Deformation problems and L_∞-algebras of Fréchet type

Arjen Baarsma

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Introduction

The study of deformations problems is a recurrent theme in mathematics in general and in geometry in particular. This arguably goes back to Riemann, who counted the number of moduli of complex curves [Rie57]. Deformation theory as we think of it today really began when the theory of deformations of compact complex manifolds of arbitrary dimension was developed about a century later by Frölicher and Nijenhuis [FN57], Kodaira and Spencer [KS58a; KS58b] and Kuranishi [Kur62; Kur65]. The study of deformations of algebras and related structures was initiated by Gerstenhaber [Ger64; Ger66; Ger68] soon after this. Research on deformations and moduli spaces has led to great progress in differential and algebraic geometry, and provided a certain understanding of common features that these problems possess.

One such underlying feature is the relationship between deformation problems and differential graded Lie algebras. This connection came to the surface in the 1960's, when similarities were observed between deformations of complex structures and deformations of associative algebras [NR66]. As far as we are aware, however, the full expected relationship between deformations and algebraic objects was first spelled out by Deligne in a letter to Millson [Del86] in 1986:

> *"The philosophy, which I had not realized before reading your paper, seems the following: in characteristic 0, a deformation problem is controlled by a differential graded Lie algebra, with quasi-isomorphic DG Lie algebras giving the same deformation theory. If the DG Lie algebra controlling a problem is "formal", i.e. quasi-isomorphic to* $\bigoplus H^i$*, then the versal (formal) deformation space is that of* $\bigoplus H^i$*, i.e. the (completion at* 0 *of the) subscheme of* H^1 *defined by the equations* $[u,u] = 0$*."*

The differential graded Lie algebra associated to a particular deformation problem is not unique, even up to isomorphism. It was soon realised that morphisms should be considered "up to homotopy", and this led to the introduction of a new algebraic object that generalises differential graded Lie algebras. These were called "strongly homotopy Lie algebras", and are now usually referred to as L_∞-algebras. The correct notion of equivalence between L_∞-algebras that determines whether they describe the same deformations is quasi-isomorphism. Up to a a choice of grading and sign conventions, they are defined as follows.

Definition.

An L_∞-algebra is a $\mathbf{Z}$-graded vector space $V = \bigoplus_{k\in\mathbf{Z}} V^k$ endowed with a sequence $\ell = (\ell_p)_{p\in\mathbf{N}}$ of graded symmetric multi-linear maps

$$\ell_p\colon \underbrace{V \times \cdots \times V}_{p \text{ copies}} \longrightarrow V$$

of degree 1 subject to the Jacobi identities

$$\sum_{p+q=r} \sum_{\sigma\in S_r} \tfrac{1}{p!q!}\, \epsilon_\sigma\, \ell_{p+1}(\ell_q(v_{\sigma(1)}, \dots, v_{\sigma(q)}), v_{\sigma(q+1)}, \dots, v_{\sigma(r)}) = 0$$

for $r \in \mathbf{N}$, where ϵ_σ denotes the Koszul sign of the permutation $\sigma \in S_r$.

Every differential graded Lie algebra can be viewed as an L_∞-algebras for which only the unary and binary "brackets" ℓ_1 and ℓ_2 are non-trivial, and every L_∞-algebra is conversely quasi-isomorphic to a differential graded Lie algebra. There is however a cost associated to replacing L_∞-algebras by quasi-isomorphic differential graded Lie algebras: while the structure may be simpler, the underlying space is potentially much larger and not as meaningful from a geometric point of view.

Deligne's insight lives on, and in a simplified form reads:

> In characteristic 0, every deformation problem is controlled by a differential graded algebra (or an L_∞-algebra).

This statement was made rigorous and proven in a formal derived setting by Lurie [Lur11] and Pridham [Pri10].

Leaving aside what it means for an algebraic object to "control" a deformation problem, examples of deformations problems for which Deligne's claim holds are plentiful: deformations of Lie algebras, Lie subalgebras, complex structures, flat connections, holomorphic vector bundles, instantons and generalised complex structures are all “controlled” by differential graded Lie algebras or L_∞-algebras. Finding a differential graded Lie algebra or L_∞-algebra that controls a given deformation problem is generally not an easy task, however, and it is often done on a case-by-case basis [Gua04; OP05; FZ15a; FZ15b]. The main objective of this thesis is to demonstrate the relation between deformations and L_∞-algebras for a large class of deformation problems.

Of course, to achieve our aim we need to introduce an abstract definition of deformation problem that should encompass all of the aforementioned examples, and hopefully more. From our point of view, the bare minimum for a deformation problem, in a rather basic (and intentionally vague) setting, consists of three components:

- a space X of “almost structures”,
- two maps $\mu, o \colon X \to Y$ to another space Y,
- a groupoid $\mathcal{G} \rightrightarrows X$ for which $S = \{x \in X \mid \mu(x) = o(x)\} \subseteq X$ is invariant.

The actual space of structures is the set S, and two elements of this space are considered *equivalent* whenever they are in the same orbit for $G \rightrightarrows X$. In an imprecise sense, deformation theory is about describing a neighbourhood in the space of orbits of the restriction $G|_S \rightrightarrows S$, which is generally a singular quotient of a singular subset. One often has to settle for just a description of an infinitesimal neighbourhood, which is the domain of formal deformation theory.

Upon closer inspection, we see that for all of the examples mentioned above, the deformation problem has additional structure. This will be our starting point:

Definition.
An equivariant deformation problem $(M \xrightarrow{\mu} E^1 \xrightarrow{\beta} E^2) \overset{\alpha}{\circlearrowleft} G$ consists of the following objects:

- a manifold X,
- a (local) Lie group G acting on X,
- two equivariant vector bundles, $E^1 \to X$ and $E^2 \to X$, over X,
- an equivariant section $\mu \colon X \to E^1$, and

- an equivariant vector bundle morphism $\beta\colon E^1 \to E^2$ covering id_X such that $\beta \circ \mu = 0$.

A simplified and imprecise version of our main theorem now becomes:

Theorem 1.
For every equivariant deformation problem $(M \xrightarrow{\mu} E^1 \xrightarrow{\beta} E^2) \overset{\alpha}{\circlearrowleft} G$ *and any solution* $x_0 \in X$ *of the equation* $\mu(x) = 0$*, there is an associated* L_∞*-algebra (unique up to isomorphism) whose homogeneous components are*

$$V^{-1} = \mathrm{Lie}(G), \qquad V^0 = \mathrm{T}_{x_0} X, \qquad V^1 = E^1_{x_0}, \qquad V^2 = E^2_{x_0},$$

and whose brackets are determined by the formal power series expansions of the vertical components of the maps α*,* μ *and* β*.*

If μ *is analytic, then a neighbourhood of* x_0 *in the zero locus of* μ *is analytically diffeomorphic to a neighbourhood of* 0 *in the set of solutions to the Maurer–Cartan equation*

$$\sum_{p=0}^{\infty} \tfrac{1}{p!} \ell_p(v, v, \ldots, v) \qquad \text{for } v \in V^0.$$

Additionally, if G *is regular and the infinitesimal action of* Lie(G) *is analytic, then gauge equivalence of Maurer–Cartan elements within this neighbourhood implies equivalence of the corresponding structures.*

A more precise version of this assertion is provided by Theorem 5.2.4 in chapter 5, but it should be clear that both the underlying space and the brackets of the L_∞-algebra produced by the theorem are very concrete and that the proof can hence take the form of a simple, albeit long, computation. At the heart of this computation lies the fact that the Jacobi identities follow from equivariance and the power series expansion of the identity $\beta \circ \mu = 0$. On the one hand, a computational proof may be satisfactory to some, particularly since it identifies clearly which features of the equivariant deformation problem correspond to the Jacobi identities and also explains to those who are not familiar with L_∞-algebras where the Jacobi identities come from. On the other hand, one may feel that there should be a conceptual reason behind those computations which, once well understood, should circumvent them.

This is indeed the case. The price to be paid, however, is an increased level of abstraction. This is because the object that is most directly related to an equivariant deformation problem is not an L_∞-algebra, but a certain type of L_∞-algebroid, which is essentially a $\mathbf{Z}$-graded vector bundle whose space of sections comes with a L_∞-algebra structure that consists of multiderivations. As we will see in section 5.2, it is in fact possible to repackage the data of an equivariant deformation problem as an L_∞-algebroid without losing any relevant information or making further choices. Once this is established, Theorem 1 above follows from the more general relation between L_∞-algebroids and L_∞-algebras described in Theorem 5.1.1.

Returning to Theorem 1, two features stand out. Firstly, we see that if μ is not analytic, any one L_∞-algebra falls short of fully describing the original deformation problem and can only be said to control it formally. Secondly, since the Maurer–Cartan equation is actually a (potentially infinite) series, it is important that these L_∞-algebras come with a topology that will allow us to talk about convergence.

The type of deformation problems we have in mind concern differential geometric structures on smooth manifolds. These are often sections of some fibre bundle that solve a differential equation, and equivalence of such structures is usually described by the action of a Lie groupoid-like object. For this reason, we allow the space X, the bundles $E^1 \to X$ and $E^2 \to X$ and the Lie group G to be Fréchet manifolds and impose a notion of smoothness that is appropriate within this context. A significant portion of this thesis is therefore dedicated to the analysis required to make sense of and prove results that would have been simple if finite-dimensionality of the involved spaces had been assumed.

This document is organised as follows. We will first introduce the two objects featured in Theorem 1: equivariant deformation problems are discussed in chapter 1 and topological L_∞-algebras are the subject of chapter 2. Smooth, analytic and holonomic families of such L_∞-algebras are treated in chapter 3, and L_∞-algebroids of Fréchet type are introduced in chapter 4. The relations between all of the aforementioned objects will finally be stated and proven in chapter 5.

CHAPTER 1.

Equivariant deformation problems

Our objective is to show that equivariant deformation problems of Fréchet type give rise to L_∞-algebras that are also of Fréchet type. The first step we need to take to achieve this goal is to make precise what the objects in this statement are. This relatively short chapter is dedicated to the first class of objects: equivariant deformation problems of Fréchet type. L_∞-algebras of Fréchet type are discussed in chapter 2 and the assertion itself can be found in chapter 5.

It is difficult to rigorously define what a deformation problem is in a general context but the idea is generally the same: given a mathematical structure of a specific type one would like to know how this structure can be deformed parametrically to obtain different (inequivalent) structures of the same type. Making this statement precise amounts to endowing (a neighbourhood in) the set of equivalence classes of structures of the desired type with some kind of geometric structure of its own. Since the types of structures we have in mind are differential geometric in nature, we would like this structure to be differential geometric as well, so that we can work with families of structures that vary smoothly in some sense.

We shall describe the space of structures of the desired type as the space of solutions to some equation on a larger space of "almost structures", which we assume does come with a differentiable structure. We moreover assume that the relevant notion of equivalence for structures is described by a Lie group acting on this larger space or, more generally, by a Lie groupoid over it. Equivariant deformation problems are obtained by requiring that the set of structures is the zero locus of an equivariant section of an equivariant vector bundle. All of these spaces are allowed to be infinite-dimensional.

Abstract and equivariant deformation problems are defined in section 1.1, and these definitions are applied to some well-known examples of deformation problems in section 1.2. Infinitesimal deformations are considered in section 1.3, and the importance of the deformation complex is discussed there as well.

1.1. The framework

The purpose of this section is to introduce the concept of an equivariant deformation problem. We will first define what we have in mind when we speak of a deformation problem in the abstract, and then modify this definition to include equivariance. Abstracted versions of the Bianchi identity (as satisfied by the curvature of a linear connection) and infinitesimal Hamiltonian automorphism (as for Poisson manifolds) will be defined as well.

The spaces considered in this section are all assumed to be Fréchet manifolds, which are smooth manifolds modelled on a Fréchet space. Finite-dimensional manifolds and Banach manifolds are examples of Fréchet manifolds, so all claims made below apply to those as well. The type of spaces we have in mind, however, are the spaces of sections of smooth fibre bundles or surjective submersions with a compact base. Some of the basics of differential calculus on Fréchet spaces can be found in appendix B.1.1.

1.1.1. Abstract deformation problems

Since we are looking at the world through a differential geometric lens, we will describe deformation problems in terms of smooth maps between manifolds. A possible definition for an abstract deformation problem would then be the following.

Definition 1.1.1.
An *abstract deformation problem* $\mathcal{G} \overset{\sigma,\tau}{\rightrightarrows} X \overset{\mu,o}{\rightrightarrows} E^1$ of Fréchet type consists of

- a Fréchet manifold X,
- a fibre bundle $\pi\colon E^1 \to X$ of Fréchet type,
- two sections $\mu, o\colon X \to E^1$ of π, and
- a (local) Fréchet Lie groupoid $\mathcal{G} \overset{\sigma,\tau}{\rightrightarrows} X$

such that the equaliser $\mathrm{Eq}(\mu, o) = \{x \in X \mid \mu(x) = o(x)\} \subseteq X$ is $\mathcal{G}$-invariant.

Given an abstract deformation problem $\mathcal{G} \overset{\sigma,\tau}{\rightrightarrows} X \overset{\mu,o}{\rightrightarrows} E^1$ in the above sense, elements of X will be referred to as *almost structures*, and elements of the equaliser $\mathrm{Eq}(\mu, o)$ are called (integrable) *structures*. We call two structures $x, y \in \mathrm{Eq}(\mu, o)$ *gauge equivalent* whenever they are in the same orbit for $\mathcal{G} \overset{\sigma,\tau}{\rightrightarrows} X$, i.e. whenever there exists an element $g \in \mathcal{G}$ such that $\sigma(g) = x$ and $\tau(g) = y$. We shall refer to μ as the *Maurer–Cartan map* for reasons that will become apparent later on. Since Definition 1.1.1 does not include a reference point x_0, the term "moduli problem" might perhaps have been more appropriate.

We do not always need a Lie groupoid to describe equivalences, and a Lie group will often do just as well. One should then replace the groupoid $\mathcal{G}$ in Definition 1.1.1 by a Lie group G that acts on X through a map $\alpha\colon X \times G \to X$, and $\mathcal{G}$-invariance should then be replaced by G-invariance. This amounts to the special case of Definition 1.1.1 where $\mathcal{G}$ is the action groupoid $G \ltimes X \rightrightarrows X$ associated to this group action.

Given an abstract deformation problem $\mathcal{G} \overset{\sigma,\tau}{\rightrightarrows} X \overset{\mu,o}{\rightrightarrows} E^1$, the only thing we are really interested in is the space of (gauge) equivalence classes of structures, i.e. the orbit space for the restriction of $\mathcal{G}$ to $\mathrm{Eq}(\mu, o)$. This is generally a singular quotient of a singular subset. Even in a finite-dimensional context, the most we can say about this space without further assumptions is essentially that it is the leaf space of a singular foliation on a closed subset of a smooth manifold. If we additionally require that μ and o are transverse at x_0, then the closed subset becomes a submanifold at this point, and if we require triviality of the isotropy group $\mathcal{G}_{x_0}$, the singular foliation becomes regular around the point x_0.

The following definition is obtained by differentiating the Lie groupoid from Definition 1.1.1.

Definition 1.1.2.
An *abstract deformation problem* with infinitesimal symmetries $A \overset{\rho}{\Rightarrow} X \overset{\mu,o}{\rightrightarrows} E^1$ of Fréchet type consists of

- a Fréchet manifold X,
- a fibre bundle $\pi\colon E^1 \to X$ of Fréchet type,
- two sections $\mu, o\colon X \to E^1$ of π, and
- a Lie algebroid $A \overset{\rho}{\Rightarrow} X$ of Fréchet type

such that $\mathrm{im}(\rho_x) \subseteq \ker(\mathrm{T}_x\mu - \mathrm{T}_x o)$ for every $x \in \mathrm{Eq}(\mu, o)$.

At this point we run into a problem because Lie algebroid structures are usually only defined on finite-dimensional vector bundles. Although it is clear that the object produced from a Lie groupoid by differentiation should be a Lie algebroid, it is not immediately apparent how such objects should be defined axiomatically. An appropriate definition for such Lie algebroids is provided in Definition 4.2.20, but the details of this definition are not important for most of this chapter.

Given an abstract deformation problem $A \overset{\rho}{\Rightarrow} X \overset{\mu,o}{\rightrightarrows} E^1$ with infinitesimal symmetries, we consider two structures $x_0, x_1 \in \mathrm{Eq}(\mu, o)$ *gauge equivalent* whenever there exists an A-path that connects them. This is a smooth path $\Gamma\colon [0,1] \to A$ covering a curve $\gamma\colon [0,1] \to \mathrm{Eq}(\mu, o)$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$, and

$$\gamma'(t) = \rho \circ \Gamma(t)$$

for all $t \in [0,1]$. The requirement that $\gamma(t) \in \mathrm{Eq}(\mu, o)$ for all $t \in [0,1]$ is redundant in the finite-dimensional case, but it is not in general.

A special case of Definition 1.1.2 occurs when a Lie algebra $\mathfrak{g}$ can be used to describe when two structures are equivalent. One should then include an infinitesimal action of this Lie algebra on X, which is a smooth bundle map $\alpha\colon \mathfrak{g} \times X \to \mathrm{T}X$ that induces a morphism $\hat{\alpha}\colon \mathfrak{g} \to \mathfrak{X}(X)$ of Lie algebras to the lie algebra of vector fields on X. Two structures $x_0, x_1 \in \mathrm{Eq}(\mu, o)$ are then equivalent whenever there exists a pair (γ, η) of paths $\gamma\colon [0,1] \to X$ and $\eta\colon [0,1] \to \mathfrak{g}$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$ and $\gamma'(t) = \hat{\alpha}(\eta(t))$ for all $t \in [0,1]$.

We can add auxiliary data to a deformation problem in the form of a Bianchi identity and Hamiltonian symmetries. These resemble the Bianchi identity satisfied by the curvature of linear connections and the Hamiltonian vector fields on a symplectic (or Poisson) manifold respectively.

Definition 1.1.3.
A *Bianchi map* for an abstract deformation problem $\mathcal{G} \overset{\sigma,\tau}{\rightrightarrows} X \overset{\mu,o}{\rightrightarrows} E^1$ or $A \overset{\rho}{\Rightarrow} X \overset{\mu,o}{\rightrightarrows} E^1$ is a fibre bundle morphism $\beta\colon E^1 \to E^2$ that covers the identity on X and for which the equation

$$\beta \circ \mu = \beta \circ o$$

holds. We refer to this equation as the *Bianchi identity*.

We will write $\mathcal{G} \overset{\sigma,\tau}{\rightrightarrows} X \overset{\mu,o}{\rightrightarrows} E^1 \xrightarrow{\beta} E^2$ to indicate that an abstract deformation problem comes with a Bianchi identity.

Definition 1.1.4.
A *Hamiltonian map* for an abstract deformation problem $A \overset{\rho}{\Rightarrow} X \overset{\mu,o}{\rightrightarrows} E^1$ is a vector bundle morphism $\eta\colon E^{-2} \to A$ such that $\rho_x \circ \eta_x = 0$ for all $x \in \mathrm{Eq}(\mu, o)$. We refer to the image $\eta(E^{-2}) \subseteq A$ as *Hamiltonian symmetries*.

We will write $E^{-2} \xrightarrow{\eta} A \overset{\rho}{\Rightarrow} X \overset{\mu,o}{\rightrightarrows} E^1$ to indicate that an abstract deformation problem comes with Hamiltonian symmetries. If it comes with both a Bianchi identity and Hamiltonian symmetries, the notation $E^{-2} \xrightarrow{\eta} A \overset{\rho}{\Rightarrow} X \overset{\mu,o}{\rightrightarrows} E^1 \xrightarrow{\beta} E^2$ will be used.

The Bianchi map and the Hamiltonian map are both optional additions. It is however always possible to add them to an abstract deformation problem by using the trivial bundles $E^{-2} = 0$ and $E^2 = 0$ whose fibres consist of a single point and then setting $\beta = 0$ and $\eta = 0$.

A few concrete examples of abstract deformation problems will be provided in section 1.2. These examples all carry additional structure that is not yet present in Definitions 1.1.1, 1.1.3 and 1.1.4, which is why we will first introduce a more restrictive version of this definition in the following section.

1.1.2. Equivariant deformation problems

We will be working with abstract deformation problems that carry some additional structure. More specifically, we will require that both $E^1 \to X$ and $E^2 \to X$ are vector bundles and that the action extends linearly to these bundles. Moreover, o is assumed to be the zero section of $E^1 \to X$ and $\beta : X \to E^2$ is be assumed to be both linear and equivariant. This leads to the following definition.

Definition 1.1.5.
An *equivariant deformation problem* $(X \xrightarrow{\mu} E^1) \overset{\alpha}{\circlearrowleft} G$ of Fréchet type consist of

- a Fréchet manifold X,
- a Fréchet Lie group G acting on X by $\alpha\colon X \times G \to X$,
- an equivariant vector bundle $E^1 \to X$ of Fréchet type, and

- an equivariant section $\mu\colon X \to E^1$ of $E^1 \to X$.

The space of structures for an equivariant deformation problem $(X \xrightarrow{\mu} E^1)\overset{\alpha}{\circlearrowleft} G$ is the zero locus $\mu^{-1}(0) = \mathrm{Eq}(\mu, 0)$, and two structures $x, y \in \mu^{-1}$ are equivalent whenever there exists a group element $g \in G$ such that $y = x \cdot g$.

We can also add a Bianchi identity and Hamiltonian symmetries to an equivariant deformation problem, which will be required to be equivariant as well.

Definition 1.1.6.
An equivariant deformation problem with *Bianchi identity* is an equivariant deformation problem $(X \xrightarrow{\mu} E^1)\overset{\alpha}{\circlearrowleft} G$ that comes with another equivariant vector bundle $E^2 \to X$ and an equivariant base-preserving vector bundle morphism $\beta\colon E^1 \to E^2$ such that $\beta \circ \mu = 0$.

We will write $(X \xrightarrow{\mu} E^1 \xrightarrow{\beta} E^2)\overset{\alpha}{\circlearrowleft} G$ to indicate that an equivariant deformation problem comes with an equivariant Bianchi identity.

Definition 1.1.7.
An equivariant deformation problem with *Hamiltonian symmetries* is an equivariant deformation problem $(X \xrightarrow{\mu} E^1)\overset{\alpha}{\circlearrowleft} G$ that comes with an equivariant vector bundle $E^{-2} \to X$ and an equivariant map $\eta\colon E^{-2} \to \mathrm{Lie}(G)$ which is linear in the fibres of E^{-2} and is such that the image $\eta(E^{-2}_x)$ is contained in the isotropy Lie subalgebra $\mathrm{Lie}(G)^x$ for any $x \in \mu^{-1}(0)$.

It is worth noting that the Hamiltonian symmetries for any given structure $x \in \mu^{-1}(0)$ form a Lie subalgebra $\eta_x(E^{-2}_x) \leq \mathrm{Lie}(G)^x$ of the corresponding isotropy Lie algebra $\mathrm{Lie}(G)^x$. Equivariance of η can in fact be used to show that it is an ideal. One could additionally ask for the existence of an (equivariant) vector bundle morphism $\gamma\colon E^1 \otimes E^{-2} \to \mathrm{T}X$ such that $\alpha_*(\eta_x(h), x) = \gamma_x(\mu(x), h)$ for all $x \in X$ and every $h \in E^{-2}_x$. This provides an explicit expression for the failure of $\eta_x(h) \in \mathrm{Lie}(G)$ to preserve an almost structure $x \in X$ for which $\mu(x) \neq 0$.

As before, the inclusion of a Bianchi identity or Hamiltonian symmetries is optional, since one can always simply use the trivial vector bundles $E^2 = 0$ and $E^{-2} = 0$.

Remark 1.1.8.

The Lie groupoid from Definition 1.1.1 has been replaced by a Lie group partially for the sake of simplicity; Definition 1.1.5 works just as well if a Lie groupoid had been used instead. Definition 1.1.7, however, does not because it is not clear what the right notion of equivariance for η would be. Definition 1.1.5 and Definition 1.1.6 of course have analogues for which the group actions are replaced by the infinitesimal actions of a Lie algebra, or by that of a Lie algebroid. △

It should be clear that a lot of generality is lost by switching from abstract deformation problems to equivariant deformation problems. Of the extra conditions that are imposed, equivariance appears to be the most harmless since one would expect symmetries of the space of structures to also be symmetries of the deformation problem as a whole. This is not always the case, however, since it may be necessary to make choices that break this symmetry in order to obtain a good description for either μ or β.

Remark 1.1.9.

Linearity of β can always be enforced without meaningfully modifying the deformation problem. Suppose for instance that $(X \xrightarrow{\mu} E^1 \xrightarrow{\beta} E^2) \circlearrowleft G$ satisfies all of the axioms described in Definition 1.1.5, except that an additional equivariant section o of $E^1 \to X$ has been provided and $\beta\colon E^1 \to E^2$ is an equivariant fibre bundle morphism, rather than a vector bundle morphism, with the property that $\beta \circ \mu = \beta \circ o$. To get rid of this section o one can simply replace the section μ by $\tilde{\mu} = \mu - o$ and consider its zero locus instead of the equaliser $\mathrm{Eq}(\mu, o)$.

It is similarly possible to replace β by a vector bundle morphism $\tilde{\beta}\colon E^1 \to E^2$ which is still equivariant and satisfies $\tilde{\beta} \circ \tilde{\mu} = 0$ using a simple trick. We can define this vector bundle morphism by setting

$$\tilde{\beta}_x = \int_0^1 \mathrm{D}\beta_x(o(x) + t\,\tilde{\mu}(x))\,\mathrm{d}t \in \mathrm{L}(E^1_x, E^2_x)$$

for every $x \in X$. Because β, o and μ are all equivariant, the resulting vector bundle morphism is as well. It moreover follows from the fundamental theorem of calculus that

$$\tilde{\beta}_x(\tilde{\mu}(x)) = \int_0^1 \mathrm{D}\beta_x(o(x) + t\,\tilde{\mu}(x))(\tilde{\mu}(x))\,\mathrm{d}t = \beta_x(\mu(x)) - \beta_x(o(x)) = 0$$

for all $x \in X$. The deformation complex of a structure $x \in \mathrm{Eq}(\mu, o) = \tilde{\mu}^{-1}(0)$, which is described in Definition 1.3.1, is unaffected by these replacements because $\mathrm{T}^{\mathrm{v}}_x \tilde{\mu} = \mathrm{T}_x \mu - \mathrm{T}_x o$ and $\tilde{\beta}_x$ is equal to $\mathrm{T}_{o(x)} \beta_x$ whenever $\tilde{\mu}(x) = 0$. △

Equivariant deformation problems will be related to curved L_∞-algebroids and families of curved L_∞-algebras in section 5.2. It this context it will be important to know whether the deformation problem is analytic in the following sense.

Definition 1.1.10.
An equivariant deformation problem $(X \xrightarrow{\mu} E^1 \xrightarrow{\beta} E^2) \overset{\alpha}{\circlearrowleft} G$ with Bianchi identity of Fréchet type is *analytic* if X, E^1 and E^2 are analytic manifolds, μ and β are analytic maps, and the infinitesimal action $\alpha_*\colon \mathrm{Lie}(G) \times X \to \mathrm{T}X$ is analytic as well.

It should be noted that a Fréchet Lie group does not generally admit an analytic structure and that the fact that this was not assumed in Definition 1.1.10 is intentional. It was for instance shown in Corollary 9.2 of [Mil84] that the diffeomorphism group of any (non-discrete) compact manifold does not admit a compatible analytic structure. It is nevertheless perfectly reasonable to ask whether the infinitesimal action of its Lie algebra, which would be the Lie algebra of vector fields in this case, is analytic. Many deformation problems that arise in differential geometry are analytic, even if the underlying manifold is not.

1.1.3. Geometric deformation problems

As we have stated before, the types of deformation problems we have in mind describe the deformations of differential geometric structures on compact manifolds. This means that we expect the space of almost structures, X, as well as the Lie group G and the bundles $E^n \to X$ to be spaces of smooth maps between finite-dimensional manifolds. The purpose of this section is to illustrate how such deformation problems might arise in general.

Suppose we are interested in the deformations of a type of structure on a compact manifold M that can be described as sections of some finite-dimensional fibre bundle $\check{\pi}_0\colon \check{E}^0 \to M$ that solve a particular differential equation. We will moreover assume that equivalence of such structures is described by the action of the group of bisections of a finite-dimensional Lie groupoid $\check{\mathcal{G}} \overset{s,t}{\rightrightarrows} M$ whose action can be described using differential operators. More concretely, we might expect the deformation problem to be of the form described below.

- **Almost structures.**

  Since the structures we are interested in are all sections of $\check{\pi}_0\colon \check{E}^0 \to M$, so we could declare all sections of this bundle to be almost structures by setting $X = \Gamma(M;\check{E}^0)$. This is a smooth Fréchet manifold.

- **The Maurer–Cartan map.**

  Since structures are elements of $X = \Gamma(M;E^0)$ that solve a differential equation we might want to postulate the existence of a finite-dimensional vector bundle vector bundle $\check{\pi}_1\colon \check{E}^1 \to \check{E}^0$ over $\check{E}^0$ and a differential operator μ on $\check{E}^0 \to M$ with values in this bundle such that an almost structure $e_0 \in X$ is a structure precisely when $\mu(e_0) = 0$. We will describe what this means and how it would fit the framework below.

  We first of all note that the projection map $\check{\pi}_0 \circ \check{\pi}_1\colon \check{E}^1 \to M$ would describe a fibre bundle over M and that its space of sections, $\Gamma(M;\check{E}^1)$, is a smooth Fréchet manifold. We shall set $E^1 = \Gamma(M;\check{E}^1)$ and observe that it is the total space of a vector bundle $\pi_1\colon \Gamma(M;\check{E}^1) \to \Gamma(M;\check{E}^0)$ with projection map $\pi_1\colon e_1 \mapsto \check{\pi}_1 \circ e_1$ whose fibre at $e_0 \in \Gamma(M;\check{E}^0)$ is canonically isomorphic to the space $\Gamma(M;e_0^*\check{E}^1)$ of sections of the pullback of $\check{E}^1 \to \check{E}^0$ by $e_0\colon M \to \check{E}^0$.

  The aforementioned differential operator is a smooth map $\check{\mu}\colon \mathrm{J}^r_M\check{E}^0 \to \check{E}^1$ which is such that $\check{\mu}(\mathrm{j}^r_x e_0) \in \check{E}^1_{e_0(x)}$ for any local section e_0 of $\check{E}^0 \to M$ and can therefore also be described as a section of the pullback of $\check{E}^1 \to \check{E}^0$ by the canonical projection map $\mathrm{J}^r\check{E}^0 \to \check{E}^0$. It induces a smooth map

$$\mu\colon \Gamma(M;\check{E}^0) \longrightarrow \Gamma(M;\check{E}^1), \qquad e_0 \mapsto \check{\mu} \circ \mathrm{j}^r e_0$$

  which is a section for $\pi_1\colon \Gamma(M;\check{E}^1) \to \Gamma(M;\check{E}^0)$. We can thus use it as the Maurer–Cartan map in Definition 1.1.5.

- **Gauge equivalence.**

  To describe when two structures are isomorphic we might use a finite-dimensional Lie groupoid $\check{\mathcal{G}} \overset{s,t}{\rightrightarrows} M$. Its space of bisections, $G = \mathrm{Bis}(\check{\mathcal{G}})$, is a regular Fréchet Lie group and composition on this group is given by $(g \cdot h) = g \circ h$ for $g, h \in G$. We will assume that this groupoid acts on the bundles $\check{E}^0$ through the q-th jet groupoid $\mathrm{J}^r\check{\mathcal{G}}$, i.e. we assume that there exist a smooth map

$$\check{\alpha}^0 \colon \check{E}^0 \,{}_{\pi_2}\!\times_t \mathrm{J}^r\check{\mathcal{G}} \longrightarrow \check{E}^0$$

such that $\check{\alpha}^0(e_0, \mathrm{j}^r_{s(g)}g) \in \check{E}^n_{s(g)}$ and $\check{\alpha}^0(\check{\alpha}^0(e_0, \mathrm{j}^r_y g), \mathrm{j}^r_x h) = \check{\alpha}^n(e_n, \mathrm{j}^r_x(g \cdot h))$ whenever these equations make sense. This induces a Lie group action

$$\alpha^0 \colon \Gamma(M; E^0) \times \mathrm{Bis}(\check{\mathcal{G}}) \longrightarrow \Gamma(M; E^0), \qquad \alpha^0(e, g)(x) = \check{\alpha}^0(e_{t(g_x)}, \mathrm{j}^r_x g),$$

which we would require to be such that $\mu(e) = 0$ for a given $e \in X$ if and only if $\mu(\alpha^0(e, g)) = 0$ for all $g \in G$.

We obtain an equivariant deformation problem if $\mathrm{J}^r\mathcal{G}$ also acts on the bundle $\check{E}^1 \to \check{E}^0$ in a way that is compatible with the linear structure on its fibres and such that μ is equivariant for the corresponding group action.

By combining these elements we obtain an equivariant deformation problem

$$\Gamma(M; \check{E}^0) \xrightarrow{\mu} \Gamma(M; \check{E}^1) \circlearrowleft^{\alpha} \mathrm{Bis}(\check{\mathcal{G}}).$$

For structures that are somehow related to the tangent bundle of M, one would expect $\check{E}^0 \to M$ and $\check{E}^1 \to M$ to be natural bundles and $\mathrm{Bis}(\check{\mathcal{G}})$ to be the diffeomorphism group of M with its natural action on these bundles. This is the group of bisections of the pair groupoid. For structures that are related to some principal bundle it seems reasonable to expect $\check{E}^0$ and $\check{E}^1$ to be associated bundles and one might expect $\mathrm{Bis}(\check{\mathcal{G}})$ to be the group of automorphisms of this bundle. Depending on whether one requires these automorphism to be base-preserving, this can either be the group of bisections of the corresponding Atiyah groupoid or the space of sections of the bundle of automorphism of the fibres.

We can augment deformation problems of this type by adding a Bianchi identity or Hamiltonian symmetries.

- **Bianchi identity.**

  The differential operator $\check{\mu} \colon \check{E}^0 \to \check{E}^1$ might itself map sections of $\check{E}^0 \to M$ to solutions of another differential equation on $\check{E}^1 \to M$. This can be expressed using another finite-dimensional vector bundle $\check{E}^2 \to \check{E}^0$ and a linear differential operator

$$\check{\beta} \colon \mathrm{J}^r\check{E}^1 \longrightarrow \check{E}^2$$

that covers the identity on $\check{E}^0$. This induces a vector bundle morphism

$$\beta\colon \Gamma(M;\check{E}^1) \longrightarrow \Gamma(M;\check{E}^2), \qquad e_1 \mapsto \check{\beta}\circ \mathrm{j}^r e_1$$

from $E^1 \to X$ to $E^2 \to X$ if we set $E^2 = \Gamma(M;\check{E}^2)$. We should additionally require that the action of $\mathrm{J}^r\check{\mathcal{G}}$ extends to a linear action on $\check{E}^2 \to \check{E}^0$ for which β is equivariant.

- **Hamiltonian symmetries.**

  Hamiltonian symmetries can be described using a finite-dimensional vector bundle $\check{E}^{-2} \to \check{E}^0$ and another linear differential operator

  $$\check{\eta}\colon \mathrm{J}^r\check{E}^{-2} \longrightarrow \mathrm{Lie}(\check{\mathcal{G}})$$

  from $\check{E}^{-2} \to \check{E}^0$ to the Lie algebroid of $\check{\mathcal{G}}$ that covers the projection map $\check{\pi}_0\colon \check{E}^0 \to M$. It induces a smooth fibrewise linear map

  $$\eta\colon \Gamma(M;\check{E}^{-2}) \longrightarrow \mathrm{Lie}(\mathrm{Bis}(\check{\mathcal{G}})) = \Gamma(M;\mathrm{Lie}(\check{\mathcal{G}})), \qquad e_{-2} \mapsto \check{\eta}\circ e_{-2}$$

  from the vector bundle $E^{-2} = \Gamma(M;\check{E}^{-2}) \to X$ to the Lie algebra of G. This map should be such that the image $\eta(E^{-2}_{e_0})$ is contained in the stabiliser subalgebra $\mathrm{Lie}(G)^{e_0}$ whenever $\mu(e_0) = 0$. For an equivariant deformation problem we should moreover demand the existence of a linear action of $\mathrm{J}^r\check{\mathcal{G}}$ on $\check{E}^{-2} \to \check{E}^0$ for which η is equivariant.

There are many ways to weaken the assumptions presented in this section and still produce an equivariant deformation problem.

1.2. Concrete examples

To illustrate the definitions in section 1.1.1, a few concrete examples are provided in this section. In each case we will identify the space of almost structures, the Maurer–Cartan map and the appropriate notion of gauge equivalence, as well as the Bianchi identity and Hamiltonian symmetries where appropriate.

The deformation problems considered in this section are all well known and well understood. Apart from Einstein metrics, these are to some extent algebraic in nature and can be described cleanly and concisely using differential graded

Lie algebras; casting them into the form of Definition 1.1.5 does not really provide any additional insight. The point of these examples is thus not to argue that deformations of these structures should be described using equivariant deformation problems, but rather that they can be.

1.2.1. Deformations of Lie algebras

Lie algebra structures on a fixed vector space $\mathfrak{g}$ can be described very cleanly using a graded Lie algebra. This graded Lie algebra, $(V, \{\cdot,\cdot\})$, consists of the $\mathbf{Z}$-graded vector space $V = \bigoplus_{n\in\mathbf{Z}} V^n$ with homogeneous components

$$V^n = \mathrm{L}(\wedge^{n+1}(\mathfrak{g}), \mathfrak{g}),$$

and a graded Lie bracket $\{\cdot,\cdot\}$ of degree 0 given by

$$\{\varphi, \psi\} = \varphi \mathbin{\square} \psi - (-1)^{|\psi||\varphi|} \psi \mathbin{\square} \varphi$$

for homogeneous $\varphi, \psi \in V$. Here $\varphi \mathbin{\square} \psi \in V$ is given by

$$(\varphi \mathbin{\square} \psi)(a_1, \ldots, a_r) = \sum_{\sigma\in S_r} \mathrm{sgn}(\sigma)\, \tfrac{1}{q!(r-q)!}\, \varphi\big(\psi(a_{\sigma(1),\ldots,\sigma(q)}), a_{\sigma(q+1),\ldots,\sigma(r)}\big)$$

if $p = |\varphi|+1$, $q = |\psi|+1$ and $r = |\varphi|+|\psi|+1$, and $\psi \mathbin{\square} \varphi$ is defined analogously.

The spaces and maps from the deformation problem are readily identifiable. They form an analytic equivariant deformation problem of finite-dimensional type.

- **Almost structures.**

  We define an "almost Lie bracket" on $\mathfrak{g}$ as a skew-symmetric bilinear map $\theta\colon \wedge^2(\mathfrak{g}) \to \mathfrak{g}$ from $\mathfrak{g}$ to itself, i.e. as an element of V^1. We therefore set $X = \mathrm{L}(\wedge^2(\mathfrak{g}), \mathfrak{g})$.

- **Vector bundles.**

  We shall consider the trivial vector bundles

  $$E^n = \mathrm{L}(\wedge^{n+2}(\mathfrak{g}), \mathfrak{g}) \times X \longrightarrow X$$

for $n \in \mathbf{Z}$ with $n \geq -2$ and note that we can define a vector bundle morphism

$$\mathrm{ad}\colon E^n \longrightarrow E^{n+1}, \qquad \varphi \in E^n_\theta \mapsto \mathrm{ad}_\theta(\varphi) = \{\theta, \varphi\} \in E^{n+1}_\theta$$

for every such n.

- **The Maurer–Cartan map.**

  The commutator $\{\theta, \theta\}$ of an almost Lie bracket $\theta \in X$ with itself is given by

  $$\{\theta, \theta\}(a, b, c) = \theta(a, \theta(b, c)) + \theta(b, \theta(c, a)) + \theta(c, \theta(a, b))$$

  for $a, b, c \in \mathfrak{g}$, which is twice the Jacobiator of θ. If the Maurer–Cartan map is thus defined as the section

  $$\mu\colon X \longrightarrow E^1, \qquad \theta \mapsto \tfrac{1}{2}\{\theta, \theta\} = \mathrm{Jac}_\theta \in E^1_\theta,$$

  its zero locus will be precisely the space of Lie algebra structures on $\mathfrak{g}$.

- **The Bianchi identity.**

  A good candidate for the Bianchi map is the vector bundle morphism

  $$\beta\colon E^1 \longrightarrow E^2, \qquad \varphi = \mathrm{ad} \in E^1_\theta \mapsto \{\theta, \varphi\} \in E^2_\theta.$$

  It satisfies $\beta \circ \mu = 0$ because

  $$\beta_\theta(\mu(\theta)) = \{\theta, \tfrac{1}{2}\{\theta, \theta\}\} = 0 \in E^2_\theta$$

  for any $\theta \in X = \mathrm{L}(\wedge^2(\mathfrak{g}), \mathfrak{g})$ since $\{\cdot, \cdot\}$ itself satisfies the Jacobi identity.

- **Gauge equivalence.**

  The gauge group $G = \mathrm{GL}(\mathfrak{g})$ acts on the space X through the map

  $$\alpha\colon X \times G \longrightarrow X, \qquad \alpha(\theta, g) = g^*\theta\colon (a, b) \mapsto g^{-1}\big(\theta(g(a), g(b))\big).$$

  This action can be extended to any trivial vector bundle of the form $E^n = \mathrm{L}(\wedge^{n+2}(\mathfrak{g}), \mathfrak{g}) \times X \to X$ by sending a form $\varphi \in E^n_\theta$ to

  $$g^*\varphi \in E^n_{g^*\theta}\colon (a_1, \ldots, a_{n+2}) \mapsto g^{-1}\big(\theta(g(a_1), \ldots, g(a_{n+2}))\big).$$

The Lie algebra of G is $\mathrm{End}(\mathfrak{g}) = \mathrm{L}(\Lambda^1(\mathfrak{g}), \mathfrak{g})$. By differentiating α we obtain an infinitesimal action $\alpha_*\colon \mathrm{Lie}(G) \to \mathrm{C}^\infty(X, \mathrm{T}X)$ given by

$$\alpha_*(A)(\theta) = \{\theta, A\}\colon (a,b) \mapsto \theta(A(a), b) + \theta(a, A(b)) - A(\theta(a,b))$$

for $A \in \mathrm{Lie}(G)$ and $\theta \in X$. The infinitesimal isotropy of a Lie bracket $\theta \in \mu^{-1}(0)$ is the Lie subalgebra $\mathrm{Der}(\theta) \leq \mathrm{Lie}(G)$ which consists of all derivations for that bracket.

- **Hamiltonian symmetries.**

  We can use inner derivations as infinitesimal Hamiltonian symmetries by defining the vector bundle morphism

$$\eta = \mathrm{ad}\colon E^{-2} \longrightarrow \mathrm{End}(\mathfrak{g}), \qquad a \in E^{-2}_\theta \mapsto \{\theta, a\} \in \mathrm{End}(\mathfrak{g}).$$

  The failure of $\eta_\theta(a)$ to be a derivation if θ is not a Lie bracket is described by μ and the (multilinear) vector bundle morphism

$$\gamma\colon E^1 \otimes E^{-2} \longrightarrow \mathrm{T}X, \qquad \varphi \otimes \psi \in E^1_\theta \otimes E^{-2}_\theta \mapsto \{\varphi, \psi\} \in \mathrm{T}_\theta X,$$

  which is such that $\alpha_*(\eta_\theta(a))(\theta) = \{\theta, \{\theta, a\}\}$ is equal to $\gamma(\mu(\theta), a) = \frac{1}{2}\{\{\theta,\theta\}, a\}$ for all $\theta \in X$ and all $a \in E^{-2}_\theta = \mathfrak{g}$.

One can readily verify that μ, β, η and γ are all equivariant.

1.2.2. Deformations of symplectic structures

Deformations of symplectic structures on a compact manifold M are relatively simple to describe in terms of the De Rham complex,

$$\Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \xrightarrow{\mathrm{d}} \Omega^3(M) \xrightarrow{\mathrm{d}} \Omega^4(M).$$

They are described by an analytic equivariant deformation problem.

- **Almost structures.**

  An almost symplectic structure on M is simply a non-degenerate 2-form on M, so we define the space of almost symplectic structures on M as

  $$X = \{\omega \in \Omega^2(M) \mid i_V \omega \neq 0 \text{ for every } V \in \mathrm{T}M \setminus \{0\}\}.$$

  This is an open subset of the space of all smooth 2-forms on M.

