof.* The only point to make is that smooth maps preserve $\mathfrak{S}$-regular sections, by definition of section functor. Then, if there is an $\mathfrak{S}$-regular leafwise map $h'$ near $h$, it can be written as $E_h^\nabla(v, w)$ for some $(v, w) \in C^{\text{const}}(h^*N\mathcal{F}) \times C^0(h^*T\mathcal{F})$. The proof that $w \in \mathfrak{S}(h^*T\mathcal{F})$ is then the same argument as Remark 5.1.4. □

Finally we can prove that the parametrizations above form a smooth atlas.

**Theorem 5.2.8.** *The maps* $E_h^\nabla$ *form a smooth atlas for* $\mathfrak{S}_0(M, \mathcal{F})$*, giving it the structure of a smooth (Banach) manifold. The inclusion map*

$$\mathfrak{S}_0(M, \mathcal{F}) \hookrightarrow \mathfrak{S}(M, X)$$

*is an embedding. Moreover, for each leaf* $F \subset X$ *the inclusion*

$$\mathfrak{S}(M, F) \hookrightarrow \mathfrak{S}(M, \mathcal{F})$$

*is an embedding.*

*Proof.* Once we note that the induced topology on $\mathfrak{S}_0(M, \mathcal{F})$ is the one making the maps $E_h^\nabla$ homeomorphisms, the proof is word by word the one given for Theorem 5.2.4. □

### 5.2.5 Manifold structure via the holonomy groupoid

Another approach to the construction of a smooth structure on the set of leafwise maps is by relating them to maps with values in the fibers of a submersion. This is done using the *holonomy groupoid* of a foliation. The submersion case is easier, due to the fact that the corresponding foliation has no holonomy, and it has transverse coordinates. In particular one can prove easily that the set of maps with values in the fibers of a submersion is a submanifold of the set of all maps, as indicated in the following lemma.

**Lemma 5.2.9.** *Let* $s : Y^m \longrightarrow B^d$ *be a smooth surjective submersion (not necessarily proper), let* $s^{-1}$ *denote the foliation given by the connected components of the fibers of* $s$*. Fix a splitting* $Ns^{-1} \oplus Ts^{-1}$*. Given* $h \in C^\infty(M, s^{-1})$*, there is a convex subset* $C^{const}(h^*Ns^{-1}) \subset C^k(h^*Ns^{-1})$*, such that* $C^{const}(h^*Ns^{-1}) \cong \mathbb{D}^d \subset T_{s(h)}B$*, and such that* $C(M, s^{-1})$ *is a smooth submanifold of* $C^k(M, Y)$*, locally modeled on* $C^{const}(h^*Ns^{-1}) \times C^k(h^*Ts^{-1})$*.*

*Proof.* Let $h \in C^k(M, s^{-1})$. Let $b$ be such that $\mathrm{im}(h) \subset s^{-1}(b)$. Now let $K \subset s^{-1}(b)$ be a compact neighbourhood of $h(M)$ in $s^{-1}(b)$. We can assume that there is a neighbourhood of $h(M)$ diffeomorphic to $\mathbb{D}^d \times K$, where $s$ is equal to the first projection (by the proof of Ehresmann's theorem). Now we can take a split Riemannian metric on $\mathbb{D}^d \times K$, and extend it to $M$. Let $C^{\mathrm{const}}(h^*Ns^{-1})$ be the set of sections that are constant in the trivialization $\mathbb{D}^d \times K$. This forms a convex set, and the map $s_* : C^{\mathrm{const}}(h^*Ns^{-1}) \longrightarrow T_{s(h)}B$ is an isomorphism onto a neighbourhood $\mathbb{D}^d \subset T_{s(h)}B$. The exponential map restricted from $C^k(h^*TX)$ to $C^{\mathrm{const}}(h^*Ns^{-1}) \times C^k(h^*T\mathcal{F})$ is then a smooth parametrization for $C^k(M, s^{-1})$ around $h$. □

**Remark 5.2.10.** The same proof would work for a general foliation, if we restrict our attention to leafwise maps $h$ such that $\mathrm{im}(h) \subset \mathcal{T} \times K$, for a transversal $\mathcal{T}$ and a compact neighbourhood $K$ of $\mathrm{im}(h)$ in its leaf. For example, this is the case for leafwise *embeddings* with trivial holonomy. However, that is in general not a closed condition.

Given a manifold $X$ with a foliation $\mathcal{F}$, one can "desingularize it to a submersion" by constructing the so-called *holonomy groupoid* $\mathrm{Hol}(\mathcal{F})$ of $\mathcal{F}$. As a set, it is defined as

$$\mathrm{Hol}(\mathcal{F}) := \{\gamma : [0,1] \longrightarrow X \,\text{smooth} \; : \dot{\gamma} \in T\mathcal{F}\}/\sim \tag{5.3}$$

where $\gamma \sim \gamma'$ if and only if $\gamma(0) = \gamma'(0)$, $\gamma(1) = \gamma'(1)$, and $\gamma^{-1} \star \gamma'$ has trivial holonomy. Equivalently, given $x = \gamma(0)$, one has the holonomy sequence $1 \longrightarrow K_x \longrightarrow \pi_1(F_x, x) \longrightarrow H_x \longrightarrow 1$; the equivalence relation is that the homotopy class $[\gamma^{-1} \star \gamma']$ belongs to $K_x$.

Concatenation and inversion of paths give $\mathrm{Hol}(\mathcal{F})$ the structure of a groupoid over $X$, with source and target maps being the evaluations $s := \mathrm{ev}_0$, $t := \mathrm{ev}_1$. One can show that $\mathrm{Hol}(\mathcal{F})$ can be naturally given the structure of a smooth $\dim(X) + \mathrm{rank}(\mathcal{F})$-dimensional manifold, for which $s$ and $t$ are submersions. By the very definition of $\mathrm{Hol}(\mathcal{F})$, one sees immediately that for each $x \in X$, $t : s^{-1}(x) \longrightarrow F_x$ is the covering space corresponding to the subgroup $K_x$. In particular, the following lemma is now obvious.

**Lemma 5.2.11.** *The leafwise maps $h : M \longrightarrow F_x \subset X$ with trivial holonomy are exactly those that admit a lift $\overline{h} : M \longrightarrow s^{-1}(x)$, so that $h = t \circ \overline{h}$.*

The correspondence is not one-to-one, in fact for each leaf $F$ one can lift a map $h : M \longrightarrow F$ to each fiber $s^{-1}(x)$ with $x \in F$. Moreover, given a fixed fiber $s^{-1}(x)$, the lift is also not unique. One can say that this lift is unique up to the action of a groupoid on $C^k_0(M, F)$.

**Definition 5.2.12.** A *groupoid action* of a groupoid $G \overset{s}{\underset{t}{\rightrightarrows}} X$ on a space $Y$ is given by:

- a moment map $\mu : Y \longrightarrow X$
- a multiplication map $\cdot : \{(y, g) : \mu(y) = s(g)\} \longrightarrow Y$

such that $\mu(g \cdot y) = t(y)$

Consider the map $s : C^k(M, s^{-1}) \longrightarrow X$, which selects the $s$-fiber. Each element $\gamma$ in the holonomy groupoid is a (holonomy class of) path(s), which acts by pointwise concatenation on $C^k(M, \mathrm{Hol}(\mathcal{F}))$, preserving $C^k(M, s^{-1})$. It is an exercise to prove the following proposition, which gives an equivalent definition of the space of leafwise maps.

**Proposition 5.2.13.** $C^k(M, \mathcal{F}) \cong C^k(M, s^{-1})/\mathit{Hol}(\mathcal{F})$.

We will construct a smooth structure on $C^k(M, F)$ by constructing local sections of the projection $\pi : C^k(M, s^{-1}) \longrightarrow C^k(M, \mathcal{F})$. We start by showing that the quotient topology coincides with the $C^k$ topology.

**Proposition 5.2.14.** *$C^k(M, \mathcal{F})$ with the quotient topology is topologically embedded in $C^k(M, X)$.*

*Proof.* The proof is elementary and we only sketch it. The quotient map coincides with post-composition with the (smooth) target map $t$. Hence, given a map $\overline{h} : M \longrightarrow \mathrm{Hol}(\mathcal{F})$, one can consider an open neighbourhood $U$ with compact closure of $\overline{h}(M)$. The $C^k$ norm of $t$ is bounded on $U$. Let $h = t \circ \overline{h}$. Given the set $\mathcal{U}$ of maps with $\varepsilon$-small $C^k$ distance from $h$, a neighbourhood of $\overline{h}$ given by $\delta$-close maps (with $\delta$ depending on $\varepsilon$ and the $C^k$ norm of $t$ on $U$) is contained in $q^{-1}(\mathcal{U})$, thus $q^{-1}(\mathcal{U})$ is open and $\mathcal{U}$ is open in the quotient topology. For the converse, if a subset $\mathcal{U}$ is such that $q^{-1}(\mathcal{U})$ is open, then an $\varepsilon$-neighbourhood of a lift $\overline{h}$ of $h$ (for each $h \in \mathcal{U}$) is contained in $q^{-1}(\mathcal{U})$. The image of such $\varepsilon$-neighbourhood coincides with a $\delta$-neighbourhood in the $C^k$ distance, for suitable $\delta$. Hence $\mathcal{U}$ is $C^k$ open. □

Take a smooth map $h$ in $\mathfrak{S}(M, F)$, and let $F_h$ be the leaf containing $h(M)$. Since $h(M)$ is compact, there is a smooth transversal $\mathcal{T} \cong \mathbb{D}$ such that $h(M) \cap \mathcal{T} = \{x\}$.

**Theorem 5.2.15.** *There is a convex subset $C^{const}(h^*N\mathcal{F}) \subset C^\infty(h^*N\mathcal{F})$, $C^{const}(h^*N\mathcal{F}) \cong \mathcal{T}$, such that the set of leafwise maps $C^k(M, \mathcal{F})$, with the quotient topology, can be given the structure of a smooth manifold, locally modeled on $C^{const}(h^*N\mathcal{F}) \times C^k(h^*T\mathcal{F})$. With this smooth structure, the quotient map is a submersion, and for each leaf $F$ of $\mathcal{F}$ the manifold $C^k(M, F)$ is a submanifold.*

*Proof.* Consider $s^{-1}(\mathcal{T}) \subset \mathrm{Hol}(\mathcal{F})$. Fixing a point $p \in M$, one can define a canonical lift $\overline{h} : M \longrightarrow s^{-1}(x) \subset s^{-1}(\mathcal{T})$ by requiring that $\overline{h}(p) = \overline{x}$, where $\overline{x}$ is the constant path at $x$. For a nearby map $h'$, one there is a unique element $x' \in \mathcal{T}$, and map $\overline{h}' : (M, p) \longrightarrow (s^{-1}(x')), \overline{x}')$ lifting $h'$. This defines a section $\mathcal{S}_{\mathcal{T},p} : C^k(M, U; \mathcal{F}) \longrightarrow C^k(M, s^{-1}|_{s^{-1}(U)})$, which is continuous (and hence an embedding) by the same arguments as in the proof of Proposition 5.2.14. The image is an open subset of $C^k(M, s^{-1}|_{s^{-1}(\mathcal{T})})$, hence it is a smooth subset, by Lemma 5.2.9.
This defines a smooth chart for $C^k(M, \mathcal{F})$ around $h$, locally given by $C^{\mathrm{const}}(\overline{h}^* N s^{-1}|_{s^{-1}(\mathcal{T})}) \times C^k(\overline{h}^* T s^{-1}_{s^{-1}(\mathcal{T})})$ as seen in Lemma 5.2.9. The map $t$ is a fiberwise covering map, and a local diffeomorphism (upon restriction to $s^{-1}(\mathcal{T})$), then $Ts^{-1} = t^*T\mathcal{F}$, and $t^*N\mathcal{F} = Ns^{-1}$. Since $t \circ \overline{h} = h$, one gets $C^k(\overline{h}^* T s^{-1}) \cong C^k(h^*T\mathcal{F})$, and $C^k(\overline{h}^* N s^{-1}) = C^k(h^*N\mathcal{F})$.

This does not depend on the choice of the transversal and of the point. Pick another transversal $\mathcal{T}'$ such that $h(M) \cap \mathcal{T}' = \{x'\}$. Given a path $\gamma$ joining $x$ to $x'$, there is a holonomy diffeomorphism from $\mathcal{T}$ to $\mathcal{T}'$ (up to restricting both $\mathcal{T}$ and $\mathcal{T}'$). Moreover, there is a continuous family of paths $\gamma_\tau$ parametrized by $\mathcal{T}$, realizing the diffeomorphism between $\mathcal{T}$ and $\mathcal{T}'$. Concatenation with $\gamma_\tau$ defines a diffeomorphism $s^{-1}(\mathcal{T}) \cong s^{-1}(\mathcal{T}')$, preserving $s$. This induces a diffeomorphism on the corresponding sets of maps. Changing base point $p$ to

$p'$ so that $h(p) = h(p')$, with the same transversal, can be dealt with as follows. Take a path $\eta$ joining $p$ to $p'$, this induces a leafwise path $\eta_{h'}$ for each leafwise function $h'$, by composition (smoothly in $h'$). Then $\eta \star S_{\mathcal{T},\,p} = S_{\mathcal{T},\,p'}$. With this is it easy to conclude that any two charts are smoothly compatible. □

**Proposition 5.2.16.** *With the smooth structure defined in Theorem 5.2.15, the inclusion map*

$$C^k(M,\,\mathcal{F}) \hookrightarrow C^k(M,\,X)$$

*is an embedding.*

*Proof.* We already know that it is a topological embedding. Moreover, it is a smooth embedding, because it is locally the composition of a section of the quotient map with the target map (which coincides with the quotient map). □

As a corollary, one one has the following.