- **Vector bundles.**

  For any $n \in \mathbf{Z}$ with $n \geq -2$ we can consider the trivial vector bundle

  $$E^n = \Omega^{n+2}(M) \times X \longrightarrow X$$

  and the vector bundle morphism $\mathrm{d}\colon E^n \to E^{n+1}$ that maps $\varphi \in E^n_\omega$ to $\mathrm{d}\varphi \in E^{n+1}_\omega$.

- **The Maurer–Cartan map.**

  We define the Maurer–Cartan map as the section

  $$\mu\colon X \longrightarrow E^1, \qquad \omega \mapsto \mathrm{d}\omega \in E^1_\omega.$$

  Its zero locus consists is precisely the set of symplectic structures on M.

- **The Bianchi identity.**

  The Bianchi map for this deformation problem is the vector bundle morphism

  $$\beta\colon E^1 \longrightarrow E^2, \qquad \varphi \in E^1_\omega \mapsto \mathrm{d}\varphi \in E^2_\omega.$$

  The Bianchi identity $\beta \circ \mu = 0$ now reads $\mathrm{d} \circ \mathrm{d}\omega = 0$ and thus expresses the fact that the exterior derivative squares to zero.

- **Gauge equivalence.**

  The gauge group for the deformations of symplectic structures is the diffeomorphism group $G = \mathrm{Diff}(M)$. A diffeomorphism $g\colon M \to M$ acts on a symplectic structure by pull-back, sending $\omega \in X$ to the form

  $$g^*\omega\colon \Lambda^2(\mathrm{T}M) \longrightarrow \mathbf{R}, \qquad V \wedge W \in \Lambda^2(\mathrm{T}_x M) \mapsto \omega_{f(x)}\big(\mathrm{T}_x f(V), \mathrm{T}_x f(W)\big)$$

  The group acts similarly on the bundles $E^n \to X$, and one can readily verify that $\mathrm{d}\colon E^n \to E^{n+1}$ is equivariant for this action.

The infinitesimal action of $\mathrm{Lie}(G) = \mathfrak{X}(M)$ on X is described by the map

$$\alpha_*\colon \mathrm{Lie}(G) \times X \longrightarrow \mathrm{T}X, \qquad (V, \omega) \mapsto \mathscr{L}_V \omega.$$

- **Hamiltonian symmetries.**

  Hamiltonian symmetries can be described using the maps

$$\eta\colon E^{-2} \longrightarrow \mathrm{Lie}(G), \qquad (h, \omega) \in E^{-2}_{\omega} \mapsto \omega^{-1}(\mathrm{d}f).$$

  and

$$\eta\colon E^{1} \otimes E^{-2} \longrightarrow \mathrm{T}X, \qquad (\varphi, h) \in E^{1}_{\omega} \otimes E^{-2}_{\omega} \mapsto i_{\omega^{-1}(\mathrm{d}h)}\varphi.$$

  Since $\mathscr{L}_{\omega^{-1}(\mathrm{d}h)}\omega = \mathrm{d}^2 h + i_{\omega^{-1}(\mathrm{d}h)}\mathrm{d}\omega$ and $\mathrm{d}^2 = 0$, these morphisms satisfy

$$\alpha_*(\eta_{\omega}(h), \omega) = \gamma(\mu(\omega), \eta_{\omega}(h))$$

  for any $\omega \in X$ and any $h \in E^{-2}_{\omega} = \mathrm{C}^{\infty}(M, \mathbf{R})$. One can readily verify that also η and γ are equivariant.

1.2.3. Deformations of complex structures

Deformations of complex structures on a compact manifold M also fit within the framework. They are described by an analytic equivariant deformation problem without Hamiltonian symmetries.

Almost complex structures on a compact manifold M can be described in many different ways, but the most commonly encountered description is probably as an endomorphism $J\colon \mathrm{T}M \to \mathrm{T}M$ of its tangent bundle such that $J^2 = -\mathrm{id}_{\mathrm{T}M}$. Any almost complex structure J on M gives rise to a decomposition $\mathrm{T}_{\mathbf{C}}M = \mathrm{T}^{1,0}_J M \oplus \mathrm{T}^{0,1}_J M$ of its complexified tangent bundle into $\pm i$-eigenspaces. It is integrable if the subbundle $\mathrm{T}^{0,1}_J M \leq \mathrm{T}_{\mathbf{C}}M$ is involutive, i.e. if the commutator $[V, W]$ of any two vector fields $V, W \in \Gamma(M; \mathrm{T}^{0,1}_J M)$ of type $(0,1)$ is itself a section of $\mathrm{T}^{0,1}_J M$. The almost complex structure can be integrated to a (unique) complex structure whenever this is the case.

The complexified cotangent bundle can be similarly decomposed as a direct sum $\mathrm{T}^*_{\mathbf{C}} = \mathrm{T}^{*1,0}_J M \oplus \mathrm{T}^{*0,1}_J M$, which allows us to consider the complex vector bundle

$\wedge^q(\mathrm{T}_J^{*0,1}M)\otimes(\mathrm{T}_J^{1,0}M)\to M$ for any $q\in\mathbf{N}_0$. We shall denote the space of smooth sections of this bundle by $\Omega_J^{0,q}(\mathrm{T}_J^{1,0}M)$ and note that one can define a differential operator $\bar\partial_J\colon\Omega_J^{0,q}(\mathrm{T}_J^{1,0}M)\to\Omega_J^{0,q+1}(\mathrm{T}_J^{1,0}M)$ which by setting

$$(\bar\partial_J\varphi)(V_0,\dots,V_q)=\sum_i(-1)^i\big[V_i,\varphi(\textstyle\bigotimes_{j\neq i}V_i)\big]_J^{1,0}$$
$$+\sum_{i<j}(-1)^{i+j}\varphi\big([V_i,V_j]\otimes\textstyle\bigotimes_{k\neq i,j}V_k\big)$$

for all $V_0,V_1,\dots,V_q\in\Gamma(M;\mathrm{T}_J^{0,1}M)$. Here we have written $V_J^{1,0}$ denotes the $(1,0)$-component of a complex vector field $V\in\Gamma(M;\mathrm{T}_{\mathbf{C}}M)=\mathrm{T}_J^{1,0}M\oplus\mathrm{T}_J^{0,1}M$. These form a sequence

$$\begin{array}{ccccccccc}
\Omega_J^{0,0}(\mathrm{T}_J^{1,0}M) & \xrightarrow{\bar\partial_J} & \Omega_J^{0,1}(\mathrm{T}_J^{1,0}M) & \xrightarrow{\bar\partial_J} & \Omega_J^{0,2}(\mathrm{T}_J^{1,0}M) & \xrightarrow{\bar\partial_J} & \Omega_J^{0,3}(\mathrm{T}_J^{1,0}M) & \xrightarrow{\bar\partial_J} & \cdots\\
\| & & \| & & \| & & \| & & \\
E_J^{-1} & & E_J^{0} & & E_J^{1} & & E_J^{2} & &
\end{array}$$

which is not generally a differential complex because $\bar\partial_J\circ\bar\partial_J$ is not guaranteed to vanish. The differential $\bar\partial_J$ does square to zero whenever J is integrable, however, and this sequence then coincides with the usual Dolbeault complex.

- **Almost structures.**

  Let $\mathcal{J}_x=\{J\in\mathrm{End}(\mathrm{T}_xM)\mid J^2=-\mathrm{id}_{\mathrm{T}_xM}\}$ denote the space of linear complex structures on T_xM for any $x\in M$. These together form a subbundle $\mathcal{J}\to M$ of the bundle $\mathrm{End}(\mathrm{T}M)\to M$ of endomorphisms of $\mathrm{T}M$ whose space of sections,

  $$X=\Gamma(M;\mathcal{J})\subseteq\Gamma(M;\mathrm{End}(\mathrm{T}M)),$$

  parametrises all almost complex structures on M.

- **Vector bundles.**

  By varying the almost complex structure $J\in X$ and considering the matching spaces $\Omega_J^{0,q}(\mathrm{T}_J^{1,0}M)$ we obtain a sequence of vector bundles

  $$E^n=\coprod_{J\in X}\Omega_J^{0,n+1}(\mathrm{T}_J^{1,0}M)\longrightarrow X,\qquad \varphi\in\Omega_J^{0,n}(\mathrm{T}_J^{1,0}M)\mapsto J$$

  connected by vector bundle morphisms $\bar\partial_J\colon E^n\to E^{n+1}$ that cover the identity on X.

- **The Maurer–Cartan map.**

  A complex structure on M is an almost complex structure that is integrable, i.e. one for which the subbundle $\mathrm{T}^{0,1}M \leq \mathrm{T}_{\mathbb{C}}M$ is involutive. This happens precisely when its Nijenhuis tensor $N_J \in \Omega_J^{0,2}(\mathrm{T}_J^{1,0}M)$ vanishes. This tensor is given by

  $$N_J(V,W) = [V,W]_J^{1,0}$$

  for $V, W \in \Gamma(M; \mathrm{T}^{0,1}M)$, where $V_J^{1,0}$ again denotes the $(1,0)$-component of a complex vector field V. We thus define the Maurer–Cartan map as the section

  $$\mu \colon X \longrightarrow E^1, \qquad J \mapsto N_J \in \Omega_J^{0,2}(\mathrm{T}_J^{1,0}M)$$

  so that $\mu^{-1}(0) \subseteq X$ becomes the space of complex structures.

- **The Bianchi identity.**

  The Nijenhuis tensor N_J of an almost complex structure J satisfies the identity $\bar{\partial}_J N_J = 0$, since one can work out that

  $$(\bar{\partial}_J N_J)(V_1, V_2, V_3) = \sum_{\sigma \in S_3} [V_{\sigma(1)}[V_{\sigma(2)}, V_{\sigma(3)}]]^{1,0} = 0$$

  for any three vector fields $V_1, V_2, V_3 \in \Gamma(M; \mathrm{T}_J^{0,1}M)$ of type $(0,1)$ because the commutator bracket on vector fields is a Lie bracket and hence satisfies the Jacobi identity. The vector bundle morphism

  $$\beta \colon E^1 \longrightarrow E^2, \qquad \varphi \in \Omega_J^{0,2}(\mathrm{T}_J^{1,0}M) \mapsto \bar{\partial}_J \varphi \in \Omega_J^{0,3}(\mathrm{T}_J^{1,0}M)$$

  is consequently such that $\beta \circ \mu = 0$.

- **Gauge equivalence.**

  Gauge equivalence in this setting is described by the diffeomorphism group $G = \mathrm{Diff}(M)$, which acts on X by pull-backs: a diffeomorphism $g \colon M \to M$ sends an almost complex structure J to the endomorphism $\mathrm{T}f^{-1} \circ J \circ \mathrm{T}f$. It acts similarly on the bundles $E^n \to X$, and one can readily verify that μ and $\bar{\partial}$ are both equivariant for this action.

1.2.4. Deformations of Einstein metrics

An Einstein metric on a compact manifold M is a Riemannian metric g whose Ricci curvature is given by $r_g = \lambda\, g$ for some number $\lambda \in \mathbf{R}$, which is called the Einstein constant. These are metrics that are critical points of the functional

$$S\colon \Gamma(M; \textstyle\bigodot^2(\mathrm{T}^*M)) \longrightarrow \mathbf{R}, \qquad g \mapsto \int_M s_g\, \mu_g,$$

where $s_g = \mathrm{tr}_g(r_g)$ denotes the scalar curvature of g and μ_g is the density (or measure) associated to g. We only consider Einstein metrics whose total volume $\mathrm{vol}_g(M) = \int_M \mu_g$ is equal to 1 since any positive scalar multiple of an Einstein metric is Einstein as well.

The deformation theory for Einstein metrics on compact manifolds was originally developed by Koiso [Koi78; Koi79; Koi80], and an overview of it can be found in chapter 12 of [Bes87]. This deformation problem is analytic and does not have Hamiltonian symmetries.

- **Almost structures.**

  An almost Einstein structure with volume 1 on a compact manifold M is just a Riemannian metric for which the total volume $\int_M \mu_g$ of M is equal to 1. The space of almost structures thus becomes

  $$X = \big\{ g \in \Gamma\big(M; \textstyle\bigodot^2(\mathrm{T}^*M)\big) \bigm| g > 0 \text{ and } \mathrm{vol}_g(M) = 1 \big\},$$

  where $\mathrm{vol}_g(M) = \int_M \mu_g$ is the total volume of g. This is a smooth codimension 1 submanifold of the space $\Gamma(M; \bigodot^2(\mathrm{T}^*M))$ of symmetric bilinear forms on $\mathrm{T}M$. Its tangent space at $g \in X$ is the space

  $$\mathrm{T}_g X = \big\{ h \in \Gamma(M; \textstyle\bigodot^2(\mathrm{T}^*M)) \bigm| \int_M \mathrm{tr}_g(h)\, \mu_g \big\}$$

  of all symmetric 2-tensors on M whose total trace relative to g is zero.

- **The Maurer–Cartan map.**

  By taking total traces on both sides of the equation $r_g = \lambda\, g$, one can work out that if g is an Einstein with total volume 1, the corresponding Einstein constant is given by $\lambda = \frac{1}{n}\int_M s_g\,\mu_g$. This means that g is Einstein if and only if $r_g - \frac{1}{n}\left(\int_M s_g\ u_g\right) g \in \mathrm{T}_g X$ vanishes.

One can moreover work out that the total trace of the 2-tensor $r_g - \frac{1}{n}\left(\int_M s_g\ u_g\right) g \in \mathrm{T}_g X$ vanishes for any metric g whose total volume is equal to one 1. This allows us to consider the Maurer–Cartan map

$$\mu\colon X \longrightarrow \mathrm{T}X, \qquad g \mapsto r_g - \tfrac{1}{n}\left(\textstyle\int_M s_g\ u_g\right) g \in \mathrm{T}_g X,$$

and view it as section of the tangent bundle $E^1 = \mathrm{T}X \to X$.

- **The Bianchi identity.**

  By taking appropriate traces of the Bianchi identity $\mathrm{d}^{\nabla^g} R^g = 0$ for the Levi-Civita connection ∇^g for g an identity for the Ricci tensor is obtained. This identity reads
  $$\delta_g r_g + \tfrac{1}{2}\,\mathrm{d}s_g = 0,$$
  where $\delta_g\colon \Gamma(M; \bigodot^2(\mathrm{T}^*M)) \to \Omega^1(M)$ denotes the divergence operator. This operator maps a symmetric 2-tensor h to the trace of the 3-tensor $-\nabla h$ with respect to the first two arguments and relative to g. It is also the formal adjoint of the operator $\delta_g^*\colon \xi \mapsto \nabla\xi \circ s$ that maps a 1-form to the symmetrisation of its covariant derivative.

  The Bianchi identity for the Ricci curvature can now be expressed using the Bianchi operator

  $$\beta\colon E^1 \longrightarrow \Omega^1(M), \qquad r \in E^1_g \mapsto \delta_g r + \tfrac{1}{2}\mathrm{d} \circ \mathrm{tr}_g(r),$$

  which we view as a vector bundle morphism from $E^1 = \mathrm{T}X$ to the trivial bundle $E^2 = \Omega^1(M) \times X \to X$. It satisfies the Bianchi identity $\beta \circ \mu = 0$ because $\beta_g(r_g) = \beta_g(g) = 0$.

- **Gauge equivalence.**

  Gauge equivalence for Einstein metrics is described by the diffeomorphism group $G = \mathrm{Diff}(M)$, which acts on the space X and the bundles $E^1 \to X$ and $E^2 \to X$ as on might expect: a diffeomorphism $f\colon M \to M$ sends a tensor $T \in \Gamma(M; \bigotimes^k(\mathrm{T}^*M))$ to the pullback f^*T. Both μ and β are equivariant for this action.

1.3. Infinitesimal deformations

The first step one should take when trying to understand an abstract deformation problem, as defined in section 1.1.1, is to consider its linearisation around a particular structure. This produces a linearised problem which is easier to solve and which may give a first glimpse of the nature of the original deformation problem. Of course, explicitly relating equivalence classes of solutions of the linearised problem to solutions of the original problem is something that can only be done under special circumstances.

Linearisation of an abstract deformation problem amounts to replacing all of the maps of which it consists by their derivatives at the structure one is interested in.

Definition 1.3.1 (Deformation complex)**.**
Let $E^{-2} \xrightarrow{\eta} A \overset{\rho}{\Rightarrow} X \overset{\mu,o}{\rightrightarrows} E^1 \xrightarrow{\beta} E^2$ be an abstract deformation problem with an infinitesimal action, an (optional) Bianchi identity and (optional) Hamiltonian symmetries. The *deformation complex* for a structure $x_0 \in \mathrm{Eq}(\mu, o)$ is the differential complex

$$\begin{array}{ccccccccc} E^{-2}_{x_0} & \xrightarrow{\eta_{x_0}} & A_{x_0} & \xrightarrow{\rho_{x_0}} & \mathrm{T}_{x_0}X & \xrightarrow{\mathrm{T}_{x_0}\mu - \mathrm{T}_{x_0}o} & \mathrm{T}_{y_0}E^1_{x_0} & \xrightarrow{\mathrm{T}_{y_0}\beta_{x_0}} & \mathrm{T}_{z_0}E^2_{x_0}, \\ \| & & \| & & \| & & \| & & \| \\ V^{-2}_{x_0} & & V^{-1}_{x_0} & & V^0_{x_0} & & V^1_{x_0} & & V^2_{x_0} \end{array}$$

where $y_0 = o(x_0) \in E^1_{x_0}$ and $z_0 = \beta(y_0) \in E^2_{x_0}$. We denote all four differentials in this complex by d_{x_0} when there is no risk of ambiguity, and will write (V_{x_0}, d_{x_0}) to refer to the complex as a whole.

The deformation complex (V_{x_0}, d_{x_0}) associated to an equivariant deformation problem $(X \xrightarrow{\mu} E^1) \overset{\alpha}{\circlearrowleft} G$ with an equivariant Bianchi identity $\beta\colon E^1 \to E^2$ and equivariant Hamiltonian symmetries $\eta\colon E^{-2} \to \mathrm{Lie}(G)$ is given by

$$\begin{array}{ccccccccc} E^{-2}_{x_0} & \xrightarrow{\eta_{x_0}} & \mathrm{Lie}(G) & \xrightarrow{\alpha_{*x_0}} & \mathrm{T}_{x_0}X & \xrightarrow{\mathrm{T}^{\mathrm{v}}_{x_0}\mu} & E^1_{x_0} & \xrightarrow{\beta_{x_0}} & E^2_{x_0}, \\ \| & & \| & & \| & & \| & & \| \\ V^{-2}_{x_0} & & V^{-1}_{x_0} & & V^0_{x_0} & & V^1_{x_0} & & V^2_{x_0} \end{array}$$

where α_* is the infinitesimal action of $\mathrm{Lie}(G)$ on X and $\mathrm{T}^{\mathrm{v}}_{x_0}\mu$ denotes the vertical component of the derivative $\mathrm{T}_{x_0}\mu\colon \mathrm{T}_{x_0}X \to \mathrm{T}_0 E^1_{x_0} \simeq E^1_{x_0} \oplus \mathrm{T}_{x_0}X$. The Bianchi map β no longer needs to be differentiated since it is linear by assumption.

Given an abstract deformation problem $E^{-2} \xrightarrow{\eta} A \overset{\rho}{\Rightarrow} X \overset{\mu,o}{\rightrightarrows} E^1 \xrightarrow{\beta} E^2$ with Bianchi identity and Hamiltonian symmetries, the differential complex (V_{x_0}, d_{x_0}) associated to a structure $x_0 \in \mu^{-1}(0)$ provides three meaningful cohomology groups. These are

$$\mathrm{H}^n(V_{x_0}, d_{x_0}) = \frac{\ker\left(d_{x_0} : V_{x_0}^n \to V_{x_0}^{n+1}\right)}{\mathrm{im}\left(d_{x_0} : V_{x_0}^{n-1} \to V_{x_0}^n\right)}$$

for $n = -1, 0, 1$. If no Bianchi identity is included then $\mathrm{H}^1(V_{x_0}, d_{x_0})$ is simply the cokernel of $\mathrm{T}_{x_0}\mu - \mathrm{T}_{x_0}o$, and $\mathrm{H}^{-1}(V_{x_0}, d_{x_0})$ is the isotropy Lie algebra $\ker(\rho_{x_0}) \leq A_{x_0}$ if Hamiltonian symmetries are omitted.

The middle cohomology group, $\mathrm{H}^0(V_{x_0}, d_{x_0})$, parametrises infinitesimal deformations (up to infinitesimal gauge equivalence), which one may want to try to extend to actual deformations. It is what we would ideally expect the tangent space of $\mathrm{Eq}(\mu, o) / \mathcal{G}$ at $[x_0]$ to be if this space were somehow nicely behaved. The upper cohomology group, $\mathrm{H}^1(V_{x_0}, d_{x_0})$, functions as an obstruction space: if one wants to extend an infinitesimal deformation to a formal family of deformations (i.e., the formal power series expansion of a curve in $\mathrm{Eq}(\mu, o)$) one order at a time, one will encounter obstructions that live in this space. The cohomology group $\mathrm{H}^{-1}(V_{x_0}, d_{x_0})$ is not considered as often, but it is describes infinitesimal automorphisms of the structure x_0 that may not carry over to nearby structures, which would cause the space $\mathrm{Eq}(\mu, o) / \mathcal{G}$ to pick up quotient singularities.

Example 1.3.2 (Lie algebras).

Deformations of Lie algebra structures are described in section 1.2.1. One can readily work out that the deformation complex associated to a given Lie bracket $\theta \in X = \mathrm{L}(\wedge^2(\mathfrak{g}), \mathfrak{g})$ is given by

$$\mathfrak{g} \xrightarrow{\{\theta, \cdot\}} \mathrm{L}(\mathfrak{g}, \mathfrak{g}) \xrightarrow{\{\theta, \cdot\}} \mathrm{L}(\wedge^2(\mathfrak{g}), \mathfrak{g}) \xrightarrow{\{\theta, \cdot\}} \mathrm{L}(\wedge^3(\mathfrak{g}), \mathfrak{g}) \xrightarrow{\{\theta, \cdot\}} \mathrm{L}(\wedge^4(\mathfrak{g}), \mathfrak{g}),$$

which is part of a larger algebraic object, namely the differential graded Lie algebra $\left(V, \{\theta, \cdot\}, \{\cdot, \cdot\}\right)$ with $V = \bigoplus_{n \in \mathbf{Z}} V^n$ and $V^n = \mathrm{L}(\wedge^{n+1}(\mathfrak{g}), \mathfrak{g})$.

It is interesting to observe that the deformation complex above is not generally a differential graded Lie subalgebra and does not inherit a natural bracket operation. It does if Hamiltonian symmetries are omitted since one can then describe it as the quotient $\bigoplus_{n \geq 0} V^n / \bigoplus_{n \geq 5} V^n$ of the differential subalgebra $\bigoplus_{n \geq 0} V^n \leq V$ by the differential ideal $\bigoplus_{n \geq 5} V^n \leq \bigoplus_{n \geq 0} V^n$. ◊

Example 1.3.3 (Complex structures).
Deformations of complex structures were described in section 1.2.3. The relevant deformation complex consists of the first four terms of the Dolbeault differential complex

$$0 \longrightarrow \Omega_J^{0,0}(\mathrm{T}_J^{1,0}M) \xrightarrow{\bar\partial} \Omega_J^{0,1}(\mathrm{T}_J^{1,0}M) \xrightarrow{\bar\partial} \Omega_J^{0,2}(\mathrm{T}_J^{1,0}M) \xrightarrow{\bar\partial} \Omega_J^{0,3}(\mathrm{T}_J^{1,0}M).$$

To obtain this complex we have had to identify $\mathrm{Lie}(G) = \mathfrak{X}(M)$ with the space $\Omega_J^{0,0}(\mathrm{T}_J^{1,0}M)$ of vector fields of type $(1,0)$ using the projection map $-\pi^{1,0}\colon X \mapsto -X^{1,0}$ of $\mathrm{T}_{\mathbf{C}}M = \mathrm{T}^{1,0}M \oplus \mathrm{T}^{0,1}M$ onto $\mathrm{T}^{1,0}M$. We have similarly used that the projection map

$$\Gamma(M; \mathrm{End}(\mathrm{T}_{\mathbf{C}}M)) \longrightarrow \Omega_J^{0,1}(\mathrm{T}_J^{1,0}M), \qquad A \mapsto \tfrac{i}{2}\pi^{1,0} \circ A|_{\mathrm{T}^{0,1}M},$$

restricts to an isomorphism from T_JX to $\Omega_J^{0,1}(\mathrm{T}_J^{1,0}M)$ to identify these two spaces. It should be noted that the first identification is not compatible with the natural Lie algebra structures on $\mathfrak{X}(M)$ and $\Omega^{0,0}(\mathrm{T}^{1,0}M)$. ◊

Example 1.3.4 (Einstein metrics).
Deformations of volume 1 Einstein metrics on a compact manifold M were described in section 1.2.4. If we choose a fixed Einstein metric $g \in X$ and differentiate the infinitesimal action and the Maurer–Cartan map at this metric, we obtain the differential complex

$$0 \longrightarrow \Omega^1(M) \xrightarrow{-\delta_g^*} \mathrm{T}_gX \xrightarrow{\mu_g'} \mathrm{T}_gX \xrightarrow{\delta_g + \frac{1}{2}\mathrm{d}\,\mathrm{tr}_g} \Omega^1(M).$$

Here, the metric g was used to identify the Lie algebra $\mathrm{Lie}(G) = \mathfrak{X}(M)$ with the space $\Omega^1(M)$ of 1-forms on M, and T_gX denotes the set of symmetric 2-tensors for which the total trace $\int_M \mathrm{tr}_g(h)\,\mu_g$ is zero. The operator δ_g^* maps a 1-form ξ to the symmetrisation $\nabla\xi \circ s$ of its covariant derivative, δ_g is the divergence operator and μ_g' denotes the restriction of the second-order differential operator

$$\mu_g'\colon h \mapsto \tfrac{1}{2}\nabla^{g*}\nabla^g h - \delta_g^*\delta_g h - \tfrac{1}{2}\nabla \mathrm{d}\,\mathrm{tr}_g(h) - \mathring{R}^g(h)$$

to $\mathrm{T}_gX \subseteq \Gamma(M; \bigodot^2(\mathrm{T}^*M))$. Here $\mathring{R}^g(h)$ is the symmetric 2-tensor which is given by $\mathring{R}^g(h)(X,Y) = \mathrm{tr}_g(h(R^g(X,\cdot)Y,\cdot))$ for $X, Y \in \mathrm{T}_xM$ and $x \in M$. More detailed descriptions of these operators can be found in [Bes87].

It is shown in [Koi80] that the obstruction space $\mathrm{H}^1(V_{x_0}, d_{x_0})$ is canonically isomorphic to the space $\mathrm{H}^0(V_{x_0}, d_{x_0})$ of infinitesimal deformations. This is unfortunate because it means that obstruction space can only vanish if the structure itself is infinitesimally rigid. ◊

The importance of these cohomology groups is illustrated by the following proposition, which applies to abstract deformation problems of finite-dimensional type.

Proposition 1.3.5.
Let $E^{-2} \xrightarrow{\eta} A \overset{\rho}{\Rightarrow} X \overset{\mu,o}{\rightrightarrows} E^1 \xrightarrow{\beta} E^2$ be an abstract deformation problem with infinitesimal symmetries, an (optional) Bianchi identity and (optional) Hamiltonian symmetries and let (V_{x_0}, d_{x_0}) denote the deformation complex associated to a structure $x_0 \in \mu^{-1}(0)$. Assume moreover that E^{-2}, A, X, E^1 and E^2 are all finite-dimensional and that we are given a decomposition $V^0_{x_0} = B \oplus H \oplus B'$ with $B = \mathrm{im}(d^{-1}_{x_0})$ and $B \oplus H = \ker(d^0_{x_0})$. There exists a smooth chart

$$\psi \colon U \subseteq X \longrightarrow V^0_{x_0} = B \oplus H \oplus B',$$

defined on a neighbourhood $U \subseteq X$ of x_0, for which the following hold.

0. *The derivative $\mathrm{T}\psi(x_0)$ of ψ at x_0 is the identity map on $V^0_{x_0}$.*

1. *The equaliser of μ and o is given by*

$$\mathrm{Eq}(\mu, o) \cap U = \psi^{-1}\big(B \times K \times \{0\}\big)$$

for some subset $K \subseteq H \simeq \mathrm{H}^0(V_{x_0}, d_{x_0})$. This subset contains 0, and it is a zero neighbourhood if $\mathrm{H}^1(V_{x_0}, d_{x_0}) = 0$.

2. *Each orbit of $A|_U$ which is contained in $\mathrm{Eq}(\mu, o)$ is of the form*

$$[x] = \psi^{-1}\big(B \times A_x \times \{0\}\big)$$

for some immersed submanifold $A_x \subseteq H$. Specifically, $A_{x_0} = \{0\}$, and if $\mathrm{H}^{-1}(V_{x_0}, d_{x_0}) = 0$ then the neighbourhood U can be chosen such that A_x consists of a single point for every $x \in \mathrm{Eq}(\mu, o) \cap U$.

The proof of Proposition 1.3.5 is not complicated and is very similar to that of Theorem 4.3.12 in chapter 4, as well as to that of Proposition 4.5 in [CSS14]. It will not be provided here in full, but the argument can essentially be divided into two steps. First, one will need to use the integrability of the singular distribution $\rho(A) \subseteq \mathrm{T}X$ spanned by the Lie algebroid. This gives rise to a local decomposition of the associated singular foliation as the product of a singular foliation on a transversal to $\rho(A_{x_0})$ and a neighbourhood in the leaf through x_0. This goes back to [Her62] (for integrability) and [Ste74a] (for the local form), and proofs of slightly stronger statement that specifically apply to Lie algebroids can be found in e.g. [Duf01], [Fer02] or [BLM16]. Afterwards, a Lyapunov–Schmidt reduction can be applied to show that the intersection of the zero locus of μ with this transversal is a subset of a submanifold of dimension $\mathrm{H}^0(V_{x_0}, d_{x_0})$, and that this subset it itself the zero locus of a smooth map with values in $\mathrm{H}^1(V_{x_0}, d_{x_0})$. If the deformation problem is analytic then ψ can be chosen analytic as well.

The embedding $\psi^{-1}|_K \colon K \subseteq \mathrm{H}^0(V_{x_0}, d_{x_0}) \hookrightarrow X$ constructed in Proposition 1.3.5 is called a Kuranishi family for the deformations of the structure $x_0 \in \mathrm{Eq}(\mu, o)$. The assertions of this proposition describe some properties of this family and of the structure x_0.

- **Local completeness.** For any sufficiently small neighbourhood $U \subseteq X$ of x_0, $\psi^{-1}(K) \cap U$ intersects every orbit of $A|_U$ that is contained in $\mu^{-1}(0)$.
- **Unobstructedness.** If $\mathrm{H}^1(V_{x_0}, d_{x_0}) = 0$, then $\mu^{-1}(0) \cap U$ is a smooth submanifold of X for any sufficiently small neighbourhood $U \subseteq X$ of x_0.
- **Rigidity.** If $\mathrm{H}^0(V_{x_0}, d_{x_0}) = 0$, then the elements of $\mathrm{Eq}(\mu, o) \cap U$ are mutually gauge equivalent for any sufficiently small neighbourhood $U \subseteq X$ of x_0.
- **Universality.** If $\mathrm{H}^{-1}(V_{x_0}, d_{x_0}) = 0$, then $\psi^{-1}(K) \cap U$ intersects every orbit of $A|_U$ exactly once for any sufficiently small neighbourhood $U \subseteq X$ of x_0.

The above terms are generally defined in using families of structures rather than using neighbourhoods in X.

Remark 1.3.6.

Proposition 1.3.5 still holds for abstract deformations of Banach type if the differential complex (V_{x_0}, d_{x_0}) is assumed to come with a continuous splitting, It generally fails for deformation problems of Fréchet type, however, mainly due to the failure of both the inverse function theorem and the existence and uniqueness theorem for ordinary differential equations in that context.

A notable class of exceptions is formed by the deformations of certain geometric structures on compact manifolds for which (part of) the deformation complex is elliptic. The same or similar statements can often be proven for these using Banach space techniques and elliptic regularity. Examples of deformation problems for which parts of Proposition 1.3.5 are known to hold (in some form) include:

- deformations of complex structures [Kur62; Kur65; Wav69; Kur69],
- deformations of symplectic structures [Mos65],
- deformations of Yang–Mills instantons [AHS78; DK90],
- deformations of Einstein metrics [Koi78; Koi79; Koi80; Bes87],
- deformations of metrics with G_2 or Spin(7) as their holonomy group [Joy96a; Joy96b; Joy96c; Joy00],
- deformations of generalised complex structures [Gua04; Gua11].

This list is not exhaustive. Compactness of the underlying manifold is required in each case. △

CHAPTER 2.

L_∞-algebras

L_∞-algebras, or strongly homotopy Lie algebras, naturally and silently appeared in the work of several researchers in the early 80's in connections with deformation theory [Sta78; SS85; SS], and in physics in connection with the BRST formalism [Zwi93]. Their definition, in a form equivalent to what we use here, was given by Lada and Stasheff in [LS93].

A common way to a define an L_∞-algebra is as a $\mathbf{Z}$-graded vector space V endowed with collection $(\ell_p)_{p\in\mathbf{N}}$ of graded symmetric multilinear maps (called brackets),

$$\ell_p\colon \underbrace{V\times\cdots\times V}_{p\text{ copies}} \longrightarrow V,$$

that are homogeneous of degree 1 and satisfy a sequence of equations, which are called *Jacobi identities* due to their similarity to the usual Jacobi identity for binary brackets. A curved L_∞-algebra also includes a zeroth bracket, $\ell_0\colon \{*\}\to V$ that is referred to as its *curvature*.

When L_∞-algebras are used in the context of deformation theory, the *Maurer–Cartan equation*,

$$\sum_{p\in\mathbf{N}_0} \tfrac{1}{p!}\,\ell_p(u,\ldots,u) = 0, \qquad (u\in V^0)$$

and its solutions, the *Maurer–Cartan elements*, play a central role. Since the Maurer–Cartan equation is expressed as an infinite series, it is only well-defined if all but finitely many terms vanish, in which case it is a polynomial equation. In general, we require either a topology on V or some type of nilpotence to make sense of this equation.

Therefore one of the tasks we need to perform in this chapter is to introduce an appropriate notion of topological L_∞-algebras. With that at hand we can formalise in which context the Maurer–Cartan equation and gauge equivalence of Maurer–Cartan elements makes sense.

In this chapter we will cover first the algebraic aspects of L_∞-algebras and then bring topology into the mix. The algebraic aspects are well settled in the literature, so that part of the material is standard. In section 2.1 we introduce the symmetric algebra and in section 2.2 we introduce L_∞-algebras, first as a sequence of brackets, $(\ell_p)_{p\in\mathbf{N}_0}$, satisfying a collection of Jacobi identities and then as degree 1 coderivation on the symmetric coalgebra of V that squares to zero. In section 2.3, we introduce topological L_∞-algebras. We then proceed to introduce the Maurer–Cartan equation and the notion of equivalence of Maurer–Cartan elements in section 2.4.

2.1. The graded symmetric setting

Before defining what a (curved) L_∞-algebra is, we will take a moment to discuss the graded symmetric algebra of a graded vector space and its relation to graded symmetric multilinear maps. We will discuss the Koszul sign rule, which explains the abundant signs that appear in most equations involving L_∞-algebras, and conclude this section by introducing the graded symmetric coalgebra.

2.1.1. The graded symmetric algebra

Let $V = \bigoplus_{k\in\mathbf{Z}} V^k$ be a real $\mathbf{Z}$-graded vector space. The tensor algebra of V is the bigraded vector space

$$\bigotimes(V) = \bigoplus_{n\in\mathbf{Z}} \bigoplus_{p\in\mathbf{N}_0} \bigotimes^p(V)^n \qquad \text{with} \qquad \bigotimes^p(V)^n = \bigoplus_{n_1+\cdots+n_p=n} \bigotimes_{j=1}^p V^{n_j},$$

on which the tensor product $\otimes$ naturally describes unital algebra structure that is compatible with both gradings. From it, we can construct the *graded symmetric algebra* of V as a quotient,

$$\bigodot(V) = \bigotimes(V) \big/ \mathscr{I},$$

where $\mathscr{I} \leq \bigotimes(V)$ denotes the (graded) ideal generated by elements of the form $u \otimes v - (-1)^{|u||v|} v \otimes u$ for homogeneous $u, v \in V$.

This quotient is again a unital algebra. We denote the induced (associative) product on $\bigodot(V)$ by $\odot$ and write

$$u_1 \odot u_2 \odot \cdots \odot u_n := u_1 \otimes u_2 \otimes \cdots \otimes u_n + \mathscr{I} \in \bigodot(V)$$

for $u_1, u_2, \ldots, u_n \in V$. Note that $\bigodot(V)$ is spanned by elements of this form and that $u \odot v = (-1)^{|u||v|} v \odot u$ by construction. Given homogeneous elements $u_1, u_2, \ldots u_n$ and an (ordered) subset $I \subseteq \{1, 2, \ldots, n\}$, we let u_I denote the ordered product

$$u_I = \bigodot\nolimits_{i \in I} u_i = u_{i_1} \odot u_{i_2} \odot \cdots \odot u_{i_{\#I}},$$

where the indices $i_1, i_2, \ldots, i_{\#I} \in I$ are arranged such that $i_1 < i_2 < \cdots < i_{\#I}$ and $\#I$ denotes the cardinality of the finite set I. In particular, $u_{\{1,\ldots,n\}}$ shall denote the product $u_1 \odot u_2 \cdots \odot u_n$ and $u_\varnothing = 1 \in \bigodot^0(V) \simeq \mathbf{R}$ is the multiplicative identity element for $\bigodot(V)$.

Like the tensor algebra, the graded symmetric algebra $\bigodot(V)$ comes with two natural gradings: an $\mathbf{N}_0$-grading by (tensor) rank and a $\mathbf{Z}$-grading by (total) degree. These gradings are such that the product $u_1 \odot u_2 \odot \cdots \odot u_p$ of p homogeneous elements of V has rank $p \in \mathbf{N}_0$ and degree $\sum_{i=1}^{p} |u_i| \in \mathbf{Z}$. Accordingly, we can decompose $\bigodot(V)$ as

$$\bigodot(V) = \bigoplus_{n \in \mathbf{Z}} \bigoplus_{p \in \mathbf{N}_0} \bigodot\nolimits^p(V)^n \qquad \text{with} \qquad \bigodot\nolimits^p(V)^n = \bigoplus_{\substack{n_1 + \cdots + n_p = n \\ n_1 \leq \cdots \leq n_p}} \bigodot\nolimits_{j=1}^{p} V^{n_j},$$

where $\bigodot_{j=1}^{p} V^{n_j} := \bigotimes_{j=1}^{p} V^{n_j} \bmod \mathscr{I} \subseteq \bigodot(V)$. When referring to the degree of an element of $\bigodot(V)$, we shall always mean its degree with respect to the $\mathbf{Z}$-grading.

We observe that $\bigodot(V)$ is just the ordinary symmetric algebra if V is concentrated in even degrees, and that it coincides with the exterior algebra of V if this space is concentrated in odd degrees, albeit with an additional $\mathbf{Z}$-grading. It is in general isomorphic to $\bigodot(V_{\text{even}}) \otimes \bigodot(V_{\text{odd}})$ if $V = V_{\text{even}} \oplus V_{\text{odd}}$ and V_{even} and V_{odd} are concentrated in even and odd degrees respectively.

2.1.2. Graded symmetric multilinear maps

Graded symmetric multilinear maps will play an important role throughout this thesis. Given two $\mathbf{Z}$-graded vector spaces V and W, a graded symmetric multilinear map of *arity* $p \in \mathbf{N}_0$ (also called a graded symmetric p-linear map) from the former to the latter is a map

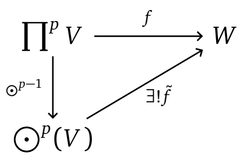
$$f\colon \prod^p V = \underbrace{V \times V \times \cdots \times V}_{p \text{ copies}} \longrightarrow W,$$

which is linear in each of its arguments and is graded symmetric in the sense that for all homogeneous $v_1, v_2, \dots, v_p \in V$ and any two consecutive indices i and $i+1$,

$$f(v_1, \dots, v_i, v_{i+1}, \dots, v_p) = (-1)^{|v_i||v_{i+1}|} f(v_1, \dots, v_{i+1}, v_i, \dots, v_p).$$

All multilinear maps we shall consider are also homogeneous of some degree $d \in \mathbf{Z}$ in the sense that $f(V^{n_1} \times \cdots \times V^{n_p}) \subseteq W^{n_1+\cdots+n_p+d}$ for all combinations of degrees $n_1, n_2, \dots, n_p \in \mathbf{Z}$. Since the product of zero copies of V is a singleton set, $\prod^0 V = \{*\}$, a graded symmetric multilinear map of arity 0 from V to W can be identified with its image $f(*) \in W^d$.

Given a non-negative integer $p \in \mathbf{N}_0$, the map $\odot^p\colon \prod^p V \to \bigodot^p(V)$ that sends $(v_1, \dots, v_p)$ to $\bigodot_{i=1}^p v_i$ is a homogeneous graded symmetric multilinear map of degree 0. This map, along with its codomain $\bigodot^p(V)$, is characterised up to isomorphism by the following universal property: for any graded symmetric multilinear map $f\colon \prod^p V \to W$ there exists a unique linear map $\tilde{f}\colon \bigodot^p(V) \to W$ for which the diagram



commutes. Since f is conversely determined by $\tilde{f}$ through this diagram, this describes a one-to-one correspondence between homogeneous linear maps from $\bigodot^p(V)$ to W and homogeneous graded symmetric p-linear maps from V to W for any $p \in \mathbf{N}_0$.

We often choose not to make a distinction between these two concepts by freely identifying the maps f and $\tilde{f}$ described above and referring to both as a graded symmetric multilinear maps. We therefore denote the space of homogeneous graded symmetric multilinear from V to W of degree d by $\operatorname{Lin}(\bigodot^p(V), W)^d$, and shall write

$$\operatorname{Lin}(\bigodot^p(V), W) = \bigoplus_{d \in \mathbf{Z}} \operatorname{Lin}(\bigodot^p(V), W)^d$$

to denote the **Z**-graded vector space obtained by combining these spaces.

We shall additionally consider homogeneous linear maps $f \colon \bigodot(V) \to W$ of a given degree $d \in \mathbf{Z}$ from the full graded symmetric algebra to W. The space $\operatorname{Lin}(\bigodot(V), W)^d$ of such maps is naturally isomorphic to the product

$$\prod_{p \in \mathbf{N}_0} \operatorname{Lin}(\bigodot^p(V), W)^d \simeq \operatorname{Lin}\big(\bigoplus_{p \in \mathbf{N}_0} \bigodot^p(V), W\big)^d,$$

whose elements are sequences $(f_p)_{p \in \mathbf{N}_0}$ of graded symmetric multilinear maps $f_p \colon \bigodot^p(V) \to W$ of degree d which are indexed by arity. This isomorphism is such that a map $f \colon \bigodot(V) \to W$ is identified with the sequence $(f_p)_{p \in \mathbf{N}_0}$ given by $f_p = f|_{\bigodot^p(V)}$. Such homogeneous linear maps of degree d or, equivalently, such sequences, together form a **Z**-graded vector space

$$\operatorname{Lin}(\bigodot(V), W) = \bigoplus_{d \in \mathbf{Z}} \operatorname{Lin}(\bigodot(V), W)^d \simeq \bigoplus_{d \in \mathbf{Z}} \prod_{p \in \mathbf{N}_0} \operatorname{Lin}(\bigodot^p(V), W)^d,$$

whose elements we call homogeneous (graded symmetric) *plurilinear maps* of degree d.

Once we have introduced a topology on the spaces V and W, as well as on their graded symmetric algebras, we will usually work with the subspace $\mathrm{L}(\bigodot(V), W)$ of continuous plurilinear maps instead. Spaces of continuous linear maps are discussed in appendix A.2.4, and the topological graded symmetric algebra is introduced in section 2.3.

2.1.3. The Koszul sign rule

When dealing with the graded symmetric algebra and related objects, one must often deal with signs that appear when commuting objects (either elements or maps). These signs are generally compatible with and can often be derived from the *Koszul sign rule*, which states that a factor $(-1)^{|a||b|}$ should be introduced whenever one switches the order in which two elements (or maps) a and b appear in an expression or equation.

Given homogeneous elements $u_1, \dots, u_p \in V$ and a permutation $\sigma \in S_p$ of their indices, the *Koszul sign* of σ is defined as

$$\epsilon_\sigma(u_1, \dots, u_p) = \prod_{\substack{\sigma(i)<\sigma(j) \\ i>j}} (-1)^{|u_i||u_j|}.$$

This sign is such that $u_{\sigma(1)} \odot \cdots \odot u_{\sigma(p)} = \epsilon_\sigma(u_1, \dots, u_p)\, u_{\{1,\dots,p\}}$ and it will show up in equations whenever the elements $u_1, \dots, u_p$ appear out of order. We can accordingly associate a sign to any two disjoint subsets I and J of $\{1, 2, \dots, p\}$ by setting

$$\epsilon_{I,J}(u_1, \dots, u_p) = \prod_{\substack{(i,j)\in I\times J \\ i>j}} (-1)^{|u_i||u_j|}, \tag{2.1.1}$$

to ensure that $u_I \odot u_J = \epsilon_{I,J}\, u_{I\sqcup J}$. The Koszul sign corresponding to a decomposition $\{1, \dots, p\} = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_m$ of $\{1, \dots, p\}$ into disjoint subsets $I_1, \dots, I_m$ is given by

$$\epsilon_{I_1,\dots,I_m}(u_1, \dots, u_p) = \prod_{1\le r<s\le m} \epsilon_{I_r,I_s}(u_1, \dots, u_p),$$

and it is such that $\bigodot_{r=1}^m u_{I_r} = \epsilon_{I_1,\dots,I_m}\, u_{\{1,\dots,p\}}$. For the sake of brevity, the homogeneous elements will usually be omitted and we will simply write ϵ_σ, $\epsilon_{I,J}$ or $\epsilon_{I_1,\dots,I_m}$ instead of these longer expressions.

To ensure consistency of the the Koszul sign rule, the definition of the tensor product of two homogeneous linear maps $f\colon V_1 \to W_1$ and $g\colon V_2 \to W_2$ between $\mathbf{Z}$-graded vector spaces needs to be modified slightly. It is the map $f \otimes g\colon V_1 \otimes V_2 \to W_1 \otimes W_2$ that is given by

$$(f \otimes g)(v_1 \otimes v_2) = (-1)^{|g||v_1|} f(v_1) \otimes g(v_2)$$

for homogeneous $v_1, v_2 \in V$. The factor $(-1)^{|g||v_1|}$ is necessary because g appears on different sides of v_1 in these expressions.

The Koszul sign rule also applies if one tries to compare the tensor algebra of a $\mathbf{Z}$-graded vector space V to that of its *desuspension* $V[1]$. This is the $\mathbf{Z}$-graded vector space given by $(V[1])^p = V^{p+1}$ for $p \in \mathbf{Z}$, and we denote the element corresponding to $v \in V^{p+1}$ by $\downarrow v \in (V[1])^p$. The *desuspension operator* $\downarrow$ induces an isomorphism $\downarrow^{\otimes p}\colon \bigotimes^p(V) \to \bigotimes^p(V[1])$ for any $p \in \mathbf{N}_0$, which is given by

$$\downarrow^{\otimes p}\left(\bigotimes_{i=1}^p v_i\right) = (-1)^{\sum_{i=1}^p (p-i)|v_i|} \bigotimes_{i=1}^p \downarrow v_i$$

for homogeneous $v_1, v_2, \ldots, v_p \in V$. These maps, or sometimes their inverses, are often collectively referred to as the *décalage isomorphism*. It induces a vector space isomorphisms between the graded skew symmetric algebra $\bigwedge(V)$ of V and the graded symmetric algebra $\bigodot(V[1])$ of its desuspension, as well as between the spaces $\bigodot(V)$ and $\bigwedge(V[1])$.

2.2. Discrete L_∞-algebras

Now that we have introduced the graded symmetric algebra of a graded vector space, we define what an L_∞-algebra is. We will first provide a version of the definition of an L_∞-algebra that generalises the conventional definition of a Lie algebra. The more conceptual definition as a self-commuting coderivation on a graded symmetric coalgebra is then presented in section 2.2.2 and we conclude with a discussion of morphisms in section 2.2.3.

2.2.1. Definition in terms of brackets

Without much additional set-up, L_∞-algebra structures can be defined as a collections of graded symmetric multilinear maps.

Definition 2.2.1.
A *curved L_∞-algebra*, (V, ℓ), is a $\mathbf{Z}$-graded vector space $V = \bigoplus_{k\in\mathbf{Z}} V^k$, together with a family $\ell = (\ell_p)_{p\in\mathbf{N}_0}$ of linear maps

$$\ell_p\colon \bigodot^p(V) \longrightarrow V$$

of degree 1 subject to the *Jacobi identities*

$$\sum_{I\sqcup J=\{1,2,\ldots,n\}} \epsilon_{I,J}\, \ell_{\#J+1}(\ell_{\#I}(u_I) \odot u_J) = 0 \qquad (2.2.1)$$

for $n \in \mathbf{N}_0$ and $u_1, u_2, \ldots, u_n \in V$ homogeneous.

The zeroth structure map, $\ell_0\colon \mathbf{R} \to V^1$, is identified with the image $\ell_0(1) \in V^1$ and is called the *curvature* of (V, ℓ). An L_∞*-algebra* is a curved L_∞-algebra for which $\ell_0 = 0$.

The first structure map, $\ell_1\colon V \to V$, is called the *differential*, and the higher structure maps ℓ_p for $p \geq 2$ are often referred to as *brackets*. Accordingly, it is not uncommon to write du for $\ell_1(u)$, and $\{u_1, \dots, u_k\}$ or $[u_1, \dots, u_k]$ instead of $\ell_k(u_1 \odot \cdots \odot u_k)$ for any $k \in \mathbf{N}$.

We shall write $\ell = (\ell_p)_{p \in \mathbf{N}_0}$ when referring to all of the structure maps ℓ_p collectively. It is often helpful to not view ℓ as a family of multilinear maps, but as a single linear map $\ell\colon \bigodot(V) \to V$ of degree 1 whose restrictions to $\bigodot^p(V)$ for $p \in \mathbf{N}_0$ are given by $\ell|_{\bigodot^p(V)} = \ell_p$. We should think of ℓ as the formal power series expansion of a vector field on V, and of the components ℓ_p for $p \in \mathbf{N}_0$ as its Taylor coefficients.

For an arbitrary sequence $\ell = (\ell_p)_{p \in \mathbf{N}_0}$ and $n \in \mathbf{N}_0$, we refer to the left-hand side of equation (2.2.1) as the n-th Jacobiator of ℓ, and write

$$\mathrm{Jac}_n(\ell)(u_1, u_2, \dots, u_n) = \sum_{I \sqcup J = \{1,2,\dots,n\}} \epsilon_{I,J}\, \ell_{\#J+1}(\ell_{\#I}(u_I)\, u_J)$$

for $u_1, u_2, \dots, u_n \in V$ homogeneous. Since this expression is graded symmetric and linear in each argument, the Jacobiator Jac_n can itself be interpreted as a homogeneous linear map $\mathrm{Jac}_n(\ell)\colon \bigodot^n(V) \to V$ of degree 2. As with the structure maps ℓ_p, we shall write $\mathrm{Jac}(\ell) = (\mathrm{Jac}_n(\ell))_{n \in \mathbf{N}_0}$ when referring to these Jacobiators collectively and again note that it can be interpreted as a single map $\mathrm{Jac}(\ell)\colon \bigodot(V) \to V$ of degree 2, which is given by

$$\mathrm{Jac}(u_{\{1,\dots,p\}}) = \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J}\, \ell(\ell(u_I) \odot u_J)$$

for homogeneous $u_1, u_2, \dots, u_p \in V$. We have left out the subscripts $\#I$ and $\#J + 1$ from the previous equation for the sake of brevity.

The zeroth Jacobi identity, $\ell_1(\ell_0) = 0$, states that the curvature of (V, ℓ) is ℓ_1-closed, which is reminiscent of the Bianchi identity for connections. The first Jacobi identity states that $\ell_1 \circ \ell_1 + \ell_2(\ell_0, \cdot) = 0$ and thus tells us that ℓ_1 squares to zero if $\ell_2(\ell_0, \cdot) = 0$. This in particular implies that the sequence

$$(V, \ell_1) = \Big(\cdots \xrightarrow{\ell_1} V^{k-2} \xrightarrow{\ell_1} V^{k-1} \xrightarrow{\ell_1} V^k \xrightarrow{\ell_1} V^{k+1} \xrightarrow{\ell_1} V^{k+2} \xrightarrow{\ell_1} \cdots \Big)$$

is a cochain complex whenever (V,ℓ) is uncurved.

The second identity expresses a version of the Leibniz rule and it is given by

$$\ell_1(\ell_2(u_1,u_2))+\ell_2(\ell_1(u_1),u_2)+(-1)^{|u_1||u_2|}\ell_2(\ell_1(u_2),u_1)+\ell_3(\ell_0,u_1,u_2)=0$$

for homogeneous $u_1,u_2\in V$. The additional final term drops out if either $\ell_3=0$ or $\ell_0=0$. Similarly, the first line in the following expression for the third Jacobi identity,

$$\begin{aligned}\ell_2(\ell_2(v_1,v_2),v_3)+\epsilon_{\{1,3\},\{2\}}\,\ell_2(\ell_2(v_1,v_3),v_2)+\epsilon_{\{2,3\},\{1\}}\,\ell_2(\ell_2(v_2,v_3),v_1)\\ +\ell_3(\ell_1(v_1),v_2,v_3)+\epsilon_{\{2\},\{1,3\}}\,\ell_3(\ell_1(v_2),v_1,v_3)+\epsilon_{\{3\},\{1,2\}}\,\ell_3(\ell_1(v_3),v_1,v_2)\\ \ell_1(\ell_3(v_1,v_2,v_3))+\ell_4(\ell_0,v_1,v_2,v_3)=0,\end{aligned}$$

resembles the usual Jacobiator. We say that the binary bracket of an uncurved L_∞-algebra satisfies the Jacobi identity up to homotopy. The last four terms drop out whenever $\ell_p=0$ for all $p\geq 3$. We say that an L_∞-algebra satisfies the Jacobi identity up to homotopy. Beyond this, there is an infinite sequence of higher Jacobi identifies $\mathrm{Jac}_p=0$ for $p\geq 4$.

Example 2.2.2 (Cochain complexes).
Any cochain complex (C,d) can, rather trivially, be interpreted as an L_∞-algebra with unary bracket $\ell_1=d\colon C\to C$ and $\ell_p=0$ for every $p\in\mathbf{N}_0$ other than $p=1$. If we choose an arbitrary closed element $c\in\ker(\ell_1\colon C^1\to C^2)$ and use $\ell_0=c$ instead of setting ℓ_0 to zero, then (C,ℓ) becomes a curved L_∞-algebra with curvature c. ◊

An L_∞-algebra can be thought of as a generalisation of a differential graded Lie algebra up to a shift in degree.

Example 2.2.3 (Differential graded Lie algebras).
Let $(V,d,[\cdot,\cdot])$ be a differential graded Lie algebra (dgla) and let $V[1]$ denote its desuspension, i.e. the graded vector space obtained by shifting all degrees down by 1. Its homogeneous components are given by $(V[1])^n=V^{n+1}$ for every $n\in\mathbf{Z}$ and we denote the element of this space corresponding to $v\in V$ by $\downarrow v\in V[1]$.

The space $V[1]$ becomes an L_∞-algebra if it is endowed with the structure maps $\ell_0,\ell_1,\ell_2,\dots$ given by $\ell_p=0$ if $p\neq 1,2$, and

$$\ell_1(\downarrow u)=\downarrow(du)\qquad\text{and}\qquad \ell_2(\downarrow u,\downarrow v)=(-1)^{|u|}\downarrow[u,v],$$

for homogeneous $u, v \in V$. Taking into account the sign conventions discussed in section 2.1.3, these identities can be rephrased as $\ell_1 \circ \downarrow = \downarrow \circ d$ and $\ell_2 \circ (\downarrow \otimes \downarrow) = \downarrow \circ [\,\cdot\,, \cdot\,]$.

The first Jacobiator for $(\downarrow V, \ell)$ is given by $\mathrm{Jac}_1(\ell)(\downarrow u) = \downarrow(d^2 u)$ and it is zero precisely when d is a differential. The second Jacobiator reads

$$\begin{aligned}\mathrm{Jac}_2(\ell)(\downarrow u, \downarrow v) &= \ell_1(\ell_2(\downarrow u, \downarrow v)) + \ell_2(\ell_1 \downarrow u, \downarrow v) + (-1)^{|\downarrow u||\downarrow v|}\ell_2(\ell_1 \downarrow v, \downarrow u) \\ &= (-1)^{|u|}\downarrow\big(d[u,v] - [du, v] - (-1)^{|u|}[u, dv]\big),\end{aligned}$$

and its vanishing thus similarly describes the property that d is a derivation for $[\,\cdot\,, \cdot\,]$. The third Jacobiator can similarly be expressed as

$$\mathrm{Jac}_3(\ell)(\downarrow u, \downarrow v, \downarrow w) = (-1)^{|v|}\downarrow\big([[u,v],w] - [u,[v,w]] - (-1)^{|v||w|}[[u,w],v]\big),$$

which is equal to the usual Jacobiator up to a global sign. The higher Jacobiators $\mathrm{Jac}_p(\ell)$ for $p \geq 4$ vanish automatically due to the absence of brackets of arity greater than 2.

This construction describes a one-to-one correspondence between differential graded Lie algebra structures on V and L_∞-algebra structures on $V[1]$ for which only the unary and binary brackets are (possibly) non-trivial. ◊

Example 2.2.4 (Formal power series).

A simple type of L_∞-algebra is one which is concentrated in degrees 0 and 1. Due to homogeneity, the only non-trivial components of the structure map of a curved L_∞-algebra (V, ℓ) with $V = V^0 \oplus V^1$ are the maps $\ell_p \colon \bigodot^p(V^0) \to V^1$ for $p \in \mathbf{N}_0$. Since the Jacobiator is homogeneous of degree two, it automatically vanishes, which means that any sequence $(\ell_p)_{p \in \mathbf{N}_0}$ of such maps describes a curved L_∞-algebra structure on V.

We should think of the sequence $(\ell_p)_{p \in \mathbf{N}_0}$ as the power series expansion of a map from V^0 to V^1. In fact, if V is finite-dimensional, we can always find a smooth map $\mu \colon V^0 \to V^1$ such that

$$\ell_p(u_1, \ldots, u_p) = \mathrm{D}^p \mu(0)(u_1, \ldots, u_p).$$

for every $p \in \mathbf{N}_0$ and $u_1, u_2, \ldots, u_p \in V^0$. The L_∞-algebra obtained from μ is uncurved precisely when $\mu(0) = 0$. ◊

Remark 2.2.5.

What we have presented as *the* definition of an L_∞-algebra in Definition 2.2.1 is in fact only one of two equivalent definitions that one can expect to encounter in the literature. The other definition, which goes back to [LS93], uses a different grading convention for which the brackets are graded skew-symmetric and have a degree that depends on their arity.

To be more precise, according to this definition an L_∞-algebra is a **Z**-graded vector space V endowed with a sequence $(l_p)_{p\in\mathbf{N}}$ of graded skew-symmetric multilinear maps $l_p\colon \otimes^p V \to V$ of degree $2-p$ such that

$$\sum_{p+q=r} \sum_{\sigma\in \mathrm{Sh}_{p,q}} \mathrm{sgn}(\sigma)\,\epsilon_\sigma l_{q+1}\big(l_p(v_{\sigma(1)},\dots,v_{\sigma(p)}),v_{\sigma(p+1)},\dots,v_{\sigma(r)}\big)=0$$

for all homogeneous $v_1,v_2,\dots,v_r\in V$, where summation runs over all (p,q)-(un)shuffle permutations.

There is a natural one-to-one correspondence between such skew symmetric L_∞-algebra structures on a graded vector space V and symmetric L_∞-algebra structures, as described in Definition 2.2.1, on the desuspension $V[1]$. The symmetric L_∞-algebra $(V[1],\ell)$ corresponding to a skew symmetric L_∞-algebra (V,l) is given by $\ell_p \circ {\downarrow}^{\otimes p} = {\downarrow} \circ l_p$, or

$$\ell_p({\downarrow}v_1,\dots,{\downarrow}v_p)=(-1)^{\sum_{i=1}^p (p-i)|v_i|}{\downarrow} l_p(v_1,\dots,v_p)$$

for homogeneous $v_1,v_2,\dots,v_p\in V$. Different sign conventions are possible.

Although both definitions are actively used, what we call an L_∞-algebra here is often instead referred to as an $L_\infty[1]$-algebra because it is the structure on the space $V[1]$ that corresponds to a traditional L_∞-algebra structure on V. The obvious advantage of the skew-symmetric definition is its compatibility with the usual grading on differential graded Lie algebras, which can be viewed as skew symmetric L_∞-algebras directly (without introducing any signs). On the other hand, the fact that the skew symmetric brackets have different degrees means that they cannot be combined into a single homogeneous operator $l\colon \bigwedge(V)\to V$. This is particularly important for the description of curved L_∞-algebra structures in terms of the canonical coalgebra structure on $\bigodot(V)$, which is the topic of section 2.2.2. △

2.2.2. Definition in terms of coderivations

We now present an alternative, and in some sense simpler, description of L_∞-algebras. The description arises by observing that $\bigodot(V)$ carries a canonical coalgebra structure, and that the structure map $\ell\colon \bigodot(V) \to V$ determines a unique degree 1 coderivation $\bar{\ell}\colon \bigodot(V) \to \bigodot(V)$. The Jacobi identities for the structure map ℓ then become equivalent to the condition $\bar{\ell} \circ \bar{\ell} = 0$. This is in fact how L_∞-algebras first appeared, and how the Jacobi identities were identified.

The starting point for the definition is the fact that $\bigodot(V)$ has a graded cocommutative, counital, coassociative coalgebra structure with comultiplication map given by

$$\Delta\colon \bigodot(V) \longrightarrow \bigodot(V) \otimes \bigodot(V), \qquad v_1 \odot v_2 \odot \cdots \odot v_k \mapsto \sum_{I \sqcup J = \{1,2,\ldots,k\}} \epsilon_{I,J}\, v_I \otimes v_J,$$

where $v_I = \bigodot_{i\in I} v_i$ and $v_J = \bigodot_{j\in J} v_j$ are the ordered products introduced before and $\epsilon_{I,J}$ is the appropriate Koszul sign. The counit is the canonical projection map $\varepsilon\colon \bigodot(V) \to \bigodot^0(V) \simeq \mathbf{R}$ sending an element of $\bigodot(V) = \bigoplus_{p\in\mathbf{N}_0} \bigodot^p(V)$ to its rank 0 component.

A morphism of counital coalgebras from $\bigodot(V)$ to $\bigodot(W)$ is a homogeneous linear map $f\colon \bigodot(V) \to \bigodot(W)$ of degree 0 such that $\Delta \circ f = (f \otimes f) \circ \Delta$ and $\varepsilon \circ f = \varepsilon$. A coderivation on $\bigodot(V)$ is a linear map $\ell\colon \bigodot(V) \to \bigodot(V)$ such that $\Delta \circ \ell = (\ell \otimes \mathrm{id} + \mathrm{id} \otimes \ell) \circ \Delta$ and $\epsilon \circ \ell = 0$ (although the latter equality follows from the former). Here, the tensor product of maps is defined to be compatible with the Koszul sign convention, which means that $(f \otimes g)(u \otimes v) = (-1)^{|g||u|} f(u) \otimes g(v)$, where $|g|$ is the degree of g. Degree 0 coderivations can be thought of as infinitesimal automorphisms, just as degree 0 derivations are thought of as infinitesimal automorphisms of graded algebras.

There is a one-to-one correspondence between coderivations on $\bigodot(V)$ and linear maps from $\bigodot(V)$ to V because any linear map $f\colon \bigodot(V) \to V$ can be lifted to a coderivation $\bar{f}\colon \bigodot(V) \to \bigodot(V)$. This coderivation is given by

$$\bar{f}(u_1 \odot \cdots \odot u_n) = \sum_{I \sqcup J = \{1,\ldots,n\}} \epsilon_{I,J}\, f(u_I) \odot u_J. \tag{2.2.2}$$

for homogeneous $u_1, u_2, \ldots, u_n \in V$. Conversely, f is recovered from $\bar{f}$ through post-composition with the canonical projection map onto $\bigodot^1(V) = V$.

Because the graded commutator $[\bar{f}, \bar{g}] = \bar{f} \circ \bar{g} - (-1)^{|\bar{f}||\bar{g}|} \bar{g} \circ \bar{f}$ of two coderivations is also a coderivation, the space of coderivations for $(\bigodot(V), \Delta)$ is a graded

Lie subalgebra of the algebra of endomorphisms of $\bigodot(V)$. The corresponding graded Lie bracket on $\mathrm{Lin}(\bigodot(V), V)$ obtained through the correspondence described above is called the *Gerstenhaber bracket* and it is given by

$$[f,g](u_1 \odot \cdots \odot u_p) = \sum_{I \sqcup J = \{1,\ldots,p\}} \epsilon_{I,J} \big(f(g(u_I) \odot u_J) - (-1)^{|f||g|} g(f(u_I) \odot u_J) \big)$$

for two homogeneous linear maps $f\colon \bigodot(V) \to V$ and $g\colon \bigodot(V) \to V$ and homogeneous $u_1, u_2, \ldots, u_p \in V$. We note the resemblance to equation (2.2.1) and observe that the Jacobiator of a homogeneous linear map $\ell\colon \bigodot(V) \to V$ of degree 1 is given by

$$\mathrm{Jac}(\ell) = \tfrac{1}{2}[\ell, \ell].$$

The Jacobi identity (2.2.1) thus reduces to $[\ell,\ell] = 0$, or $[\bar\ell, \bar\ell] = 0$.

This enables a very succinct reformulation of Definition 2.2.1 in terms of the coalgebra structure on $\bigodot(V)$.

Definition 2.2.6 (Alternative)**.**
A *curved L_∞-algebra* is a **Z**-graded vector space $V = \bigoplus_{n \in \mathbf{Z}} V^n$, together with a degree 1 coderivation $\bar\ell\colon \bigodot(V) \to \bigodot(V)$ such that $[\bar\ell, \bar\ell] = 0$. It is uncurved if $\bar\ell(1) = 0$.

We observe that $\frac{1}{2}[\bar\ell, \bar\ell]$ is equal to $\bar\ell \circ \bar\ell$ since $\bar\ell$ is homogeneous of degree 1. The condition $[\bar\ell, \bar\ell]$ can thus be replaced by $\bar\ell \circ \bar\ell = 0$.

Definition 2.2.6 tells us that we can think of a curved L_∞-algebra (V, ℓ) as a structure that lives on the coalgebra $(\bigodot(V), \Delta)$, rather than on the graded vector space V. This suggests that intrinsic properties of curved L_∞-algebras should be invariant under coalgebra isomorphisms.

In light of this, the appearance of the unit element $1 \in \bigodot(V)$ in the definition of an (uncurved) L_∞-algebra may seem a bit out of place. We recall that 1 is the unit element for the algebra structure on $\bigodot(V)$ and note that this algebra structure is not itself invariant under coalgebra automorphisms. Nevertheless, 1 can be recovered from the coalgebra structure on $\bigodot(V)$ alone since it is the only non-zero element of $\bigodot(V)$ with the property that $\Delta(1) = 1 \otimes 1$.

The coalgebra structure on $(\bigodot(V), \Delta)$ and the element $1 \in \bigodot(V)$ together induce an increasing filtration

$$F^0 \subseteq F^1 \subseteq F^2 \subseteq \bigcup_{r \in \mathbf{N}_0} F^r = \bigodot(V).$$

by graded subspaces given by $F^0 = \mathbf{R} \cdot 1$ and

$$F^r = \left\{ x \in \bigodot(V) \,\middle|\, \Delta(x) - x \otimes 1 - 1 \otimes x \in F^{r-1} \otimes F^{r-1} \right\}$$

for $r \geq 1$. We can in fact work out that for every $r \in \mathbf{N}_0$, F^r is equal to the space $\bigoplus_{p=0}^r \bigodot^p(V)$ of elements of rank at most r for every $r \in \mathbf{N}_0$, which tells us that even though the algebra structure on $\bigodot(V)$ is not invariant under coalgebra automorphisms, the induced filtration by rank is.

Of the subspaces $\bigodot^p(V) \subseteq \bigodot(V)$ for $p \in \mathbf{N}_0$, only $\bigodot^0(V)$ and $\bigodot^1(V)$ are completely determined by the coalgebra structure on $\bigodot(V)$: the first is equal to F^0, and the second is equal the intersection $F^1 \cap \ker(\varepsilon)$ of F^1 with the kernel of the counit. This in particular implies that the differential complex (V, ℓ_1) associated to an L_∞-algebra (V, ℓ) is intrinsically determined by the differential $\bar{\ell}$ on the coalgebra $(\bigodot(V), \Delta)$.

Remark 2.2.7.
A more intrinsic definition of a (curved) L_∞-algebra would be as a $\mathbf{Z}$-graded cofree connected cocommutative counital coalgebra (C, δ, ε), endowed with a homogeneous self-commuting coderivation $\bar{\ell}\colon C \to C$ of degree 1. Any such coalgebra is non-canonically isomorphic to the graded symmetric coalgebra

$$\left(\bigoplus_{r \in \mathbf{N}_0} (F^r / F^{r-1}), \delta\right) \simeq (\bigodot(V), \Delta)$$

of $V = \bigoplus_{n \in \mathbf{N}_0} F^1 / F^0$, where $(F_r)_{r \in \mathbf{N}_0}$ is the canonical filtration of C. What we called a (curved) L_∞-algebra in Definition 2.2.6 is then strictly speaking a *split (curved) L_∞-algebra*, since finding an isomorphism between C and $\bigodot(V)$ amounts to choosing a splitting for this filtration. △

2.2.3. Morphisms

To complete our description of the category of curved L_∞-algebras, we should discuss morphisms. While one might expect a morphism of L_∞-algebras to be a morphism of the underlying graded vector spaces that satisfies some compatibility condition, the actual definition is a little more involved.

Definition 2.2.8.
A *morphism* $f : (V, \ell) \to (W, \ell')$ of curved L_∞-algebras is a homogeneous linear

map $f\colon \bigodot(V) \to W$ of degree 0 such that $f(1) = 0$ and

$$\sum_{I \sqcup J = \{1,2,\dots,p\}} \epsilon_{I,J}\, f(\ell(u_I) \odot u_J) = \sum_{I_1,\dots,I_m} \tfrac{1}{m!}\, \epsilon_{I_1,\dots,I_m}\, \ell'\big(\textstyle\bigodot_{j=1}^m f(u_{I_j})\big) \tag{2.2.3}$$

for all $p \in \mathbf{N}_0$ and all homogeneous $v_1, v_2, \dots, v_p \in V$.

In equation (2.2.3), and from this point onward, $\sum_{I_1,\dots,I_m}$ indicates summation over all $m \in \mathbf{N}_0$ and all ordered partitions of $\{1, 2, \dots, p\}$ into disjoint subsets $I_1, I_2, \dots, I_m$. The factor $\frac{1}{m!}$ compensates for the fact that every partition is represented $m!$ times in this sum (the number of ways in which the sets $I_1, I_2, \dots, I_m$ can be ordered).

If (V, ℓ) is an uncurved L_∞-algebra and $f\colon (V, \ell) \to (W, \ell')$ is a morphism of curved L_∞-algebras, then also (W, ℓ') is uncurved since $\ell'_0 = f(\ell_0) = 0$ as part of the definition. We moreover observe that the linear component of f then describes a morphism $f_1\colon (V, \ell_1) \to (W, \ell'_1)$ of cochain complexes since equation (2.2.3) for $p = 1$ states that $f \circ \ell_1 = \ell'_1 \circ f$ if (V, ℓ) is uncurved.

When expressed in terms of the coalgebra description from Definition 2.2.6, the notion of a morphism of curved L_∞-algebras becomes significantly simpler. Below, $\bar\ell\colon \bigodot(V) \to \bigodot(V)$ and $\bar\ell'\colon \bigodot(W) \to \bigodot(W)$ denote the lifts of ℓ and ℓ' as coderivations, which are explicitly given by equation (2.2.2).

Definition 2.2.9 (Alternative).
A *morphism* $\bar f\colon (V, \ell) \to (W, \ell')$ of curved L_∞-algebras is a counital coalgebra morphism $\bar f\colon \bigodot(V) \to \bigodot(W)$ of degree 0 such that $\bar f \circ \bar\ell = \bar\ell' \circ \bar f$.

To relate these two definitions, we note that any morphism $\bar f\colon \bigodot(V) \to \bigodot(W)$ of coalgebras is completely determined by the map $f\colon \bigodot(V) \to W$ obtained by post-composing with the projection map onto $\bigodot^1(W) = W$. In terms of this map, $\bar f$ is given by

$$\bar f(u_1 \odot \cdots \odot u_n) = \sum_{I_1,\dots,I_m} \tfrac{1}{m!}\, \epsilon_{I_1,\dots,I_m}\, \textstyle\bigodot_{j=1}^m f(u_{I_j}) \tag{2.2.4}$$

for homogeneous $u_1, u_2, \dots, u_n \in V$.

We observe that any morphism $\bar f\colon (\bigodot(V), \Delta) \to (\bigodot(W), \Delta)$ of graded symmetric coalgebras automatically satisfies $\bar f(1) = 1$, even though this was not explicitly imposed in the definition, and that this corresponds precisely to the condition that $f(1)=0$. Using equation (2.2.4), the identity $\bar{f}\circ\bar{\ell}=\bar{\ell}'\circ\bar{f}$ can then be expanded to

$$\sum_{I\sqcup J\sqcup K=\{1,2,\dots,p\}} \epsilon_{I,J,K}\, f(\ell(u_I)\odot u_J)\odot \bar{f}(u_K) \\ = \sum_{I_1,\dots,I_m,K} \tfrac{1}{m!}\,\epsilon_{I_1,\dots,I_m,K}\,\ell'\big(\textstyle\bigodot_{j=1}^m f(u_{I_j})\big)\odot \bar{f}(u_K),$$

which is equivalent to equation (2.2.3).

One of the advantages of the interpretation of an L_∞-algebra as a structure on the graded symmetric coalgebra is that it becomes evident what the composition of two L_∞-algebra morphisms should be. Given two morphisms $\bar{f}\colon (V,\ell)\to(W,\ell')$ and $\bar{g}\colon (W,\ell')\to(Z,\ell'')$ of curved L_∞-algebras in the sense of Definition 2.2.9, the composition $\bar{g}\circ\bar{f}\colon \bigodot(V)\to\bigodot(Z)$ is clearly again such a morphism. If we work out what this composition looks like in terms of the more elementary objects from Definition 2.2.8, we obtain the expression

$$(g\diamond f)(v_1\odot\cdots\odot v_p)=\sum_{I_1,\dots,I_m}\tfrac{1}{m!}\,\epsilon_{I_1,\dots,I_m}\,g\big(\textstyle\bigodot_{j=1}^m f(I_j)\big) \tag{2.2.5}$$

for homogeneous $v_1,v_2,\dots,v_p\in V$. We moreover see that the identity morphism $\overline{\mathrm{id}}\colon(\bigodot(V),\Delta)\to(\bigodot(V),\Delta)$ corresponds to the map $i\colon\bigodot(V)\to V$ given by $i|_V=\mathrm{id}_V$ and $i|_{\bigodot^p(V)}=0$ for $p\neq 1$.

Example 2.2.10 (Taylor series)**.**

Let V and W be finite-dimensional vector spaces and let $(V,0)$ and $(W,0)$ be the L_∞-algebras concentrated in degree 0 obtained by endowing these spaces with the zero maps $0\colon\bigodot(V)\to V$ and $0\colon\bigodot(W)\to W$. The Taylor series $\mathrm{j}^\infty f=(\mathrm{D}^p f)_{p\in\mathbb{N}_0}$ of any smooth map $f\colon U\to W$ for which $f(0)=0$ from a zero neighbourhood U in V is a morphism of L_∞-algebras from $(V,0)$ to $(W,0)$.

Given two such maps $f\colon U_V\subseteq V\to U_W\subseteq W$ and $g\colon U_W\subseteq W\to Z$, the derivatives of the composition $g\circ f\colon U_V\to Z$ at 0 are given by

$$\mathrm{D}^k h(0)(\dot{v}_1,\dots,\dot{v}_k)=\sum_{I_1,\dots,I_m}\tfrac{1}{m!}\mathrm{D}^m g(0)\big(\textstyle\bigodot_{j=1}^m \mathrm{D}^{\#I_j}f(0)(\dot{v}_{I_j})\big)$$

for $\dot{v}_1,\dot{v}_2,\dots,\dot{v}_k\in\mathrm{T}_0U_V\simeq V$. Consequently, the Taylor series $\mathrm{j}^\infty(g\circ f)(0)$ of their composition coincides with the composition $(\mathrm{j}^\infty g)\diamond(\mathrm{j}^\infty f)$ of their Taylor series as L_∞-algebra morphisms. ◊

Example 2.2.11.

Let V and W be finite dimensional L_∞-algebras concentrated in degrees 0 and 1, as in Example 2.2.4. For this example it is worth regarding $V^0 \times V^1$ and $W^0 \times W^1$ as trivial vector bundles over V^0 and W^0 respectively.

Then the brackets on V and W correspond to the power series expansion of functions $L_V \colon V^0 \to V^1$ and $L_W \colon W^0 \to W^1$, i.e., sections of the bundles $V^0 \times V^1 \to V^0$ and $W^0 \times W^1 \to W^0$. A morphism between V and W is a pair of (families of) maps,

$$\varphi_p \colon \bigodot^p V^0 \longrightarrow W^0, \qquad \Phi_p \colon \bigodot^{p-1} V^0 \odot V^1 \longrightarrow W^1.$$

We can interpret the collection φ_p above as the power series expansion of a map $\varphi \colon V^0 \to W^0$ and the maps Φ_p together as the power series expansion of a bundle map $\Phi \colon V^0 \times V^1 \to W^0 \times W^1$. In this light the condition that the pair (φ, Φ) is a map of L_∞-algebras corresponds to the statement that $\Phi \circ L_V$ and $L_W \circ \varphi$ have the same infinite jet at 0. ◊

As we saw in Example 2.2.3, every differential graded Lie algebra is also an L_∞-algebra. It is interesting to study morphisms between differential graded Lie algebras from the L_∞ point of view. If $(A, d, [\cdot,\cdot])$ and $(B, d, [\cdot,\cdot])$ are differential graded Lie algebras an L_∞-morphism $f : A \to B$ is a collection of maps $f_p : \bigodot^p(A) \to B$ which are required to satisfy

$$\begin{aligned} d(f_p(v_{\{1,\dots,p\}})) + \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J} [f_{\#I}(v_I), f_{\#J}(v_J)] \\ = \sum_{\{i\} \sqcup J = \{1,\dots,p\}} f_{\#J+1}((dv_i) \odot v_J) \\ + \sum_{\{i\} \sqcup \{j\} \sqcup K = \{1,\dots,p\}} \epsilon_{\{i\},\{j\},K} f_{\#K+1}([v_i, v_j] \odot v_K), \end{aligned}$$

where the grading used is the one making A into an L_∞-algebra, as in Example 2.2.3

If the only nontrivial map in f was f_1 this would translate into to the statement that f is a usual morphism of differential graded Lie algebras. The presence of the remaining maps, f_2, f_3,... leaves space for a number of other possibilities. The next example illustrates this feature.

Example 2.2.12.
Every morphism $\phi\colon (A, d, [\cdot, \cdot]) \to (B, d, [\cdot, \cdot])$ of differential graded Lie algebras induces a linear morphism $f\colon (A[1], \ell) \to (B[1], \ell')$ between the corresponding L_∞-algebras for which $f_1 = \downarrow \circ \phi \circ \uparrow$ and $f_p = 0$ for $p \neq 1$. Not all morphisms between such L_∞-algebras are of this form. ◊

Finally, we mention that in this algebraic context, there is a natural generalisation of the inverse function theorem for formal power series.

Proposition 2.2.13 (Inverse function theorem for L_∞-algebras).
A morphism $f\colon (V, \ell) \to (W, \ell')$ of curved L_∞-algebras is invertible if and only if its linear component, $f_1\colon V \to W$, is an isomorphism of graded vector spaces.

The inverse $g\colon (W, \ell') \to (V, \ell)$ provided by Proposition 2.2.13 is unique and its first component is given by $g_1 = f_1^{-1}\colon W \to V$. The remaining components, $g_p\colon \bigodot^p(W) \to V$ for $p \geq 2$, can be constructed recursively. A complete proof of this proposition can be found in the literature (e.g. [Mar12, Theorem 7.5] or [KS, Theorem 1.3.1]), and a version that takes into account topology is provided in Proposition 2.3.9 below.

A morphism $f\colon (V, \ell) \to (W, \ell')$ of L_∞-algebras is called a *quasi-isomorphism* whenever its linear component, $f_1\colon (V, \ell_1) \to (W, \ell'_1)$, is a quasi-isomorphism of differential complexes. This means that the induced map in cohomology, $\mathrm{H}(f_1)\colon \mathrm{H}(V, \ell_1) \to \mathrm{H}(W, \ell'_1)$, is an isomorphism of graded vector spaces. A quasi-inverse for g is a morphism $g\colon (W, \ell') \to (V, \ell)$ in the opposite direction such that that $\mathrm{H}(f_1)$ and $\mathrm{H}(g_1)$ are mutually inverse.

Proposition 2.2.14.
A morphism $f\colon (V, \ell) \to (W, \ell')$ of L_∞-algebras is a quasi-isomorphism if and only if it admits a quasi-inverse $g\colon (W, \ell') \to (V, \ell)$.

A proof of Proposition 2.2.14 can for instance be found in [KS], where it is Corollary 3.0.11. It is proven by first showing that every L_∞-algebra is quasi-isomorphic to a minimal L_∞-algebra, i.e. one for which $\ell_1 = 0$. The underlying graded vector space of this so-called "minimal model" consists of the cohomology groups of the original L_∞-algebra. Morally, finding a minimal model is similar to finding a formal Kuranishi family for an abstract deformation problem.

Example 2.2.15.

In this example we will only consider the case where f is linear (i.e., $f_p = 0$ for all $p \geq 2$) and (W, ℓ') is minimal (i.e., $\ell'_1 = 0$). Below, we will therefore assume that both of these statements hold. We will construct the components g_p for $p \in \mathbf{N}$ recursively by considering the component of $\ell \diamond g = g \mathbin{\square} \ell$ one at a time.

The first component of this equation simply reads $\ell_1 \circ g_1 = 0$ due to the assumption that $\ell'_1 = 0$. For the first component of g we can therefore pick any linear section $g_1 \colon W \to \ker(\ell_1) \subseteq V$ of $f|_{\ker(\ell_1)}$.

The arity 2 component of the equation reads

$$\ell_1(g_2(v_1, v_2)) + \ell_2(g_1(v_1), g_1(v_2)) = g_1(\ell_2(v_1, v_2))$$

for $v_1, v_2 \in W$. An element $g_2(v_1, v_2) \in V$ solving this equation exists if and only if $z = g_1(\ell_2(v_1, v_2)) - \ell_2(g_1(v_1), g_1(v_2))$ is ℓ_1-exact. This element is automatically ℓ_1-closed because $\operatorname{im}(g_1) \subseteq \ker(\ell_1)$ by construction and the second Jacobi identity tells us that ℓ_2 maps pairs of closed elements to closed elements. If we apply f_1 to this element, we obtain

$$f_1(z) = (f_1 \circ g_1)(\ell'_2(v_1, v_2)) - \ell_2(f_1 \circ g_1(v_1), f_1 \circ g_1(v_2)) = 0$$

by using that f is a morphism of L_∞-algebras and that g_1 is a right inverse of f_1. Since f_1 is a quasi-isomorphism and z is closed, this shows that z is exact. The equation $\ell \diamond g = g \mathbin{\square} \ell'$ is therefore solved up to arity 2 if we set

$$g_2(v_1, v_2) = P\big(g_1(\ell_2(v_1, v_2)) - \ell_2(g_1(v_1), g_1(v_2))\big)$$

for all $v_1, v_2 \in W$.

This argument can be continued to recursively define the components g_r for every $r \in \mathbf{N}$. Assume that $\ell \diamond g - g \mathbin{\square} \ell'$ vanishes on $\bigodot^s(W)$ and that $\operatorname{im}(g_s) \subseteq \ker(\ell_1)$ for every $s \leq r$. If we apply ℓ_1 to the arity $r+1$ component of $\ell \diamond g - g \mathbin{\square} \ell'$ we obtain

$$\begin{aligned}\ell_1 \circ (\ell \circ \bar{g}_{\mathrm{mor}} - g \circ \bar{\ell}'_{\mathrm{cod}})\big|_{\bigodot^{r+1}(W)} &= \ell \circ (\bar{\ell}_{\mathrm{cod}} \circ \bar{g}_{\mathrm{mor}} - \bar{g}_{\mathrm{mor}} \circ \bar{\ell}'_{\mathrm{cod}})\big|_{\bigodot^{r+1}(W)} \\ &\quad - \sum_{p \geq 2} \ell_p \circ (\bar{\ell} \circ \bar{g} - \bar{g} \circ \bar{\ell}')\big|_{\bigodot^{r+1}(W)}.\end{aligned}$$

The sum on the second line vanishes identically due to the induction hypothesis. We are thus left with

$$\ell \circ \bar{\ell}_{\mathrm{cod}} \circ \bar{g}_{\mathrm{mor}} - \ell \circ \bar{g}_{\mathrm{mor}} \circ \bar{\ell}'_{\mathrm{cod}}\big|_{\bigodot^{n+1}(W)} = \ell \circ \bar{\ell}_{\mathrm{cod}} \circ \bar{g}_{\mathrm{mor}} - g \circ \bar{\ell}'_{\mathrm{cod}} \circ \bar{\ell}'_{\mathrm{cod}}\big|_{\bigodot^{n+1}(W)},$$

where the equality follows from the induction hypothesis and the fact that $\bar{\ell}'(\bigodot^{n+1}(W)) \subseteq \bigodot^n(W)$ due to the assumption that $\ell'_1 = 0$. The first is zero because (V,ℓ) is an L_∞-algebra and the second because (W,ℓ') is, so we see that $\ell \circ \bar{g}_{\mathrm{mor}} - g \circ \bar{\ell}'_{\mathrm{cod}}$ is indeed ℓ_1-closed.

To show that this element is exact, we should again verify that it is in the kernel of f_1 by applying this map to it. Since $f = f_1$ is a morphism of L_∞-algebras, we have that

$$f_1 \circ (\ell \diamond g - g \mathbin{\square} \ell)\big|_{\bigodot^{r+1}(W)} = \big(\ell \circ f \circ \bar{g}_{\mathrm{mor}} - f \circ g \mathbin{\square} \bar{\ell}_{\mathrm{cod}}\big)\big|_{\bigodot^{r+1}(W)},$$

which is zero because g_1 is a right inverse for f_1 and $\mathrm{im}(g_p) \subseteq \ker(f_1)$ for all $p \geq 2$. We can therefore set

$$g_{r+1}(v_{\{1,\dots,r+1\}}) = P\Big(- \sum_{\substack{I_1,\dots,I_m \\ m\geq 2}} \epsilon_{I_1,\dots,I+m}\, \ell\big(\textstyle\bigodot_{j=1}^m\big) g(v_{I_j}) \\ + \sum_{I \sqcup J = \{1,\dots,r+1\}} \epsilon_{I,J}\, g(\ell(v_I) \odot v_J).$$

This map satisfies $\mathrm{im}(g_{r+1}) \subseteq \ker(f_1)$ and $\big(\ell \diamond g - g \mathbin{\square} \ell'\big)\big|_{\bigodot^r(W)} = 0$. By applying this construction recursively, we obtain a map $g = (g_p)_{p\in\mathbf{N}} \colon \bigodot(W) \to V$ which satisfies $\ell \diamond g = g \mathbin{\square} \ell'$ and is consequently a morphism of L_∞-algebras. It is a quasi-inverse for f because the linear component g_1 was chosen to be such that . ◊

2.2.4. A second look at composition of maps

In the previous section we discussed morphisms between L_∞-algebras as well as their composition. It is worth taking a step back and re-evaluating the process that justified Definition 2.2.8 and led to the composition rule (2.2.5).