**Proposition 5.2.17.** *The smooth structures on $C^k(M,\,\mathcal{F})$ defined in Theorem 5.2.4 and Theorem 5.2.15 are diffeomorphic.*

*Proof.* This is a consequence of the fact that a topological subspace of a manifold carries at most one differentiable structure turning the inclusion into a smooth embedding. □

It is straightforward to extend the discussion above to define the space of *leafwise $\mathfrak{S}$-regular maps*, as defined in Definition 5.2.5.

**Theorem 5.2.18.** *There is a convex subset $C^{const}(h^*N\mathcal{F}) \subset C^\infty(h^*N\mathcal{F})$, $C^{const}(h^*N\mathcal{F}) \cong \mathcal{T}$, such that the set of leafwise maps $\mathfrak{S}(M,\,\mathcal{F})$, with the quotient topology, can be given the structure of a smooth submanifold of $\mathfrak{S}(M,\,X)$, locally modeled on $C^{const}(h^*N\mathcal{F}) \times \mathfrak{S}(h^*T\mathcal{F})$. With this smooth structure, the quotient map is a submersion, and for each leaf $F$ of $\mathcal{F}$ the manifold $\mathfrak{S}(M,\,F)$ is a submanifold of $\mathfrak{S}(M,\,\mathcal{F})$.*

*Proof.* The proof is totally analogous to the one of Theorem 5.2.15 and Proposition 5.2.16, exploiting the fact that the transition maps in Theorem 5.2.15 are induced by smooth maps at the vector bundle level, hence induce smooth maps at the level of $\mathfrak{S}$-regular sections Lemma 5.1.2. The only things that need to be checked is that the topologies are the right ones, i.e., that the quotient topology agrees with the subspace topology, and that the sections $S_{\mathcal{T},\,p'}$ are topological embeddings (the latter fact is a consequence of the first).
Equivalently, one can show that the projection $t : \mathfrak{S}(M,\,s^{-1}) \longrightarrow \mathfrak{S}(M,\,\mathcal{F})$, with the topology defined in Section 5.2.4, is a quotient map. In order to do so, we will show that it is continuous and open. This follows from the fact that in local coordinates around $h$ and a lift $\overline{h}$, the map

$$t : C^{\text{const}}(\overline{h}^* N s^{-1}) \times \mathfrak{S}(\overline{h}^* T s^{-1}) \longrightarrow C^{\text{const}}(h^*N\mathcal{F}) \times \mathfrak{S}(h^*T\mathcal{F})$$

is a submersion between finite dimensional spaces on the first components, and an isomorphism on the second components, hence continuous and open. □

### 5.2.6 Boundary constraints

The above construction can be performed including a boundary constraint for the domain. More precisely, one can consider a submanifold $Y \subset X$, such that $X \pitchfork \mathcal{F}$, and consider the space of maps

$$\mathfrak{S}(M, \mathcal{F})^Y := \{h : M \longrightarrow X \,:\, h \in \mathfrak{S}(M, \mathcal{F}),\, h(\partial M) \subset Y\} \tag{5.4}$$

Such maps are locally parametrized, around a smooth map $h$, by

$$C^{\text{const}} \times \mathfrak{S}(h^*T\mathcal{F};\, h^*(TY \cap T\mathcal{F})) \tag{5.5}$$

where

$$\mathfrak{S}(h^*T\mathcal{F};\, h^*(TY \cap T\mathcal{F})) := \{s \in \mathfrak{S}(h^*T\mathcal{F}) \,:\, s|_{\partial M} \in C^0(h|^*_{\partial M}(TY \cap T\mathcal{F}))\} \tag{5.6}$$

This is a closed subspace of $\mathfrak{S}(h^*T\mathcal{F})$, which implies that $\mathfrak{S}(M, \mathcal{F})^Y$ with the subspace topology inherits the structure of a submanifold of $\mathfrak{S}(M, \mathcal{F})$. This would be useful when considering holomorphic curves with constrained boundary, where the boundary is mapped to a transversal with Lagrangian (or totally real) intersection with each leaf.

## 5.3 The Cauchy-Riemann operator

Let now $(X, \mathcal{F}, \omega)$ be a symplectic foliation. We would like to consider spaces of pseudo-holomorphic curves with values in the leaves of $\mathcal{F}$. In order to do this, consider a smooth $\omega$–compatible complex structure $J$ on the bundle $T\mathcal{F} \longrightarrow X$, turning the leaves of $\mathcal{F}$ into almost complex manifolds. In this context, it makes sense to ask for a map $u : \Sigma \longrightarrow X$ with values in a leaf to be $J$-holomorphic. This amounts to the vanishing of the Cauchy-Riemann operator $\overline{\partial}_J$. One can formulate this in the following global fashion.

Let $u$ be a map as above, and let $W^{k,p}$, $kp > 2$ be the manifold model of Sobolev sections with $k$ $p$–integrable derivatives. Consider the space $\mathcal{E}_u := W^{k-1,p}(\bigwedge^{0,1}(T\Sigma) \otimes u^*T\mathcal{F})$. We know from the previous discussion that the space $W_0^{k,p}(\Sigma, \mathcal{F})$ of Sobolev maps with trivial holonomy is a smooth Banach submanifold of the manifold $W^{k,p}(\Sigma, X)$ of all Sobolev maps. The bundle of Banach spaces $\pi : \mathcal{E} := \coprod_u \mathcal{E}_u \longrightarrow W_0^{k,p}(\Sigma, \mathcal{F})$ is in fact a Banach space bundle, as can be deduced from Theorem 6.1 in [12] (the key point in loc. cit. is that the sections defining the charts for the base manifold are exactly those that preserve $T\mathcal{F}$).
The Cauchy-Riemann operator is a (smooth) section of this bundle (whenever $J$ is smooth). Letting $u$ be a solution of the Cauchy-Riemann equation, we can naturally define the vertical differential of $\overline{\partial}_J$ at $u$. This can be seen as a map $D_u := d_u^V(\overline{\partial}_J) : C^{\text{const}}(u^*N\mathcal{F}) \times W^{k,p}(u^*TF) \longrightarrow W^{k-1,p}(\bigwedge^{0,1}(T\Sigma) \otimes u^*T\mathcal{F})$.

### 5.3.1 The linearized operator

We compute here the linear operator $D_u := d_u^V(\overline{\partial}_J)$ (the vertical derivative of the section $\overline{\partial}_J$), for a given complex structure $J$, at a solution $u$. This means that for $u : \Sigma \longrightarrow X$ a leafwise holomorphic curve with trivial holonomy, and $\eta \in T_u W^{k,p}(\Sigma, \mathcal{F}) \cong C^{\text{const}}(u^*N\mathcal{F}) \times \mathfrak{S}(u^*T\mathcal{F})$, we want to take a path $u_\lambda$ such that $u_0 = u$, and $\frac{d}{d\lambda}(u_\lambda)\big|_{\lambda=0} = \eta$. The operator $D_u(\overline{\partial}_J)$ applied to $\eta$ is $D_u(\overline{\partial}_J)(\eta) = \frac{d}{d\lambda}(\overline{\partial}_J(u_\lambda))\big|_{\lambda=0}$.

**Notation 5.3.1.** Before giving the statement, we introduce the following notation. Given a map $u : \Sigma \longrightarrow X$, we denote by $F_u$ the leaf of $\mathcal{F}$ such that $\text{im}(u) \subset F_u$. For a complex structure $J$ on $T\mathcal{F}$ and a leaf $F$, we denote with $J_F$ the restriction of $J$ to $F$. In particular one has $\overline{\partial}_J|_{W^{k,p}(\Sigma,\, F)} = \overline{\partial}_{J_F}$. We denote with $D_{u,F_u}$ the linearization of the operator $\overline{\partial}_{J_F}$.

Let $\mathcal{T}$ be a transversal through $u(\Sigma)$, let $\nabla^N = \nabla^{Ns^{-1}}$ be a connection on the normal bundle to $s^{-1}$ in $s^{-1}(\mathcal{T})$ such that the elements of $C^{\text{const}}(u^*N\mathcal{F})$ are flat with respect to the pull-back connection.

**Proposition 5.3.2.** *The linearized Cauchy-Riemann operator $D_u$ at $u$ is given by*

$$D_u(\eta) = D_{u,F_u}\eta^{\mathcal{F}} + \nabla^N_{\eta^N} J(u) du \circ j = (\nabla\eta^{\mathcal{F}})^{0,1} + \nabla_\eta J du \circ j \tag{5.7}$$

*Proof.* Consider a lift $\overline{u}$ of $u$ to $s^{-1}(\mathcal{T}) \subset \text{Hol}(\mathcal{F})$, for some suitable choice of transversal $\mathcal{T}$. Lift the complex structure $J$ (along $t$)to $\overline{J}$; then $\overline{u}$ is $\overline{J}$-holomorphic. Since the smooth structure can be defined by lifting maps to $s^{-1}(\mathcal{T})$ (Section 5.2.5), the linearization of $\overline{\partial}_J$ at $u$ coincides with the linearization of $\overline{\partial}_{\overline{J}}$ at $\overline{u}$. A neighbourhood $U$ of $\text{im}(u)$ is isomorphic to a product $\mathbb{D}^q \times K$ ($K$ is a compact set in a fiber), and $s$ is the projection. Take a connection $\nabla$ on $s^{-1}(\mathcal{T})$ such that:

- $\nabla|_U = \begin{pmatrix} \nabla^{Ns^{-1}} & \\ & \nabla^{\mathcal{F}} \end{pmatrix}$
- $\eta \in C^{\text{const}}(u^*N\mathcal{F})$ implies $\nabla^N\eta = 0$
- $\nabla^{\mathcal{F}}$ is the pull-back of a connection on $T\mathcal{F}$

Let $\eta \in T_0C^{\text{const}}(u^*N\mathcal{F}) \times \mathfrak{S}(u^*T\mathcal{F})$, and let $u_\lambda$ be a 1-parameter family of maps such that $u_0 = u$, $\frac{d}{d\lambda}\overline{u}_\lambda|_{\lambda=0} = \eta$. Assume first that $\eta = 0 + \eta_{\mathcal{F}}$. Then

$$D_u\eta = D_u\eta^{\mathcal{F}} = D_{u,\,F_u}\eta^{\mathcal{F}} = \nabla^{\mathcal{F}}\eta^{\mathcal{F}} + J\nabla^{\mathcal{F}}\eta^{\mathcal{F}} \circ j + (\nabla^{\mathcal{F}}_\eta J du \circ j)$$

($j$ is the complex structure on $\Sigma$) ([41], Section 2.4).

Assume now that $\eta = \eta^N + 0_{\mathcal{F}}$. Then

$$\frac{d}{d\lambda}\overline{\partial}_J(u_\lambda)(\partial_t) = \frac{d}{d\lambda}(du_\lambda + J(u_\lambda)du_\lambda j)(\partial_t) = \nabla_t\eta + \nabla_\eta J du + J(u)\nabla_s\eta = \nabla^N_\eta J du$$

(as $\nabla^N\eta^N = 0$ by our choice of connection) □

**Corollary 5.3.3.** *For each $\eta^N \in C^{const}(u^*N\mathcal{F})$, the operator $D_u^{\mathcal{F}}(\eta^{\mathcal{F}}) = D_u(\eta^N + \eta^{\mathcal{F}})$ is a real-linear Cauchy-Riemann operator.*

*Proof.* A real-linear Cauchy-Riemann operator is by definition (see [28], Definition C.1.5) a complex-antilinear perturbation of a complex-linear Cauchy-Riemann operator. The operator $D_{u,\,F_u}$ is a real-linear Cauchy-Riemann operator (by the general theory) and the term $\nabla^N_{\eta^N} J(u) du \circ j$ is complex antilinear, because $du$ is complex linear and $\nabla^{\mathcal{F}}(J^2) = 0$ implies that $\nabla^{\mathcal{F}}_\eta J$ is $J$-antilinear for each $\eta$. □

Let now $\Sigma$ be a closed surface.

**Corollary 5.3.4.** *The linearized Cauchy-Riemann operator is Fredholm, with index $ind(u) = (n-q)\chi(\Sigma) + 2c_1([u]) + q$.*

*Proof.* Since $D_{u,\,F_u} : \Gamma(u^*T\mathcal{F}) \longrightarrow \Omega^{0,1}(u^*T\mathcal{F})$ is Fredholm of index $(n-q)\chi(\Sigma)+2c_1([u])$ given by the Riemann-Roch formula, the same operator viewed as $D_{u,\,F_u} : C^{\text{const}}(u^*N\mathcal{F}) \times \Gamma(u^*T\mathcal{F}) \longrightarrow \Omega^{0,1}(u^*T\mathcal{F})$ is Fredholm of index $(n-q)\chi(\Sigma) + 2c_1([u]) + q$. Since $D_u(\eta) = D_{u,\,F_u}\eta^{\mathcal{F}} + \nabla^N_\eta J(u) du \circ j$, and the latter summand is of finite rank (hence compact), one has

$$\text{ind}(D_u) = \text{ind}(D_{u,\,F_u}) = (n-q)\chi(\Sigma) + 2c_1([u]) + q$$