As argued in the previous section, the natural way to regard an L_∞-algebra structure on V is as a self-commuting degree 1 coderivation on the symmetric coalgebra $\bigodot(V)$. From this point of view the notions of morphisms and their composition agrees with one's intuition. For computational purposes, it is often useful to have the less intuitive formulas involving sequence of multilinear maps, such as (2.2.3) and (2.2.5). In this short section we take a second look at the process that gave produces those equations.

Both coderivations and morphisms are represented by similar types of objects. For a coderivation this is a linear map

$$a\colon \bigodot(V) \longrightarrow V,$$

which always admits a unique lift to a coderivation of $(\bigodot(V), \Delta)$. In fact, as we argued before, there is a one-to-one correspondence between coderivations and such maps. The inverse of this lifting procedure corresponds to composing a coderivation $\bar{a}$ with a projection onto V:

$$\begin{array}{ccc} & & \bigodot(V) \\ & \overset{\bar{a}}{\nearrow} & \big\downarrow \mathrm{pr}_V \\ \bigodot(V) & \xrightarrow{\quad a \quad} & V. \end{array}$$

For a morphism this object is a degree zero linear map

$$a\colon \bigodot(V) \longrightarrow W,$$

which admits a unique lift to a morphism from $\bigodot(V)$ to $\bigodot(W)$ if and only if $a_0 = 0$.

In particular we see that there is little difference in the object describing a coderivation of $\bigodot(V)$ and the object describing an endomorphism of $\bigodot(V)$. A difference only becomes evident when we try to lift these sequences, since the lifts as coderivation and as morphism are very different. It is therefore useful to make a visual distinction between these two possible lifts. We write $\bar{a}_{\mathrm{cod}}$ and $\bar{a}_{\mathrm{mor}}$ for the lifts of $(a_p)_{p\in\mathbf{N}_0}$ as coderivation and morphism, repectively.

When it comes to composing sequences of maps $(a_p)_{p\in\mathbf{N}_0}$ and $(b_p)_{p\in\mathbf{N}_0}$ it is important to understand what type of object these represent, i.e., if they are intended to describe morphisms or coderivations. Once the appropriate lift has been chosen, composition is just the usual composition of functions. Therefore, in principle, starting with two linear maps $a, b\colon \bigodot(V) \to V$, there are four possible ways to compose them. For each of a and b we must specify whether we want to lift it as a coderivation or a morphism, then we can compose the lifted maps and project back to V:

$$\begin{aligned} &\mathrm{pr}_V \circ \bar{a}_{\mathrm{mor}} \circ \bar{b}_{\mathrm{mor}}, && \mathrm{pr}_V \circ \bar{a}_{\mathrm{mor}} \circ \bar{b}_{\mathrm{cod}}, \\ &\mathrm{pr}_V \circ \bar{a}_{\mathrm{cod}} \circ \bar{b}_{\mathrm{mor}}, && \mathrm{pr}_V \circ \bar{a}_{\mathrm{cod}} \circ \bar{b}_{\mathrm{cod}}. \end{aligned}$$

Since $\mathrm{pr}_V \circ \bar{a}_{\mathrm{cod}} = \mathrm{pr}_V \circ \bar{a}_{\mathrm{mor}} = a$ the final expression for the maps above only depends on how b is lifted, but, this map does not carry in itself any information on whether it represents a morphism or a coderivation (along a morphism) or anything at all. This gives rise to two composition rules for linear maps from $\bigodot(V)$ to V:

$$a \,\square\, b = \mathrm{pr}_V \circ \bar{a}_* \circ \bar{b}_{\mathrm{cod}} = a \circ \bar{b}_{\mathrm{cod}}$$
$$a \diamond b = \mathrm{pr}_V \circ \bar{a}_* \circ \bar{b}_{\mathrm{mor}} = a \circ \bar{b}_{\mathrm{mor}},$$

where $\bar{a}_*$ can be either $\bar{a}_{\mathrm{cod}}$ or $\bar{a}_{\mathrm{mor}}$. More concretely, we obtain the expressions

$$a \,\square\, b(v_{\{1,\ldots,p\}}) = \sum_{I \sqcup J = \{1,\ldots,p\}} \epsilon_{I,J}\, a(b(v_I) \odot v_J) \tag{2.2.6}$$

and

$$a \diamond b(v_{\{1,\ldots,p\}}) = \sum_{I_1,\ldots,I_m} \epsilon_{I_1,\ldots,I_m}\, a\big(\textstyle\bigodot_{j=1}^m b(v_{I_j})\big). \tag{2.2.7}$$

by applying these to homogeneous elements $v_1, v_2, \ldots, v_p \in V$.

These same composition rules are also meaningful for maps $a\colon \bigodot(W) \to Z$ and $b\colon \bigodot(V) \to W$ between different graded vector spaces. The only restrictions are that the composition $a \,\square\, b$ is only defined $V = W$ and that $a \diamond b$ is only sensible if b is homogeneous of degree 0.

Both composition rules are also meaningful if a is a map from $\bigodot(W)$ to Z for two different $\mathbf{Z}$-graded vector spaces W and Z, and the latter is moreover well-defined.

With this notation at hand, we see that a number of equations we dealt with in the previous sections acquire a simpler form. For example the graded Lie bracket discussed in section 2.2.2 is given by

$$[\ell, \ell'] = \ell \,\square\, \ell' - (-1)^{|\ell||\ell'|} \ell' \,\square\, \ell.$$

In particular $\ell\colon \bigodot(V) \to V$ is an L_∞-algebra structure if ℓ has degree 1 and $[\ell, \ell] = 2\ell \,\square\, \ell = 0$. Phrased differently, a degree 1 map ℓ is an L_∞-algebra structure if and only if ℓ is a Maurer–Cartan element for the graded Lie algebra $(\mathrm{Lin}(\bigodot(V), V), [\,\cdot\,,\cdot\,])$. Furthermore, if (V, ℓ) and (W, ℓ') are L_∞-algebras, a map $f\colon \bigodot(V) \to W$ is a morphism of curved L_∞-algebras if and only if $\ell' \diamond f = f \,\square\, \ell$ and the composition of L_∞-algebra morphisms $f\colon \bigodot(V) \to W$ and $g\colon \bigodot(W) \to Z$ is $g \diamond f\colon \bigodot(V) \to Z$.

2.3. Topological L_∞-algebras

Although we have thus far viewed L_∞-algebras as purely algebraic objects, the ones we are interested come from geometry and thus carry additional (topological) information. We will incorporate this information by providing a topological version of Definition 2.2.1. This allows us to make sense of the Maurer–Cartan equation and gauge equivalence in cases where the number of summands is infinite.

The minimum we need to assume is that V has a locally convex topology since this allows us to define an appropriate topology on the graded symmetric algebra, $\bigodot(V)$, and use the corresponding notion of continuity for the structure maps ℓ_p. Some choices need to be made for the topology on tensor products and direct sums of locally convex vector spaces because the graded symmetric algebra $\bigodot(V)$ was defined as a direct sum of symmetric tensor products. These choices are only briefly mentioned here, and the reader is referred to appendix A.2 for further details.

Families of Maurer–Cartan elements are expected to give rise to analytic families of L_∞-algebras. For this to be true we have to sharpen our requirements and impose that each of the components V^n is in fact a Fréchet space. This is in line with the applications we have in mind, since many deformation problems that arise from geometry are formulated in terms of sections of vector bundles and differential operators.

2.3.1. Definition

Because we require topological versions of the direct sum and tensor product operations, which are discussed in appendix A.2, we will only work with locally convex vector spaces. These are topological vector spaces whose topology is generated by a collection of semi-norms and which we will moreover assume to be Hausdorff. Most interesting topological vector spaces that occur in geometry are locally convex, including finite-dimensional vector spaces, Banach spaces, Fréchet spaces and many spaces that can be obtained from these using standard constructions (such as subspaces, quotients (if Hausdorff), products, direct sums, limits and topological duals).

A locally convex $\mathbf{Z}$-graded vector space is a $\mathbf{Z}$-graded vector space $V = \bigoplus_{n\in\mathbf{Z}} V^n$ whose homogeneous components V^n for $n \in \mathbf{Z}$ are locally convex vector spaces. We endow the full space V with the locally convex direct sum topology induced by these subspaces, and the graded symmetric algebra $\bigodot(V)$ with the locally convex direct sum topology for the decomposition

$$\bigodot(V) = \bigoplus_{p\in\mathbf{N}_0} \bigoplus_{n_1\le\cdots\le n_p} V^{n_1} \odot V^{n_2} \odot \cdots \odot V^{n_p}, \tag{2.3.1}$$

where the symmetric tensor product $\bigodot_{j=1}^{p} V^{n_p}$ comes with the projective tensor product topology. A more detailed discussion of these topological constructions can be found in appendix A.2.

We note that both of the decompositions

$$\bigodot(V) = \bigoplus_{p\in\mathbf{N}_0} \bigodot^p(V) \qquad \text{and} \qquad \bigodot(V) = \bigoplus_{n\in\mathbf{Z}} \bigodot(V)^n, \tag{2.3.2}$$

corresponding to the $\mathbf{N}_0$-grading (by rank) and the $\mathbf{Z}$-grading (by total degree) on $\bigodot(V)$ respectively, are locally convex direct sums with respect to this topology.

Remark 2.3.1.
One can show that the canonical coalgebra structure $\Delta\colon \bigodot(V) \to \bigodot(V) \otimes \bigodot(V)$ described in section 2.2.2 is continuous if the latter space is endowed with the projective $\mathbf{Z}$-graded tensor product topology. Since also the counit $\varepsilon\colon \bigodot(V) \to \mathbf{R}$ is continuous, $(\bigodot(V), \Delta)$ is a topological $\mathbf{Z}$-graded coalgebra. △

We say that a graded vector space $V = \bigoplus_{n\in\mathbf{Z}} V^n$ is of Fréchet type if each of its homogeneous components, V^n for $n \in \mathbf{Z}$, is a Fréchet space. Similarly, we call V of finite-dimensional type, of Banach type or of nuclear Fréchet type if V^n is a space of the corresponding type for every $n \in \mathbf{Z}$. This does not imply that the direct sum topology on $\bigoplus_{n\in\mathbf{Z}} V^n$ is of the specified type (unless only finitely many of the spaces V^n are non-trivial).

This brings us to the types of L_∞-algebra we wish to consider.

Definition 2.3.2.
A (curved) L_∞-algebra of *locally convex type* of is a (curved) L_∞-algebra (V, ℓ) which consists of a $\mathbf{Z}$-graded locally convex vector space $V = \bigoplus_{n\in\mathbf{Z}} V^n$ and a continuous structure map $\ell\colon \bigodot(V) \to V$.

For us, the most important class of topological L_∞-algebras will be those that live on a graded vector space of Fréchet type.

Definition 2.3.3.
A (curved) L_∞-algebra of *Fréchet type* is a (curved) L_∞-algebra (V, ℓ) consisting of a $\mathbf{Z}$-graded vector space $V = \bigoplus_{n\in\mathbf{Z}} V^n$ of Fréchet type and a continuous structure map $\ell\colon \bigodot(V) \to V$.

L_∞-algebras of *finite-dimensional type*, of *Banach type*, and of *nuclear Fréchet type* type are defined analogously. Since these are all special cases of Definition 2.3.3, they will not be explicitly discussed in this chapter. For readers who are not familiar with general locally convex vector spaces, it may be useful to keep in mind that finite-dimensional spaces are also Banach spaces, that Banach spaces are also Fréchet spaces and that Fréchet spaces are by definition locally convex. The space of smooth sections of any finite-dimensional vector bundle is a nuclear Fréchet space.

There are several equivalent ways to formulate the compatibility of the structure map $\ell\colon \bigodot(V) \to V$ with the topology on V.

Proposition 2.3.4.
Let $V = \bigoplus_{n\in\mathbf{Z}} V^n$ be a $\mathbf{Z}$-graded locally convex vector space and let $\ell\colon \bigodot(V) \to V$ be a homogeneous linear map of degree $d \in \mathbf{Z}$. The following statements are equivalent.

(i) ***Continuity:*** *the map $\ell\colon \bigodot(V) \to V$ is continuous.*

(ii) ***Continuity by rank:*** *the restriction $\ell_p\colon \bigodot^p(V) \to V$ is continuous for each $p \in \mathbf{N}_0$.*

(iii) ***Continuity by degree:*** *the restriction $\ell|_{\bigodot(V)^n}\colon \bigodot(V)^n \to V^{n+d}$ is continuous for each $n \in \mathbf{Z}$.*

(iv) ***Continuity by component:*** *the restriction*

$$\ell_p\Big|_{\bigodot_{i=1}^p V^{n_i}}\colon \bigodot_{i=1}^p V^{n_i} \longrightarrow V^{n_1+\cdots+n_p+d}$$

is continuous for any sequence of degrees $n_1, n_2, \ldots, n_p \in \mathbf{Z}$.

(v) ***Continuity of multilinear maps:*** *the graded symmetric multilinear map*

$$\ell_p\colon V^{n_1} \times \cdots \times V^{n_p} \longrightarrow V^{n_1+\cdots+n_p+d}$$

is continuous for any sequence of degrees $n_1, n_2, \ldots, n_p \in \mathbf{Z}$.

(vi) ***Continuity of the coderivation:*** *the lift of ℓ as a coderivation $\bar{\ell}$: on $(\bigodot(V), \Delta)$ is continuous.*

Proof: The spaces $\bigodot(V)$ can be decomposed by rank, by degree or by component, as described in equations (2.3.1) and (2.3.2). Equivalence of statements (i), (ii), (iii) and (iv) thus follow directly from the universal property for the locally convex direct sum, as stated in Proposition A.2.14. Also statements (iv) and (v) are equivalent due to the universal property of the projective tensor product, which is described in Proposition A.2.36.

Statement (vi) implies statement (i) because the projection map $\mathrm{pr}_V : \bigodot(V) \to V$ is continuous, so we only need to show that one of the first five statements implies the last. Note that by the same argument used before, continuity of $\bar{\ell}\colon \bigodot(V) \to \bigodot(V)$ is equivalent to continuity of its restriction to $V^{n_1} \odot \cdots \odot V^{n_p}$ for every sequence of integers $n_1, n_2, \ldots, n_p \in \mathbf{Z}$ due to the nature of the direct sum topology on $\bigodot(V)$.

Now assume that statement (iv) holds and suppose we are given a sequence of degrees $n_1, n_2, \ldots, n_p \in \mathbf{Z}$. The coderivation $\bar{\ell}$ is described by equation (2.2.2), so its restriction to $\bigodot_{j=1}^{p} V^{n_j}$ can be written as a finite sum $\sum_{I \sqcup J = \{1,\ldots,p\}} \epsilon_{I,J}\, \bar{\ell}_{I,J}$, where $\bar{\ell}_{I,J}$ is the composition of

$$\ell_{\#I} \odot \mathrm{id}\colon \Big(\bigodot_{i\in I} V^{n_i}\Big) \odot \Big(\bigodot_{j\in J} V^{n_j}\Big) \longrightarrow V^{n_I + d} \odot \Big(\bigodot_{j\in J} V^{n_j}\Big)$$

and the continuous inclusion map $V^{n_I+d} \odot \big(\bigodot_{j\in J} V^{n_j}\big) \hookrightarrow \bigodot(V)$ for $n_I = \sum_{i\in I} n_i$. Since $\ell_{\#I}\colon \bigodot_{i\in I} V^{n_i} \to V^{n_I+d}$ is continuous by assumption, the maps $\bar{\ell}_{I,J}$ described above are as well by Proposition A.2.38. As a finite sum of such maps, also $\bar{\ell}|_{V^{n_1} \odot \cdots V^{n_p}}$ is continuous for any given sequence of integers, as is therefore the coderivation $\bar{\ell}\colon \bigodot(V) \to \bigodot(V)$.

We conclude that all six statements are equivalent. □

Using this proposition, we can also describe a locally convex (curved) L_∞-algebra in terms of a coderivation for the canonical coalgebra structure on $\bigodot(V)$.

Definition 2.3.5 (Alternative).
A *locally convex curved L_∞-algebra* is a $\mathbf{Z}$-graded locally convex vector space $V = \bigoplus_{n\in\mathbf{Z}} V^n$ endowed with a coderivation $\bar{\ell}\colon (\bigodot(V), \Delta) \to (\bigodot(V), \Delta)$ of degree 1 which is continuous and satisfies $[\bar{\ell}, \bar{\ell}] = 0$. It is uncurved if $\bar{\ell}(1) = 0$.

Both the space V and its topology can be recovered from the topological coalgebra $(\bigodot(V), \Delta)$ through the canonical embedding $V \hookrightarrow \bigodot(V)$ mentioned above Remark 2.2.7. We could thus have chosen to define a locally convex curved L_∞-algebra as a topological coalgebra which is (non-canonically) isomorphic to $(\bigodot(V), \Delta)$ and comes endowed with a degree 1 self-commuting coderivation.

Remark 2.3.6.
The tensor product $V \otimes W$ of two complete locally convex vector spaces is not generally complete (unless either V or W is finite-dimensional). This may be undesirable since one might, for instance, like the tensor product of two Fréchet to also be a Fréchet space. For this reason, the completed tensor product $V \bar{\otimes} W := \overline{V \otimes W}$ is often used instead. It is determined by the same universal property as the usual projective tensor product, but within the category of complete locally convex vector spaces.

One can similarly define the completed graded symmetric tensor products $\bigodot_{j=1}^{p} V$ of a complete locally convex vector space with itself and, subsequently, the completed graded symmetric algebra

$$\overline{\bigodot}(V) = \bigoplus_{n_1 \leq \cdots \leq n_p} \overline{\bigodot}_{j=1}^{p} V^{n_j}.$$

For the purpose of describing continuous multilinear or plurilinear maps of Fréchet type, it generally makes little difference which tensor product is used since the spaces $\mathrm{L}(\bigodot(V), W)$ and $\mathrm{L}(\overline{\bigodot}(V), W)$ are canonically isomorphic (as topological vector spaces). It should be noted, however, that $\overline{\bigodot}(V)$ does not carry a canonical coalgebra structure. $\triangle$

2.3.2. Morphisms

At this point, the definition of a morphisms of locally convex L_∞-algebras should not be surprising. We can simply take the original definition and impose compatibility of the underlying map with the chosen topologies.

Definition 2.3.7.
A *morphism* $f : (V, \ell) \to (W, \ell')$ of locally convex L_∞-algebras is a continuous homogeneous linear map $f : \bigodot(V) \to W$ such that $f(1) = 0$ and $f \mathbin{\square} \ell = \ell' \diamond f$.

Again, continuity of f can be expressed in many different ways. The proof of the following proposition is largely similar to that of Proposition 2.3.4.

Proposition 2.3.8.
Let V and W be $\mathbf{Z}$-graded locally convex vector spaces and let $f\colon \bigodot(V) \to W$ be a homogeneous linear map of degree 0 such that $f(1) = 1$. The following statements are equivalent.

(i) ***Continuity:*** *the map $f\colon \bigodot(V) \to W$ is continuous.*

(ii) ***Continuity by rank:*** *the restriction $f_p\colon \bigodot(V)^p \to W$ is continuous for each $p \in \mathbf{N}_0$.*

(iii) ***Continuity by degree:*** *the restriction $f|_{\bigodot(V)^n}\colon \bigodot(V)^n \to W^n$ is continuous for each $n \in \mathbf{Z}$.*

(iv) ***Continuity by component:*** *the restriction*

$$f_p\Big|_{\bigodot_{i=1}^p V^{n_i}}\colon \bigodot_{i=1}^p V^{n_i} \longrightarrow W^{n_1+\cdots+n_p}$$

is continuous for any sequence of degrees $n_1, n_2, \ldots, n_p \in \mathbf{Z}$.

(v) ***Continuity of multilinear maps:*** *the graded symmetric multilinear map*

$$f_p\colon V^{n_1} \times \cdots \times V^{n_p} \longrightarrow W^{n_1+\cdots+n_p}$$

is continuous for any sequence of degrees $n_1, n_2, \ldots, n_p \in \mathbf{Z}$.

(vi) ***Continuity of the morphism:*** *the lift of f as a coalgebra morphism $\bar{f}$ from $(\bigodot(V), \Delta)$ to $(\bigodot(W), \Delta)$ is continuous.*

Proof: Equivalence of conditions (i)–(v) was already proven in Proposition 2.3.4 and the proof of the implication (vi) ⇒ (i) can be copied verbatim from there. We will demonstrate the implication (iv) ⇒ (vi).

Assume that statement (iv) holds and suppose we are given a sequence of degrees $n_1, n_2, \ldots, n_p \in \mathbf{Z}$. The morphism $\bar{f}$ is described by equation (2.2.4), so its restriction to $\bigodot_{j=1}^p V^{n_j}$ can be written as a finite sum $\sum_{I_1,\ldots,I_m} \epsilon_{I_1,\ldots,I_m} \bar{f}_{I_1,\ldots,I_m}$, where summation is over the partitions of $\{1,\ldots,p\}$ and $\bar{f}_{I_1,\ldots,I_m}$ denotes the composition of

$$\bigodot_{j=1}^m f_{\#I_j}\colon \bigodot_{j=1}^m\Big(\bigodot_{i\in I} V^{n_i}\Big) \longrightarrow \bigodot_{j=1}^m W^{n_{I_j}}$$

with the continuous inclusion map $\bigodot_{j=1}^m W^{n_{I_j}} \hookrightarrow \bigodot(W)$, where $n_I = \sum_{i\in I} n_i$ for any subset $I \subseteq \{1,\ldots,p\}$. Since $f_{\#I}\colon \bigodot_{i\in I} V^{n_i} \to W^{n_I}$ is continuous by assumption, the maps $\bar{f}_{I_1,\dots,I_m}$ described above are as well by Proposition A.2.38. As a finite sum of such maps, also $\bar{f}|_{V^{n_1}\odot\cdots V^{n_p}}$ is continuous for any given sequence of integers, as is therefore the morphism $\bar{f}\colon \bigodot(V) \to \bigodot(W)$.

We conclude that all six statements are equivalent. □

Proposition 2.2.13 has a topological analogue.

Proposition 2.3.9 (Inverse function theorem)**.**
Let (V,ℓ) and (W,ℓ') be curved locally convex L_∞-algebras and let $f\colon (V,\ell) \to (W,\ell')$ be a continuous L_∞-algebra morphism. Then f is invertible if and only if its linear component, $f_1\colon V \to W$, is a topological isomorphism.

Proof: Let $f\colon (V,\ell) \to (W,\ell')$ be a morphism of curved L_∞-algebras such that $f_1\colon V \to W$ is a topological isomorphism. An inverse to f is a continuous linear map $g\colon \bigodot(W) \to V$ such that $\bar{g}_{\text{mor}} \circ \bar{f}_{\text{mor}} = \text{id}_{\bigodot(V)}$ and $\bar{f}_{\text{mor}} \circ \bar{g}_{\text{mor}} = \text{id}_{\bigodot(W)}$. Any such map is automatically itself a morphism of curved L_∞-algebras because the equations $\bar{f}_{\text{mor}} \circ \bar{\ell}_{\text{cod}} = \bar{\ell}'_{\text{cod}} \circ \bar{f}_{\text{mor}}$ and $\bar{\ell}_{\text{cod}} \circ \bar{f}^{-1}_{\text{mor}} = \bar{f}^{-1}_{\text{mor}} \circ \bar{\ell}'_{\text{cod}}$ are equivalent whenever $\bar{f}_{\text{mor}}$ is invertible.

We will first show that f admits a right inverse, for which we need to solve the equation $\bar{f}_{\text{mor}} \circ \bar{g}_{\text{mor}} = \text{id}_{\bigodot(W)}$ or, equivalently, $f \diamond g = \text{pr}_W$. This can be done recursively by considering the restrictions to $\bigodot^p(W)$ for $p \in \mathbf{N}$. For $p = 1$, the resulting equation states that $f_1 \circ g_1 = \text{id}_W$, which is uniquely solved by the (continuous) map $g_1 = f_1^{-1}$. For subsequent $p \in \mathbf{N}$, it reads

$$f_1\big(g_p(w_{\{1,\dots,p\}})\big) + \sum_{\substack{I_1,\dots,I_m\\ m\geq 2}} \tfrac{1}{m!}\,\epsilon_{I_1,\dots,I_m}\, f_m\big(\textstyle\bigodot_{j=1}^m g_{\#I_j}(w_{I_j})\big) = 0$$

for homogeneous $w_1, w_2, \dots, w_p \in W$ with $p \geq 2$. Since $\#I_1, \dots, \#I_m < p$ for every term of this sum and f_1 is a bijection, g_p is completely determined by the lower arity components $g_{p'}$ with $p' < p$. We can in fact read off that g_p for $p \geq 2$ is given by

$$g_p = -\sum_{\substack{I_1,\dots,I_m\\ m\geq 2}} f_1^{-1} \circ f_m \circ \big(\textstyle\bigodot_{j=1}^m g_{\#I_j}\big)$$

and thus obtain a unique sequence $g = (g_p)_{p\in\mathbf{N}}$ satisfying $\bar{f}_{\text{mor}} \circ \bar{g}_{\text{mor}} = \text{id}_{\bigodot(W)}$.

For any given $p \in \mathbf{N}$, continuity of g_p can be derived from continuity of f_1^{-1} and of $g_{p'}$ with $p' < p$ because compositions, tensor products and sums of continuous linear maps are continuous. Continuity of g therefore follows by induction. By interchanging the roles of f and g, a right inverse to $\bar{g}_{\mathrm{mor}}\colon \bigodot(W) \to \bigodot(V)$ can also be found. Since this inverse necessarily coincides with $\bar{f}_{\mathrm{mor}}$, we conclude that f is invertible and that g is its inverse.

If the linear component f_1 is not a topological isomorphism, then f is not invertible because the first order components of the equations $g \diamond f = \mathrm{pr}_V$ and $f \diamond g = \mathrm{pr}_W$ state that the continuous maps f_1 and g_1 must be mutually inverse. □

2.4. The Maurer–Cartan locus

In this section we will describe the Maurer–Cartan equation for L_∞-algebras of Fréchet type, as well as an appropriate notion of gauge equivalence for solutions to this equation. Before presenting the Maurer–Cartan equation itself in section 2.4.3, it makes sense to first discuss the twisting procedure for such L_∞-algebras. This is done in section 2.4.1 and section 2.4.2. A few results from chapter 3 are used in this section.

2.4.1. Twisting of L_∞-algebra structures

We will now discuss how to twist a curved L_∞-algebra (V, ℓ) of Fréchet type by a degree 0 element $u \in V^0$ to obtain a new curved L_∞-algebra (V, ℓ^u) of Fréchet type. As we will see, (V, ℓ^u) has vanishing curvature precisely when u is a Maurer–Cartan element.

One can readily verify that for any $u \in V^0$, the map

$$\overline{m}_u\colon \bigodot(V) \longrightarrow \bigodot(V), \qquad v_1 \odot \cdots \odot v_p \mapsto u \odot v_1 \odot \cdots \odot v_p$$

describing multiplication by u is a degree 0 coderivation. Heuristically, we can try to integrate it to a coalgebra automorphism,

$$e^{\overline{m}_u}\colon \bigodot(V) \longrightarrow \bigodot(V), \qquad v_1 \odot \cdots \odot v_p \mapsto \sum_{k \in \mathbf{N}_0} \tfrac{1}{k!} \bigodot^k u \odot v_1 \odot \cdots \odot v_p,$$

and use that to produce a new coderivation by conjugation: $\bar{\ell}^u = e^{-\overline{m}_u} \circ \bar{\ell} \circ e^{\overline{m}_u}$. We refer to this procedure as the *twisting* of the L_∞-algebra (V, ℓ) by u.

By expanding the associated map $\ell^u\colon \bigodot(V) \to V$ in its components, we obtain new structure maps $\ell^u_p\colon \bigodot(V) \to V$, given by

$$\ell^u_p(v_1 \odot \cdots \odot v_p) = \sum_{q \in \mathbf{N}_0} \tfrac{1}{q!} \ell_{p+q}(\textstyle\bigodot^q u \odot v_1 \odot \cdots \odot v_p) \tag{2.4.1}$$

for $v_1, v_2, \ldots, v_p \in V$. A version of this definition for nilpotent L_∞-algebras can for instance be found in section 4 of [Get09].

The problem with the twisting procedure, as it is outlined above, is that $e^{\overline{m}_u}$ is not a well-defined automorphism of $\bigodot(V)$ since it takes values in the product $\prod_{p \in \mathbf{N}_0} \bigodot^p(V)$, rather than in $\bigodot(V)$ itself. We can nevertheless use equation (2.4.1) to define new brackets on V if this series converges.

Proposition 2.4.1.
For any L_∞-algebra (V, ℓ) of Fréchet type and any $u \in V^0$, the following statements are equivalent.

- **(i)** *The series* (2.4.1) *converges for all $v_1, v_2, \ldots, v_p \in V$.*
- **(ii)** *The series*

$$\sum_{q \in \mathbf{N}_0} \tfrac{1}{q!} \ell(\textstyle\bigodot^q u \odot \bullet) \tag{2.4.2}$$

converges in $\mathrm{L}(\bigodot(V), V)$ *to a continuous linear map* $\ell^u\colon \bigodot(V) \to V$.

Before proving this proposition, we briefly recall that $\mathrm{L}(\bigodot(V), V)$ was defined, as a topological vector space, as the locally convex direct sum

$$\mathrm{L}(\textstyle\bigodot(V), V) = \displaystyle\bigoplus_{d \in \mathbf{Z}} \mathrm{L}(\textstyle\bigodot(V), V)^d$$

and that $\mathrm{L}(\bigodot(V), V)^d$ carries the topology of uniform convergence on precompact subsets, a definition for which can be found in Definition A.2.20. This topology can be characterised by the fact that the canonical map

$$\mathrm{L}(\textstyle\bigodot(V), V)^d \xrightarrow{\sim} \displaystyle\prod_{n_1 \leq \ldots \leq n_p} \mathrm{L}\big(\textstyle\bigodot_{j=1}^p V^{n_j}, V^{n_1 + \cdots + n_p + d}\big) \tag{2.4.3}$$

is an isomorphism of topological vector spaces for every $d \in \mathbf{Z}$.

Proof: The second statement implies the first because uniform convergence on precompact subsets is a stronger condition than pointwise convergence.

To demonstrate the converse, we first of all note that any element of $\bigodot(V)$ can be written as a linear combination of decomposable elements, i.e. elements of the form $v_{\{1,\dots,p\}}$ for homogeneous $v_1, v_2, \dots, v_p \in V$. Although this tells us that the series (2.4.2) converges pointwise to a linear map from $\bigodot(V)$ to V, we still need to show that it is convergent in $\mathrm{L}(\bigodot(V), V)$.

Note that each term in the series (2.4.2) is an element of the closed subspace $\mathrm{L}(\bigodot(V), V)^1 \subseteq \mathrm{L}(\bigodot(V), V)$. Because the map from equation (2.4.3) is a topological isomorphism, it therefore suffices to demonstrate convergence of the series

$$\sum_{q\in\mathbf{N}_0} \frac{1}{q!}\,\ell(\textstyle\bigodot^q u \odot \cdot)\big|_{\bigodot_{j=1}^p V^{n_j}}$$

in $\mathrm{L}(\bigodot_{j=1}^p V^{n_j}, V^{n_1+\dots+n_p+1})$ for arbitrary $n_1, n_2, \dots, n_p \in \mathbf{Z}$.

Since the components V^n for $n \in \mathbf{Z}$ are Fréchet spaces, they are in particular metrisable barrelled vector spaces. Consequently, the projective tensor product $\bigotimes_{j=1}^p V^{n_j}$ is metrisable and barrelled as well for any sequence $n_1, n_2, \dots, n_p \in \mathbf{Z}$ due to Proposition A.2.43. Since $\bigodot_{j=1}^p V^{n_j}$ is a quotient of $\bigotimes_{j=1}^p V^{n_j}$, also this space is barrelled.

Now assume that statement (i) holds. Since $\bigodot_{j=1}^p V^{n_j}$ is a barrelled vector space, the Banach–Steinhaus theorem (Theorem A.2.28) tells us that the series (2.4.2) converges with respect to the topology of uniform convergence on precompact subsets of $\bigodot_{j=1}^p V^{n_j}$. We conclude that the original series is convergent in $\mathrm{L}(\bigodot(V), V)^1 \subseteq \mathrm{L}(\bigodot(V), V)$. □

Due to the nature of its construction, it seems reasonable to expect the twisted bracket ℓ^u to satisfy the Jacobi identity $[\ell^u, \ell^u] = 0$, which would mean that (V, ℓ^u) is a also curved L_∞-algebra of Fréchet type. We will see that this indeed the case in Proposition 2.4.4.

2.4.2. Analytic dependence on the twisting parameter

Now that we have seen that the twisting procedure provides us with new structure maps $\ell^u\colon \bigodot(V) \to V$ for some $u \in V^0$, it seems natural to ask how ℓ^u

depends on the parameter $u \in V^0$. We will see that these twisted structure maps form a real analytic family of curved L_∞-algebras.

Definition 2.4.2.
The *domain of convergence* $D_\ell \subseteq V^0$ of a curved L_∞-algebra (V, ℓ) of Fréchet type is the interior of the set of all $u \in V^0$ for which the series

$$\ell^u = \sum_{q \in \mathbf{N}_0} \tfrac{1}{q!}\, \ell(\textstyle\bigodot^q u \odot \bullet)$$

converges. We call the (V, ℓ) *convergent* if $D_\ell \neq \varnothing$ and *entire* if $D_\ell = V^0$.

When we say that the series defining ℓ^u converges we mean that it converges in $\mathrm{L}(\bigodot(V), V)$ with respect to the topology of uniform convergence on precompact subsets of $\bigodot(V)$. We have however seen in Proposition 2.4.1 that this is equivalent to pointwise convergence. The set D_ℓ thus consists of all $u \in V^0$ that admit a neighbourhood $U \subseteq V^0$ such that the series (2.4.1) converges for all $u' \in U$ and all $v_1, v_2, \ldots, v_p \in V$.

If the domain of convergence is non-empty it is a neighbourhood of $0 \in V^0$ due to Proposition B.1.19. If only finitely many of the brackets $(\ell_p)_{p \in \mathbf{N}_0}$ are non-trivial, which is not uncommon, then the series (2.4.1) terminates and (V, ℓ) is consequently entire.

Proposition 2.4.3.
For any curved L_∞-algebra (V, ℓ) of Fréchet type with domain of convergence $D_\ell \subseteq V^0$, the map

$$\mathfrak{l} \colon D_\ell \longrightarrow \mathrm{L}(\textstyle\bigodot(V), V), \qquad u \mapsto \ell^u$$

is real analytic (and therefore smooth) and its derivatives are given by

$$\mathrm{D}^k \mathfrak{l}(u)(\textstyle\bigodot^k \dot{u})(v_{\{v_1,\ldots,v_p\}}) = \mathfrak{l}(u)(\textstyle\bigodot^k \dot{u} \odot v_{\{1,\ldots,p\}}) \tag{2.4.4}$$

for $k \in \mathbf{N}_0$, $u \in D_\ell$, $\dot{u} \in V^0$ and $v_1, \ldots, v_p \in V$.

Proof: Although the map $\mathfrak{l} \colon D_\ell \to \mathrm{L}(\bigodot(V), V)$ was defined as the limit of a formal power series, we have not yet shown that the individual terms of this series depend continuously on the parameter u. If we can prove this, analyticity of $\mathfrak{l}$ will follow from Proposition B.1.22.

By employing the embedding from equation (2.4.3), we see that continuity of the q-th term of the series (2.4.2) defining $\mathfrak{l}$ is equivalent to continuity of the map

$$\mathfrak{l}_{q;n_1,\dots,n_p} : D_\ell \longrightarrow \mathrm{L}(\textstyle\bigodot_{j=1}^p V^{n_j}, V^{n+1}), \qquad u \mapsto \ell_{p+q}(\textstyle\bigodot^q u \odot \bullet)$$

for all sequences of degrees $n_1, n_2, \dots, n_p \in \mathbf{Z}$ and $n = \sum_{j=1}^p n_j$. This in turn can be viewed as a consequence of the fact that ℓ_{p+q} is continuous as a map from $\bigodot^q V^0 \otimes \bigodot_{j=1}^p V^{n_j}$ to V^{n+1} and the fact that the spaces

$$\mathrm{L}\big(\textstyle\bigodot^q V^0 \otimes \bigodot_{j=1}^p V^{n_j}, V^{n+1}\big) \simeq \mathrm{L}\big(\textstyle\bigodot^q V^0, \mathrm{L}(\bigodot_{j=1}^p V^{n_j}, V^{n+1})\big)$$

are canonically isomorphic as topological vector spaces (cf. Corollary A.2.48).

Because $\mathfrak{l}$ is analytic, Proposition B.1.25 tells us that it is smooth and that its derivatives can be computed by adding up the derivatives of the individual terms in the defining power series. Since $u \mapsto \ell(\bigodot^q u \odot \bullet)$ is a homogeneous polynomial of degree q, its derivatives are easy to compute. The k-th order derivative at u in the direction $\dot{u}$ is given by

$$\tfrac{q!}{(q-k)!}\, \ell_{p+q}(\textstyle\bigodot^k \dot{u} \odot \bigodot^{q-k} u \odot \bullet).$$

We deduce that the derivatives of $\mathfrak{l}$ are given by

$$\begin{aligned} \mathrm{D}^k \mathfrak{l}(u)(\textstyle\bigodot^k \dot{u})(v_{\{1,\dots,p\}}) &= \displaystyle\sum_{q\in\mathbf{N}_0} \tfrac{1}{(q-k)!}\, \ell_{p+q}(u^q \odot \textstyle\bigodot^k \dot{u} \odot v_1 \odot \dots \odot v_p) \\ &= \mathfrak{l}(u)(\textstyle\bigodot^k \dot{u} \odot v_1 \odot \dots \odot v_p). \end{aligned}$$

for any $u \in D_\ell$, any $\dot{u} \in V^0$ and $v_1, v_2, \dots, v_p \in V$. ☐

We shall refer to the identity (2.4.4) as the *derivative property*. In terms of the coalgebra description from section 2.2.2, it states that the derivative of $u \mapsto \bar{\ell}^u$ in the direction $\dot{u} \in V^0$ is $[\bar{\ell}^u, \bar{m}_{\dot{u}}]$, the graded commutator of the coderivations $\bar{\ell}^u$ and $\bar{m}_{\dot{u}}$. This is essentially how the twisted structure map $\bar{\ell}^u$ was (heuristically) defined at the start of this section.

The derivative of the Jacobiator $\mathrm{Jac}(\bar{\ell}^u) = [\bar{\ell}^u, \bar{\ell}^u]$ satisfies the same equation, since

$$\tfrac{\mathrm{d}}{\mathrm{d}t}[\bar{\ell}^{u_t}, \bar{\ell}^{u_t}] = [[\bar{\ell}^{u_t}, \bar{m}_{\dot{u}_t}], \bar{\ell}^{u_t}] + [\bar{\ell}^{u_t}, [\bar{\ell}^{u_t}, \bar{m}_{\dot{u}_t}]] = [[\bar{\ell}^{u_t}, \bar{\ell}^{u_t}], m_{\dot{u}_t}]$$

for any path $t \mapsto u_t$ in V^0 with derivative $\dot{u}_t = \frac{\mathrm{d}}{\mathrm{d}t} u_t$. To make this argument rigorous we would need to choose an appropriate topology on the subspace of $\mathrm{L}(\bigodot(V), \bigodot(V))$ in which these equations take values. We instead use it merely as a motivation for the proof of the following proposition.

Proposition 2.4.4.
For any (curved) L_∞-algebra (V, ℓ) of Fréchet type and any $u \in D_\ell$, the twisted brackets ℓ_p^u define a curved L_∞-algebra structure on V.

Proof: To prove this statement we will show that derivative property for the structure maps implies that the associated Jacobiators have the same property. Analyticity of the maps involved will then allow us to conclude that the twisted structure maps also satisfy the Jacobi identities.

For fixed indices $n_1, n_2, \dots, n_p \in \mathbf{Z}$, consider the map $\mathrm{Jac}(\check{\mathfrak{l}})_p \colon U \times \bigodot_{j=1}^n V^{i_j} \to V^n$ with $n = \sum_{i=1i}^n n_i$ given by

$$\mathrm{Jac}(\check{\mathfrak{l}})_p(u, v_1 \odot \cdots \odot v_n) = \sum_{I \sqcup J = \{1,\dots,n\}} \epsilon_{I,J}\, \mathfrak{l}(u)\big(\mathfrak{l}(u)(u_I)\, u_J\big)$$

for $u \in D_\ell$ and $v_i \in V^{n_i}$ for $i = 1, 2, \dots, p$. It follows from Proposition 3.1.14 and Proposition B.1.28 that this map is analytic, and from Proposition B.1.25 that it is smooth. Its derivative with respect to the first argument is given by

$$\begin{aligned}\mathrm{D}_1\mathrm{Jac}(\check{\mathfrak{l}})_p(u, v_1 \odot \cdots \odot v_p)(v_0) &= \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J} \Big(\mathfrak{l}(u)\big(v_0 \odot \mathfrak{l}(u)(v_I) \odot v_J\big) \\ &\qquad\qquad + \mathfrak{l}(u)\big(\mathfrak{l}(u)(v_0 \odot v_I) \odot v_J\big)\Big) \\ &= \sum_{\tilde{I} \sqcup \tilde{J} = \{0,1,\dots,p\}} \epsilon_{\tilde{I},\tilde{J}}\, \mathfrak{l}(u)\big(\mathfrak{l}(u)(v_{\tilde{I}})\, v_{\tilde{J}}\big),\end{aligned}$$

because $\mathfrak{l}$ has the derivative property. This coincides with $\mathrm{Jac}(\check{\mathfrak{l}})_{p+1}(u, v_0 \odot v_1 \odot \cdots \odot v_p)$, so it follows by induction on k that

$$\mathrm{D}_1^k\mathrm{Jac}(\check{\mathfrak{l}})_p(u, v_1 \odot \cdots \odot v_p)(\dot{u}) = \mathrm{Jac}(\check{\mathfrak{l}})_{p+k}(u, \textstyle\bigodot^k \dot{u} \odot v_1 \odot \cdots \odot v_p)$$

for every $p, k \in \mathbf{N}_0$, $u \in D_\ell$ and $v_0 \in V^0$ and any p-tuple $v_1, v_2, \dots, v_p \in V$.

The fact that (V, ℓ) is a curved L_∞-algebra means that $\mathrm{Jac}(\check{\mathfrak{l}})_p(0, \cdot) = 0$ for every $p \in \mathbf{N}_0$. Consequently, also the derivatives $\mathrm{D}_1^k\mathrm{Jac}(\check{\mathfrak{l}})_p$ vanish at 0 for every $p \in \mathbf{N}_0$ and all $k \in \mathbf{N}_0$. By analyticity, it now follows that $\mathrm{Jac}(\check{\mathfrak{l}})_p(u, v_1 \odot \cdots \odot v_p) = 0$ for every $u \in D_\ell$ and all $v_1, v_2, \dots, v_p \in V$. ☐

The proof of Proposition 2.4.4 relies on the analytic dependence of ℓ^u on u. An entirely algebraic proof can also be provided if the series (2.4.1) terminates for every $p \in \mathbf{N}_0$.

2.4.3. The Maurer–Cartan equation

The Maurer–Cartan equation arises naturally when one examines the twists of any given curved L_∞-algebra. For $p = 0$, equation (2.4.1) describes a real analytic function $\mu\colon D_\ell \to V^1$ which is given by $\mu(u) = \ell_0^u$ or, more concretely, by

$$\mu(u) = \sum_{p=0}^{\infty} \tfrac{1}{p!} \ell_p(\textstyle\bigodot^p u). \tag{2.4.5}$$

This map will henceforth be referred to as the *Maurer–Cartan map*. It assigns to a point u in D_ℓ the curvature of the curved L_∞-algebra (V, ℓ^u) obtained from (V, ℓ) through twisting by u.

Definition 2.4.5.

A *Maurer–Cartan element* for (V, ℓ) is an element $u \in D_\ell \subseteq V^0$ satisfying the *Maurer–Cartan equation*,

$$\mu(u) := \sum_{p \in \mathbf{N}_0} \tfrac{1}{p!} \ell_p(\textstyle\bigodot^p u) = 0. \tag{2.4.6}$$

We denote the set of Maurer–Cartan elements by $\mathrm{MC}(V, \ell) := \mu^{-1}(0)$.

We observe that the zero element $0 \in V^0$ satisfies the Maurer–Cartan equation precisely when (V, ℓ) is uncurved. It is however only a Maurer–Cartan element in the sense of Definition 2.4.5 if the domain of converge is non-empty.