□

**Corollary 5.3.5.** *If a closed holomorphic curve $u : \Sigma \longrightarrow X$ is Fredholm regular as a holomorphic curve $u : \Sigma \longrightarrow F_u$, then it is Fredholm regular as a leafwise holomorphic curve.*

*Proof.* One only needs to notice that $u^*T\mathcal{F} = u^*TF_u$, and look at Proposition 5.3.2. □

Let us also mention a formula for the linearization of the Cauchy-Riemann operator with varying complex structure. Such operator is the map

$$\overline{\partial} : (u,\, J) \mapsto \overline{\partial}_J u = \frac{1}{2}(du + Jduj) \tag{5.8}$$

which we can view as a section of the vector bundle

$$\pi : \mathcal{E} \longrightarrow \mathcal{J}(\mathcal{F}) \times W^{k,p}_0(\Sigma,\, \mathcal{F})$$

The space $\mathcal{J}(\mathcal{F})$ is embedded in the linear space of all endomorphisms $\text{End}(T\mathcal{F})$ of $T\mathcal{F}$, and the map $\overline{\partial}_J u$, extended to $\text{End}(T\mathcal{F})$ is linear. Hence it can be identified with its differential. This proves the following proposition.

**Proposition 5.3.6.** *The linearized Cauchy-Riemann operator with varying $J$ is the operator*

$$D_{(u,\,J)} : T_u W^{k,p}(\Sigma,\, \mathcal{F}) \times T_J(\mathcal{J}^k(\mathcal{F})) \longrightarrow W^{k-1,p}(\bigwedge^{0,1}_J T^*\Sigma \otimes u^*T\mathcal{F}) \tag{5.9}$$

*given by*

$$D_{(u,\,J)}(\eta,\,Y) = D_u\eta + \frac{1}{2}Y du j = D_{u,\,F_u}\eta^{\mathcal{F}} + \nabla^N_{\eta^N} J(u) du \circ j + \frac{1}{2} Y du j \quad (5.10)$$

Finally, since we want to construct a smooth structure on a space of holomorphic curves where the domain complex structure is allowed to vary, we should consider the full linearization of the Cauchy-Riemann operator, taking into account also variations of $j$. This is not straightforward, as the space of equivalence classes of complex structures on a Riemann surface is not a manifold, but a (finite dimensional) orbifold. A solution to this is to consider certain orbifold charts given by *Teichmüller slices*, and make computations on such charts (explained below). Eventually, we will construct a smooth structure on the space of somewhere injective curves, and such curves have no automorphisms – this will prevent the resulting moduli space from having singularities.

Since there is no difference in this respect between the foliated case and the usual symplectic case, we will only briefly sketch a description of the setup, and we refer to [41] for an account. The set of holomorphic structures on a smooth oriented surface $\Sigma$ is the quotient $\mathcal{J}(\Sigma)/\mathrm{Diff}_+(\Sigma)$ of all the complex structures modulo the orientation preserving diffeomorphisms. It turns out that the identity component $\mathrm{Diff}_0(\Sigma)$ of $\mathrm{Diff}_+(\Sigma)$ acts freely and properly whenever $g \geq 2$ (more generally, if one considered marked points, one would want that the stability condition $\chi(\Sigma \setminus \{p_1, \ldots, p_m\}) < 0$ to be satisfied). The quotient $\mathcal{J}(\Sigma)/\mathrm{Diff}_0(\Sigma) = \mathrm{Teich}(\Sigma)$ is a finite dimensional smooth manifold.
A smooth structure on $\mathcal{J}(\Sigma)/\mathrm{Diff}_0$ is given by constructing certain slices of the quotient maps, called *Teichmüller slices*, which are smooth subsets of the space of all complex structures on $T\Sigma$.

The smooth structure is invariant under different choices of slices. The space of holomorphic structures, then, is locally a discrete quotient of $\mathrm{Teich}(\Sigma)$, which in fact can be shown to be locally a finite quotient. In particular, the set of holomorphic structures on $\Sigma$ is locally a finite quotient of the Teichmüller slices.
Note that in the case $g < 2$ (the unstable case) one can describe explicitly the space of complex structures and the stabilizers of the action of $\mathrm{Diff}_+(\Sigma)$ on $\mathcal{J}(\Sigma)$. Thus the whole discussion goes through with minor modifications (which include eventually quotienting out the moduli space of holomorphic curves by the remaining automorphisms of the domain).

The same discussion that lead to the proof of Proposition 5.3.6 can be used to prove the following.

**Proposition 5.3.7.** *Let $\mathcal{S}_{T,j}$ be a Teichmüller slice around a complex structure $j$. Let $T_j\mathcal{S}_{T,j}$ be its tangent space. The linearization of the Cauchy-Riemann operator, with complex structure on $\Sigma$ varying in $\mathcal{S}_{T,j}$, is an operator*

$$D_{(u,\,j,\,J)} : T_u W^{k,p}(\Sigma,\,\mathcal{F}) \times T_j\mathcal{S}_{T,j} \times T_J(\mathcal{J}^k(\mathcal{F})) \longrightarrow W^{k-1,p}(\textstyle\bigwedge_{j,J}^{0,1} T^*\Sigma \otimes u^*T\mathcal{F})) \quad (5.11)$$

*of the form*

$$D_{(u,\, j,\, J)}(\eta,\, y,\, Y) = D_{(u,\, J)}(\eta,\, Y) + Jdu \circ y \tag{5.12}$$

## 5.4 Generic transversality

In this section we show that the linearized operator is surjective at solutions for generic choice of compatible complex structure $J$ on $T\mathcal{F}$. The proof is the same as the one for holomorphic curves in symplectic manifolds, as described in [28]. We provide a proof for completeness. Denote with $\mathcal{J}(\mathcal{F})$ the set of tame or compatible almost complex structures on $T\mathcal{F}$. We start by constructing a *universal moduli space* (of simply covered curves), i.e. we want to construct a manifold structure on

$$\mathcal{M}^*(\mathcal{F}) := \{(u,\, j,\, J) : \overline{\partial}_{j,J} u := \frac{1}{2}(du + Jdu \circ j) = 0,\, u \text{ simply covered}\} \subset C_0^\infty(\Sigma,\, \mathcal{F}) \times \mathcal{M}_g \times \mathcal{J}(\mathcal{F}) \tag{5.13}$$

For functional analytic reasons we will consider the set $\mathcal{J}^k(\mathcal{F})$ of $k$-times differentiable $J$-holomorphic curves, with respect to $C^k$ complex structures:

$$\mathcal{M}^*(\mathcal{F})^{(k)} := \{(u,\, j,\, J) : \overline{\partial}_{j,J} u = 0,\, u \text{ simply covered}\} \subset W_0^{k,p}(\Sigma,\, \mathcal{F}) \times \mathcal{M}_g \times \mathcal{J}^k(\mathcal{F}) \tag{5.14}$$

for $k$ sufficiently big. That is a consequence of the following theorem.

**Theorem 5.4.1.** *The linearized Cauchy-Riemann operator (5.10) is surjective at solutions, for $p > 2$.*

*Proof.* By Proposition 5.3.6, $D_{(u,\, J)}(\eta,\, Y) = D_u\eta + \frac{1}{2}Yduj$. Since $D_u$ is Fredholm, then $D_{(u,\, J)}$ has closed image.
In order to prove that it is surjective it is enough to show that the image is dense in $\mathcal{E}_{(u,\, J)} = W^{k-1,p}(\bigwedge^{0,1}(T\Sigma) \otimes u^*T\mathcal{F})$. If $k = 1$, then we need to show that there is no non-zero $\rho \in L^{p'}(\bigwedge^{0,1}(T\Sigma) \otimes u^*T\mathcal{F})$ such that $\langle \rho,\, D_u\eta\rangle = \langle \rho,\, Y(u)duj\rangle = 0$. Such $\rho$ would be a zero of the adjoints of the real Cauchy-Riemann operators defined in Corollary 5.3.3, and be of class $W^{1,\, p}$, and in particular continuous ([28], Proposition 3.1.10).
Let $z_0 \in \Sigma$ be an injective point, i.e. such that $du_{z_0} \neq 0$ and $u^{-1}(u(z_0)) = \{z_0\}$. In a neighbourhood of $z_0$ there is an endomorphism $Y$ such that $\langle \rho,\, Y(u(z_0))du_{z_0}j\rangle > 0$ (see the formula in Lemma 3.2.2 in [28], which can be applied locally). One can extend the endomorphism $Y$ to the entire manifold using a bump function. The function $\langle \rho,\, Y(u(z_0))du_{z_0}j\rangle$ thus constructed is non-negative, and in particular has strictly positive integral. Thus $\rho = 0$ on the dense set of injective points, which implies $\rho \equiv 0$ by continuity. The surjectivity for all $k$ follows from an elliptic bootstrapping argument. □

**Theorem 5.4.2.** *The components of $\mathcal{M}^*(\mathcal{F})^{(k)}$ are smooth Banach manifolds. Moreover, there exists a comeager subset $\mathcal{J}^k(\mathcal{F})^{reg} \subset \mathcal{J}(\mathcal{F})$ of $C^k$ complex* *structures such that for each* $J \in \mathcal{J}^k(\mathcal{F})^{reg}$*, each simple* $J$*-holomorphic* $u : \Sigma \longrightarrow X$ *is Fredholm regular, for* $k$ *sufficiently big.*

*Proof.* The fact that $\mathcal{M}^*(\mathcal{F})^{(k)}$ is smooth follows from the inverse function theorem, and Theorem 5.4.1. Consider the projection map $\pi_{\mathcal{J}} : W_0^{k,p}(\Sigma, \mathcal{F}) \times \mathcal{M}_g \times \mathcal{J}(\mathcal{F}) \longrightarrow \mathcal{J}(\mathcal{F})$ restricted to $\mathcal{M}^*(\mathcal{F})^{(k)}$. Its differential is the projection $(\eta, y, Y) \mapsto Y$, hence the kernel of $d\pi_{\mathcal{J}}|_{\mathcal{M}(\mathcal{F})}$ on $\pi_J^{-1}(J)$ coincides with $\ker D_{(u,j,J)}$. In particular, it is finite dimensional. By Lemma A.3.6 in [28], $d\pi_{\mathcal{J}}|_{\mathcal{M}^*(\mathcal{F})^{(k)}}$ and $D_{(u,J)}$ have isomorphic cokernel. In particular the latter is surjective exactly when the projection is surjective. By Sard-Smale theorem, this is the case for a comeager set of values (for $k$ sufficiently big. See Theorem A.5.1 in [28]). This proves the theorem. □

In particular one has the following corollary.

**Corollary 5.4.3.** *Given a homology class* $A$*, there is a comeager set* $\mathcal{J}^{reg}$ *of (smooth) complex structures such that for all* $J \in \mathcal{J}^{reg}$*, the space* $\mathcal{M}_g^*(A, J)$ *of simply covered* $J$*-holomorphic curves representing the homology class* $A$ *is a smooth manifold of dimension* $2(n-q-3)(2-2g) + 2c_1(A) + q$*.*

*Proof.* In order to prove this, one needs to extend Theorem 5.4.2 to the $C^\infty$ case. This is done using "Taubes's trick", as explained for example at the end of the proof of Theorem 3.1.6 (II) in [28]. The dimension count follows from the computation of the index of $D_{(u,j,J)}$. Such index is the number given in Corollary 5.3.4 plus the dimension of $\mathcal{M}_g$. Finally, in the unstable case one should subtract the dimension of the automorphism group to the dimension formula. □

**Remark 5.4.4.** Note that one can use Theorem 5.4.1 and the proof of Lemma 2.2.9 to prove that leafwise holomorphic curves are stable under deformations of complex structures on the bundle $T\mathcal{F}$.

The discussion above also proves the uniqueness of the moduli space up to cobordism.

**Theorem 5.4.5.** *Let* $J_0, J_1 \in \mathcal{J}^l(T\mathcal{F})^{reg}$*. Let* $\mathcal{J}_\lambda(J_0, J_1; T\mathcal{F})$ *be the set of paths of complex structures on* $T\mathcal{F}$*, joining* $J_0$ *and* $J_1$*. Given a homology class* $A$*, there is a comeager set of regular homotopies* $\mathcal{J}_\lambda^{reg}$ *(possibly depending on* $A$*), such that for each path* $J_\lambda \in \mathcal{J}_\lambda^{reg}$ *the set* $\mathcal{M}_g(A, J_\lambda)^*$ *of simple* $J_\lambda$*-holomorphic curves is a smooth manifold, with boundary* $\mathcal{M}_g(A, J_0)^* \sqcup \mathcal{M}_g(A, J_1)^*$*.*

*Proof.* Consider the manifold with boundary $[0, 1] \times X$, with the foliation $\tilde{F}$ given by $\{\star\} \times F$. Then $\mathcal{J}_\lambda(J_0, J_1; T\mathcal{F})$ is exactly the space of complex structures on $T\tilde{F}$, that coincide with $J_0$, $J_1$ on the boundary. For suitable regularity $\mathfrak{S}$, one has $\mathfrak{S}(\Sigma, \tilde{F}) = [0, 1] \times \mathfrak{S}(\Sigma, \mathcal{F})$, a manifold with boundary. One can then apply the same argument as in Theorem 5.4.1, Theorem 5.4.2 and the corollaries that follow. □

## 5.5 Compactness

In symplectic manifolds, the strength of holomorphic curve theory lies in the compactness properties of the moduli space. A compactness result for closed leafwise holomorphic curves analogous to the standard one, hold for symplectically foliated manifolds. The author is aware of such result from a talk by Klaus Niederkrüger [30]. The theorem is the following.