Because we know from Proposition 2.4.3 that the derivative of the Maurer–Cartan map is given by

$$\mathrm{D}\mu(u)(\dot{u}) = \ell_1^u(\dot{u}),$$

for $u \in D_\ell$ and $\dot{u} \in V^0$, we can think of the kernel of $\ell_1^u\colon V^0 \to V^1$ as (a candidate for) the tangent space to $\mathrm{MC}(V, \ell)$ at u. Although the tangent space to the Maurer–Cartan locus is contained in this kernel at any point where it is a manifold, it need not coincide.

2.4.4. Gauge equivalence

The Maurer–Cartan locus of a curved L_∞-algebra (V,ℓ) of Fréchet type is partitioned by gauge equivalence classes. These equivalence classes are essentially the leaves of the foliation corresponding to the distribution Δ on $\mathrm{MC}(V,\ell)$ given by $\Delta_u = \mathrm{im}(\ell_1^u|_{V^{-1}}) \subseteq V^0$ at $u \in \mathrm{MC}(V,\ell)$.

While this can be made precise for L_∞-algebras of finite-dimensional type, we will for now only use this as motivation for the following definition.

Definition 2.4.6.
A smooth *homotopy* of Maurer–Cartan elements is a pair $\gamma + \eta\, \mathrm{d}t$ of smooth paths,

$$\gamma\colon [0,1] \longrightarrow \mathrm{MC}(V,\ell) \qquad \text{and} \qquad \eta\colon [0,1] \longrightarrow V^{-1},$$

satisfying the differential equation

$$\gamma'(t) + \ell_1^{\gamma(t)} \eta(t) = 0$$

for all $t \in [0,1]$. Two Maurer–Cartan elements u_0 and u_1 are *gauge equivalent* if there exist a homotopy $\gamma + \eta\, \mathrm{d}t$ with end points $\gamma(0) = u_0$ and $\gamma(1) = u_1$.

This definition states that two Maurer–Cartan elements are gauge equivalent whenever they can be connected by a path that is tangent to the distribution described above and for which a corresponding family of primitives can be found.

Instead of the standard interval $[0,1]$, any other closed interval $[a,b]$ with $a < b$ could have been used. We observe that homotopies can be reparametrised: given a homotopy $u + v\, \mathrm{d}t$ and a smooth map $f : [a,b] \to [c,d]$ with $f(a) = c$ and $f(b) = d$, the pull-back $f^*(\gamma + \eta\, \mathrm{d}t) := \gamma \circ f + \eta \circ f \ \cdot f'\, \mathrm{d}t$ is a homotopy with the same end points.

Before moving on, we will quickly verify that gauge equivalence, as defined above, is an equivalence relation. Reflexivity, symmetry and transitivity correspond to the following facts:

- The constant path $t \mapsto u + 0\, \mathrm{d}t$ is a homotopy from u to itself for any $u \in \mathrm{MC}(V,\ell)$.
- If $\gamma + \eta\, \mathrm{d}t$ is a homotopy from u_0 to u_1 and $f : [0,1] \to [0,1]$ is given by $f(t) = 1 - t$, then $f^*(\gamma + \eta\, \mathrm{d}t)$ is a homotopy from u_1 to u_0.

- If $\gamma + \eta\,\mathrm{d}t$ is a homotopy and $f\colon [0,1] \to [0,1]$ is such that $f(0) = 0$, $f(1) = 1$ and all derivatives of f vanish at 0 and 1, then $\tilde{\gamma} + \tilde{\eta}\,\mathrm{d}t = f^*(\gamma + \eta\,\mathrm{d}t)$ is such that $\tilde{\gamma}(t) = \gamma(t)$ and $\tilde{\eta}(t) = 0$ for $t = 0, 1$ and all derivatives of both $\tilde{\gamma}$ and $\tilde{\eta}$ vanish at the endpoints. Homotopies of this type can be concatenated.

One could alternatively work with piecewise smooth homotopies to make the argument for transitivity more straightforward. These are pairs $\gamma + \eta\,\mathrm{d}t$ consisting of a continuous piece-wise smooth path $\gamma\colon [a,b] \to \mathrm{MC}(V,\ell)$ and a piece-wise smooth (but not necessarily continuous) path $\eta\colon [a,b] \to V^{-1}$ satisfying the equation $\gamma'(t) + \ell_1^{\gamma(t)}\eta(t)$ in all but finitely many points.

Given a Maurer–Cartan element $u_0 \in \mathrm{MC}(V,\ell)$, we shall refer to the equivalence class $[u_0] \subseteq \mathrm{MC}(V,\ell)$ as the (gauge) *orbit* of u_0.

Example 2.4.7 (Differential graded Lie algebras)**.**
If the L_∞-algebra (V,ℓ) corresponds to a differential graded Lie algebra, then (V^{-1},ℓ_2) is a Lie algebra and

$$\ell_1\colon V^{-1} \times D_\ell \longrightarrow V^0,$$

describes a Lie algebra action. If V is of finite-dimensional type, then this action integrates to a local action of the simply connected Lie group integrating (V^{-1},ℓ_2) on D_ℓ and two Maurer–Cartan elements are gauge equivalent if and only if they are in the same orbit for this action. ◊

CHAPTER 3.

Holonomic families of L_∞-algebras

We wish to demonstrate that every equivariant deformation problem of Fréchet type gives rise to an L_∞-algebra that can be said to control its deformations in some sense. We will not just obtain a single L_∞-algebra, however, but a family of curved L_∞-algebras that change smoothly as the reference structure is varied. The families that can arise in this way can be viewed as jets of some of their components, which is why we call them holonomic. To understand the relation between deformation problems and L_∞-algebras, it will be helpful to first study families of this type.

We have already seen a holonomic family of curved L_∞-algebras in chapter 2, since the family of twists of a curved L_∞-algebra has this property. Holonomic families will generally resemble this example, and we will see that every analytic holonomic family of curved L_∞-algebras can in fact locally be realised as the family of twists of one of its members. It will eventually be proven in chapter 5 that holonomic families of curved L_∞-algebras of Fréchet type are locally equivalent to curved L_∞-algebroids of the same type.

Before we can talk about holonomic families, however, we will need to have a clear understanding of what it means for a family of multilinear maps between locally convex vector spaces to be smooth or analytic. This is the subject of section 3.1. While this theory may be neither very surprising nor very exciting, it is important to be thorough here, since it will lay the foundation for things to come in the upcoming chapters. Once we have dealt with this, we can introduce holonomic families of plurilinear maps and discuss their description as families of coderivations and coalgebra morphisms in section 3.2. The category of holonomic families of curved L_∞-algebras of Fréchet type is introduced in section 3.3, where also different types of morphisms between its objects are discussed.

3.1. Families of linear maps

In this section we will develop the tools that are necessary to be able to work with differentiable families of continuous linear maps. By a family of continuous linear maps from a locally convex vector space V to another such space W we mean a function

$$\alpha\colon X \longrightarrow \mathrm{L}(V,W)$$

from a set X to the space $\mathrm{L}(V,W)$ of continuous linear maps from V to W. We endow this space with the topology of precompact convergence of V, which is defined in Definition A.2.19, and use the corresponding notions of continuity, differentiability and analyticity whenever X is a space of an appropriate type (either a topological space or a smooth or analytic Fréchet manifold).

We shall start this section with a number of results for continuous families of continuous linear maps, which we are not interested in per se, but which we need to be able to prove similar assertions for differentiable families. We will show that there are several equivalent ways to characterise smooth and analytic families of continuous linear maps, and that smoothness and analyticity are preserved by most elementary operations. Families of multilinear maps are discussed as well, since these will be used frequently in the remainder of this chapter, as well as in chapters 4 and 5.

3.1.1. Continuous families of linear maps

Let X be a topological space and let V and W be arbitrary locally convex vector spaces. The space $\mathrm{L}(V,W)$ of continuous linear maps from V to W is endowed with the topology of precompact convergence from Definition A.2.19. A basis of zero neighbourhoods for this topology is given by the subsets

$$\mathscr{N}(K_V,U_W) = \left\{A \in \mathrm{L}(V,W) \,\middle|\, A(K_V) \subseteq U_W\right\}$$

determined by precompact subsets $K_V \subseteq V$ and (absolutely convex) zero neighbourhoods $U_W \subseteq W$. A family $\alpha\colon X \to \mathrm{L}(V,W)$ of continuous linear maps is consequently continuous at $x \in X$ if and only if for every zero neighbourhood $U_W \subseteq W$ and every precompact subset $K_V \subseteq V$ there exists a neighbourhood $U_X \subseteq X$ of x such that $\alpha(x)(v) \in U_W$ for every $x \in K_U$ and every $v \in K_V$.

The space $\mathrm{C}^0(X, \mathrm{L}(V, W))$ of continuous families of linear maps parametrised by X is endowed with the compact-open topology. This is a locally convex vector space topology, and a basis of zero neighbourhoods for it is provided by the subsets $\mathscr{N}(K_X, \mathscr{N}(K_V, N_W))$ for $K_X \subseteq X$ compact, $K_V \subseteq V$ precompact and for zero neighbourhoods $N_W \subseteq W$.

If both X and V are metrisable, continuity of a family of continuous linear maps from V to W parametrised by X can be characterised in several reasonable and equivalent ways.

Lemma 3.1.1.
Let X be a metrisable space, let V be a metrisable locally convex vector space and let W be a locally convex vector space. For any function $\alpha\colon X \to \mathrm{L}(V, W)$, the following statements are equivalent:

- **(i)** *α is continuous;*
- **(ii)** *the map $\check{\alpha}\colon X \times V \to W$ given by $\check{\alpha}(x, v) = \alpha(x)(v)$ is continuous;*
- **(iii)** *there exists a (unique) continuous linear map $\hat{\alpha}\colon \mathrm{C}^0(X, V) \to \mathrm{C}^0(X, W)$ such that $\hat{\alpha}(e)(x) = \alpha(x)(e(x))$ for all $e \in \mathrm{C}^0(X, V)$ and every $x \in X$.*

Proof: We will first show that statements (i) and (ii) are equivalent.

Assume that $\check{\alpha}$ is continuous, pick a point $x \in X$ and let $\mathscr{N}(K_V, U_W) \subseteq \mathrm{C}^0(V, W)$ be a zero neighbourhood that is determined by some precompact subset $K_V \subseteq V$ and an absolutely convex open zero neighbourhood $U_W \subseteq W$. To show that α is continuous, we need to find a neighbourhood of $U_X \subseteq X$ of x such that $\alpha(U_X) \subseteq \alpha(x) + \mathscr{N}(K_V, U_W)$.

The continuity of $\check{\alpha}$, along with the fact that $\check{\alpha}(x, 0) = 0$, provides us with a neighbourhood $U_{X,0} \times U_{V,0} \subseteq X \times V$ of $(x, 0)$ such that $\check{\alpha}(U_X^0 \times U_V^0) \subseteq \frac{1}{3}\, U_W$. The fact that K_V is precompact then allows us to pick elements $y_1, y_2, \ldots, y_N \in K_V$ for some $N \in \mathbf{N}_0$ such that $K_V \subseteq \bigcup_{i=1}^N y_i + U_V^0$. Let the neighbourhoods $U_X^{y_i} \subseteq X$ of x be such that $\check{\alpha}(U_X^{y_i} \times \{y_i\}) \subseteq \frac{1}{3}\, U_W$ for $i = 1, 2, \ldots, N$ and define $U_X = U_{X,0} \cap \bigcap_{i=1}^N U_X^{y_i}$ as the intersection of all of these neighbourhoods.

Given $x' \in U_X$ and $y \in K_V$, let $i \in \{1, 2, \dots, N\}$ be such that $y \in y_i + U_{V,0}$. Now, since $\check{\alpha}$ is linear in its second argument,

$$\big(\alpha(x') - \alpha(x)\big)(y) = \check{\alpha}(x', y - y_i) - \check{\alpha}(x, y - y_i) + \big(\check{\alpha}(x', y_i) - \check{\alpha}(x, y_i)\big).$$

We observe that $\check{\alpha}(x', y - y_i)$ and $\check{\alpha}(x, y - y_i)$ are both elements of $\frac{1}{3}\,U_W$ since $(x', y - y_i), (x, y - y_i) \in U_{X,0} \times U_{V,0}$ and that $\check{\alpha}(x', y_i) - \check{\alpha}(x, y_i) \in \frac{1}{3}\,U_W$ because $x' \in U_X^{y_i}$. Consequently,

$$\big(\alpha(x') - \alpha(x)\big)(y) \in \tfrac{1}{3}\,U_W - \tfrac{1}{3}\,U_W + \tfrac{1}{3}\,U_W \subseteq U_W$$

for all $x' \in U_X$ and all $y \in K_V$, which implies that $\alpha(x') - \alpha(x) \in \mathscr{N}(K_V, U_W)$ for all $x' \in U_X$. Since $x \in X$ was arbitrary and any neighbourhood of $\alpha(x)$ contains one of the form $\alpha(x) + \mathscr{N}(K_V, U_W)$, continuity of α follows.

To prove the converse, assume that α is continuous and let $(x_n, y_n)_{n \in \mathbf{N}}$ be a convergent sequence in $X \times V$ with limit (x, y). Let $U_W \subseteq W$ be a zero neighbourhood in W and let $K_V = \{y_n \,|\, n \in \mathbf{N}\} \cup \{y\}$. Since $K_V \subseteq V$ is compact it is also precompact, and we can thus find an index $n_0 \in \mathbf{N}$ such that $\alpha(x_n) - \alpha(x) \in \mathscr{N}(K_V, \frac{1}{2}\,U_W)$ for all $n \geq n_0$, which in particular implies that $\check{\alpha}(x_n, y_n) - \check{\alpha}(x, y_n) \in \frac{1}{2}\,U_W$ for such n. Continuity of $\alpha(x)$ moreover tells us that we can choose an $n_1 \geq n_0$ such that $\alpha(x)(y_n) - \alpha(x)(y) \in \frac{1}{2}\,U_W$ whenever $n \geq n_1$. Now, given $n \in \mathbf{N}$,

$$\check{\alpha}(x_n, y_n) - \check{\alpha}(x, y) = \big(\check{\alpha}(x_n, y_n) - \check{\alpha}(x, y_n)\big) + \big(\check{\alpha}(x, y_n) - \check{\alpha}(x, y)\big),$$

which is an element of $\frac{1}{2}\,U_W + \frac{1}{2}\,U_W \subseteq U_W$ if $n \geq n_1$. The neighbourhood U_W was arbitrary, so $\check{\alpha}(x_n, y_n)$ in fact converges to $\check{\alpha}(x, y)$ as $n \to \infty$. Because $X \times V$ is metrisable and $\check{\alpha}$ maps convergent sequences in $X \times V$ to convergent sequences in W, we deduce that $\check{\alpha}$ is indeed continuous.

We will now show that statement (ii) is also equivalent to statement (iii).

Assume that the $\check{\alpha}$ is continuous, then the identity $\hat{\alpha}(e)(x) = \alpha(x, e(x))$ describes a function $\hat{\alpha}$ from $\mathrm{C}^0(X, V)$ to $\mathrm{C}^0(X, W)$ for elementary reasons. Let $\mathscr{N}(K_X, U_W) \subseteq \mathrm{C}^0(X, W)$ be the zero neighbourhood determined by some compact subset $K_X \subseteq X$ and a convex open zero neighbourhood $U_W \subseteq W$. For any $x \in K_X$ we can find a neighbourhood $U_X^x \times U_V^x \subseteq X \times V$ of $(x, 0)$ such that $\check{\alpha}(U_X^x \times U_V^x) \subseteq U_W$. Since K_X is compact, we can pick a finite number of point

$x_1, x_2, \dots, x_N$ such that $K_X \subseteq \bigcup_{i=1}^{N} U_X^{x_i}$ and consider the corresponding intersection $U_V := \bigcap_{i=1}^{N} U_V^{x_i}$. Now, $\check{\alpha}(x, v) \in U_W$ for every $x \in K_X$ and every $v \in U_V$, which means that

$$\hat{\alpha}(\varphi)(x) = \check{\alpha}(x, \varphi(x)) \in U_W$$

for all $x \in K_X$ and all $\varphi \in \mathcal{N}(K_X, U_V)$. We infer that $\hat{\alpha}(\varphi) \in \mathcal{N}(K_X, U_W)$ whenever $\varphi \in \mathcal{N}(K_X, U_V)$ and hence that $\hat{\alpha}$ is continuous, proving that statement (ii) implies statement (iii).

Now assume that statement (iii) holds and consider the map $\check{\alpha}_V : V \to \mathrm{C}^0(X, V)$ given by $\check{\alpha}_V(v)(x) = \hat{\alpha}(c_v)$, where $c_v \in \mathrm{C}^0(X, V)$ denotes the constant map $x \mapsto v$. This map is continuous because both $\hat{\alpha}$ and the inclusion map $v \mapsto c_v$ are. Continuity of $\check{\alpha}$ now follows from Proposition B.2.1.

We conclude that all three statements are equivalent. ☐

Equivalence of the first two statements in Lemma 3.1.1 can also be deduced from Lemma 0.1.2 and Theorem 0.3.8 of [Kel74]. It is worth noting the metrisability of V was not used to prove either of the implications (ii) ⇒ (i) and (ii) ⇒ (iii) in this lemma.

An even simpler characterisation of the continuity of α can be used if V is barrelled. The following proposition essentially coincides with Corollary A.2.29 to the Banach–Steinhaus theorem.

Proposition 3.1.2.
Let X be a metrisable space, let V be a barrelled locally convex vector space (e.g. a Fréchet space) and let W be a locally convex vector space. A function $\alpha : X \to \mathrm{L}(V, W)$ is continuous if and only if the map

$$\alpha(\cdot)(v) : X \longrightarrow W, \qquad x \mapsto \alpha(x)(v)$$

is continuous for every $v \in V$.

Even though the composition map $\circ : \mathrm{L}(W, Z) \times \mathrm{L}(V, W) \to \mathrm{L}(V, Z)$ is seldom continuous, the composition of two continuous families is continuous whenever the parameter space is metrisable.

Proposition 3.1.3.

Let X be a metrisable space and let V, W and Z be locally convex vector spaces. Given two continuous maps $\alpha\colon X \to \mathrm{L}(V,W)$ and $\beta\colon X \to \mathrm{L}(W,Z)$, the composition

$$\beta\circ\alpha\colon X \longrightarrow \mathrm{L}(V,Z), \qquad (\beta\circ\alpha)(x)(v) = \big(\beta(x)\circ\alpha(x)\big)(v)$$

is continuous as well. Moreover, the composition map

$$\circ\colon \mathrm{C}^0(X,\mathrm{L}(W,Z))\times\mathrm{C}^0(X,\mathrm{L}(V,W)) \longrightarrow \mathrm{C}^0(X,\mathrm{L}(V,Z))$$

is always continuous in its first argument. It is continuous in its second argument if W is either metrisable or barrelled.

Proof: Let $(x_n)_{n\in\mathbf{N}}$ be a convergent sequence in X with limit x and consider the zero neighbourhood $\mathscr{N}(K_V,U_Z)\subseteq\mathrm{L}(V,Z)$ determined by some precompact subset $K_V\subseteq V$ and a convex open zero neighbourhood $U_Z\subseteq Z$. We claim that the subset

$$K_W = \{0\}\cup\alpha(x)(K_V)\cup\bigcup_{n\in\mathbf{N}}\big(\alpha(x_n)-\alpha(x)\big)(K_V)\subseteq W$$

is precompact

To demonstrate this, pick an arbitrary absolutely convex zero neighbourhood $U_W\subseteq W$. Because α is continuous, there is an $N\in\mathbf{N}$ such that $\alpha(x_n)-\alpha(x)\in\mathscr{N}(K_V,U_W)$ whenever $n>N$ and such that consequently

$$\bigcup_{n>N}\big(\alpha(x_n)-\alpha(x)\big)(K_V)\subseteq U_W.$$

Each of the subsets $\big(\alpha(x_n)-\alpha(x)\big)(K_V)$ for $n\in\{1,\dots,N\}$ and $\alpha(x)(K_V)$ of W is the image of the precompact subset $K_V\subseteq V$ for a a continuous linear map and is therefore also precompact, as is consequently their union. Consequently,

$$\bigcup_{n=1}^{N}\big(\alpha(x_n)-\alpha(x)\big)(K_V)\subseteq\bigcup_{i=1}^{M}(w_i+U_W)$$

for some finite collection of points $w_1,w_2,\dots,w_M\in W$. Since also $0\in U_W$, we infer that K_W is covered a finite collection of neighbourhoods of the form $w+U_W$ and that, since U_W was arbitrary, K_W is indeed precompact.

Now choose an open zero neighbourhood $U'_W \subseteq W$ such that $\beta(U'_W) \subseteq \frac{1}{3}U_Z$. We can write the difference $(\beta \circ \alpha)(x_n) - (\beta \circ \alpha)(x)$ for any $n \in \mathbf{N}$ as

$$\beta(x_n)\circ\alpha(x_n) - \beta(x)\circ\alpha(x) = \beta(x)\circ\big(\alpha(x_n)-\alpha(x)\big) + \big(\beta(x_n)-\beta(x)\big)\circ\alpha(x) + \big(\beta(x_n)-\beta(x)\big)\circ\big(\alpha(x_n)-\alpha(x)\big).$$

Note that both $\alpha(x)(K_V)$ and $(\alpha(x_n)-\alpha(x))(K_V)$ are contained in K_W by construction. If we can now show that $(\alpha(x_n)-\alpha(x))(K_V) \subseteq U'_W$ and that also $(\beta(x_n)-\beta(x))(K_W) \subseteq \frac{1}{3}U_Z$ for all sufficiently large $n \in \mathbf{N}$, it would follow that $(\beta\circ\alpha)(x_n) - (\beta\circ\alpha)(x)$ is an element of $\mathcal{N}(K_V, \frac{1}{3}U_Z + \frac{1}{3}U_Z + \frac{1}{3}U_Z) \subseteq \mathcal{N}(K_V, U_Z)$ for such n.

Because α and β are continuous, we know that $\alpha(x_n) \to \alpha(x)$ and $\beta(x_n) \to \beta(x)$ as $n \to \infty$ and we can thus choose an $N \in \mathbf{N}$ such that $\alpha(x_n) \in \alpha(x) + \mathcal{N}(K_V, U'_W)$ and $\beta(x_n) \in \beta(x) + \mathcal{N}(K_W, \frac{1}{3}U_Z)$ for all $n > N$. For such $n \in \mathbf{N}$, we do have $(\alpha(x_n)-\alpha(x))(K_V) \subseteq U'_W$ and $(\beta(x_n)-\beta(x))(K_W) \subseteq \frac{1}{3}U_Z$ and consequently $(\beta\circ\alpha)(x_n) \in (\beta\circ\alpha)(x) + \mathcal{N}(K_V, U_Z)$. Because any zero neighbourhood in $\mathrm{L}(V,Z)$ contains one of the form $\mathcal{N}(K_V, U_Z)$, we deduce that $(\beta\circ\alpha)(x_n) \to (\beta\circ\alpha)(x)$ as $n \to \infty$.

We have shown that the composition $\beta\circ\alpha\colon U \to \mathrm{L}(V,Z)$ maps convergent sequences in X to convergent sequences in $\mathrm{L}(V,Z)$. Since X is metrisable, $\beta\circ\alpha$ is therefore continuous.

To show that precomposition by a fixed map $\alpha \in \mathrm{C}^0(X, \mathrm{L}(V,W))$ is continuous, pick a zero neighbourhood $\mathcal{N}(K_X, \mathcal{N}(K_V, N_Z))$, which is determined by a compact subset $K_X \subseteq X$, a precompact subset $K_V \subseteq V$ and a zero neighbourhood $N_Z \subseteq Z$. Since α is continuous and K_X is compact, also $\alpha(K_X) \subseteq \mathrm{L}(V,W)$ is compact, and thus in particular precompact. Lemma A.2.26 therefore tells us that also $K_W := \alpha(K_X)(K_V) \subseteq W$ is precompact. Given a map $\beta \in \mathcal{N}(K_X, \mathcal{N}(K_W, N_Z))$ and two elements $x \in K_X$ and $v \in K_V$ it now follows that

$$(\beta\circ\alpha)(x)(v) = \beta(x)\circ\alpha(x)(v) \in \beta(K_X)(K_W) \subseteq N_Z,$$

so $\beta\circ\alpha \in \mathcal{N}(K_X, \mathcal{N}(K_V, N_Z))$ whenever $\beta \in \mathcal{N}(K_X, \mathcal{N}(K_W, N_Z))$. Any neighbourhood in $\mathrm{C}^0(X, \mathrm{L}(V,Z))$ contains one of the form $\mathcal{N}(K_X, \mathcal{N}(K_V, N_Z))$, so continuity of $\beta \mapsto \beta\circ\alpha$ follows.

To show that postcomposition by $\beta \in \mathrm{C}^0(X, \mathrm{L}(W,Z))$ is continuous if W is either metrisable or barrelled, we first assume that W is metrisable. We then consider the zero neighbourhood $\mathcal{N}(K_X, \mathcal{N}(K_V, N_Z))$ in $\mathrm{C}^0(X, \mathrm{L}(V,Z))$ which is determined by the compact subset $K_X \subseteq X$, the precompact subset $K_V \subseteq V$ and the zero neighbourhood $N_Z \subseteq Z$. Because X and W are metrisable, β induces a continuous map from $X \times W$ to Z by Lemma 3.1.1, which tells us that there exists a neighbourhood $N_X^x \times N_W^x \subseteq X \times W$ of $(x, 0)$ for any $x \in K_X$ such that $\beta(N_X^x)(N_W^x) \subseteq N_Z$. Because K_X is compact, we can pick a finite subset $F_X \subseteq X$ such that such that $K_X \subseteq \bigcup_{x \in F_X} N_X^x$ and consider the zero neighbourhood $N_W = \bigcap_{x \in F_X} N_W^x \subseteq W$. For any $\alpha \in \mathscr{N}(K_X, \mathscr{N}(K_V, N_W))$, we now have

$$(\beta \circ \alpha)(x)(v) = \beta(x) \circ \alpha(x)(v) \in \beta(K_X)(N_W) \subseteq N_Z,$$

for all $x \in K_X$ and all $v \in K_V$, and thus $\beta \circ \alpha \in \mathscr{N}(K_X, \mathscr{N}(K_V, N_Z))$. Because any neighbourhood in $\mathrm{C}^0(X, \mathrm{L}(V, Z))$ contains one of the form $\mathscr{N}(K_X, \mathscr{N}(K_V, N_Z))$, continuity of $\alpha \mapsto \beta \circ \alpha$ follows.

Now assume that W is barrelled and let $\mathscr{N}(K_X, \mathscr{N}(K_V, N_Z))$ be as described above. Because β is continuous, $\beta(K_X) \subseteq \mathrm{L}(W, Z)$ is compact and consequently also bounded for the topology of precompact convergence. Since this implies boundedness for the topology of simple convergence, the uniform boundedness principle (Theorem A.2.27) tells us that $\beta(K_X)$ is equicontinuous. There therefore exists a zero neighbourhood $N_W \subseteq W$ such that $\beta(K_X)(N_W) \subseteq N_Z$. For any $\alpha \in \mathscr{N}(K_X, \mathscr{N}(K_V, N_W))$ we now have

$$(\beta \circ \alpha)(x)(v) = \beta(x) \circ \alpha(x)(v) \in \beta(K_X)(N_W) \subseteq N_Z$$

for all $x \in K_X$ and all $v \in K_V$. It follows that $\beta \circ \alpha \in \mathscr{N}(K_X, \mathscr{N}(K_V, N_Z))$, and continuity of $\alpha \mapsto \beta \circ \alpha$ follows. ☐

A similar statement for the evaluation map follows because $W \simeq \mathrm{L}(\mathbf{R}, W)$ as a topological vector space for any locally convex vector space W. Note that the evaluation map $\mathrm{ev}\colon \mathrm{L}(V, W) \times V \to W$ itself is never continuous unless either V is normable or $W = 0$.

Corollary 3.1.4.

Let X be a metrisable space and let V and W be locally convex vector spaces. Given two continuous maps $e\colon X \to V$ and $\beta\colon X \to \mathrm{L}(V, W)$, the composition

$$\beta \circ e\colon X \longrightarrow W, \qquad (\beta \circ e)(x)(v) = \beta(x)(e(x))$$

is continuous as well. The evaluation map

$$\mathrm{ev}\colon \mathrm{C}^0(X, \mathrm{L}(V, W)) \times \mathrm{C}^0(X, V) \longrightarrow \mathrm{C}^0(X, W), \qquad (\beta, e) \mapsto \beta \circ e$$

is always continuous in its first argument, and it is continuous in its second argument if V is metrisable or barrelled.

Given two (continuous) linear maps $\alpha_1 \in \mathrm{L}(V_1, W_1)$ and $\alpha_2 \in \mathrm{L}(V_2, W_2)$, we can consider their tensor product

$$\alpha_1 \otimes \alpha_2 \colon V_1 \otimes V_2 \longrightarrow W_1 \otimes W_2, \qquad v_1 \otimes v_2 \mapsto \alpha_1(v_1) \otimes \alpha_2(v_2),$$

which is again a continuous linear map. The following proposition is a generalisation of the usual product rule from Proposition B.1.7.

Proposition 3.1.5.
Let X be a metrisable locally convex vector space and let V_1, V_2, W_1 and W_2 be locally convex vector spaces. The tensor product

$$\alpha_1 \otimes \alpha_2 \colon X \longrightarrow \mathrm{L}(V_1 \otimes V_2, W_1 \otimes W), \qquad x \mapsto \alpha_1(x) \otimes \alpha_2(x)$$

of two continuous maps $\alpha_1 \colon X \to \mathrm{L}(V_1, W_1)$ and $\alpha_2 \colon X \to \mathrm{L}(V_2, W_2)$ is continuous. Moreover, the resulting operator

$$\otimes \colon \mathrm{C}^0(X, \mathrm{L}(V_1, W_1)) \times \mathrm{C}^0(X, \mathrm{L}(V_2, W_2)) \longrightarrow \mathrm{C}^0(X, \mathrm{L}(V_1 \otimes V_2, W_1 \otimes W_2))$$

is continuous whenever V_1 and V_2 are metrisable.

Proof: The tensor product is continuous as a map from $\mathrm{L}(V_1, W_1) \times \mathrm{L}(V_2, W_2)$ to $\mathrm{L}(V_1, W_1) \otimes \mathrm{L}(V_2, W_2)$ by definition, so continuity of $\alpha_1 \otimes \alpha_2$ follows from the first half of Proposition A.2.38. When combined with Lemma B.2.4, the second half of this proposition tells us that this describes a continuous operation whenever the spaces V_1 and V_2 are metrisable. ☐

3.1.2. Differentiable families of linear maps

Each of the statements from section 3.1.1 has an analogue for differentiable families of linear maps parametrised by a Fréchet domain. As in Definition B.2.2, a Fréchet domain is defined as a connected open subset $X \subseteq T$ of a Fréchet space T, and we endow the space $\mathrm{C}^k(X, V)$ of maps of class C^k from this domain to a locally convex vector space V with the compact-open C^k topology from appendix B.2.2. The space $\mathrm{L}(V,W)$ of continuous linear maps from V to W still comes with the relevant topology of precompact convergence.

For the sake of presentation, Proposition 3.1.6 and Proposition 3.1.8 appear out of order in the sense that the latter is required to prove part of the former. The part in question is the implication (i) ⇒ (iii) from Proposition 3.1.6, since it relies on the smoothness of a composition of smooth families of linear maps.

Proposition 3.1.6.
Let T and V be metrisable locally convex vector spaces, let W be a locally convex vector space and let $X \subseteq T$ be an open subset. For any function $\alpha\colon X \to \mathrm{L}(V,W)$ and any $k \in \mathbf{N}_0 \cup \{\infty\}$, the following statements are equivalent:

- **(i)** *α is of class C^k;*
- **(ii)** *the map $\check{\alpha}\colon X \times V \to W$ given by $\check{\alpha}(x,v) = \alpha(x)(v)$ is of class C^k;*
- **(iii)** *there exists a (unique) continuous linear map*

$$\hat{\alpha}_k\colon \mathrm{C}^k(X,V) \longrightarrow \mathrm{C}^k(X,W) \qquad \text{with} \qquad \hat{\alpha}_k(e)(x) = \alpha(x)(e(x))$$

for all $e \in \mathrm{C}^k(X,V)$ and every $x \in X$.

Proof: We will prove the implications (i) ⇒ (iii), (iii) ⇒ (ii) and (ii) ⇒ (i), in that order.

We first prove that (i) implies statement (iii) and therefore assume that α is of class C^k for some $k \in \mathbf{N}_0$. Then Proposition 3.1.8 tells us that also the composition $\hat{\alpha}_k(e) = \alpha \circ e$ is of class C^k for any $e \in \mathrm{C}^k(X,V)$, and that its derivatives are described by the commuting diagram

$$\begin{array}{ccc} \mathrm{C}^k(X,V) & \xrightarrow{\hat{\alpha}_k} & \mathrm{C}^k(X,W) \\ \big\downarrow \mathrm{j}^k & & \big\downarrow \mathrm{j}^k \\ \prod_{p=0}^k \mathrm{C}^0(X,\mathrm{L}(\bigodot^p(T),V)) & \xrightarrow{\sum_{0\le p\le q\le k} \hat{F}_{p,q}} & \prod_{q=0}^k \mathrm{C}^0(X,\mathrm{L}(\bigodot^q(T),W)). \end{array}$$

Here j^k is the k-th jet map introduced in appendix B.1.2 and $\hat{F}_{p,q}$ denotes the map

$$\hat{F}_{p,q}\colon \mathrm{C}^0(X,\mathrm{L}(\textstyle\bigodot^p(T),V)) \longrightarrow \mathrm{C}^0(X,\mathrm{L}(\textstyle\bigodot^q(T),V)),$$

given by

$$\hat{F}_{p,q}(E_p)(x)(\dot{x}_{\{1,\ldots,q\}}) = \sum_{\substack{I \sqcup J = \{1,\ldots,q\} \\ \#I = q-p, \#J = p}} \mathrm{D}^{q-p}\alpha(x)(\dot{x}_I)\big(E_p(x)(\dot{x}_J)\big)$$

for $E_p \in \mathrm{C}^0(X, \mathrm{L}(\bigodot^p(T), V))$, $x \in X$ and $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_q \in T$. Since the topologies on $\mathrm{C}^k(X,V)$ and $\mathrm{C}^k(X,W)$ were defined to be such that these two jet maps are topological embeddings, we only need to show that $\hat{F}_{p,q}$ is continuous for all appropriate $p, q \in \mathbf{N}_0$.

To prove that $\hat{F}_{p,q}$ is continuous, we observe that $\hat{F}_{p,q}(E_p)$ can be expressed as

$$\hat{F}_{p,q}(E_p) = \mathrm{D}^{q-p}\alpha \circ (\mathrm{id}_{\bigodot^{q-p}(T)} \otimes E) \circ \Delta_{p,q}$$

for any $E \in \mathrm{C}^0(X, \mathrm{L}^{q-p}(T,V))$, where $\Delta_{p,q} \colon \bigodot^q(T) \to \bigodot^{q-p}(T) \otimes \bigodot^p(T)$ is the map given by

$$\Delta_{p,q}\big(\textstyle\bigotimes_{i=1}^q \dot{x}_i\big) = \displaystyle\sum_{\substack{I \sqcup J = \{1,\ldots,q\} \\ \#I = q-p, \#J = p}} \dot{x}_I \otimes \dot{x}_J$$

for $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_q \in T$. Since both $\mathrm{D}^{q-p}\alpha$ and $\Delta_{p,q}$ are continuous and $\bigodot^q(T)$ and $V \otimes \bigotimes^{q-p}(T)$ are metrisable, it now follows from Proposition 3.1.3 and Proposition 3.1.5 that $\hat{F}_{p,q}(E_p)$ depends continuously on E_p. Consequently, the map $\hat{\alpha}_k$ described above is continuous for a given $k \in \mathbf{N}_0$ whenever α is of class C^k.

The spaces of C^k maps from X to V and W for $k \in \mathbf{N}_0$ together form two projective systems of locally convex vector spaces whose limits are the spaces $\mathrm{C}^\infty(X,V)$ and $\mathrm{C}^\infty(X,W)$ of smooth sections. If α is of class C^∞, then the maps $\hat{\alpha}_l$ for $l \in \mathbf{N}_0$ together form a morphism between these projective systems and therefore induce a continuous map $\alpha_\infty \colon \mathrm{C}^\infty(X,V) \to \mathrm{C}^\infty(X,W)$. We conclude that the implication (i) ⇒ (iii) therefore also holds when $k = \infty$.

To prove the implication (iii) ⇒ (ii), we shall assume that the map $\hat{\alpha}_k$ exists and is continuous for a given $k \in \mathbf{N}_0 \cup \{\infty\}$. Now $\check{\alpha}$ can be described in terms of $\hat{\alpha}_k$ by the equation $\check{\alpha}(x,v) = \hat{\alpha}_k(c_v)(x)$, where $c_v \colon X \to V$ denotes the constant map $x \mapsto v$ (which is of class C^k). Consequently, the restriction $\check{\alpha}|_{X \times \{v\}}$ is of class C^k for every $v \in V$. We will use Proposition B.1.3 to show that $\check{\alpha}$ is of class C^k.

We observe that for any $p \in \mathbf{N}_0$ such that $p \leq k$ and $(x, v) \in X \times V$, the partial derivative $\mathrm{D}_1^p \check{\alpha}(x, v)$ is equal to $(\mathrm{D}^p \circ \hat{\alpha}_k(c_v))(x)$. We deduce that $\mathrm{D}_1^p \alpha(x, v)$ is continuous, as well as linear, as a function of v because

$$\mathrm{D}^p \circ \hat{\alpha}_k \circ c \colon V \longrightarrow \mathrm{C}^{k-p}(X, \mathrm{L}^p(T, W)), \qquad v \mapsto \mathrm{D}^p(\hat{\alpha}_k(c_v)),$$

is a composition of continuous linear maps, and evaluation at $x \in X$ is both continuous and linear as well. This in particular means that $\mathrm{D}_1^p \check{\alpha}|_{\{x\}\times V}$ is of class C^{k-P} for any $x \in X$ and that $\mathrm{D}_2 \mathrm{D}_1^p \check{\alpha}(x, v) = \mathrm{D}_1^p \check{\alpha}(x, \cdot)$ and $\mathrm{D}_2^q \mathrm{D}_1^p \check{\alpha}(x, v) = 0$ for all $q \geq 2$ and $(x, v) \in X \times V$.

The continuity of $\mathrm{D}^p \circ \hat{\alpha}_k \circ c$ also tells us, through Lemma 3.1.1, that $\mathrm{D}_1^p \check{\alpha}$ is continuous as a map from $X \times V$ to $\mathrm{L}^p(X, W)$. By applying Lemma 3.1.1 again, we see that also the derivative $\mathrm{D}_2 \mathrm{D}_1^p \check{\alpha} \colon (x, v) \mapsto \mathrm{D}_1^p \check{\alpha}(x, \cdot)$ is continuous. Since all of its hypotheses are satisfied, we now conclude from Proposition B.1.3 that $\check{\alpha}$ is itself of class C^k.

We can prove that the implication (ii) $\Rightarrow$ (i) holds for all $k \in \mathbf{N}_0$ by induction. The case $k = 0$ was already proven in Lemma 3.1.1. Assume that this claim holds for a given $k \in \mathbf{N}_0$ and that $\check{\alpha}$ is of class C^{k+1}, then the induction hypothesis tells us α is of class C^k. It follows from Lemma 3.1.1 and Corollary A.2.48 that the $(k+1)$-st derivative,

$$\mathrm{D}_1^{k+1} \check{\alpha} \colon X \times V \longrightarrow \mathrm{L}^k(T, V),$$

of $\check{\alpha}$ with respect to the first argument induces a continuous map,

$$A_{k+1} \colon X \longrightarrow \mathrm{L}\big(V, \mathrm{L}^{k+1}(T, W)\big) \simeq \mathrm{L}^{k+1}\big(T, \mathrm{L}(V, W)\big),$$

which is given by $A_{k+1}(x)(\dot{x}_1, \ldots, \dot{x}_p)(v) = \mathrm{D}_1^{k+1} \check{\alpha}(x, v)(\dot{x}_1, \ldots, \dot{x}_p)$ for $(x, v) \in X \times V$ and $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_p \in T$. We claim that this is the derivative of $\mathrm{D}^k \alpha$.

Suppose we are given a point $x \in X$ and a vector $\dot{x} \in T$ and let $\mathscr{N}(K_V, U_{T,W})$ be the zero neighbourhood in $\mathrm{L}(\bigodot^k(T), \mathrm{L}(V, W)) \simeq \mathrm{L}(V, \mathrm{L}(\bigodot^k(T), W))$ determined by some precompact subset $K_V \subseteq V$ and a convex open zero neighbourhood $U_{T,W} \subseteq \mathrm{L}(\bigodot^k(T), W)$. We need to show that

$$\tfrac{1}{t}\big(\mathrm{D}^k \alpha(x + t\,\dot{x}) - \mathrm{D}^k \alpha(x)\big) \in A_{k+1}(x)(\dot{x}) + \mathscr{N}(K_V, U_{T,W})$$

for all $t \in \mathbf{R}$ sufficiently close to 0.

Since $\mathrm{D}_1^{k+1}\check{\alpha}$ is continuous, we can choose a convex neighbourhood $U_X \times U_V \subseteq X \times V$ of $(x,0)$ such that $\mathrm{D}^{k+1}\check{\alpha}(x',v') \in \frac{1}{4}\,U_{T,W}$ for all $(x',v') \in U_X \times U_V$. We can similarly choose a neighbourhood $U_{x,v}$ for any $v \in K_V$ such that $\mathrm{D}_1^{k+1}\check{\alpha}(x',v) \in \mathrm{D}_1^{k+1}\check{\alpha}(x,v) + \frac{1}{4}U_{T,W}$ for all $x' \in U_{x,v}$. The precompactness of K_V then allows us to choose $v_1, v_2, \dots, v_N \in K_V$ such that $K_V \subseteq \bigcup_{i=1}^N v_i + U_V$ and consider the intersection $U_X \coloneqq \bigcup_{i=1}^N U_{X,v_i}$. For every $x' \in U_X$ and every $v \in K_V$ we can now pick an $i \in \{1,\dots,N\}$ such that $v \in v_i + U_{v,v_i}$ and such that therefore

$$\begin{aligned}\mathrm{D}_1^{k+1}\check{\alpha}(x',v) - \mathrm{D}_1^{k+1}\check{\alpha}(x,v) = \mathrm{D}_1^{k+1}\check{\alpha}(x',v-v_i) + \big(\mathrm{D}_1^{k+1}\check{\alpha}(x,v_i-v)\big)\\ + \big(\mathrm{D}_1^{k+1}\check{\alpha}(x',v_i) - \mathrm{D}_1^{k+1}\check{\alpha}(x',v_i)\big)\end{aligned}$$

is an element of $\frac{1}{4}\,U_{T,W} + \frac{1}{4}\,U_{T,W} + \frac{1}{4}\,U_{T,W} \subseteq \frac{3}{4}\,U_{T,W}$.

We observe that since $\mathrm{D}_1^k\check{\alpha}$ is differentiable,

$$\begin{aligned}&\mathrm{D}^k\alpha(x+t\,\dot{x})(\bullet)(v) - \mathrm{D}^k\alpha(x)(\bullet)(v) - t\,A_{k+1}(x)(\dot{x}\odot\bullet)(v)\\ &\qquad = \mathrm{D}_1^k\check{\alpha}(x+t\,\dot{x},v) - \mathrm{D}_1^k\check{\alpha}(x,v) - t\,\mathrm{D}_1^{k+1}\check{\alpha}(x,v)(t\,\dot{x})\\ &\qquad = \int_0^t \big(\mathrm{D}_1^{k+1}\check{\alpha}(x+s\,\dot{x},v) - \mathrm{D}_1^{k+1}\check{\alpha}(x,v)\big)(\dot{x})\,\mathrm{d}s\end{aligned}$$

for every $t \in \mathbf{R}$ and every $v \in K_V$. Since we have just shown that the integrand is an element of $\frac{3}{4}\,U_{T,W}$ for all $s \in \mathbf{R}$ such that $x + s\,\dot{x} \in U_X$ and U_X is convex, it follows that this expression describes an element of $\frac{3}{4}\,t\,\overline{U_{T,W}} \subseteq t\,U_{T,W}$ for every $t \in \mathbf{R}$ such that $x + t\,\dot{x} \in U_X$ and every $v \in K_V$. Therefore,

$$\tfrac{1}{t}\big(\mathrm{D}^k\alpha(x+t\,\dot{x}) - \mathrm{D}^k\alpha(x) - A_{k+1}(x)(t\,\dot{x}\odot\bullet)\big) \in \mathscr{N}\big(K_V, U_{T,W}\big).$$

for every $t \in \mathbf{R}$ such that $x + t\,\dot{x} \in U_X$. Since any zero neighbourhood in $\mathrm{L}(V, \mathrm{L}^k(T,W))$ contains one of this form and x and $\dot{x}$ were arbitrary, we conclude that $\mathrm{D}^k\alpha$ is differentiable and that A_{k+1} is its (continuous) derivative. It follows by induction that the implication (ii) ⇒ (i) holds for all $k \in \mathbf{N}_0$, and therefore also for $k = \infty$.