**Theorem 5.5.1.** *Let $(X, \mathcal{F}, J)$ be a manifold with an almost complex foliation, and let $h$ be an hermitian metric. Let $u_k$ be a sequence of leafwise $J$–holomorphic maps, with uniformly bounded $h$-area. Then there exists a subsequence of $u_k$ converging to a nodal $J$–holomorphic curve with values in a leaf of $\mathcal{F}$.*

Convergence needs to be interpreted in the sense of Gromov, see [28]. For the definition of nodal holomorphic curve we refer back to Chapter 2. In order to keep the exposition self-contained, we prove such result in a simplified case, which will be sufficient for proving our main result (Theorem 5.0.1).

### 5.5.1 A simpler case: stably complex normal bundle

We prove here the compactness theorem in case the normal bundle to the foliation is stably complex, i.e. there exists a number $h$ such that $N\mathcal{F} \oplus \mathbb{R}^h$ admits an almost complex structure.

**Theorem 5.5.2.** *Let $(X, \mathcal{F}, J)$ be a closed manifold with an almost complex foliation. Assume that $N\mathcal{F}$ is stably complex. Let $u_k$ be a sequence of holomorphic curves with values in leaves of a foliation, with uniformly bounded energy with respect to some Hermitian metric $g_X$. Then there is a subsequence that converges to a stable nodal curve.*

*Proof.* The key observation is that if $N\mathcal{F} \oplus \mathbb{R}^h$ has an almost complex structure, then the manifold $X \times (S^1)^h$ is a closed almost complex manifold. One can identify $X$ with $X \times \{(1, \ldots, 1)\}$. Then $u_k$ can be viewed as a sequence in $X \times \{(1, \ldots, 1)\} \subset X \times (S^1)^h$ with bounded energy (with respect to a choice of split Hermitian metric extending $g_X$), which converges to a nodal curve by the standard Gromov compactness. The limit will again be a leafwise nodal curve in $X \times \{(1, \ldots, 1)\}$, by closedness. □

Let us point out some interesting examples to which Theorem 5.5.2 applies.

**Corollary 5.5.3.** *Let $(X, \mathcal{F}, J)$ be a closed manifold with an almost complex cooriented codimension–1 foliation. Then a sequence of leafwise holomorphic curves with bounded energy converges to a leafwise nodal curve.*

**Corollary 5.5.4.** *Let $X$ be an almost complex manifold, and $(\mathcal{F}, J)$ be an almost complex foliation on $X$, such that $T\mathcal{F}$ is a complex subbundle of $TX$. Then a sequence of leafwise holomorphic curves with bounded energy converges to a leafwise nodal curve.*

A class of manifold with a symplectic foliation and almost complex normal bundle is that of *regular generalized complex manifolds* ([15]).

**Corollary 5.5.5.** *Let* $(X, \mathcal{F}, \omega)$ *be a closed manifold with a symplectic foliation induced by a generalized complex structure. Let* $J$ *be an almost complex structure on* $\mathcal{F}$ *compatible with the symplectic form. Then a sequence of leafwise holomorphic curves with bounded energy converges to a leafwise nodal curve.*

### 5.5.2 Homological energy bounds

In symplectic geometry, the energy identity implies that the energy of a holomorphic curve only depends on its homology class. Thus one can get a uniform energy bound by considering holomorphic curves representing the same homology class. This is the case also in Poisson geometry, if we assume the manifold to be compact.

**Proposition 5.5.6.** *Let* $(X, \mathcal{F}, \omega)$ *be a manifold with a symplectic foliation, with compact leaf space. Let* $A \in H_2(X; \mathbb{Z})$ *be a homology class. Then there is a constant* $C(A) > 0$ *such that* $E(u) \leq C(A)$ *for each holomorphic* $u$ *such that* $[u] = A$.

*Proof.* Consider the space of leafwise maps $C^\infty(\Sigma, X, \mathcal{F}; A)$ representing the homology class $A$ (this is a smooth space, as it is a union of components of $C^\infty(\Sigma, X, \mathcal{F})$). Extend $\omega$ to a smooth form on $X$, and define a continuous map $I_\omega$ as

$$I_\omega(u) = \int_\Sigma u^*\omega$$

Since $\omega$ is leafwise closed, the value of $I_\omega(u)$ depends only on the leaf that $u$ is mapped to. Hence there is a diagram

$$\begin{array}{ccc} C^\infty(\Sigma, X, \mathcal{F}; A) & \xrightarrow{I_\omega} & \mathbb{R} \\ \downarrow & \nearrow_{\overline{I}_\omega} & \\ X/\mathcal{F} & & \end{array} \tag{5.15}$$

where the map $\overline{I}_\omega$ is continuous, by definition of quotient topology. The assumption that $X/\mathcal{F}$ is compact ensures that $I_\omega$ is bounded above. □

**Remark 5.5.7.** If $\omega$ could be extended to a *closed* form on $X$, then $I_\omega$ would be constant.

## 5.6 An application: foliations with a ruled leaf

We outline one application of the theory above, related to ruled surfaces. Essentially we will try to apply the arguments from [25], which we used extensively in Chapter 3, to the foliated case, using the compact(ifiable) moduli spaces just constructed.

We will consider a closed 5–manifold $X$, with a cooriented codimension–1 symplectic foliation $(\mathcal{F}, \omega)$. Then the leaves are symplectic 4–manifolds, though not necessarily compact. This situation is special for two reasons, the first is that given that the leaves are 4–dimensional one can apply the automatic transversality results from [17], and the adjunction formula. The second is that in codimension 1 it is easy to find bounds for the index of a holomorphic curve.

**Notation 5.6.1.** Let $(X, \mathcal{F}, J)$ be an almost complex foliation of codimension $q$ on a $n$-dimensional manifold. It is going to be useful to introduce the following terminology. Let $u : \Sigma \longrightarrow F \subset X$. The *index of $u$ in $X$*, is the index of $u$ as a leafwise map, i.e. the number

$$\mathrm{ind}_X(u) = \mathrm{ind}(u) = (n - q - 3)\chi(\Sigma) + 2c_1([u]) + q \tag{5.16}$$

while the *leafwise index of $u$* is the index of $u$ as a holomorphic map in the almost complex manifold $F$, i.e.

$$\mathrm{ind}_F(u) = (n - q - 3)\chi(\Sigma) + 2c_1([u]) = \mathrm{ind}(u) - q \tag{5.17}$$

We say that a holomorphic curves is *leafwise Fredholm regular* if it is Fredholm regular as a holomorphic curve $u : \Sigma \longrightarrow F_u$.

**Proposition 5.6.2.** *Let $(X, \mathcal{F}, J)$ be a manifold with a codimension–1 almost complex foliation. If $J$ is Fredholm regular, then there are no simple holomorphic curves with negative leafwise index.*

*Proof.* By Fredholm regularity of $J$, for each simple $u$ one has $\mathrm{ind}(u) \geq 0$, hence $\mathrm{ind}_F(u) \geq -1$. But the index of a holomorphic curve in an almost complex manifold is an even number, thus $\mathrm{ind}_F(u) \geq 0$. □

This can be strengthened if the foliation has rank 4, using the Riemann-Hurwitz formula, which states that, if $\varphi : \tilde{\Sigma} \longrightarrow \Sigma$ is a branched cover of degree $d$, then

$$-\chi(\tilde{\Sigma}) + d\chi(\Sigma) = Z(d\varphi) \tag{5.18}$$

where $Z(d\varphi)$ is the number of zeros of $d\varphi$, with multiplicity. The result is

**Proposition 5.6.3.** *Let $(X, \mathcal{F}, J)$ be a manifold with a codimension–1 almost complex foliation of rank 4. If $J$ is Fredholm regular, then there are no holomorphic curves with negative leafwise index.*

*Proof.* If $\tilde{u} = u \circ \varphi$ is a multiple cover of degree $d$ of a simple holomorphic curve $u$, one computes

$$\mathrm{ind}_F(\tilde{u}) = d\,\mathrm{ind}_F(u) - (2 - 3)Z(d\varphi) \geq 0 \tag{5.19}$$

□

Further, applying the automatic transversality Theorem 2.2.12 from [17], one gets

**Corollary 5.6.4.** *Let $(X, \mathcal{F}, J)$ be a manifold with a codimension–1 almost complex foliation of rank 4. If $J$ is Fredholm regular, then all immersed holomorphic spheres are leafwise Fredholm regular.*

**Remark 5.6.5.** One could also prove that the set of immersed spheres is dense, adapting a transversality theorem from [39], Appendix A (see also [40]).

**Proposition 5.6.6.** *Let $(X, \mathcal{F}, \omega)$ be a 5–dimensional manifold with a codimension–1 symplectic foliation. Let $A$ be a homology class represented by an embedded symplectic sphere. Then, for generic $J$, the moduli space $\mathcal{M}_0(A, J)^*$ of simple $J$–holomorphic spheres homologous to $A$ is a smooth 3–dimensional manifold, and all its elements have leafwise index 2. The compactification $\overline{\mathcal{M}}_0(A, J)$ consists of*

- *curves in $\mathcal{M}_0(A, J)^*$*
- *double covers of holomorphic spheres with leafwise index 0*
- *nodal curves with two irreducible components, both of leafwise index 0*

*Proof.* Since $A$ can be represented by a symplectic sphere, one can realize it (up to deformation) as a $J$–holomorphic sphere $u$, with $J$ Fredholm regular. By the adjunction formula, $\mathrm{ind}(u) = 3$. In particular the leafwise index of each curve in $\mathcal{M}_0(A, J)$ is 2. The index computations in [42], section 4.2, and in particular Lemma 4.12, directly imply the result. □

**Theorem 5.6.7.** *Let $(X, \mathcal{F}, \omega)$ be a 5–dimensional manifold with a codimension–1 symplectic foliation. Assume that there exists a leaf $F \subset X$ containing a symplectically embedded sphere $S$ with 0 self-intersection. Then*

- *for every point $p \in X$ there exists a non-constant holomorphic sphere passing through $p$*
- *for every leaf $F$ there exists a leafwise Fredholm regular holomorphic sphere $u : S^2 \longrightarrow F$ such that $ind(u) \geq 2$*

*Proof.* Let $A \in H_2(X)$ be the homology class representing the embedded sphere $S$ with 0 self-intersection. It is a standard fact that we can find a generic complex structure $J$ and an embedded holomorphic sphere $u$ in the homology class $A$. By Proposition 5.6.6, the moduli space $\mathcal{M}_{0,1}(A, J)^*$ is a (non-empty) 5–dimensional manifold. We claim that each point in $X$ belongs to the image of a holomorphic curve in $\overline{\mathcal{M}}_0(A, J)$. In order to prove this, it is enough to show that the set of points in $X$ belonging to the image of a curve in $\overline{\mathcal{M}}_0(A, J) \setminus \mathcal{M}_0(A, J)^*$ has codimension 2. Indeed, let $Y \subset X$ be the set of points that belong to the image of a curve in $\overline{\mathcal{M}}_0(A, J) \setminus \mathcal{M}_0(A, J)^*$. If $p \in Y$, in particular it belongs to the image of a holomorphic curve, and there is nothing to prove. If the codimension of $Y$ is 2, then each $p \in X \setminus Y$ can be joined to a point in $S$ via a path in $X \setminus Y$. Using the compactness of the moduli space over a neighbourhood of such path, one can prove that the evaluation map is surjective onto such neighbourhood (by a degree computation). This step is totally analogous to the strategy of [25], which we explained in Chapter 3. The resulting holomorphic curve through $p$ would belong to $\mathcal{M}_0(A, J)^*$, thus it would have index 2. In particular, leafwise Fredholm regular by Theorem 2.2.12.

In order to prove that the codimension of $Y$ is at least 2, we use Proposition 5.6.6. Indeed, it follows from Proposition 5.6.6 that $Y$ can be realized as a union of images of the evaluation map with domain a 3–dimensiona moduli space (if the leafwse index of a curve is 0, then the dimension of the 1–pointed moduli space is 3). If we can prove that there is only a finite number of cohomology classes in $H_2(X)$ that could be $J$–holomorphic summands of $A$, then we are done. This follows from the same argument as McDuff's Lemma 4.6 in [25]: a holomorphic summand of $A$ has area $\leq E(A) = \omega(A)$. If there were infinitely many summands of $A$ represented by a holomorphic curve, we would find a sequence with energy bounded by $E(A)$, which would admit a convergent subsequence by Theorem 5.5.2, and in particular the corresponding sequence of homology classes would converge. Since the set of homology classes is discrete, this is impossible. □

The following corollary is an immediate consequence of Theorem 5.6.7.

**Corollary 5.6.8.** *Let $(X, \mathcal{F}, \omega)$ be as above. Then for each leaf $F$ one has $\pi_2(F) \neq 0$, and $[\omega|_F] \neq 0 \in H^2(F)$.*

In case the leaves of the foliation are compact, one can apply the classification of rational and ruled closed symplectic 4–manifolds, and get the following strengthening of Theorem 5.6.7.

**Corollary 5.6.9.** *Let $(X, \mathcal{F}, \omega)$ be as above. Assume further that all leaves are compact. Then all leaves are rational or blown-up ruled surfaces.*

*Proof.* Since each leaf contains an index–2 holomorphic sphere, one can apply Theorem 7.3 in [40] (originally appeared in [24]) to conclude that each leaf is rational or a blow-up of a ruled surface. □

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