We conclude that these three statements are equivalent for any $k \in \mathbf{N}_0 \cup \{\infty\}$. □

For barrelled spaces we can again do even better.

Lemma 3.1.7.

Let T be a metrisable locally convex vector space, let V and W be locally convex vector spaces of which the first is barrelled and let $X \subseteq T$ be an open subset. A function $\alpha\colon X \to \mathrm{L}(V,W)$ is of class C^k for a given $k \in \mathbf{N}_0 \cup \{\infty\}$ if and only if the map

$$\alpha_v := \alpha(\cdot)(v)\colon X \longrightarrow W, \qquad x \mapsto \alpha(x)(v)$$

is of class C^k for every $v \in V$.

Proof: Since the evaluation map $\mathrm{ev}_v\colon \mathrm{L}(V,W) \to W$ is continuous and linear for any fixed $v \in V$, $\alpha_v = \mathrm{ev}_v \circ \alpha$ is of class C^k whenever α is. We will prove the converse implication by induction on $k \in \mathbf{N}$.

Suppose that α is of class C^{k-1} for some $k \in \mathbf{N}$ and that α_v is of class C^k for every $v \in V$. This means that for all $x \in X$, every $v \in V$ and any $\dot{x} \in T$,

$$\lim_{t\to 0} \tfrac{1}{t}\left(\mathrm{D}^{k-1}\alpha_v(x + t\,\dot{x}) - \mathrm{D}^{k-1}\alpha_v(x + \dot{x})\right) = \mathrm{D}^k\alpha_v(x)(\dot{x}) \in \mathrm{L}^{k-1}(T,W),$$

where $\mathrm{D}^k\alpha_v(x) \in \mathrm{L}^k(T,W)$ depends continuously on $x \in X$. One can demonstrate that the derivatives $\mathrm{D}^p\alpha(x)(\dot{x}_{\{1,\dots,k-1\}})(v)$ of α for $p = 0,1,\dots,k-1$ coincide with by $\mathrm{D}^{k-1}\alpha_v(x)(\dot{x}_{\{1,\dots,k-1\}})$, and also for $p = k$ if this derivative exists.

Since V is barrelled, the Banach–Steinhaus theorem (Theorem A.2.28) tells us that the aforementioned limit is uniform on precompact subsets of V and thus induces a similar limit in the space $\mathrm{L}(V,\mathrm{L}^{k-1}(T,W))$. The same corollary can also be used to show that there exists a canonical injective map $\mathrm{L}(V,\mathrm{L}^{k-1}(V,W)) \hookrightarrow \mathrm{L}^{k-1}(T,\mathrm{L}(V,W))$, which is easily shown to be a topological embedding. It follows that the limit

$$A_k(x)(\dot{x}) = \lim_{t\to 0} \tfrac{1}{t}\left(\mathrm{D}^{k-1}\alpha(x + t\,\dot{x}) - \mathrm{D}^{k-1}\alpha(x + \dot{x})\right) \in \mathrm{L}^{k-1}(T,\mathrm{L}(V,W)),$$

exists for every $x \in X$ and every $\dot{x} \in T$. The only thing that remains to be shown that $A_k(x)$ is an element of $\mathrm{L}^k(T,\mathrm{L}(V,W))$ that depends continuously on $x \in X$.

One can readily verify that $A_k(x)(\dot{x}_{\{1,\dots,k\}})(v) = \mathrm{D}^k\alpha_v(\dot{x}_{\{1,\dots,k\}})$ for all $x \in X$, $v \in V$ and $\dot{x}_1, \dot{x}_2, \dots, \dot{x}_k \in T$. Continuous dependence of this expression on the parameters $(x, \dot{x}_1, \dots, \dot{x}_k) \in X \times \prod^k T$ follows from Lemma 3.1.1 because $\mathrm{D}^k\alpha_v$ is continuous by assumption and both X and T are metrisable. Since V is barrelled, we learn from Proposition 3.1.2 that also $A_k(x)(\dot{x}_1, \dots, \dot{x}_k) \in \mathrm{L}(V,W)$ depends continuously on these parameters, and finally deduce from

Lemma 3.1.1 that A_k is a continuous map from X to $\mathrm{L}^k(T,\mathrm{L}(V,W))$. We conclude that α is of class C^k.

Since the case $k=0$ was already covered in Lemma 3.1.7, it now follows by induction that for any $k \in \mathbf{N}_0$, α is of class C^k whenever all maps α_v for $v \in V$ are. Given this fact, the case $k=\infty$ is immediate. □

Proposition 3.1.8 and its corollaries are versions of the usual chain rule for families of continuous linear maps.

Proposition 3.1.8.
Let T, be a metrisable locally convex vector space, let V, W and Z be locally convex vector spaces and let $X \subseteq T$ be an open subset. Given two maps $\alpha\colon X \to \mathrm{L}(V,W)$ and $\beta\colon X \to \mathrm{L}(W,Z)$ of class C^k for some $k \in \mathbf{N}_0 \cup \{\infty\}$, the composition

$$\beta \circ \alpha\colon X \longrightarrow \mathrm{L}(V,Z), \qquad (\beta \circ \alpha)(x)(v) = \big(\beta(x) \circ \alpha(x)\big)(v)$$

is of class C^k as well. Its derivatives are given by

$$\mathrm{D}^p(\beta \circ \alpha)(x)(\dot{x}_1,\ldots,\dot{x}_p) = \sum_{I \sqcup J = \{1,\ldots,p\}} \big(\mathrm{D}^{\#I}\beta(x)(\dot{x}_I) \circ \mathrm{D}^{\#J}\alpha(x)(\dot{x}_J)\big) \qquad (3.1.1)$$

for all $x \in X$ and $\dot{x}_1,\ldots,\dot{x}_p \in T$ with $p \in \mathbf{N}_0$.
The composition map,

$$\circ\colon \mathrm{C}^k(X,\mathrm{L}(W,Z)) \times \mathrm{C}^k(X,\mathrm{L}(V,W)) \longrightarrow \mathrm{C}^k(X,\mathrm{L}(V,Z)), \qquad (\beta,\alpha) \mapsto \beta \circ \alpha$$

is always continuous in its first argument, and continuous in its second argument if W is metrisable or barrelled.

Proof: We shall first consider the case $k=1$ and then do the general case using an inductive argument.

Assume that both β and α are of class C^1 and pick a point $x \in X$ and a vector $\dot{x} \in T$. Because both β and α are differentiable, there exist continuous maps $Q_\alpha\colon \mathbf{R} \to \mathrm{L}(V,W)$ and $Q_\beta\colon \mathbf{R} \to \mathrm{L}(W,Z)$ that are given by

$$Q_\alpha(t) = \begin{cases} \frac{1}{t}\big(\alpha(x + t\,\dot{x}) - \alpha(x)\big) & \text{if } t \neq 0 \\ \mathrm{D}\alpha(x)(\dot{x}) & \text{if } t = 0 \end{cases}$$

and

$$Q_\beta(t) = \begin{cases} \frac{1}{t}\big(\beta(x+t\,\dot{x}) - \beta(x)\big) & \text{if } t \neq 0 \\ \mathrm{D}\beta(x)(\dot{x}) & \text{if } t = 0 \end{cases}$$

for $t \in \mathbf{R}$. In terms of these, the difference quotient for $t \mapsto (\beta \circ \alpha)(x + t\,\dot{x})$ can be expressed as

$$\tfrac{1}{t}\big((\beta\circ\alpha)(x+t\,\dot{x}) - (\beta\circ\alpha)(x)\big) = Q_\beta(t)\circ\alpha(x+t\,\dot{x}) + \beta(x)\circ Q_\alpha(t).$$

Now, Proposition 3.1.3 tells us that the limit of this expression for $t \to 0$ exists and is given by

$$\lim_{t\to 0}\tfrac{1}{t}\big((\beta\circ\alpha)(x+t\,\dot{x}) - (\beta\circ\alpha)(x)\big) = \mathrm{D}\beta(x)(\dot{x})\circ\alpha(x) + \beta(x)\circ\mathrm{D}\alpha(x)(\dot{x}).$$

which is compatible with equation (3.1.1). We shall denote the right-hand side of this equation by $\mathrm{D}(\beta\circ\alpha)(x)(\dot{x})$.

To show that this derivative depends continuously on the base point $x \in X$, consider the following maps:

$$\begin{aligned} \alpha_{X\times T}\colon X\times T &\longrightarrow \mathrm{L}(V,W), && (x,\dot{x}) \mapsto \alpha(x) \\ \beta_{X\times T}\colon X\times T &\longrightarrow \mathrm{L}(W,Z), && (x,\dot{x}) \mapsto \beta(x) \\ (\mathrm{D}\alpha)_{X\times T}\colon X\times T &\longrightarrow \mathrm{L}(V,W), && (x,\dot{x}) \mapsto \mathrm{D}\alpha(x)(\dot{x}) \\ (\mathrm{D}\beta)_{X\times T}\colon X\times T &\longrightarrow \mathrm{L}(W,Z), && (x,\dot{x}) \mapsto \mathrm{D}\beta(x)(\dot{x}). \end{aligned}$$

Because X and T are metrisable, Lemma 3.1.1 tells us that each of these maps is continuous, which means that continuity of the composition

$$\mathrm{D}(\beta\circ\alpha)_{X\times T} = (\mathrm{D}\beta)_{X\times T}\circ\alpha_{X\times T} + \beta_{X\times T}\circ(\mathrm{D}\alpha)_{X\times T}\colon X\times T \longrightarrow \mathrm{L}(V,Z)$$

now follows from Lemma 3.1.1. By applying Lemma 3.1.1 again, we deduce that the derivative

$$\mathrm{D}(\beta\circ\alpha)\colon X \longrightarrow \mathrm{L}(T,\mathrm{L}(V,Z))$$

is continuous. We conclude that $\beta\circ\alpha$ is of class C^1 and that its derivative is given by equation (3.1.1) for $p = 1$.

Now assume that the statement of this proposition is true for some $k \in \mathbf{N}$ and that both α and β are of class C^{k+1}. We know that the derivative of $\beta\circ\alpha$ is given by $D\beta \circ \alpha + \beta \circ D\alpha$ and that α, β, $D\alpha$ and $D\beta$ are all of class C^k, so the induction hypothesis tells us that $D(\beta \circ \alpha)$ is of class C^k. It moreover follows that the k-th order derivative of $D(\beta \circ \alpha)$ is given by

$$D^k(D(\beta \circ \alpha))(x)(\dot{x}_{\{1,\dots,k\}}) = \sum_{I \sqcup J = \{1,\dots,k\}} \begin{aligned}[t] &D^{\#I}(D\beta)(x)(\dot{x}_I) \circ D^{\#J}\alpha(x)(\dot{x}_J) \\ &+ D^{\#I}\beta(x)(\dot{x}_I) \circ D^{\#J}(D\alpha)(x)(\dot{x}_J)\end{aligned}$$

for any $x \in X$ and all $\dot{x}_1, \dot{x}_2, \dots, \dot{x}_k \in T$.

Since $D(\beta \circ \alpha)$ is of class C^k, we infer that $\beta \circ \alpha$ is in fact of class C^{k+1}, and that its $(k+1)$-st order derivative is given by

$$\begin{aligned} D^{k+1}(\beta \circ \alpha)(x)(\dot{x}_{\{0,\dots,k\}}) &= D^k\big(D(\beta \circ \alpha)\big)(x)(\dot{x}_{\{1,\dots,k\}})(\dot{x}_0) \\ &= \sum_{I \sqcup J = \{1,\dots,k\}} \begin{aligned}[t] &(D^{\#I+1}\beta(x)(\dot{x}_0 \odot \dot{x}_I) \circ D^{\#J}\alpha(x)(\dot{x}_J)) \\ &+ (D^{\#I}\beta(x)(\dot{x}_I) \circ D^{\#J+1}\alpha(x)(\dot{x}_0 \odot \dot{x}_J))\end{aligned} \\ &= \sum_{\tilde{I} \sqcup \tilde{J} = \{0,1,\dots,k\}} (D^{\#\tilde{I}}\beta(x)(\dot{x}_{\tilde{I}}) \circ D^{\#\tilde{J}}\alpha(x)(\dot{x}_{\tilde{J}})), \end{aligned}$$

for any $x \in X$ and $\dot{x}_0, \dot{x}_1, \dots, \dot{x}_k \in T$.

We conclude that the statment of this proposition is true for all $k \in \mathbf{N}$, and that it therefore also holds for $k = \infty$.

Continuity of pre- and postcomposition can be proven in the same way as in Proposition 3.1.3 by using the canonical embedding

$$C^k\big(X, L(V', W')\big) \longleftrightarrow \prod_{p=0}^{k} C^0\big(X, L\big(\textstyle\bigodot^p(T), L(V', W')\big)\big)$$

for any two locally convex vector spaces V' and W', as well as the fact that $L(\bigodot^p(T), V')$ is isomorphic to $L(\bigotimes^r(T), L(\bigotimes^s(T)V'))$ whenever $p = r+s$. These facts are discussed in Proposition A.2.46 and Proposition A.2.47. □

The following corollary follows from Proposition 3.1.8 and the basic fact that $L(\mathbf{R}, V)$ is isomorphic to V for any locally convex vector space V.

Corollary 3.1.9.

Let T, be a metrisable locally convex vector space, let V and W be locally convex vector spaces and let $X \subseteq T$ be an open subset. Given two maps $e\colon X \to V$ and $\beta\colon X \to \mathrm{L}(V,W)$ of class C^k for some $k \in \mathbf{N}_0 \cup \{\infty\}$, the composition

$$\beta \circ e\colon X \longrightarrow W, \qquad (\beta \circ e)(x)(v) = \beta(x)(e(x))$$

is of class C^k as well. Its derivatives are given by

$$\mathrm{D}^p(\beta \circ e)(x)(\dot{x}_1, \ldots, \dot{x}_p)(v) = \sum_{I \sqcup J = \{1,\ldots,p\}} \mathrm{D}^{\#I}\beta(x)(\dot{x}_I) \circ \mathrm{D}^{\#J} e(x)(\dot{x}_J) \tag{3.1.2}$$

for all $x \in X$, all $v \in V$ and $\dot{x}_1, \ldots, \dot{x}_p \in T$ with $p \in \mathbf{N}_0$. The evaluation map

$$\mathrm{ev}\colon \mathrm{C}^k(X, \mathrm{L}(V,W)) \times \mathrm{C}^k(X,V) \longrightarrow \mathrm{C}^k(X,W), \qquad (\beta, e) \mapsto \beta \circ e$$

is always continuous in its first argument, and it is continuous in its second argument if W is metrisable or barrelled.

Proof: This is a special case of Proposition 3.1.8 since the space V is naturally isomorphic to $\mathrm{L}(\mathbf{R}, V)$ for any locally convex vector space V. ☐

The next corollary, essentially describes the composition of vector bundle morphisms that need not be base-preserving.

Corollary 3.1.10.

Let S and T be metrisable locally convex vector spaces, let V, W and Z be locally convex vector spaces and let $X \subseteq S$ and $Y \subseteq T$ be open subsets. Given three maps $\alpha\colon X \to \mathrm{L}(V,W)$, $\beta\colon Y \to \mathrm{L}(W,Z)$ and $f\colon X \to Y$ of class C^k, with $k \in \mathbf{N}_0 \cup \{\infty\}$, the composition

$$\beta \circ (\alpha, f)\colon X \longrightarrow \mathrm{L}(V,Z), \qquad x \mapsto \beta(f(x)) \circ \alpha(x)$$

is of class C^k as well. For any $p \in \mathbf{N}_0$, its p-th order derivative is given by

$$\mathrm{D}^p(\beta \circ (\alpha, f))(x)(\dot{x}_{\{1,\ldots,p\}}) = \sum_{I \sqcup J = \{1,\ldots,p\}} \mathrm{D}^m \beta(f(x))\big(\overline{\mathrm{D}}^I f(x)(\dot{x}_I)\big) \circ \mathrm{D}^{\#J} \alpha(x)(\dot{x}_J)$$

for all $x \in X$ and $\dot{x}_1, \ldots, \dot{x}_p \in T$, where $\overline{\mathrm{D}}^I f(x)(\dot{x}_I)$ denotes the sum

$$\overline{\mathrm{D}}^I f(x)(\dot{x}_I) := \sum_{I_1,\ldots,I_m} \tfrac{1}{m!} \textstyle\bigodot_{j=1}^m \mathrm{D}^{\#I_j} f(x)(\dot{x}_{I_j})$$

over all subsets $I_1, I_2, \ldots, I_m \subseteq I$ that form an (ordered) partition of I.

Proof: Since $\beta \circ (\alpha, f) = (\beta \circ f) \circ \alpha$ by definition, this corollary is essentially a combination of Proposition B.1.6 and Proposition 3.1.8. □

Given families $\alpha_i \colon X \to \mathrm{L}(V_i, W_i)$ of continuous linear maps for $i = 1, 2, \ldots, q$, we may consider the product

$$\bigotimes_{i=1}^{q} \alpha_i \colon X \longrightarrow \mathrm{L}\big(\bigotimes_{i=1}^{q} V_i, \bigotimes_{j=1}^{q} W_j\big),$$

which is the family of continuous linear maps given by

$$\big(\bigotimes_{i=1}^{q} \alpha_i\big)(x)(v_1 \otimes \cdots \otimes v_q) = \bigotimes_{i=1}^{q} \alpha_i(x)(v_i)$$

when applied to a decomposable element $v_1 \otimes v_2 \cdots \otimes v_q \in \bigotimes_{i=1}^{q} V_i$. The product rule from Proposition B.1.7 can be adapted to apply to tensor products of this type.

Proposition 3.1.11 (Product rule).
Let $X \subseteq T$ be an open subset of a metrisable locally convex vector space T, and let $V_1, V_2, \ldots, V_q$ and $W_1, W_2, \ldots, W_q$ be locally convex vector spaces. Given maps $\alpha_i \colon X \to \mathrm{L}(V_i, W_i)$ for $i = 1, 2, \ldots, q$ of class C^k with $k \in \mathbf{N}_0 \cup \{\infty\}$, also the tensor products $\bigotimes_{i=1}^{q} \alpha_i$ is of class C^k and its derivatives are given by

$$\mathrm{D}^p\big(\bigotimes_{i=1}^{q} \alpha_i\big)(x)(\dot{x}_{\{1,\ldots,p\}})(v_{\{1,\ldots,q\}}) = \sum_{I_1,\ldots,I_q} \bigotimes_{i=1}^{q} \mathrm{D}^{\#I_i} \alpha_i(x)(\dot{x}_{I_i})(v_i)$$

for $p \in \mathbf{N}_0$, $x \in X$ and all $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_p \in T$ and $v_i \in V_i$ for $i = 1, 2, \ldots, q$. Moreover, the operator

$$\otimes \colon \prod_{i=1}^{q} \mathrm{C}^k(X, \mathrm{L}(V_i, W_i)) \longrightarrow \mathrm{C}^k\big(X, \mathrm{L}\big(\bigotimes_{i=1}^{q} V_i, \bigotimes_{j=1}^{q} W_j\big)\big)$$

that maps $(\alpha_1, \ldots, \alpha_q)$ to $\bigotimes_{i=1}^{q} \alpha_i$ is continuous.

Proof: This is nearly a direct consequence of Proposition B.1.7. The key observation is the existence of a canonical continuous linear map from $\bigotimes_{i=1}^{p} \mathrm{L}(V_i, W_i)$ to $\mathrm{L}\big(\bigotimes_{i=1}^{q} V_i, \bigotimes_{j=1}^{q} W_j\big)$ that relates the tensor product defined above to that from Proposition B.1.7. Postcomposition by this map produces another map

$$\mathrm{C}^k\big(X, \bigotimes_{i=1}^{q} \mathrm{L}(V_i, W_i)\big) \longrightarrow \mathrm{C}^k\big(X, \mathrm{L}\big(\bigotimes_{i=1}^{q} V_i, \bigotimes_{j=1}^{q} W_j\big)\big),$$

whose continuity is readily verified. Continuity of the tensor product operator described above then follows.. □

Proposition 3.1.12 (Direct sum).
Let $X \subseteq T$ be an open subset of a metrisable locally convex vector space T, and let $(V_\lambda)_{\lambda\in\Lambda}$ be a collection of locally convex vector spaces. Given maps $\alpha_\lambda\colon X \to \mathrm{L}(V_\lambda, W_\lambda)$ for $\lambda \in \Lambda$ of class C^k with $k \in \mathbf{N}_0 \cup \{\infty\}$, also the sum

$$\bigoplus_{\lambda\in\Lambda} \alpha_\lambda\colon X \longrightarrow \mathrm{L}\big(\textstyle\bigoplus_{\kappa\in\Lambda} V_\kappa, \bigoplus_{\lambda\in\Lambda} W_\lambda\big), \qquad \sum_{\lambda\in A} v_\lambda \mapsto \sum_{\lambda\in A} \alpha_\lambda(v_\lambda)$$

is of class C^k. The induced operator

$$\oplus\colon \prod_{\lambda\in\Lambda} \mathrm{C}^k(X, \mathrm{L}(V_\lambda, W_\lambda)) \longrightarrow \mathrm{C}^k\big(X, \mathrm{L}\big(\textstyle\bigoplus_{\kappa\in\Lambda} V_\kappa, \bigoplus_{\lambda\in\Lambda} W_\lambda\big)\big)$$

is a topological isomorphism onto its image.

Proof: The collection $(\alpha_\lambda)_{\lambda\in\Lambda}$ can be viewed as a single map of class C^k from X to the product $\prod_{\lambda\in\Lambda} \mathrm{L}(V_\lambda, W_\lambda)$, and the sum $\bigoplus_{\lambda\in\Lambda} \alpha_\lambda$ is the composition of this map with the canonical maps

$$\prod_{\lambda\in\Lambda} \mathrm{L}(V_\lambda, W_\lambda) \hookrightarrow \prod_{\kappa\in\Lambda} \mathrm{L}\big(V_\kappa, \textstyle\bigoplus_{\lambda\in\Lambda} W_\lambda\big) \xrightarrow{\sim} \mathrm{L}\big(\textstyle\bigoplus_{\kappa\in\Lambda} V_\kappa, \bigoplus_{\lambda\in\Lambda} W_\lambda\big).$$

The first map is clearly a topological embedding, and Proposition A.2.25 tells us that the second map is a topological isomorphism. From this, we not only learn that that $\bigoplus_{\lambda\in\Lambda} \alpha_\lambda$ is of class C^k, but also that the corresponding inclusion of $\mathrm{C}^k(X, \mathrm{L}(V_\lambda, W_\lambda))$ into $\mathrm{C}^k\big(X, \mathrm{L}\big(\bigoplus_\kappa V_\kappa, \bigoplus_\lambda W_\lambda\big)\big)$ is a topological isomorphism onto its image. ☐

Although we have stated Proposition 3.1.11 using the uncompleted projective tensor product, the exact same statements would also hold if the spaces $\bigotimes_{i=1}^p V_i$ and $\bigotimes_{j=1}^q W_j$ had been replaced by their completions. This follows from the observation in Remark A.2.42 that the space $\mathrm{L}\big(\bigotimes_{i=1}^q V_i, \bigotimes_{j=1}^q W_j\big)$ can be (canonically) embedded into $\mathrm{L}\big(\overline{\bigotimes}_{i=1}^q V_i, \overline{\bigotimes}_{j=1}^q W_j\big)$ if the spaces V_i for $i = 1, 2, \dots, q$ are metrisable.

Proposition 3.1.13 (Transposition).
Let $X \subseteq T$ be an open subset of a metrisable locally convex vector space T, and let V and W be locally convex vector spaces of which the second is barrelled. The transpose $\alpha^\vee\colon X \to \mathrm{L}(W^\vee, V^\vee)$ of a map $\alpha\colon X \to \mathrm{L}(V, W)$ is of class C^k with $k \in \mathbf{N}_0 \cup \{\infty\}$ whenever α is.

Proof: This is a consequence of Proposition A.2.30 and the fact that the composition of map of class C^k with a continuous linear map is of class C^k. ☐

3.1.3. Analytic families of linear maps

The last type of families we consider are analytic families of linear maps. We will see that statements similar to those in section 3.1.1 and section 3.1.2 hold for this class as well. Given a Fréchet domain $X \subseteq T$ and two locally convex vector spaces V and W, the space $\mathrm{C}^\omega(X, \mathrm{L}(V, W))$ of analytic maps is a subspace of $\mathrm{C}^\infty(X, \mathrm{L}(V, W))$, and will be endowed with the corresponding subspace topology.

The next proposition describes a number of equivalent characterisations of analytic families of families of continuous linear maps.

Proposition 3.1.14.
Let $X \subseteq T$ be a Fréchet domain, let V be a barrelled, metrisable locally convex vector space and let W be an arbitrary locally convex vector space. For any function $\alpha\colon X \to \mathrm{L}(V, W)$, the following statements are equivalent:

- **(i)** *α is analytic;*
- **(ii)** *the map $\check{\alpha}\colon X \times V \to W$ given by $\check{\alpha}(x, v) = \alpha(x)(v)$ is analytic;*
- **(iii)** *there exists a (unique) continuous linear map*

$$\hat{\alpha}_\omega\colon \mathrm{C}^\omega(X, V) \longrightarrow \mathrm{C}^\omega(X, W) \qquad \text{with} \qquad \hat{\alpha}_\omega(e)(x) = \alpha(x)(e(x))$$

  for all $e \in \mathrm{C}^\omega(X, V)$ and every $x \in X$.
- **(iv)** *the map $\alpha(\cdot)(v)\colon X \to W$ is of class C^ω for every $v \in V$;*

Proof: It is easy to see that statement (iv) is implied by each of the other three statements. Equivalence of this conditions to statement (i) follows from the Banach–Steinhaus theorem, Theorem A.2.28, which directly states that a sequence in $\mathrm{L}(V, W)$ is convergent for the topology of precompact convergence if and only if it is pointwise convergent.

It is also relatively simple to see that condition (ii) holds whenever condition (i) does because evaluation at v is a continuous operation and convergence of a sequence converging to $\alpha(x) \in \mathrm{L}(V, W)$ therefore implies convergence to

$\alpha(x)(v)$ of the corresponding sequence in $\mathrm{L}(W)$ for every $v \in V$. This proposition would therefore be proven if we can show that condition (iii) is implied by statement (i).

To demonstrate this last implication, we need to do two things: we need to show that $\hat{\alpha}_\omega(e)$ is analytic for every $e \in \mathrm{C}^\omega(X, V)$ and that $\hat{\alpha}_\omega$ is continuous. The first fact is a form of the chain rule that is important on its own and is stated and proven separately in Corollary 3.1.16 below. The first fact is proven in Corollary 3.1.16 below, which we state and prove separately because it is important in its own right. Once it is established that α_ω is well-defined, its continuity is an immediate consequence of the similar claim from Proposition 3.1.6 because $\mathrm{C}^\omega(X, V)$ and $\mathrm{C}^\omega(X, W)$ are embedded subspaces of the corresponding spaces of smooth sections. □

Proposition 3.1.15 and its corollaries are versions of the usual chain rule for families of continuous linear maps.

Proposition 3.1.15.
Let $X \subseteq T$ be a Fréchet domain and let V, W and Z be arbitrary locally convex vector spaces. Given two analytic maps $\alpha\colon X \to \mathrm{L}(V, W)$ and $\beta\colon X \to \mathrm{L}(W, Z)$, the composition

$$\beta \circ \alpha\colon X \longrightarrow \mathrm{L}(V, Z), \qquad (\beta \circ \alpha)(x)(v) = \big(\beta(x) \circ \alpha(x)\big)(v)$$

is analytic as well.

Proof: Proposition B.1.23 tells us that there exists a complex Fréchet domain $\tilde{X} \subseteq T \otimes \mathbf{C}$ containing X and analytic maps $\tilde{\alpha}\colon \tilde{X} \to \mathrm{L}(V \otimes \mathbf{C}, \overline{W} \otimes \mathbf{C})$ and $\tilde{\beta}\colon \tilde{X} \to \mathrm{L}(W \otimes \mathbf{C}, \overline{Z} \otimes \mathbf{C})$ that extend α and β. We know from Proposition B.1.24 that both of these maps satisfy the Cauchy–Riemann equation.

It follows from Proposition 3.1.8 that the composition $\tilde{\beta} \circ \tilde{\alpha}$ is smooth when viewed as a map between real vector spaces. Its derivative is given by $\mathrm{D}(\tilde{\beta} \circ \tilde{\alpha}) = \mathrm{D}\tilde{\beta} \circ \tilde{\alpha} + \tilde{\beta} \circ \mathrm{D}\tilde{\alpha}$ and one can readily verify that it also satisfies the Cauchy–Riemann equation. We can thus deduce from Proposition B.1.24 that $\tilde{\beta} \circ \tilde{\alpha}$ is complex analytic, and that $\beta \circ \alpha = \tilde{\beta} \circ \tilde{\alpha}|_X$ is therefore analytic as well. □

Corollary 3.1.16.

Let $X \subseteq T$ be a Fréchet domain, and let W and Z be locally convex vector spaces. Given two analytic maps $e\colon X \to W$ and $\beta\colon X \to \mathrm{L}(W, Z)$, the composition

$$\beta \circ e\colon X \longrightarrow Z, \qquad (\beta \circ e)(x)(v) = \beta(x)(e(x))$$

is analytic as well.

By combining Proposition 3.1.15 and Proposition B.1.28, the following corollary is obtained.

Corollary 3.1.17.

Let $X \subseteq S$ and $Y \subseteq T$ be Fréchet domains and let V, W and Z be arbitrary locally convex vector spaces. Given three analytic maps $\alpha\colon X \to \mathrm{L}(V, W)$, $\beta\colon Y \to \mathrm{L}(W, Z)$ and $f\colon X \to Y$, the composition

$$\beta \circ (\alpha, f)\colon X \longrightarrow \mathrm{L}(V, Z), \qquad x \mapsto \beta(f(x)) \circ \alpha(x)$$

is also analytic.

We finally formulate analytic versions of the product, direct sum and transposition rules Proposition 3.1.11 and Proposition 3.1.13 for analytic maps.

Proposition 3.1.18 (Product rule)**.**

Let $X \subseteq T$ be a Fréchet domain, and let $V_1, V_2, \ldots, V_q$ and $W_1, W_2, \ldots, W_q$ be locally convex vector spaces. Given analytic maps $\alpha_i\colon X \to \mathrm{L}(V_i, W_i)$ for $i = 1, 2, \ldots, q$, also the tensor product $\bigotimes_{i=1}^q \alpha_i\colon X \to \mathrm{L}\big(\bigotimes_{i=1}^q V_i, \bigotimes_{j=1}^q W_q\big)$ is analytic.

Proof: This follows from the product rule in Proposition B.1.28, along with the continuity of the canonical map from $\bigotimes_{i=1}^p \mathrm{L}(V_i, W_i)$ to $\mathrm{L}\big(\bigotimes_{i=1}^q V_i, \bigotimes_{j=1}^q W_j\big)$. □

Proposition 3.1.19 (Direct sum)**.**

Let $X \subseteq T$ be an open subset of a metrisable locally convex vector space T, and let $(V_\lambda)_{\lambda\in\Lambda}$ be a collection of locally convex vector spaces. Given analytic maps $\alpha_\lambda\colon X \to \mathrm{L}(V_\lambda, W_\lambda)$ for $\lambda \in \Lambda$, the sum

$$\bigoplus_{\lambda\in\Lambda} \alpha_\lambda\colon X \longrightarrow \mathrm{L}\big(\textstyle\bigoplus_{\kappa\in\Lambda} V_\kappa, \bigoplus_{\lambda\in\Lambda} W_\lambda\big), \qquad \textstyle\sum_{\lambda\in A} v_\lambda \mapsto \sum_{\lambda\in A} \alpha_\lambda(v_\lambda)$$

is analytic as well.

Proof: The argument from Proposition 3.1.12 can nearly be copied verbatim since the composition of an analytic map with a continuous linear map is analytic and the space of analytic maps from X to a locally convex vector space is an embedded subspace of the corresponding space of smooth maps. ☐

Proposition 3.1.20 (Transposition).
Let $X \subseteq T$ be an open subset of a metrisable locally convex vector space T, and let V and W be locally convex vector spaces of which the second is barrelled. The transpose $\alpha^\vee\colon X to \mathrm{L}(W^\vee, V^\vee)$ of a map $\alpha\colon X \to \mathrm{L}(V, W)$ is analytic whenever α is. Transposition is continuous as a map from $\mathrm{C}^k(X, \mathrm{L}(V, W))$ to $\mathrm{C}^k(W^\vee, V^\vee)$.

Proof: Analyticity of the transpose $\alpha^\vee$ can be demonstrated by applying Proposition A.2.30, and continuity of transposition is a consequence of Proposition 3.1.13. ☐

3.1.4. Families of multilinear maps

A continuous multilinear map from the product of locally convex vector spaces $V_1, V_2, \ldots, V_p$ to another such space W is a continuous map from $\prod_{i=1}^p V_i$ to W which is linear in each argument. We denote the space of such maps by $\mathrm{L}(V_1, V_2, \ldots, V_p; W)$ and recall from Proposition A.2.46 that the tensor product operator $\chi\colon \prod_{i=1}^p V_i \to \bigotimes_{i=1}^p V_i$ induces a canonical isomorphism

$$\mathrm{L}\big(\textstyle\bigotimes_{i=1}^p V_i, W\big) \xrightarrow{\sim} \mathrm{L}(V_1, \ldots, V_p; W), \qquad \alpha \mapsto \alpha \circ \chi \tag{3.1.3}$$

of locally convex vector spaces whenever each of the spaces V_i is metrisable.

We define a family of continuous multilinear maps from the product of locally convex vector spaces $V_1, V_2, \ldots, V_p$ to W, parametrised by a space X, as a function

$$\alpha\colon X \longrightarrow \mathrm{L}\big(\textstyle\bigotimes_{i=1}^p V_i, W\big),$$

and declare this family to be of class C^k for $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$ whenever this map is of class C^k for the aforementioned topology. Most of the statements from sections 3.1.2 and 3.1.3 have multilinear analogues, which are presented below.

Lemma 3.1.21.
Let $X \subseteq T$ be a Fréchet domain, let $V_1, V_2, \ldots, V_p$ be barrelled, metrisable locally convex vector spaces and let W be an arbitrary locally convex vector space. For any function $\alpha\colon X \to \mathrm{L}\big(\bigotimes_{i=1}^p V_i, W\big)$ and any $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$, the following statements are equivalent:

(i) *α is of class C^k;*

(ii) *the map $\check{\alpha}\colon X \times \prod_{i=1}^p V_i \to W$ given by $\check{\alpha}(x, v_1, \ldots, v_p) = \alpha(x)(v_1, \ldots, v_p)$ is of class C^k;*

(iii) *there exists a (unique) continuous multilinear map*

$$\hat{\alpha}_k\colon \bigotimes_{i=1}^p \mathrm{C}^k(X, V_i) \longrightarrow \mathrm{C}^k(X, W),$$

which is given by

$$\hat{\alpha}_k\big(\bigotimes_{i=1}^p e_i\big)(x) = \alpha(x)\big(\bigotimes_{i=1}^p e_i(x)\big)$$

for all $e_i \in \mathrm{C}^k(X, V_i)$ with $i = 1, 2, \ldots, p$ and every $x \in X$.

(iv) *for all $(v_1, v_2, \ldots, v_p) \in \prod_{i=1}^p V_i$, the map*

$$\alpha_{v_1,\ldots,v_p} = \alpha(\cdot)(v_1, \ldots, v_p)\colon X \longrightarrow W, \qquad x \mapsto \alpha(x)(v_1, \ldots, v_p)$$

is of class C^k.

Proof: Because the canonical map from $\prod_{i=1}^p V_i$ to $\bigotimes_{i=1}^p V_i$ is continuous and multilinear it is in particular of class C^k for any $k \in \mathbf{N}_0 \cup \{\infty\}$ (resp. of class C^ω). Consequently, Proposition 3.1.6 (resp. Proposition 3.1.14) and Proposition 3.1.8 (resp. Proposition 3.1.15) can be used to demonstrate the implication (i) ⇒ (ii). Continuity of the multilinear map from $\prod_{i=1}^p \mathrm{C}^k(X, V_i)$ to $\mathrm{C}^k(X, \bigotimes_{i=1}^p V_i)$, which was shown in Proposition B.1.7 (resp. Proposition B.1.28), similarly allows us to use tells us to use Proposition 3.1.6 (resp. Proposition 3.1.14) to prove the implication (i) ⇒ (iii).

It is clear that both statements (i) and (iii) imply statement (iv), so the only implication that remains to be demonstrated is (iv) ⇒ (i). This can be done by applying Lemma 3.1.7 if $k \in \mathbf{N}_0 \cup \{\infty\}$ and Proposition 3.1.14 if $k = \omega$.

Assume that $\alpha_{v_1,\ldots,v_p}$ is of class C^k for all $(v_1, v_2, \ldots, v_p) \in \prod_{i=1}^p V_i$ and some fixed $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. For any element $\overline{v} \in \bigotimes_{i=1^p} V_i$, the corresponding map $\alpha_{\overline{v}}$ from X to W is a linear combination of maps of the form $\alpha_{v_1,\ldots,v_p}$ with

$(v_1, v_2, \ldots, v_p) \in \prod_{i=1}^p V_i$ and is therefore of class C^k. As a projective tensor product of barrelled metrisable locally convex vector spaces, $\bigotimes_{i=1^p} V_i$ is both metrisable and barreled due to Proposition A.2.43(iv). Therefore, Lemma 3.1.7 (resp. Proposition 3.1.14) tells us that α is of class C^k as well. ☐

Proposition 3.1.22.

Let $X \subseteq T$ be a Fréchet domain, let $V_1, V_2, \ldots, V_p$, $W_1, W_2, \ldots, W_q$ be metrisable locally convex vector spaces and let Z_1 and Z_2 be arbitrary locally convex vector spaces. Consider two maps $\alpha\colon X \to \mathrm{L}\big(\bigotimes_{i=1}^p V_i, Z_1\big)$ and $\beta\colon X \to \mathrm{L}\big(\bigotimes_{j=1}^q W_j, Z_2\big)$ which are of class C^k for some $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. The map

$$\alpha \otimes \beta\colon X \longrightarrow \mathrm{L}\big(\textstyle\bigotimes_{i=1}^p V_i \otimes \bigotimes_{j=1}^q W_j, Z_1 \otimes Z_2\big),$$

which is given by

$$(\alpha \otimes \beta)(x)(v_1, \ldots, v_p, w_1, \ldots, w_q) = \alpha(x)(v_1, \ldots, v_p) \otimes \beta(x)(w_1, \ldots, w_q),$$

for $x \in X$, $v_i \in V_i$ and $w_j \in W_j$, is of class C^k as well.

Proof: This follows directly from Proposition 3.1.11 and Proposition 3.1.18. ☐

Whenever we have a family $\beta\colon X \to \mathrm{L}\big(\bigotimes_{j=1}^p W_j, Z\big)$ of continuous multilinear maps and a p-tuple $(\alpha_j)_{j=1}^p$ of similar families $\alpha_j\colon X \to \mathrm{L}\big(\bigotimes_{i=1}^{q_j} V_{i,j}, W_j\big)$, we can consider the composition

$$\beta \circ (\alpha_1, \ldots, \alpha_p)\colon X \longrightarrow \mathrm{L}\big(\textstyle\bigotimes_{i=1}^p \bigotimes_{j=1}^{q_i} V_{i,j}, Z\big),$$

which is given by

$$(\beta \circ \alpha)(x)\big(\textstyle\bigotimes_{i=1}^p \bigotimes_{j=1}^{q_i} v_{i,j}\big) = \beta(x)\big(\bigotimes_{i=1}^p \alpha_1(x)\big(\bigotimes_{j=1}^{q_i} v_{i,j}\big)\big)$$

for $x \in X$ and $v_{i,j} \in V_{i,j}$ with $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q_i$.

Proposition 3.1.23.

Let $X \subseteq T$ be Fréchet domain, let $V_{i,j}$ and W_j for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q_p$ be locally convex vector spaces and assume that each of the spaces $V_{i,j}$ is metrisable. Let the maps α_i and β for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q_p$ be as described above. If each of the maps α_i and β is of class C^k for some fixed $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$, then so is the composition $\beta \circ (\alpha_1, \ldots, \alpha_p)$.

Proof: The map $\beta \circ (\alpha_1, \ldots, \alpha_0)$ can be expressed as a composition

$$\beta \circ (\alpha_1, \ldots, \alpha_p) = \beta \circ \left(\bigotimes_{i=1}^{p} \alpha_i\right)$$

of β with the tensor products of the maps α_i. It therefore follows from Proposition 3.1.22, along with Proposition 3.1.8 and Proposition 3.1.15, that $\beta \circ (\alpha_1, \ldots, \alpha_p)$ is of class C^k whenever each of the maps β and α_i is. ☐

There is a multitude of other ways to compose families of multilinear maps, but these can generally be viewed as special cases of Proposition 3.1.23. Two such compositions are discussed in and section 3.2.3.

3.2. Holonomic families

In this section, we will introduce families of continuous plurilinear maps and, more importantly, the concept of a holonomic family of such maps. We have borrowed the term "holonomicity" from the theory of jet bundles and use it to describe families of plurilinear maps that contain their own derivatives and can thus be viewed as infinite jets of other families of plurilinear maps. We will describe what holonomicity means for the families of coderivations and coalgebra morphisms that a family of plurilinear maps represents, and will show that this property is respected by the compositions described in section 2.2.4.

3.2.1. Families of plurilinear maps

We recall from section 2.1.2 that a homogeneous continuous plurilinear map from a **Z**-graded metrisable locally convex vector space V to another such space W of degree d is a continuous linear map from the graded symmetric algebra $\bigodot(V)$ to W of the same degree. We endow the space $L(\bigodot(V), W)$ with the topology of precompact convergence and note that since $\bigodot(V)$ is a locally convex direct sum of spaces of the form $\bigodot_{i=1}^{p} V^{n_i}$, Proposition A.2.25 tells us that this space decomposes as a product

$$L(\bigodot(V), W)^d \simeq \prod_{p \in \mathbf{N}_0} \prod_{n_1 \leq \cdots \leq n_p} L\left(\bigodot_{i=1}^{p} V^{n_i}, W^{n_1 + \cdots + n_p + d}\right),$$

of topological vector spaces.

A family of homogeneous plurilinear maps from V to W parametrised by a space X can therefore be defined as either a single function

$$\mathfrak{f}\colon X \longrightarrow \mathrm{L}(\bigodot(V), W)^d$$

for some $d \in \mathbf{Z}$, or as a collection $(\mathfrak{f}_{n_1,\dots,n_p})_{n_1 \le n_2 \le \dots \le n_p}$ of families

$$\mathfrak{f}_{n_1,\dots,n_p}\colon X \longrightarrow \mathrm{L}\big(\textstyle\bigodot_{i=1}^p V^{n_i}, W^{n_1+\dots+n_p+d}\big)$$

of continuous linear maps. We can also think of $\mathfrak{f}$ as a collection $(\mathfrak{f}_p)_{p\in\mathbf{N}_0}$ of families $\mathfrak{f}_p\colon X \to \mathrm{L}(\bigodot^p(V), W)^d$ of homogeneous multilinear maps, given by $\mathfrak{f}_p(x) = \mathfrak{f}(x)|_{\bigodot^p(V)}$.

If X is a Fréchet domain, we declare the map $\mathfrak{f}$ described above smooth or analytic if it is smooth in the sense of Definition B.1.1 or analytic in the sense of Definition B.1.21 respectively. The following proposition is a culmination of some of the results from section 3.1.

Proposition 3.2.1.
Let $X \subseteq T$ be a Fréchet domain, let V be a $\mathbf{Z}$-graded vector space of Fréchet type and let W be a $\mathbf{Z}$-graded locally convex vector space. For any family $\alpha\colon X \to \mathrm{L}(\bigodot(V), W)^d$ of plurilinear maps of degree d and any $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$, the following statements are equivalent:

- **(i)** *α is of class C^k as a map from X to $\mathrm{L}(\bigodot(V), W)$;*
- **(ii)** *for any sequence $n_1, n_2, \dots, n_p \in \mathbf{Z}$ of degrees and $n = \sum_{i=1}^p n_i$, the map*

$$\check{\alpha}_{n_1,\dots,n_p}\colon X \times \prod_{i=1}^p V^{n_i} \longrightarrow W^{n+d}, \qquad (x, v_1, \dots, v_p) \mapsto \alpha(x)\big(\textstyle\bigodot_{i=1}^p v_i\big)$$

  is of class C^k;
- **(iii)** *for any sequence $v_1, v_2, \dots, v_p \in V$ of homogeneous elements, the map*

$$\alpha_{v_1,\dots,v_p}\colon X \longrightarrow W^{|v_1|+\dots+|v_p|+d}, \qquad x \mapsto \alpha(x)\big(\textstyle\bigodot_{i=1}^p v_i\big)$$

  is of class C^k;

(iv) *there exists a continuous multilinear map*

$$\hat{\alpha}_k\colon \bigodot(\mathrm{C}^k(X,V)) \longrightarrow \mathrm{C}^k(X,W),$$

which is such that $\hat{\alpha}_k\big(\bigodot_{i=1}^p e_i\big)(x) = \alpha(x)\big(\bigodot_{i=1}^p e_i(x)\big)$ *for all functions* $e_1, e_2, \ldots, e_p \in \mathrm{C}^k(X,V)$ *and any* $x \in X$.

Proof: Since the space $\mathrm{L}(\bigodot(V),V)^d$ is canonically isomorphic to the product of $\mathrm{L}(\bigodot_{i=1}^p V^{n_i}, W)^d$ for $n_1 \leq \cdots \leq n_p \in \mathbf{Z}$, α is of class C^k for a given $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$ precisely when the components

$$\alpha_{n_1,\ldots,n_p} = \pi_{n_1,\ldots,n_p} \circ \alpha\colon X \longrightarrow \mathrm{L}\big(\bigodot_{i=1}^p V^{n_i}, W^{n+d}\big)$$

are of class C^k for every sequence $n_1.n_2,\ldots,n_p \in \mathbf{Z}$ of degrees and $n = \sum_{i=1}^p n_i$. Since the canonical inclusion map $\mathrm{L}\big(\bigodot_{i=1}^p V^{n_i}, W\big) \hookrightarrow \mathrm{L}\big(\bigotimes_{i=1}^p V^{n_i}, W\big)$ is an embedding, this in turn happens if and only if the corresponding map from X to $\mathrm{L}\big(\bigotimes_{i=1}^p V^{n_k}, W^{n+d}\big)$ is of class C^k. Equivalence of statements (i), (ii) and (iii) consequently follows from Lemma 3.1.21.

Lemma 3.1.21 moreover tells us that these statements are therefore equivalent to existence and continuity of the operators

$$\hat{\alpha}_{n_1,\ldots,n_p}\colon \bigotimes_{i=1}^p \mathrm{C}^k(X,V^{n_i}) \longrightarrow \mathrm{C}^k(X,W^{n+d}),$$

given by $\hat{\alpha}_{n_1,\ldots,n_p}(\bigotimes_{i=1}^p e_i)(x) = \alpha(x)(\bigotimes_{i=1}^p e_i(x))$ for $x \in X$ and $e_i \in \mathrm{C}^k(X,V^{n_i})$ with $i = 1,2,\ldots,p$, and $n_1,n_2,\ldots,n_p \in \mathbf{Z}$. Because this map is necessarily graded symmetric, this is equivalent to existence and continuity of a similar map from $\bigodot_{i=1}^p \mathrm{C}^k(X,V^{n_i})$ to $\mathrm{C}^k(X,W^{n+d})$. Since the graded symmetric algebra from statement (iv) is a locally convex direct sum

$$\bigodot(\mathrm{C}^k(X,V)) = \bigoplus_{\substack{n_1\leq\cdots\leq n_p\\ n_1+\cdots+n_p=n}} \bigodot_{i=1}^p \mathrm{C}^k(X,V^{n_i}),$$

of the domains of these maps, the universal property of the locally convex direct sum tells us that this last statement is also equivalent to the first three. ☐

We recall from Definition B.1.16 that the infinite jet of a smooth map $f : X \to W$ defined on an open subset $X \subseteq T$ of a locally convex vector space T is the map

$$\mathrm{j}^\infty f : X \longrightarrow \mathrm{L}(\bigodot(T), W),$$

given by

$$\mathrm{j}^\infty f(x)\big(\textstyle\bigodot_{i=1}^p \dot{x}_i\big) = \mathrm{D}^p f(x)\big(\textstyle\bigodot_{i=1}^p \dot{x}_i\big)$$

for $x \in X$ and $\dot{x}_1, \dot{x}_2, \dots, \dot{x}_p \in T$. As in Definition B.1.18, we call a general map $\tilde{f}$ from X to the space $\mathrm{L}(\bigodot(T), W)$ holonomic whenever it is of this form.

Below, we will consider a $\mathbf{Z}$-graded vector space V of Fréchet type and the subspace $V^{\neq 0} = \bigoplus_{n \neq 0} V^n$ consisting of all homogeneous components of V apart from V^0. We observe that the graded symmetric algebra $\bigodot(V)$ is algebraically isomorphic to the tensor product $\bigodot(V^0) \otimes \bigodot(V^{\neq 0})$, and that

$$\mathrm{L}(\textstyle\bigodot(V), W)^d \simeq \mathrm{L}\big(\bigodot(V^0), \mathrm{L}(\bigodot(V^{\neq 0}), W)\big)^d,$$

as a topological vector space, for any $d \in \mathbf{Z}$.

Definition 3.2.2.
Let V and W be $\mathbf{Z}$-graded vector spaces of Fréchet type and let $X \subseteq V^0$ be an open subset. A smooth map from X to $\mathrm{L}(\bigodot(V), W)$ is *holonomic* if it is of the form

$$\mathrm{j}^\infty \tilde{\mathfrak{f}} \colon X \longrightarrow \prod_{q \in \mathbf{N}_0} \mathrm{L}\big(\textstyle\bigodot^q(V^0), \mathrm{L}(\bigodot(V^{\neq 0}), W)^d\big) \simeq \mathrm{L}(\bigodot(V), W)$$

for some smooth map $\tilde{\mathfrak{f}} \colon X \to \mathrm{L}(\bigodot(V^{\neq 0}), W)^d$.

We shall write $\mathrm{C}^\infty_{\mathrm{hol}}\big(X, \mathrm{L}(\bigodot(V), W)\big)$ to denote the space of holonomic maps from X to $\mathrm{L}(\bigodot(V), W)$. Because the restriction of $\mathrm{j}^\infty \tilde{\mathfrak{f}}(x)$ to $\bigodot(V^{\neq 0})$ coincides with $\tilde{\mathfrak{f}}(x)$ for any $x \in X$, the maps

$$\mathrm{C}^\infty_{\mathrm{hol}}\big(X, \mathrm{L}(\textstyle\bigodot(V), W)\big) \underset{\mathrm{j}^\infty}{\overset{\bullet|_{\bigodot(V^{\neq 0})}}{\rightleftarrows}} \mathrm{C}^\infty\big(X, \mathrm{L}(\bigodot(V^{\neq 0}), W)\big), \tag{3.2.1}$$

are mutually inverse isomorphisms of graded vector spaces.

We will often use the following characterisation of holonomic maps instead of directly working with Definition 3.2.2.

Proposition 3.2.3.
Let V and W be $\mathbf{Z}$-graded vector spaces of Fréchet type and let $X \subseteq V^0$ be a

Fréchet domain. A smooth map $\mathfrak{f}\colon X \to \mathrm{L}(\bigodot(V), W)$ *is holonomic if and only if its derivative is given by*

$$\mathrm{D}\mathfrak{f}(x)(\dot{x})(v_{\{1,\dots,p\}}) = \mathfrak{f}(x)(\dot{x} \odot v_{\{1,\dots,p\}}). \tag{3.2.2}$$

for every $x \in X$*, every* $\dot{x} \in V^0$ *and all* $v_1, v_2, \dots, v_p \in V$*.*

Proof: If we assume that equation (3.2.2) holds, it can be applied repeatedly, along with Corollary 3.1.9, to deduce that

$$\mathrm{D}^q\mathfrak{f}(x)(\dot{x}_{\{1,\dots,q\}})(v_{\{1,\dots,p\}}) = \mathfrak{f}(x)(\dot{x}_{\{1,\dots,q\}} \odot v_{\{1,\dots,p\}}).$$

for any $x \in X$ and all $\dot{x}_1, \dot{x}_2, \dots, \dot{x}_q \in V^0$ and $v_1, v_2, \dots, v_p \in V$. This tells us that $\mathfrak{f}$ is of the form $\mathrm{j}^\infty\tilde{\mathfrak{f}}$ for $\tilde{\mathfrak{f}} = \mathfrak{f}|_{\bigodot(V^{\neq 0})}$ and that it is therefore holonomic.

If, on the other hand, $\mathfrak{f}$ is holonomic and $\tilde{\mathfrak{f}}\colon X \to \mathrm{L}(\bigodot(V^{\neq 0}), W)$ is such that $\mathfrak{f} = \mathrm{j}^\infty\tilde{\mathfrak{f}}$, then the derivative of $\mathfrak{f}$ derivative can be expressed as

$$\begin{aligned}\mathrm{D}\mathfrak{f}(x)(\dot{x}_0)(\dot{x}_{\{1,\dots,q\}} \odot v_{\{1,\dots,p\}}) &= \mathrm{D}\big(\mathrm{D}^q\tilde{\mathfrak{f}}(x)\big)(\dot{x}_0)(\dot{x}_{\{1,\dots,q\}})(v_{\{1,\dots,p\}}) \\ &= \mathrm{D}^{q+1}\tilde{\mathfrak{f}}(x)(\dot{x}_0 \odot \dot{x}_{\{1,\dots,q\}})(v_{\{1,\dots,p\}}) \\ &= \mathfrak{f}(x)(\dot{x}_0 \odot \dot{x}_{\{1,\dots,q\}} \odot v_{\{1,\dots,p\}})\end{aligned}$$

for $x \in X$, $\dot{x}_0, \dot{x}_1, \dots, \dot{x}_q \in V^0$ (with $q \in \mathbf{N}_0$) and homogeneous $v_1, \dots, v_p \in V$ of non-zero degree. We conclude that equation (3.2.2) holds in this case. ☐

We have encountered the condition (3.2.2) before in Proposition 2.4.3, where it was referred to as the *derivative property*. In that proposition we had shown that the family of twists of a curved L_∞-algebra (V, ℓ),

$$\mathfrak{l}\colon D_\ell \subseteq V^0 \longrightarrow \mathrm{L}(\bigodot(V), V), \qquad u \mapsto \sum_{p \in \mathbf{N}_0} \ell(\bigodot^p u \odot \bullet),$$

has this property. Proposition 3.2.3 thus allows us to conclude that this family is in fact holonomic. We had used the following fact in the proof of Proposition 2.4.4.

Proposition 3.2.4.
Let V *and* W *be* $\mathbf{Z}$*-graded vector spaces of Fréchet type, let* $X \subseteq V^0$ *be a Fréchet domain and let* $x_0 \in X$ *be arbitrary. If two analytic holonomic families* $\mathfrak{f}\colon X \to \mathrm{L}(\bigodot(V), W)$ *and* $\mathfrak{g}\colon X \to \mathrm{L}(\bigodot(V), W)$ *are such that* $\mathfrak{f}(x_0) = \mathfrak{g}(x_0)$*, then* $\mathfrak{f} = \mathfrak{g}$*.*

Proof: Let $\mathfrak{f}$ and $\mathfrak{g}$ be as described as above, then there exist analytic functions f and g from X to $\mathrm{L}(\bigodot(V^{\neq 0}), W)$ such that $\mathfrak{f} = \mathrm{j}^\infty f$ and $\mathfrak{g} = \mathrm{j}^\infty g$. The infinite jets of f and g at x_0 coincide because $\mathfrak{f}(x_0) = \mathfrak{g}(x_0)$, so Corollary B.1.26 tells us that that $f = g$ on X. It follows that also $\mathfrak{f} = \mathfrak{g}$. □

3.2.2. Families of coderivations and morphisms

Although we started this section by introducing smooth and holonomic families of plurilinear maps, it is really the corresponding families of coderivations or morphisms that we would like to describe. In this section, we show that also these families are smooth, and describe conditions that are equivalent to holonomicity for the corresponding families of plurilinear maps.

Families of coderivations

Recall from section 2.2.2 that every homogeneous linear map $\ell\colon \bigodot(V) \to V$ can be lifted to a unique coderivation $\bar{\ell}_{\mathrm{cod}}\colon \bigodot(V) \to \bigodot(V)$ on $(\bigodot(V), \Delta)$. This coderivation is given by

$$\bar{\ell}_{\mathrm{cod}}(v_{\{1,\dots,p\}}) = \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J}\, \ell(v_I) \odot v_J.$$

for $v_1, v_2, \dots, v_p \in V$. By applying this construction at every point in its domain, a smooth family of linear maps $\mathfrak{l}\colon X \to \mathrm{L}(\bigodot(V), V)$ can thus be lifted to a family of coderivations,

$$\bar{\mathfrak{l}}_{\mathrm{cod}}\colon X \longrightarrow \mathrm{L}(\bigodot(V), \bigodot(V)), \qquad x \mapsto \overline{\mathfrak{l}(x)}_{\mathrm{cod}}.$$

Proposition 3.2.5.

Let V be a $\mathbf{Z}$-graded vector space of Fréchet type, let $X \subseteq V^0$ be an open subset and let $\mathfrak{l}\colon X \to \mathrm{L}(\bigodot(V), V)$ be a homogeneous family of plurilinear maps (not necessarily smooth). The associated family of coderivations,

$$\bar{\mathfrak{l}}_{\mathrm{cod}}\colon X \longrightarrow \mathrm{L}(\bigodot(V), \bigodot(V)), \qquad x \mapsto \overline{\mathfrak{l}(x)}_{\mathrm{cod}},$$

is smooth or analytic if and only if $\mathfrak{l}$ is. Moreover, it satisfies the equation

$$\mathrm{D}\bar{\mathfrak{l}}_{\mathrm{cod}}(x)(\dot{x}) = [\bar{\mathfrak{l}}_{\mathrm{cod}}(x), \bar{m}_{\dot{x}}], \tag{3.2.3}$$

for every $x \in X$ and every $\dot{x} \in V^0$, if and only if $\mathfrak{l}$ is holonomic.

In equation (3.2.3), $\overline{m}_{\dot{x}}$ denotes the coderivation on $\bigodot(V)$ that describes multiplication by $\dot{x} \in V^0$, and is given by $\overline{m}_{\dot{x}}(v_{\{1,\dots,p\}}) = \dot{x} \odot v_{\{1,\dots,p\}}$. This equation therefore states that

$$\mathrm{D}\bar{\mathfrak{l}}_{\mathrm{cod}}(x)(\dot{x})(v_{\{1,\dots,p\}}) = \bar{\mathfrak{l}}_{\mathrm{cod}}(x)(\dot{x} \odot v_{\{1,\dots,p\}}) - \dot{x} \odot \bar{\mathfrak{l}}_{\mathrm{cod}}(x)(v_{\{1,\dots,p\}})$$

for every $x \in X$, every $\dot{x} \in V^0$ and $v_1, v_2, \dots, v_p \in V$.

Proof: We first assume that $\mathfrak{l}$ is smooth (resp. analytic). To prove that also $\bar{\mathfrak{l}}_{\mathrm{cod}}$ is smooth, it is sufficient to show that $\bar{\mathfrak{l}}_x(\cdot)(v_{\{1,\dots,p\}})$ depends smoothly (resp. analytically) on x for an arbitrary sequence of homogeneous elements $v_1, v_2, \dots, v_p \in V$, in light of to Lemma 3.1.21.

Given such homogeneous elements, we can write $\bar{\mathfrak{l}}_{\mathrm{cod}}(\cdot)(v_{\{1,\dots,p\}})$ as a finite sum $\sum_{I,J} \bar{\mathfrak{l}}_{I,J}$, where $\bar{\mathfrak{l}}_{I,J}$ denotes the map

$$\bar{\mathfrak{l}}_{I,J} : X \longrightarrow V^{|\mathfrak{l}|+|v_I|} \odot \bigodot_{j \in J} V^{|v_j|}, \qquad x \mapsto \epsilon_{I,J}\, \mathfrak{l}(x)(v_I) \odot v_J.$$

and I and J range over all decompositions of $\{1, \dots, p\}$ into pairs of disjoint subsets. Smoothness (resp. analyticity) of each of these maps, and thereby of $\bar{\mathfrak{l}}_{\mathrm{cod}}$, follows from the assumption that $\mathfrak{l}$ is smooth and the fact that $w \mapsto w \odot v_J$ is a continuous linear map.

Conversely, smoothness (or analyticity) of $\bar{\mathfrak{l}}_{\mathrm{cod}}$ implies smoothness of $\mathfrak{l}$ due to the continuity of the canonical projection map $\mathrm{L}(\bigodot(V), \bigodot(V)) \to \mathrm{L}(\bigodot(V), V)$.

If we now work out the derivative of $\bar{\mathfrak{l}}_{\mathrm{cod}}$, we obtain the expression

$$\mathrm{D}\bar{\mathfrak{l}}_{\mathrm{cod}}(x)(\dot{x})(v_{\{1,\dots,p\}}) = \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J}\, \mathrm{D}\mathfrak{l}(x)(\dot{x})(v_I) \odot v_J, \tag{3.2.4}$$

while on the other hand

$$\begin{aligned} [\bar{\mathfrak{l}}_{\mathrm{cod}}(x), m_{\dot{x}}](v_{\{1,\dots,p\}}) &= \bar{\mathfrak{l}}_{\mathrm{cod}}(x)(\dot{x} \odot v_{\{1,\dots,p\}}) - \dot{x} \odot \bar{\mathfrak{l}}_{\mathrm{cod}}(x)(\dot{x} \odot v_{\{1,\dots,p\}}) \\ &= \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J}\, \mathfrak{l}(x)(\dot{x} \odot v_I) \odot v_J \end{aligned} \tag{3.2.5}$$

for all $x \in X$ and all $\dot{x} \in V^0$. These expressions coincide if $\mathfrak{l}$ is holonomic because Proposition 3.2.3 provides us with the identity $\mathrm{D}\mathfrak{l}(x)(\dot{x})(v_I) = \mathfrak{l}(x)(\dot{x} \odot v_I)$ in this case.

Conversely, if equation (3.2.3) holds, then the rank 1 components in equations (3.2.4) and (3.2.5) (the terms for which $J = \varnothing$) must also coincide. The identity $\mathrm{D}\mathfrak{l}(x)(\dot{x})(v_{\{1,\dots,p\}}) = \mathfrak{l}(x)(\dot{x} \odot v_{\{1,\dots,p\}})$ thus necessarily holds, which allows us to conclude that $\mathfrak{l}$ is holonomic by again applying Proposition 3.2.3. ☐

A more conceptual argument for the second part of Proposition 3.2.5 uses that the derivative $\mathrm{D}\bar{\mathfrak{l}}_{\mathrm{cod}}(x)(\dot{x})$ and the graded commutator $[\bar{\mathfrak{l}}_{\mathrm{cod}}, \overline{m}_{\dot{x}}]$ are both smooth families of coderivations, regardless of whether $\mathfrak{l}$ is holonomic. These are determined by the fact that they extend $\mathrm{D}\mathfrak{l}(x)(\dot{x})$ and $\mathfrak{l}(x)(\dot{x} \odot \cdot)$ respectively, so, by Proposition 3.2.3, they coincide if and only if $\mathfrak{l}$ is holonomic.

We will henceforth refer to a family $\mathfrak{l}\colon X \to \mathrm{L}(\bigodot(V), \bigodot(V))$ as holonomic whenever it satisfies equation (3.2.3) for all $x \in X$ and any $\dot{x} \in V^0$.

Families of morphisms

We can similarly lift a plurilinear map $f\colon \bigodot(V) \to W$ of degree 0 to a (unique) coalgebra morphism $\bar{f}_{\mathrm{mor}}\colon (\bigodot(V), \Delta) \to (\bigodot(W), \Delta)$ if $f_0 := f(1)$ vanishes. While this is not possible if $f_0 \neq 0$, we can always define $\bar{f}_{\mathrm{mor}}$ as the lift $f - f_0$, so that

$$\bar{f}_{\mathrm{mor}}(v_1 \odot \cdots \odot v_p) = \sum_{I_1,\dots,I_m \neq \varnothing} \epsilon_{I_1,\dots,I_m} \tfrac{1}{m!} \bigodot_{j=1}^{m} f(v_{I_j}), \tag{3.2.6}$$

for homogeneous $v_1, v_2, \dots, v_p \in V$. The sum on the right-hand side of this equation is over all subsets $I_1, I_2, \dots, I_m \subseteq \{1,\dots,p\}$ that form an (ordered) partition of $\{1,\dots,p\}$. Note that $f_0 = f(v_\varnothing)$ does not appear in this sum since each of the subsets $I_j \subseteq \{1,\dots,p\}$ is by assumption non-empty.

Since the coalgebra morphism $\bar{f}_{\mathrm{mor}}$ is uniquely determined by the fact that it extends $f - f_0\colon \bigodot(V) \to W$, the pair $(\bar{f}_{\mathrm{mor}}, f_0)$ encodes the same information as the original linear map $f\colon \bigodot(V) \to W$. We can apply the same procedure to holonomic families of plurilinear maps.

Proposition 3.2.6.
Let V and W be a $\mathbf{Z}$-graded vector spaces of Fréchet type, let $X \subseteq V^0$ be a Fréchet domain and let $\mathfrak{f}\colon X \to \mathrm{L}(\bigodot(V), W)$ be a family of plurilinear maps of degree 0.

Then $\mathfrak{f}$ is smooth (or analytic) if and only if both $\mathfrak{f}_0\colon X \to W^0$ and the associated family of coalgebra morphisms,

$$\bar{\mathfrak{f}}_{\mathrm{mor}}\colon X \longrightarrow \mathrm{L}(\bigodot(V), \bigodot(W)), \qquad x \mapsto \overline{\mathfrak{f}(x)}_{\mathrm{mor}},$$

are smooth (resp. analytic). Moreover, $\mathfrak{f}$ is holonomic if and only if the pair $(\bar{\mathfrak{f}}_{\mathrm{mor}}, \mathfrak{f}_0)$ satisfies

$$\mathrm{D}\bar{\mathfrak{f}}_{\mathrm{mor}}(x)(\dot{x}) = \bar{\mathfrak{f}}_{\mathrm{mor}}(x) \circ \overline{m}_{\dot{x}} - \overline{m}_{\mathrm{D}\mathfrak{f}_0(x)(\dot{x})} \circ \bar{\mathfrak{f}}_{\mathrm{mor}}(x) \tag{3.2.7}$$

for every $x \in X$ and every $\dot{x} \in V^0$.

Proof: Assume that $\mathfrak{f}$ is smooth or analytic and that we are given homogeneous elements $v_1, v_2, \ldots, v_p \in V$, let $n_i = |v_i|$ for $i = 1, 2, \ldots, p$ and let $n_I = |v_I|$ for $I \subseteq \{1, \ldots, p\}$. We can express $\bar{\mathfrak{f}}_{\mathrm{mor}}(\cdot)(v_{\{1,\ldots,p\}})$ as a sum $\sum_{I_1,\ldots,I_m} \bar{\mathfrak{f}}_{I_1,\ldots,I_m}$ of maps

$$\bar{\mathfrak{f}}_{I_1,\ldots,I_m}\colon X \longrightarrow \bigodot_{j=1}^m W^{n_{I_j}}, \qquad x \mapsto \epsilon_{I_1,\ldots,I_m} \tfrac{1}{m!} \bigodot_{j=1}^m \mathfrak{f}(x)(v_{I_j}),$$

where $I_1, \ldots, I_m$ range over all ordered partitions of $\{1, 2, \ldots, p\}$. If we can show that these maps are smooth (resp. analytic), this will imply that also $\bar{\mathfrak{f}}_{\mathrm{mor}}(\cdot)(v_{\{1,\ldots,p\}})$ is smooth (resp. analytic) and thereby that $\bar{\mathfrak{f}}_{\mathrm{mor}}$ is smooth due to Lemma 3.1.21.

We know that $\mathfrak{f}(\cdot)(v_I)\colon X \to W^{n_I}$ is smooth (resp. analytic) for each subset $I \subseteq \{1, \ldots, p\}$, so the same is true for the product $\big(\mathfrak{f}(\cdot)(v_{I_j})\big)_{j=1}^m\colon X \to \prod_{j=1}^m W^{n_{I_j}}$. Smoothness (resp. analyticity) of $\bar{\mathfrak{f}}_{I_1,\ldots,I_m}$, and thereby of $\bar{\mathfrak{f}}_{\mathrm{mor}}$, therefore follows from the fact that $\odot\colon \prod_{j=1}^m V^{n_{I_j}} \to \bigodot_{j=1}^m V^{n_{I_j}}$ is a continuous multilinear map.

Conversely, smoothness or analyticity of $\bar{\mathfrak{f}}_{\mathrm{mor}}$ implies that $\mathfrak{f}$ has the same property due to the continuity of the canonical projection map $\mathrm{L}(\bigodot(V), \bigodot(V)) \to \mathrm{L}(\bigodot(V), V)$.

To prove the second statement, we observe that equation (3.2.7) reads

$$\mathrm{D}\bar{\mathfrak{f}}_{\mathrm{mor}}(x)(\dot{x})(v_{\{1,\ldots,p\}}) = \mathfrak{f}(x)(\dot{x} \odot v_{\{1,\ldots,p\}}) - \mathrm{D}\mathfrak{f}_0(x)(\dot{x}) \odot \bar{\mathfrak{f}}_{\mathrm{mor}}(x)(v_{\{1,\ldots,p\}}).$$

when applied to to elements $v_1, v_2, \ldots, v_p \in V$.

By projecting onto $W = \bigodot^1(W)$, we obtain the identities

$$0 = \mathfrak{f}(x)(\dot{x}) - \mathrm{D}\mathfrak{f}(x)(\dot{x})(1)$$

if $p = 0$ (since $v_\varnothing = 1$), and

$$\mathrm{D}\mathfrak{f}(x)(\dot{x})(v_{\{1,\dots,p\}}) = \mathfrak{f}(x)(\dot{x} \odot v_{\{1,\dots,p\}}) - 0$$

otherwise. Through Proposition 3.2.3, these equations together imply that $\mathfrak{f}$ is holonomic.

Now assume that $\mathfrak{f}$ is holonomic and let $v_1, v_2, \dots, v_p \in V$ again be homogeneous with $p \geq 1$. The derivative $\mathrm{D}\bar{\mathfrak{f}}_{\mathrm{mor}}(x)(\dot{x})(v_{\{1,\dots,p\}})$ is now given by

$$\sum_{I_1,\dots,I_m \neq \varnothing} \epsilon_{I_1,\dots,I_m} \tfrac{1}{m!} \sum_{i=1}^{m} \bigodot_{j<i} \mathfrak{f}(x)(v_{I_j}) \odot \mathrm{D}\mathfrak{f}(x)(\dot{x})_{v_{I_j}} \odot \bigodot_{j>i} \mathfrak{f}(x)(v_{I_j})$$
$$= \sum_{I_1,\dots,I_m \neq \varnothing} \epsilon_{I_1,\dots,I_m} \tfrac{1}{(m-1)!} \mathfrak{f}(x)(\dot{x} \odot v_{I_1}) \odot \bigodot_{j=2}^{m} \mathfrak{f}(x)(v_{I_j}).$$

By writing $v_0 = \dot{x}$, we can rewrite this as

$$\sum_{I_1,\dots,I_m \neq \varnothing} \epsilon_{I_1,\dots,I_m} \tfrac{1}{(m-1)!} \mathfrak{f}(x)(v_{\{0\}\cup I_1}) \odot \bigodot_{j=2}^{m} \mathfrak{f}(x)(v_{I_j}),$$
$$= \sum_{\tilde{I}_1,\dots,\tilde{I}_m \neq \varnothing} \epsilon_{\tilde{I}_1,\dots,\tilde{I}_m} \tfrac{1}{m!} \bigodot_{j=1}^{m} \mathfrak{f}(x)(v_{\tilde{I}_j}),$$
$$- \sum_{I_2,\dots,I_m \neq \varnothing} \epsilon_{\tilde{I}_1,\dots,\tilde{I}_m} \tfrac{1}{(m-1)!} \mathrm{D}\mathfrak{f}_0(x)(\dot{x}) \odot \bigodot_{j=2}^{m} \mathfrak{f}(x)(v_{I_j}),$$
$$= \bar{\mathfrak{f}}_{\mathrm{mor}}(x)(\dot{x} \odot v_{\{1,\dots,p\}}) - \mathrm{D}\mathfrak{f}_0(x)(\dot{x}) \odot \bar{\mathfrak{f}}_{\mathrm{mor}}(x)(v_{\{1,\dots,p\}}),$$

where the second sum is over all subsets $\tilde{I}_1, \tilde{I}_2, \dots, \tilde{I}_m \subseteq \{0, 1, \dots, p\}$ that form a partition of $\{0, \dots, p\}$. We thus see that

$$\mathrm{D}\bar{\mathfrak{f}}_{\mathrm{mor}}(x)(\dot{x})(v_{\{1,\dots,p\}}) = \bar{\mathfrak{f}}_{\mathrm{mor}}(x)(\dot{x} \odot v_{\{1,\dots,p\}}) - \mathrm{D}\mathfrak{f}_0(x)(\dot{x}) \odot \bar{\mathfrak{f}}_{\mathrm{mor}}(x)(v_{\{1,\dots,p\}})$$

whenever $p \geq 1$. The same equation also holds if $p = 0$, since in this case the left-hand side reads $\mathrm{D}\bar{\mathfrak{f}}_{\mathrm{mor}}(x)(\dot{x})(1)$, which is zero, and the right-hand side reads $\mathfrak{f}(x)(\dot{x}) - \mathrm{D}\mathfrak{f}_0(x)(\dot{x})$, which also vanishes by assumption. □

Proposition 3.2.6 effectively states that there is a canonical one-to-one correspondence between holonomic families of plurilinear maps from V to W, parametrised by $X \subseteq V^0$, and pairs $(\bar{\mathfrak{f}}_{\mathrm{mor}}, \mathfrak{f}_0)$ consisting of a smooth map $\mathfrak{f}_0 \colon X \to W^0$

and smooth families of coalgebra morphisms $\bar{\mathfrak{f}}_{\mathrm{mor}}: X \to \mathrm{L}(\bigodot(V), \bigodot(W))$ subject to equation (3.2.7).

If we view $\bigodot(V)$ and $\bigodot(W)$ as trivial vector bundles over $X \subseteq V^0$ and W^0 respectively, the pair $(\bar{\mathfrak{f}}_{\mathrm{mor}}, \mathfrak{f}_0)$ can be interpreted as a bundle morphism that covers $\mathfrak{f}_0: X \to W^0$ and is compatible with the coalgebra structures on the fibres.

3.2.3. Composition of plurilinear maps

Since families of plurilinear maps can represent either families of coderivations or families morphisms, there are several different ways to compose them. These correspond to the different types of composition discussed in section 2.2.4, now performed fibrewise.

If we are given homogeneous smooth maps $\mathfrak{l}: X \to \mathrm{L}(\bigodot(V), V)$ and $\mathfrak{f}: X \to \mathrm{L}(\bigodot(V), W)$ we can define $\mathfrak{f} \mathbin{\square} \mathfrak{l}$ as the map

$$\mathfrak{f} \mathbin{\square} \mathfrak{l}: X \longrightarrow \mathrm{L}(\bigodot(V), W), \qquad x \mapsto \mathfrak{f}(x) \mathbin{\square} \mathfrak{l}(x) := \mathfrak{f}(x) \circ \bar{\mathfrak{l}}_{\mathrm{cod}}(x).$$

Explicitly, this composition is given by

$$(\mathfrak{f} \mathbin{\square} \mathfrak{l})(x)(v_{\{1,\dots,p\}}) = \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J}\, \mathfrak{f}(x)\big(\mathfrak{l}(x)(v_I) \odot v_J\big),$$

for $x \in X$ and $v_1, v_2, \dots, v_p \in V$ homogeneous.

Proposition 3.2.7.
Let V and W be $\mathbf{Z}$-graded vector spaces of Fréchet type and let $X \subseteq V^0$ be an open subset. For any two smooth or analytic maps $\mathfrak{l}: X \to \mathrm{L}(\bigodot(V), V)$ and $\mathfrak{f}: X \to \mathrm{L}(\bigodot(V), W)$, the composition $\mathfrak{f} \mathbin{\square} \mathfrak{l}$ is also smooth (resp. analytic). If both $\mathfrak{f}$ and $\mathfrak{l}$ are holonomic, then so is $\mathfrak{f} \mathbin{\square} \mathfrak{l}$.

Proof: Let $\mathfrak{l}$ and $\mathfrak{f}$ be as described in this proposition and note that the family of coderivations $\bar{\mathfrak{l}}_{\mathrm{cod}}: X \to \mathrm{L}(\bigodot(V), \bigodot(V))$ associated to $\mathfrak{l}$ is smooth (resp. analytic) as well due to Proposition 3.2.5. Proposition 3.1.8 (resp. Proposition 3.1.15) therefore tells us that the composition $\mathfrak{f} \mathbin{\square} \mathfrak{l} = \mathfrak{f} \circ \bar{\mathfrak{l}}_{\mathrm{cod}}$ is smooth (resp. analytic) and that its derivative is given by

$$\mathrm{D}(\mathfrak{f} \mathbin{\square} \mathfrak{l})(x)(\dot{x}) = \mathrm{D}\mathfrak{f}(x)(\dot{x}) \circ \bar{\mathfrak{l}}_{\mathrm{cod}}(x) + \mathfrak{f}(x) \circ \mathrm{D}\bar{\mathfrak{l}}_{\mathrm{cod}}(x)(\dot{x})$$

for $x \in X$ and $\dot{x} \in V^0$.

If $\mathfrak{l}$ and $\mathfrak{f}$ are holonomic, then we can apply Proposition 3.2.3 and Proposition 3.2.5 to rewrite this as

$$\begin{aligned} \mathrm{D}(\mathfrak{f} \mathbin{\square} \mathfrak{l})(x)(\dot{x}) &= \mathfrak{f}(x)\big(\dot{x} \odot \bar{\mathfrak{l}}_{\mathrm{cod}}(x)(\cdot)\big) + \mathfrak{f}(x)\big(\bar{\mathfrak{l}}_{\mathrm{cod}}(x)(\dot{x} \odot \cdot) - \dot{x} \odot \bar{\mathfrak{l}}_{\mathrm{cod}}(x)(\cdot)\big) \\ &= \big(\mathfrak{f}(x) \circ \bar{\mathfrak{l}}_{\mathrm{cod}}(x)\big)(\dot{x} \odot \cdot), \end{aligned}$$

which is equal to $(\mathfrak{f} \mathbin{\square} \mathfrak{l})(x)(\dot{x} \odot \cdot)$. We conclude that also $\mathfrak{f} \mathbin{\square} \mathfrak{l}$ is holonomic. ☐

The following proposition can be proven in exactly the same way as Proposition 3.2.7, but it can also be derived from it by using Proposition 3.2.5.

Proposition 3.2.8.
Let V and W be $\mathbf{Z}$-graded vector spaces of Fréchet type and let $X \subseteq V^0$ be an open subset. Given a smooth or analytic family of coderivations $\bar{\mathfrak{l}}\colon X \to \mathrm{L}(\bigodot(V), \bigodot(V))$ and a smooth (resp. analytic) family $\bar{\mathfrak{f}}\colon X \to \mathrm{L}(\bigodot(V), \bigodot(W))$ of coalgebra morphisms, the composition $\bar{\mathfrak{f}} \circ \bar{\mathfrak{l}}$ of is a smooth (resp. analytic) family of coalgebra derivations along $(\bar{\mathfrak{f}}, \mathfrak{f}_0)$.

If $\bar{\mathfrak{l}}$ and $(\bar{\mathfrak{f}}, \mathfrak{f}_0)$ are such that $\mathrm{D}\bar{\mathfrak{l}}(x)(\dot{x}) = [\bar{\mathfrak{l}}(x), \overline{m}_{\dot{x}}]$ and $\mathrm{D}\bar{\mathfrak{f}}(x)(\dot{x}) = \overline{m}_{\mathrm{D}\mathfrak{f}_0(x)(\dot{x})} - \bar{\mathfrak{f}} \circ \overline{m}_{\dot{x}}$, then $\bar{\mathfrak{h}} = \bar{\mathfrak{f}} \circ \bar{\mathfrak{l}}$ satisfies

$$\mathrm{D}\bar{\mathfrak{h}}(x)(\dot{x}) = \overline{m}_{\mathrm{D}\mathfrak{f}_0(x)(\dot{x})} - \bar{\mathfrak{h}} \circ \overline{m}_{\dot{x}}$$

for all $x \in X$ and all $\dot{x} \in V^0$.

The graded Lie algebra structure on $\mathrm{L}(\bigodot(V), V)$ was defined in section 2.2.2 through the vector space isomorphism $\ell \mapsto \bar{\ell}_{\mathrm{cod}}$ between this space and the space of coderivations on $(\bigodot(V), \Delta)$. It can also be described using the operator $\square$ as

$$[\ell, \ell'] = \ell \mathbin{\square} \ell' - (-1)^{|\ell||\ell'|} \ell' \mathbin{\square} \ell.$$

for homogeneous $\ell, \ell' \in \mathrm{L}(\bigodot(V), V)$. The following corollary is therefore an immediate consequence of Proposition 3.2.7.

Corollary 3.2.9.
Let V be a $\mathbf{Z}$-graded vector space of Fréchet type and let $X \subseteq V^0$ be an open

subset. For any two smooth or analytic maps $\mathfrak{l}'\colon X \to \mathrm{L}(\bigodot(V), V)$ *and* $\mathfrak{l}\colon X \to \mathrm{L}(\bigodot(V), V)$*, the graded commutator*

$$[\mathfrak{l}, \mathfrak{l}']\colon X \longrightarrow \mathrm{L}(\bigodot(V), V), \qquad x \mapsto [\mathfrak{l}(x), \mathfrak{l}'(x)]$$

is smooth (resp. analytic). It is holonomic whenever both $\mathfrak{l}$ *and* $\mathfrak{l}'$ *are.*

The equivalent statement for the lifts of $\mathfrak{l}$ and $\mathfrak{l}'$ as families of coderivations follows directly from Corollary 3.2.9 and Proposition 3.2.5.

Corollary 3.2.10.
Let V *be a* $\mathbf{Z}$*-graded vector space of Fréchet type and let* $X \subseteq V^0$ *be an open subset. Given two smooth or analytic families of coderivations* $\bar{\mathfrak{l}}\colon X \to \mathrm{L}(\bigodot(V), \bigodot(V))$ *and* $\bar{\mathfrak{l}}'\colon X \to \mathrm{L}(\bigodot(V), \bigodot(V))$*, the graded commutator* $[\bar{\mathfrak{l}}, \bar{\mathfrak{l}}']$ *is a smooth (resp. analytic) family of coderivations.*

If $\bar{\mathfrak{l}}$ *and* $\bar{\mathfrak{l}}'$ *are such that* $\mathrm{D}\bar{\mathfrak{l}}(x)(\dot{x}) = [\bar{\mathfrak{l}}(x), \overline{m}_{\dot{x}}]$ *and* $\mathrm{D}\bar{\mathfrak{l}}'(x)(\dot{x}) = [\bar{\mathfrak{l}}'(x), \overline{m}_{\dot{x}}]$*, then* $\bar{\mathfrak{h}} = [\mathfrak{l}, \mathfrak{l}']$ *satisfies*

$$\mathrm{D}\bar{\mathfrak{h}}(x)(\dot{x}) = [\bar{\mathfrak{h}}(x), \overline{m}_{\dot{x}}]$$

for all $x \in X$ *and all* $\dot{x} \in V^0$*.*

Now assume we are given two homogeneous smooth maps $\mathfrak{f}\colon X \to \mathrm{L}(\bigodot(V), W)$ and $\mathfrak{g}\colon X' \to \mathrm{L}(\bigodot(W), Z)$ such that $\mathfrak{f}$ has degree 0 and $\mathfrak{f}(X) \subseteq X'$. We can define $\mathfrak{g} \diamond \mathfrak{f}$ as the map

$$\mathfrak{g} \diamond \mathfrak{f}\colon X \longrightarrow \mathrm{L}(\bigodot(V), Z), \qquad x \mapsto \mathfrak{g}(\mathfrak{f}_0(x)) \diamond \mathfrak{f}(x) := \mathfrak{g}(\mathfrak{f}_0(x)) \circ \bar{\mathfrak{f}}_{\mathrm{mor}}(x).$$

Thus,

$$(\mathfrak{g} \diamond \mathfrak{f})(x)(v_{\{1,\ldots,p\}}) = \sum_{I_1,\ldots,I_m \neq \varnothing} \epsilon_{I_1,\ldots,I_m} \tfrac{1}{m!}\, \mathfrak{g}(\mathfrak{f}_0(x))\big(\textstyle\bigodot_{j=1}^{m} \mathfrak{f}(x)(v_{I_j})\big),$$

for $x \in X$ and $v_1, v_2, \ldots, v_p \in V$ homogeneous.

Proposition 3.2.11.
Let V*,* W *and* Z *be* $\mathbf{Z}$*-graded vector spaces of Fréchet type and let* $X \subseteq V^0$ *and* $X' \subseteq W^0$ *be open subsets. Given two homogeneous smooth maps* $\mathfrak{f}\colon X \to \mathrm{L}(\bigodot(V), W)$ *and* $\mathfrak{g}\colon X' \to \mathrm{L}(\bigodot(W), Z)$ *such that* $|\mathfrak{f}| = 0$ *and* $\mathfrak{f}_0(X) \subseteq X'$*, the composition* $\mathfrak{g} \diamond \mathfrak{f}$ *is smooth. It is holonomic whenever both* $\mathfrak{f}$ *and* $\mathfrak{g}$ *are.*

Proof: Let $\mathfrak{f}$ and $\mathfrak{g}$ be as described in this proposition and note that the associated maps $\mathfrak{f}_0\colon X \to \bigodot(W)$ and $\bar{\mathfrak{f}}_{\mathrm{mor}}\colon X \to \mathrm{L}(\bigodot(V), \bigodot(W))$ are smooth due to Proposition 3.2.6. The three maps $\bar{\mathfrak{f}}_{\mathrm{mor}}\colon X \to \mathrm{L}(\bigodot(V), \bigodot(W))$, $\mathfrak{g}\colon X \to \mathrm{L}(\bigodot(W), Z)$ and $\mathfrak{f}_0\colon X \to X'$ thus satisfy the hypotheses of Corollary 3.1.10, which tells us that $\mathfrak{g} \diamond \mathfrak{f} = \mathfrak{g} \circ (\bar{\mathfrak{f}}_{\mathrm{mor}}, \mathfrak{f}_0)$ is again smooth and that its derivative is given by

$$\mathrm{D}(\mathfrak{g} \diamond \mathfrak{f})(x)(\dot{x}) = \mathrm{D}\mathfrak{g}(\mathfrak{f}_0(x))\big(\mathrm{D}\mathfrak{f}_0(x)(\dot{x})\big) \circ \bar{\mathfrak{f}}_{\mathrm{mor}}(x) + \mathfrak{g}(\mathfrak{f}_0(x)) \circ \mathrm{D}\bar{\mathfrak{f}}_{\mathrm{mor}}(x)(\dot{x})$$

for $x \in X$ and $\dot{x} \in V^0$.

If $\mathfrak{f}$ and $\mathfrak{g}$ are holonomic, then we can apply Proposition 3.2.3 and Proposition 3.2.6 to rewrite this as

$$\begin{aligned}
\mathrm{D}(\mathfrak{g} \diamond \mathfrak{f})(x)(\dot{x}) &= \mathfrak{g}(\mathfrak{f}_0(x))\big(\mathrm{D}\mathfrak{f}_0(x)(\dot{x}) \odot \bar{\mathfrak{f}}_{\mathrm{mor}}(x)(\cdot)\big) \\
&\qquad + \mathfrak{g}(\mathfrak{f}_0(x))\big(\bar{\mathfrak{f}}_{\mathrm{mor}}(x)(\dot{x} \odot \cdot) - \mathrm{D}\mathfrak{f}_0(x)(\dot{x}) \odot \bar{\mathfrak{f}}_{\mathrm{mor}}(x)(\cdot)\big) \\
&= \big(\mathfrak{g}(\mathfrak{f}_0(x)) \circ \bar{\mathfrak{f}}_{\mathrm{mor}}(x)\big)(\dot{x} \odot \cdot), \\
&= (\mathfrak{g} \diamond \mathfrak{f})(x)(\dot{x} \odot \cdot).
\end{aligned}$$

From this we can conclude that also $g \diamond \mathfrak{f}$ is holonomic. □

We observe that the composition, $\mathfrak{g} \diamond \mathfrak{f}$ of two smooth maps $\mathfrak{f}\colon X \to \mathrm{L}(\bigodot(V), W)$ and $\mathfrak{g}\colon X' \to \mathrm{L}(\bigodot(W), Z)$ of degree 0 such that $\mathfrak{f}_0(X) \subseteq X'$ is uniquely defined by the property that

$$\big(\overline{\mathfrak{g} \diamond \mathfrak{f}}_{\mathrm{mor}}(x), (\mathfrak{g} \diamond \mathfrak{f})_0(x)\big) = \big(\bar{\mathfrak{g}}_{\mathrm{mor}}(\mathfrak{f}_0(x)) \circ \bar{\mathfrak{f}}_{\mathrm{mor}}(x), \mathfrak{g}_0(\mathfrak{f}_0(x))\big)$$

for all $x \in X$. This is in line with the earlier observation that families of coalgebra morphisms can be interpreted as bundle maps, since the expression above describes the composition of such maps.

Corollary 3.2.12.

Let V, W and Z be $\mathbf{Z}$-graded vector spaces of Fréchet type and let $X \subseteq V^0$, $X' \subseteq W^0$ and $X'' \subseteq Z^0$ be open subsets. For any two smooth families $(\bar{\mathfrak{f}}, \mathfrak{f}_0)\colon X \to \mathrm{L}(\bigodot(V), \bigodot(W)) \times X'$ and $(\bar{\mathfrak{g}}, \mathfrak{g}_0)\colon X \to \mathrm{L}(\bigodot(W), \bigodot(Z)) \times X''$ of coalgebra morphisms, the composition $((\bar{\mathfrak{g}} \circ \mathfrak{f}_0) \circ \bar{\mathfrak{f}}, \mathfrak{g}_0 \circ \mathfrak{f}_0)$ is a smooth family of coalgebra morphisms.

If both $(\bar{\mathfrak{f}}, \mathfrak{f}_0)$ and $(\bar{\mathfrak{g}}, \mathfrak{g}_0)$ have the derivative property from Proposition 3.2.6, then so does $((\bar{\mathfrak{g}} \circ \mathfrak{f}_0) \circ \bar{\mathfrak{f}}, \mathfrak{g}_0 \circ \mathfrak{f}_0)$.

In this description, it is clear that the composition of holonomic families of plurilinear maps of degree 0 is an associative operation. For any **Z**-graded vector space V of Fréchet type and any open subset $X \subseteq V^0$ there moreover exists a corresponding identity map, which is the map $i_V : X \to \mathrm{L}(\bigodot(V), V)$ that extends to the pair $(x \mapsto \mathrm{id}_{\bigodot(V)}, \mathrm{id}_X)$.

3.3. Families of L_∞-algebras

We can combine what we know from sections 2.3 and 2.4 about curved L_∞-algebras of Fréchet type with our newly acquired knowledge about holonomic families to describe the category of holonomic families of curved L_∞-algebras of Fréchet type. As we will see in section 5.1, holonomic families of curved L_∞-algebras are in some sense locally equivalent to curved L_∞-algebroids. The natural notions of morphisms for these categories however do not match: morphisms of L_∞-algebroids correspond to a subset of all morphisms between the corresponding holonomic families which we call fibre preserving. These morphisms will also be described in this section.

3.3.1. Holonomic families of curved L_∞-algebras

Now that we have discussed holonomic families of plurilinear maps, the definition of a holonomic family of curved L_∞-algebras becomes rather straightforward.

Definition 3.3.1.
Let V be a **Z**-graded vector space of Fréchet type and let $X \subseteq V^0$ be an open subset. A *holonomic family of curved L_∞-algebra structures* on V is a holonomic map $\mathfrak{l} \colon X \to \mathrm{L}(\bigodot(V), V)$ of degree 1 such that $(V, \mathfrak{l}(x))$ is a curved L_∞-algebra for every $x \in X$.

The family of twists of a curved L_∞-algebra from section 2.4 is an example of a holonomic family of curved L_∞-algebras. Up to the choice of domain, this family is characterised by the fact that it is both analytic and holonomic and has the original L_∞-algebra as a member, as the following proposition shows.

Proposition 3.3.2 (Families of twists).
A curved L_∞-algebra (V,ℓ) of Fréchet type is convergent if and only if there exists a holonomic family $\mathfrak{l}\colon X \subseteq V^0 \to \mathrm{L}(\bigodot(V),V)$ of curved L_∞-algebras, defined on a neighbourhood X of $0 \in V^0$, which is analytic and such that $\mathfrak{l}(0) = \ell$.

At most one such family exists for every connected open subset $X \subseteq V^0$ that contains 0. It coincides with the family $u \mapsto \ell^u$ of twists of (V,ℓ) on some neighbourhood of $0 \in X$.

Proof: Convergence of the curved L_∞-algebra (V,ℓ) means precisely that there exists a family $u \mapsto \ell^u$ of twists parametrised by a non-empty domain of convergence $D_\ell \subseteq V^0$. Analyticity of this family was demonstrated in Proposition 2.4.3, and holonomicity follows from this proposition through Proposition 3.2.3. That (V,ℓ^u) is a curved L_∞-algebra for every $u \in D_\ell$ is verified in Proposition 2.4.4. A family matching the above specifications thus exists whenever (V,ℓ) is integrable.

Uniqueness of such families on a given domain is a consequence of Proposition 3.2.4. If a holonomic family $\mathfrak{l}\colon X \to \mathrm{L}(\bigodot(V),V)$ as described above exists, analyticity tells us that it is locally given by its Taylor series, and that consequently

$$\mathfrak{l}(u)(v_{\{1,\dots,p\}}) = \sum_{k=0}^{\infty} \tfrac{1}{k!}\,\mathrm{D}^k\mathfrak{l}(0)(\textstyle\bigodot^k u)(v_{\{1,\dots,p\}}) = \sum_{k=0}^{\infty} \tfrac{1}{k!}\,\mathfrak{l}(0)\big(\textstyle\bigodot^k u \odot v_{\{1,\dots,p\}}\big)$$

for all u sufficiently close to $0 \in V^0$ and all $v_1, v_2, \dots, v_p \in V$. Since this is precisely how the twisted structure ℓ^u is defined, existence of such families therefore implies convergence of the curved L_∞-algebra. □

It should at this point not be surprising that we can describe holonomic families of curved L_∞-algebra structures as Maurer–Cartan elements of a graded Lie algebra.

Proposition 3.3.3.
Let V be a $\mathbf{Z}$-graded vector space of Fréchet type and let $X \subseteq V^0$ be an open subset. The space $\mathrm{C}^\infty_{\mathrm{hol}}\big(X,\mathrm{L}(\bigodot(V),V)\big)$ comes with a graded Lie algebra structure given by

$$[\mathfrak{l},\mathfrak{l}'](x) = [\mathfrak{l}(x),\mathfrak{l}'(x)]$$

for any two holonomic maps $\mathfrak{l},\mathfrak{l}'\colon X \to \mathrm{L}(\bigodot(V),V)$ and $x \in X$.

The Maurer–Cartan locus of $\big(\mathrm{C}^\infty_{\mathrm{hol}}\big(X, \mathrm{L}(\bigodot(V), V)\big), [\cdot,\cdot]\big)$ *is the set of holonomic families of curved* L_∞*-algebra structures on* V *parametrised by* X*.*

Proof: It was shown in Corollary 3.2.9 that the graded Lie bracket $[\mathfrak{l}, \mathfrak{l}']$ of two holonomic families is holonomic. It defines a graded Lie algebra structure because the Jacobi identity can be verified pointwise.

In section 2.2.2, we had observed that a linear map $\ell\colon \bigodot(V) \to V$ defines a curved L_∞-algebra structure on V precisely if it is homogeneous of degree 1 and satisfies $[\ell, \ell] = 0$. A homogeneous smooth map $\mathfrak{l}\colon X \to \mathrm{L}(\bigodot(V), V)$ of degree 1 therefore describes a family of curved L_∞-algebras if and only if it satisfies the Maurer–Cartan equation $\frac{1}{2}[\mathfrak{l}, \mathfrak{l}] = 0$. □

The first part of Proposition 3.3.3 also holds for the space of analytic holonomic families, $\mathrm{C}^\omega_{\mathrm{hol}}\big(X, \mathrm{L}(\bigodot(V), V)\big)$. If this space is used, the Maurer–Cartan locus consists of all analytic holonomic families of curved L_∞-algebra structures on V parametrised by X.

We can turn holonomic families of curved L_∞-algebras into a category by adding morphisms. Below, let $\mathfrak{l}\colon X \to \mathrm{L}(\bigodot(V), V)$ and $\mathfrak{l}'\colon Y \to \mathrm{L}(\bigodot(W), W)$ denote two holonomic families of curved L_∞-algebras of Fréchet type parametrised by open subsets $X \subseteq V^0$ and $Y \subseteq W^0$ respectively.

Definition 3.3.4.
Given holonomic families of curved L_∞-algebras $\mathfrak{l}\colon X \to \mathrm{L}(\bigodot(V), V)$ and $\mathfrak{l}'\colon Y \to \mathrm{L}(\bigodot(W), W)$, a *morphism* from $\mathfrak{l}$ to $\mathfrak{l}'$ is a holonomic map $\mathfrak{f}\colon X \to \mathrm{L}(\bigodot(V), W)$ of degree 0 such that $\mathfrak{f}_0(X) \subseteq Y$ and $(\mathfrak{f} - \mathfrak{f}_0)(x)$ is a morphism of curved L_∞-algebras from $(V, \mathfrak{l}(x))$ to $(W, \mathfrak{l}(\mathfrak{f}_0(x)))$ for every $x \in X$.

In light of the discussion at the end of section 3.2.2, a morphism of holonomic families of curved L_∞-algebras can be repackaged as a pair $(\mathfrak{f}_0, \bar{\mathfrak{f}}_{\mathrm{mor}})$ consisting of a smooth map $\mathfrak{f}_0\colon X \to Y$ and a smooth family $\bar{\mathfrak{f}}_{\mathrm{mor}}\colon X \to \mathrm{Mor}_{\mathrm{coalg}}(\bigodot(V), \bigodot(W))$ of coalgebra morphisms such that

$$\bar{\mathfrak{f}}_{\mathrm{mor}}(x) \circ \bar{\mathfrak{l}}_{\mathrm{cod}}(x) = \bar{\mathfrak{l}}'_{\mathrm{cod}}(\mathfrak{f}_0(x)) \circ \bar{\mathfrak{f}}_{\mathrm{mor}}(x)$$

for every $x \in X$. In terms of the operators $\square$ and $\diamond$ from Proposition 3.2.7 and Proposition 3.2.11, this condition can be formulated very concisely as $\mathfrak{f} \square \mathfrak{l} = \mathfrak{l}' \diamond \mathfrak{f}$.

Proposition 3.3.5.
Holonomic families of curved L_∞-algebras of Fréchet type form a category if the composition of two composable morphisms $\mathfrak{f}$ and $\mathfrak{g}$ is defined as $\mathfrak{g} \diamond \mathfrak{f}$.

Proof: Let $\mathfrak{l}_i \colon X_i \to \mathrm{L}(\bigodot(V_i), V_i)$, be holonomic families of curved L_∞-algebras of Fréchet type for $i = 1, 2, 3$. Let $\mathfrak{f}$ be a morphism from $\mathfrak{l}_1$ to $\mathfrak{l}_2$ and let $\mathfrak{g}$ be a morphism from $\mathfrak{l}_2$ to $\mathfrak{l}_3$. The composition $\mathfrak{g} \diamond \mathfrak{f}$ is such that and $(\mathfrak{g} \diamond \mathfrak{f})_0 = \mathfrak{g}_0 \circ \mathfrak{f}_0$ and $\mathfrak{g} \diamond \mathfrak{f} - (\mathfrak{g} \diamond \mathfrak{f})_0 = (\mathfrak{g} - \mathfrak{g}_0) \diamond (\mathfrak{f} - \mathfrak{f}_0)$, as discussed just above Corollary 3.2.12. It is a morphism of holonomic families of curved L_∞-algebras because the first observation tells us that $(\mathfrak{g} \diamond \mathfrak{f})_0(X_1) \subseteq X_3$ and the second tells us that that $\mathfrak{g} \diamond \mathfrak{f} - (\mathfrak{g} \diamond \mathfrak{f})_0 \colon \bigodot(V_1) \to V_3$ is a morphism of L_∞-algebras from $(V_1, \mathfrak{l}_1(x))$ to $(V_3, \mathfrak{l}_3(z))$, where $z = (\mathfrak{g} \diamond \mathfrak{f})_0(x)$.

The associativity of $\diamond$ follows from the fact that both $\mathfrak{h} \diamond (\mathfrak{g} \diamond \mathfrak{f})$ and $(\mathfrak{h} \diamond \mathfrak{g}) \diamond \mathfrak{f}$ lift to the morphism $\mathfrak{h}_{\mathrm{mor}} \circ \mathfrak{g}_{\mathrm{mor}} \circ \mathfrak{f}_{\mathrm{mor}}$. Two simple computations moreover show that the identity morphism for a given holonomic family of curved L_∞-algebras, $\mathfrak{l} \colon X_1 \to \mathrm{L}(\bigodot(V_1), V_1)$, is the map $\mathfrak{f} \colon X_1 \to \mathrm{L}(\bigodot(V_1), V_1)$ for which $\mathfrak{f}_0 = \mathrm{id}_{X_1}$ and $\bar{\mathfrak{f}}_{\mathrm{mor}} = \mathrm{id}_{\bigodot(V_1)}$. □

3.3.2. Different types of morphisms

Although the definition of a morphism of holonomic families of curved L_∞-algebras provided in Definition 3.3.4 seems reasonable, it is not the type of morphism that we will need in chapter 5. The morphisms encountered in that chapter satisfy an additional compatibility condition, which we describe here.

Before we can state this condition, let us remark that the graded symmetric coalgebra $(\bigodot(V), \Delta)$ is part of an exact sequence

$$0 \longrightarrow (\bigodot(V^{\neq 0}), \Delta) \xrightarrow{\bar{\iota}} (\bigodot(V), \Delta) \xrightarrow{\bar{\pi}} (\bigodot(V^0), \Delta) \longrightarrow 0$$

in the category of cocommutative coalgebras. We would like to think of $\bigodot(V)$ as a bundle over $\bigodot(V^0)$ with fibre $\bigodot(V^{\neq 0})$, and consider those morphisms that are compatible with this “bundle structure”.

Definition 3.3.6.
Let V and W be $\mathbf{Z}$-graded vector space of Fréchet and let $X \subseteq V^0$ be a Fréchet

domain. A holonomic family $\bar{\mathfrak{f}}\colon X \to \mathrm{L}(\bigodot(V), \bigodot(W))$ of coalgebra morphisms is *fibre preserving* if $\bar{\mathfrak{f}}(x)$ is part of a commuting diagram

$$\begin{array}{ccccc} \big(\bigodot(V^{\neq 0}), \Delta\big) & \xrightarrow{\bar{\imath}} & \big(\bigodot(V), \Delta\big) & \xrightarrow{\bar{\pi}} & \big(\bigodot(V^0), \Delta\big) \\ \big\downarrow{\scriptstyle \bar{\mathfrak{f}}'(x)} & & \big\downarrow{\scriptstyle \bar{\mathfrak{f}}(x)} & & \big\downarrow{\scriptstyle \bar{\mathfrak{f}}''(x)} \\ \big(\bigodot(W^{\neq 0}), \Delta\big) & \xrightarrow{\bar{\imath}} & \big(\bigodot(W), \Delta\big) & \xrightarrow{\bar{\pi}} & \big(\bigodot(W^0), \Delta\big). \end{array} \tag{3.3.1}$$

of cocommutative coalgebras for every $x \in X$.

Although the coalgebra morphisms $\bar{\mathfrak{f}}'(x)$ and $\bar{\mathfrak{f}}''(x)$ in Definition 3.3.6 were only required to exist, they are in fact uniquely determined by $\bar{\mathfrak{f}}(x)$ due to the injectivity of $\bar{\imath}$ and the surjectivity of $\bar{\pi}$. We observe in particular that $\bar{\mathfrak{f}}''(x) = \overline{(\mathrm{j}^\infty_x \mathfrak{f}_0)}_{\mathrm{mor}}$ because $\mathfrak{f}$ was required to be holonomic.

We will call a family $\mathfrak{f}\colon X \to \mathrm{L}(\bigodot(V), W)$ of plurilinear maps *fibre preserving* if it lifts to a fibre-preserving family $\bar{\mathfrak{f}}_{\mathrm{mor}}$ of coalgebra morphisms. This definition can also be applied to morphisms of holonomic families of curved L_∞-algebras.

Definition 3.3.7.

Let $\mathfrak{l}\colon X \to \mathrm{L}(\bigodot(V), V)$ and $\mathfrak{l}'\colon Y \to \mathrm{L}(\bigodot(W), W)$ be holonomic families of curved L_∞-algebras of Fréchet type. A morphism $\mathfrak{f}\colon X \to \mathrm{L}(\bigodot(V), W)$ from $\mathfrak{l}$ to $\mathfrak{l}'$ is *fibre preserving* if $\bar{\mathfrak{f}}_{\mathrm{mor}}$ is a fibre-preserving family of coalgebra morphisms.

It should be noted that the maps in diagram (3.3.1) are in general not morphisms of curved L_∞-algebras and that this would not be a meaningful requirement. The problem is the fact that the spaces $V^{\neq 0}$ and V^0 do not generally come with a L_∞-algebra structure that is compatible with the inclusion map $\bar{\imath}\colon \bigodot(V^{\neq 0}) \to \bigodot(V)$ and the projection map $\bar{\pi}\colon \bigodot(V) \to \bigodot(V^0)$.

Fibre-preserving families of plurilinear maps that are also holonomic can be characterised more conveniently.

Proposition 3.3.8.

Let V and W be **Z***-graded vector spaces of Fréchet type, let $X \subseteq V^0$ be a Fréchet domain and consider a holonomic family $\mathfrak{f}\colon X \to \mathrm{L}(\bigodot(V), W)$ of plurilinear maps of degree* 0*. The following statements are equivalent:*

- **(i)** $\bar{\mathfrak{f}}_{\mathrm{mor}}$ *is fibre preserving, as defined in Definition 3.3.6;*

(ii) *there exists a (unique) smooth map* $\mathfrak{g}\colon X \to \mathrm{L}\big(\bigodot(V^{\neq 0}), W^{\neq 0}\big)$ *of degree* 0 *such that* $\mathfrak{f} = \mathrm{j}^\infty(\mathfrak{f}_0 + \mathfrak{g})$*, i.e.*

$$\mathfrak{f}(x)(\dot{x}_{\{1,\ldots,q\}} \odot v_{\{1,\ldots,p\}}) = \begin{cases} \mathrm{D}^q \mathfrak{f}_0(x)(\dot{x}_{\{1,\ldots,q\}}) & \text{if } p = 0 \\ \mathrm{D}^q \mathfrak{g}(x)(\dot{x}_{\{1,\ldots,q\}})(v_{\{1,\ldots,p\}}) & \text{if } p \geq 1 \end{cases}$$

for every $x \in X$ *and all* $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_q \in V^0$ *and* $v_1, v_2, \ldots, v_p \in V^{\neq 0}$.

Proof: Because $\mathfrak{f}$ is holonomic by assumption, there exists a unique smooth map $\tilde{\mathfrak{f}}\colon X \to \mathrm{L}(\bigodot(V^{\neq 0}), W)$ such that $\mathfrak{f} = \mathrm{j}^\infty \tilde{\mathfrak{f}}$. Thus, $\mathfrak{f}$ is equal to $\mathrm{j}^\infty(\mathfrak{f}_0 + \mathfrak{g})$ for a unique map $\mathfrak{g}\colon X \to \mathrm{L}(\bigodot(V^{\neq 0}), W)$, which is given by $\mathfrak{g} = \tilde{\mathfrak{f}} - \mathfrak{f}_0$. Condition (ii) now holds precisely when $\mathfrak{f}$ takes values in the subspace $W^{\neq 0} \subseteq W$.

If $\mathfrak{g}$ does take values in $W^{\neq 0}$, then one can readily verify that the diagram (3.3.6) commutes if $\bar{\mathfrak{f}}' = \overline{\mathfrak{g}(x)}_{\mathrm{mor}}$ and $\bar{\mathfrak{f}}'' = \overline{(\mathrm{j}^\infty_x \mathfrak{f}_0)}$. This tells us that $\mathfrak{f}$ is fibre preserving in this case.

Conversely, if a map $\bar{\mathfrak{f}}'\colon \bigodot(V^{\neq 0}) \to \bigodot(W^{\neq 0})$ as in equation (3.3.1) exists, then it is equal to $\overline{\mathfrak{f}(x)}_{\mathrm{mor}}|_{\bigodot(V^{\neq 0})} = \bar{\mathfrak{g}}$. This then implies that $\mathfrak{g}$ takes values in $W^{\neq 0}$. ☐

We can consider an even more restrictive type of morphisms.

Definition 3.3.9.
Let V and W be $\mathbf{Z}$-graded vector space of Fréchet and let $X \subseteq V^0$ be a Fréchet domain. A holonomic family $\bar{\mathfrak{f}}\colon X \to \mathrm{L}(\bigodot(V), \bigodot(W))$ of coalgebra morphisms is *fibrewise-linear* if $\bar{\mathfrak{f}}(x)$ is part of a commuting diagram

$$\begin{array}{ccccc} \big(\bigodot(V^{\neq 0}), \Delta\big) & \xrightarrow{\bar{\iota}} & \big(\bigodot(V), \Delta\big) & \xrightarrow{\bar{\pi}} & \big(\bigodot(V^0), \Delta\big) \\ \big\downarrow{\scriptstyle \bar{\mathfrak{f}}'(x)} & & \big\downarrow{\scriptstyle \bar{\mathfrak{f}}(x)} & & \big\downarrow{\scriptstyle \bar{\mathfrak{f}}''(x)} \\ \big(\bigodot(W^{\neq 0}), \Delta\big) & \xrightarrow{\bar{\iota}} & \big(\bigodot(W), \Delta\big) & \xrightarrow{\bar{\pi}} & \big(\bigodot(W^0), \Delta\big). \end{array} \tag{3.3.2}$$

of cocommutative coalgebras for every $x \in X$ and $\bar{\mathfrak{f}}'(x)\colon \bigodot(V^{\neq 0}) \to \bigodot(W^{\neq 0})$ is linear in the sense that its only non-trivial component is $\bar{\mathfrak{f}}'(x)_1\colon V \to W$.

We will call a holonomic family $\mathfrak{f}\colon X \to \mathrm{L}(\bigodot(V), W)$ of plurilinear maps fibrewise linear if it lifts to a fibrewise linear family $\bar{\mathfrak{f}}_{\mathrm{mor}}$ of coalgebra morphisms.

Proposition 3.3.10.

Let V and W be $\mathbf{Z}$-graded vector spaces of Fréchet type, let $X \subseteq V^0$ be a Fréchet domain and consider a holonomic family $\mathfrak{f}\colon X \to \mathrm{L}(\bigodot(V), W)$ of plurilinear maps of degree 0. The following statements are equivalent:

- **(i)** $\bar{\mathfrak{f}}_{\mathrm{mor}}$ *is fibrewise linear, as defined in Definition 3.3.9;*
- **(ii)** *there exists a (unique) homogeneous smooth map $\mathfrak{g}\colon X \to \mathrm{L}\big(V, W^{\neq 0}\big)$ of degree 0 such that $\mathfrak{f} = \mathrm{j}^\infty(\mathfrak{f}_0 + \mathfrak{g})$, i.e.*

$$\mathfrak{f}(x)(\dot{x}_{\{1,\dots,q\}} \odot v_{\{1,\dots,p\}}) = \begin{cases} \mathrm{D}^q\mathfrak{f}_0(x)(\dot{x}_{\{1,\dots,q\}}) & \text{if } p = 0 \\ \mathrm{D}^q\mathfrak{g}(x)(\dot{x}_{\{1,\dots,q\}})(v_1) & \text{if } p = 1 \\ 0 & \text{if } p \geq 2 \end{cases}$$

for every $x \in X$ and all $\dot{x}_1, \dot{x}_2, \dots, \dot{x}_q \in V^0$ and $v_1, v_2, \dots, v_p \in V^{\neq 0}$;

Proof: This proposition is a simple adaptation of Proposition 3.3.8. □

Remark 3.3.11.

Holonomic families of curved L_∞-algebras can be glued together over a manifold using any of the three types of morphisms described above. Suppose we are given a $\mathbf{Z}$-graded vector space V of Fréchet type and a smooth Fréchet manifold M modelled on V^0 endowed with an atlas $(U_i, \kappa_i)_{i\in I}$ consisting of open subsets $U_i \subseteq M$ and charts $\kappa_i\colon U_i \to V^0$. A holonomic family of L_∞-algebras over M could be described as a collection $(\mathfrak{l}^{(i)})_{i\in I}$ of holonomic families $\mathfrak{l}^{(i)}\colon \kappa_i(U_i) \to \mathrm{L}(\bigodot(V), V)$ of curved L_∞-algebras along with a collection $(\mathfrak{f}^{(j,i)})_{i,j\in I}$ of isomorphisms $\mathfrak{f}^{(j,i)}$ from $\mathfrak{l}^{(i)}$ to $\mathfrak{l}^{(j)}$. These morphisms should satisfy three conditions:

1. $\mathfrak{f}_0^{(j,i)} \circ \kappa^{(i)}|_{U_i\cap U_j} = \kappa^{(j)}|_{U_i\cap U_j}$ for all $i, j \in I$;
2. $\bar{\mathfrak{f}}^{(i,i)}(x) = \mathrm{id}_{\bigodot(V)}$ for all $i \in I$ and all $x \in \kappa_i(U_i)$;
3. $\bar{\mathfrak{f}}^{(k,j)} \circ \bar{\mathfrak{f}}^{(j,i)}|_{U_i\cap U_j\cap U_k} = \bar{\mathfrak{f}}^{(k,i)}|_{U_i\cap U_j\cap U_k}$ for all $i, j, k \in I$.

We say that the resulting object is of fibre bundle type if each of the morphisms $\mathfrak{f}_{ji}$ is fibre preserving, and that it is of vector bundle type if each of these morphisms is fibrewise linear. Morphisms between these objects are collections of morphisms of the relevant type that agree on intersections.

Holonomic families of curved L_∞-algebras of vector bundle type over a smooth manifold are essentially equivalent to curved split L_∞-algebroids, which are defined in Definition 4.3.1. Holonomic families of curved L_∞-algebras of general type are essentially the same thing as differential graded manifolds, which will not be discussed here. △

CHAPTER 4.

L_∞-algebroids

The aim of this chapter is to define curved L_∞-algebroids of Fréchet type. This concept is central to this thesis, as it allows us to bridge the gap between equivariant deformation problems and L_∞-algebras. As we will see in section 5.2, L_∞-algebroids can be used to repackage (nearly) all of the information contained in an equivariant deformation problem, and give rise to equivalence classes of L_∞-algebras that can be said to formally control deformations of the corresponding structures.

Curved L_∞-algebroids of Fréchet type are **Z**-graded vector bundles of Fréchet type whose spaces of sections come with curved L_∞-algebra structures that consist of smooth multiderivations. Before we can properly state this definition, however, we will need to set up a theory of smooth derivations and multiderivations on such bundles. We will therefore start this chapter by discussing locally convex vector bundles over Fréchet manifolds, as well as their morphisms and sheaves of sections, in section 4.1. We then provide an appropriate definition for smooth derivations and multiderivations in 4.2, which also deals with linear connections and Lie algebroids on vector bundles of Fréchet type. Curved L_∞-algebroids will finally be introduced in section 4.3.

This chapter builds on material that is covered in the appendices and chapter 3 and will use of a lot of the notation and conventions introduced there. This in particular applies to choices of topology for the space $\mathrm{L}(V,W)$ of continuous linear maps between two vector space V and W, and the space $\mathrm{C}^\infty(X,V)$ of smooth maps from a Fréchet domain $X \subseteq T$ to a locally convex vector space V. More specifically, $\mathrm{L}(V,W)$ comes with the topology of precompact convergence from Definition A.2.20, and $C^\infty(X, V)$ shall be endowed with the compact-open C^∞-topology from Definition B.2.2.

4.1. Locally convex vector bundles

Before we can talk about L_∞-algebroids in an infinite-dimensional context, infinite-dimensional vector bundles need to be discussed. For vector bundles whose base and fibres are both Fréchet manifolds, most of the usual definitions from the finite-dimensional setting just work and are equivalent to most reasonable alternatives one might come up with. Most of the vector bundles we consider are of this type, but for those that are not a few choices will need to be made.

An overview of the theory of Fréchet manifolds can for instance be found in [Ham82], and manifolds modelled on arbitrary locally convex are for instance considered in [Mil84], [Glö02] and [Nee06]. A definition can also be found in appendix B.1.1 of this document. While the definitions provided here are equivalent to those in all four references for bundles of Fréchet type, they are distinct for bundles modelled on arbitrary locally convex vector spaces.

We will provide a definition of a locally convex vector bundle over a Fréchet manifold that is equivalent to the usual definition whenever the fibres are Fréchet as well. Afterwards we will topologise the corresponding sheaves of sections, as well as the sheaves of base-preserves morphisms between such vector bundles, and we will finally describe what modifications need to be made to these definitions in the **Z**-graded case. Many of the statements in section 4.1 are simple corollaries to similar statements in section 3.1.

4.1.1. Vector bundles

We define a discrete bundle of locally convex vector spaces as a disjoint union $E = \coprod_{x \in M} E_x$ of locally convex vector spaces endowed with the canonical projection map $\pi\colon E \to M$ for which $\pi(E_x) = \{x\}$. We shall assume that M is a smooth Fréchet manifold and will define a concept of smoothness for sections of this bundle without imposing a manifold structure its total space.

A local *trivialisation* on $E \to M$ is a function $\tilde{\Phi}\colon U \to \coprod_{x\in U} \mathrm{L}(E_x, V)$ for some locally convex vector space an open subset $U \subseteq M$, such that $\tilde{\Phi}(x)$ is an invertible element of $\mathrm{L}(E_x, V)$ for every $x \in U$. We declare two such trivialisations $\tilde{\Phi}\colon U \to \coprod_{x\in U} \mathrm{L}(E_x, V)$ and $\tilde{\Phi}\colon U' \to \coprod_{x\in U'} \mathrm{L}(E_x, W)$ smoothly compatible whenever the transition functions

$$\tilde{\Psi} \circ \tilde{\Phi}^{-1}\colon U \cap U' \longrightarrow \mathrm{L}(V, W), \qquad \text{and} \qquad \tilde{\Phi} \circ \tilde{\Psi}^{-1}\colon U \cap U' \longrightarrow \mathrm{L}(W, V),$$

are both smooth. We shall often identify a trivialisation $\tilde{\Phi}\colon U \to \coprod_{x\in U} \mathrm{L}(E_x, V)$ with the corresponding map of total spaces

$$\check{\Phi}\colon E|_U = \pi^{-1}(U) \longrightarrow V \times U \qquad \text{given by} \qquad \check{\Phi}(e_x) = (\tilde{\Phi}(x)(e_x), x)$$

for $e_x \in E_x$ with $x \in U$.

We shall refer to a collection $(\tilde{\Phi}_\lambda)_{\lambda\in\Lambda}$ of local trivialisations as a smooth trivialisation atlas whenever its elements are mutually compatible in the above sense. A *trivialisation atlas* is then called smooth with respect to this atlas whenever it is compatible with each of its elements, and two smooth trivialisation atlases are considered equivalent whenever all elements of the first are mutually compatible with all elements of the second. That this is indeed an equivalence relation can easily be verified: reflexivity and symmetry are trivial and transitivity follows from Proposition 3.1.8.

Definition 4.1.1.
A smooth (or analytic) *locally convex vector bundle* over a smooth (resp. analytic) Fréchet manifold M is a discrete locally convex vector bundle $E \to M$ endowed with an equivalence class of smooth (resp. analytic) trivialisation atlases.

A section $e\colon M \to E$ of $E \to M$ is smooth (resp. analytic) when the composition $\tilde{\Phi} \circ e\colon M \to V$ is smooth (resp. analytic) for every trivialisation chart $\tilde{\Phi}$ that is an element of a member of this equivalence class of trivialisation atlases.

It follows from Corollary 3.1.9 that for a given trivialisation $\tilde{\Phi}\colon E|_U \to V$, smoothness of a local section $e\colon U \to E|_U$ defined on U is equivalent to smoothness of $\tilde{\Phi} \circ e\colon U \to V$, and that smoothness is in fact a local property. The same holds for analyticity, but Corollary 3.1.16 should then be used instead of Corollary 3.1.9.

Definition 4.1.2.
A *morphism* of smooth (or analytic) locally convex vector bundles from $E \to$

M to $F \to N$ is a pair $\alpha = (\tilde{\alpha}, f)$ consisting of a smooth (resp. analytic) map $f: M \to N$ and a function $\tilde{\alpha}: M \to \coprod_{x \in M} \mathrm{L}(E_x, F_{f(x)})$ such that $\tilde{\alpha}(x) \in \mathrm{L}(E_x, F_{f(x)})$ for every $x \in M$ and such that the composition

$$\tilde{\Psi} \circ \tilde{\alpha} \circ \tilde{\Phi}^{-1}: U \cap f^{-1}(U') \longrightarrow \mathrm{L}(V, W), \qquad x \mapsto \tilde{\Psi}_{f(x)} \circ \alpha_x \circ (\tilde{\Phi}_x)^{-1}$$

is smooth (resp. analytic) for any pair $(\tilde{\Phi}, \tilde{\Psi})$ of smooth (resp. analytic) trivialisations $\tilde{\Phi}: U \to \coprod_{x \in M} \mathrm{L}(E_x, V)$ and $\tilde{\Psi}: U' \to \coprod_{y \in N} \mathrm{L}(F_y, W)$.

Like with sections, it is sufficient to verify smoothness of vector bundle morphisms locally, and smoothness of the restriction $\tilde{\alpha}|_U$ for a sufficiently small neighbourhood can be verified using just a single pair of trivialisations. This follows from Corollary 3.1.10. The analytic case is again analogous and uses Corollary 3.1.17 instead.

As we had done for trivialisations, we will often identify a vector bundle morphism $\alpha: M \to \coprod_{x \in M} \mathrm{L}(E_x, F_{f(x)})$ with the corresponding map of total spaces

$$\check{\alpha}: E \longrightarrow F \qquad \text{given by} \qquad \check{\alpha}(e_x) = \alpha(x)(e_x)$$

for $e_x \in E_x$ with $x \in M$.

We define a *trivialisation chart* for $E \to M$ as a vector bundle isomorphism of the restriction $E|_U$ to an open subset $U \subseteq M$ onto a trivial vector bundle $V \times X \to X$ over a Fréchet domain $X \subseteq T$. In other words, it is a local vector bundle isomorphisms of the form $(\tilde{\Phi}, \phi)$ for some chart $\phi: U \xrightarrow{\sim} X \subseteq T$ and a trivialisation $\tilde{\Phi}: U \to \coprod_{x \in M} \mathrm{L}(E_x, V)$.

A smooth Fréchet vector bundle is traditionally defined as a smooth map $\pi: E \to M$ of Fréchet manifolds whose fibres come with a linear structure and which admits an atlas of smooth vector bundle charts. A smooth vector bundle chart for $E \to M$ is a chart $\check{\Phi}: E|_U = \pi^{-1}(U) \to V \times T$ that takes values in the product $V \times T$ of two Fréchet spaces V and T and is such that $\mathrm{pr}_2 \circ \check{\Phi} = \phi \circ \pi$ for some chart $\phi: U \to T$ for M and the restriction $\check{\Phi}|_{E_x}: E_x \to V \times \{\phi(x)\}$ is a linear isomorphism for every $x \in U$.

Proposition 4.1.3 (Vector bundles of Fréchet type).
Let $E \to M$ be a smooth (or analytic) locally convex vector bundle over a Fréchet manifold M whose fibres are Fréchet spaces. The total space E admits a smooth

(resp. analytic) manifold structure with a smooth atlas that consists of charts of the form

$$\check{\Phi}\colon E|_U \longrightarrow V \times T, \qquad e_x \in E_x \mapsto (\tilde{\Phi}(x)(e_x), \phi(x)),$$

where $\phi\colon U \subseteq M \to T$ *is a chart for the base manifold and* $\tilde{\Phi}\colon U \to \coprod_{x\in U} \mathrm{L}(E_x, V)$ *is a trivialisation of* $E|_U$*. This gives* $E \to M$ *the structure of a smooth (resp. analytic) Fréchet vector bundle in the traditional sense.*

For any smooth vector bundle morphism (α, f) *between two such vector bundles* $E \to M$ *and* $F \to N$*, the induced map* $\check{\alpha}\colon E \to F$ *is a smooth map of Fréchet manifolds and is consequently a morphism of Fréchet vector bundles.*

Every Fréchet vector bundle in the traditional sense and every morphism between such bundles can be obtained in this way.

Proof: Given an atlas $(\phi_\lambda)_{\lambda\in\Lambda}$ and an atlas $(\tilde{\Phi}_\lambda)_{\lambda\in\Lambda}$ of trivialisations with matching domains, it is clear that the corresponding charts $(\check{\Phi}_\lambda)_{\lambda\in\Lambda}$ cover E. Smoothness of the transition functions $\check{\Phi}_\lambda \circ \check{\Phi}_\kappa$ is a consequence of Proposition 3.1.6, which can in fact be used to deduce the converse claim as well. The resulting space is Hausdorff because both its base and its fibres are. For the analytic case, Proposition 3.1.14 can be used instead of Proposition 3.1.6.

Whether or not a fibrewise linear function $\check{\alpha}\colon E \to F$ describes a vector bundle in either sense can be verified locally, on the domain of a trivialisation chart. Proposition 3.1.6 therefore tells us that $\check{\alpha}$ is a smooth map of Fréchet manifolds if and only if (α, f) is a vector bundle morphism in the sense of Definition 4.1.2. The analytic case is again analogous.

□

In addition to consistency with the theory from section 3.1, one of the advantages of choosing Definition 4.1.1 over the traditional definition is that products, locally convex direct sums and tensor products of locally convex vector bundles are locally convex vector bundles as well. The same is true for the topological dual if barrelledness of the fibres is assumed. The category of vector bundles of Fréchet type is not closed under these operations.

Proposition 4.1.4.

Let $E_1, E_2, \ldots, E_p \to M$ *be smooth (or analytic) locally convex vector bundles* *over a Fréchet manifold M. The tensor product*

$$\bigotimes_{i=1}^{p} E_i = \coprod_{x\in M} \bigotimes_{i=1}^{p} E_{i,x} \longrightarrow M$$

admits a unique smooth (resp. analytic) locally convex vector bundle structure which is determined by trivialisations of the form

$$\bigotimes_{i=1}^{p} \tilde{\Phi}_i \colon U \longrightarrow \mathrm{L}\big(\bigotimes_{i=1}^{p} E_{i,x}, \bigotimes_{i=1}^{p} V_{i,x}\big)$$

for smooth (resp. analytic) trivialisations $\tilde{\Phi}_i \colon U \to \coprod_{x\in U} \mathrm{L}(E_{i,x}, V_i)$ with a common domain for $i = 1, 2, \ldots, p$.

Proof: This follows from Proposition 3.1.11 and Proposition 3.1.18. □

Proposition 4.1.5.
Let $(E_i)_{i\in I}$ be a collection of smooth (or analytic) locally convex vector bundles over a Fréchet manifold M. If M admits an open cover $(U_\lambda)_{\lambda\in\Lambda}$ such that $E_i|_{U_\lambda}$ is trivialisable for every $i \in I$, then the locally convex direct sum

$$\bigoplus_{i\in I} E_\lambda = \coprod_{x\in M} \bigoplus_{i\in I} E_{i,x} \longrightarrow M$$

admits a unique smooth (resp. analytic) locally convex vector bundle structure which it determined by trivialisations of the form

$$\sum_{i\in I} \tilde{\Phi}_i \colon U \longrightarrow \mathrm{L}\big(\bigoplus_{i\in I} E_{i,x}, \bigoplus_{i\in I} V_i\big)$$

for smooth (resp. analytic) trivialisations $\tilde{\Phi}_i \colon U \to \coprod_{x\in U} \mathrm{L}(E_{i,x}, V_i)$ with a common domain for every $i \in I$.

For any other locally convex vector bundle $F \to M$, the assignment $\alpha \mapsto (\alpha|_{E_i})_{i\in I}$ describes a one-to-one correspondence between vector bundle morphisms from E to F and collections $(\alpha_i)_{i\in I}$ of vector bundle morphisms $\alpha_i \colon E_i \to F$.

Proof: This follows from Proposition 3.1.12 and Proposition 3.1.19. The existence of neighbourhoods on which the bundles $E_i \to M$ for $i \in I$ are simultaneously trivialisable is necessary to be able to define trivialisations of the form described in this proposition.

We know from Proposition A.2.25 that there exists a canonical isomorphism between the locally convex vector spaces $\mathrm{L}\big(\bigoplus_{i\in I} V_i, W\big)$ and $\prod_{i\in I} \mathrm{L}\big(V_i, W\big)$. This tells us that vector bundle morphisms on E to F can locally be described as collections of vector bundle morphisms defined from E_i to F indexed by $i \in I$, which in turn implies global equivalence of these descriptions. □

Proposition 4.1.6.
Let $E \to M$ be smooth (or analytic) locally convex vector bundle over a Fréchet manifold M with barrelled fibres. The topological dual $E^{\check{}} = \coprod_{x\in M} E_x^{\check{}}$ admits a unique smooth (resp. analytic) locally convex vector bundle structure which it determined by trivialisations of the form $\tilde{\Phi}^{\check{}}\colon U \to \coprod_{x\in U} \mathrm{L}\big(E_x^{\check{}}, V^{\check{}}\big)$ for a smooth (resp. analytic) trivialisation $\tilde{\Phi}\colon U \to \coprod_{x\in U} \mathrm{L}(E_x, V)$.

Proof: This follows from Proposition 3.1.13 and Proposition 3.1.20. □

The usual sum, product and chain rules for vector bundles are all valid in the present context.

Proposition 4.1.7.
If α and β are both assumed smooth (resp. analytic), then the following hold:

(i) *given locally convex vector bundles $E \to M$ and $F \to M$ and two morphisms $\alpha, \beta\colon E \to F$, the sum $\alpha + \beta$ is smooth (resp. analytic) as a vector bundle morphism from $E \to M$ of $F \to M$;*

(ii) *given locally convex vector bundles $E_\lambda \to M$ and $F_\lambda \to M$ for $\lambda \in \Lambda$ and morphisms $\alpha_\lambda, E_\lambda \to F_\lambda$, the direct sum $\bigoplus_{\lambda\in\Lambda} \alpha_\lambda$ is smooth (resp. analytic) as a vector bundle morphism from $\bigoplus_{\lambda\in\Lambda} E_\lambda \to M$ of $\bigoplus_{\lambda\in\Lambda} F_\lambda \to M$;*

(iii) *given locally convex vector bundles $E_i \to M$ and $F_i \to M$ for $i = 1, 2$ and morphisms $\alpha\colon E_1 \to F_1$ and $\beta\colon E_2 \to F_2$, the product $\beta \otimes \alpha$ is smooth (resp. analytic) as a vector bundle morphism from $E_1 \otimes E_2 \to M$ to $F_1 \otimes F_2 \to N$;*

(iv) *given three locally convex vector bundles $E_i \to M_i$ for $i = 1, 2, 3$ and morphisms $\alpha\colon E_1 \to E_2$ and $\beta\colon E_2 \to E_3$, the composition $\beta \circ \alpha$ is smooth (resp. analytic) as a vector bundle morphism from $E_1 \to M_1$ to $E_3 \to M_3$;*

(v) *given a morphism $\alpha\colon E \to F$ of locally convex vector bundles from $E \to M$ to $F \to M$ of with barrelled fibres, the transpose $\alpha^{\check{}}$ is smooth (resp. analytic) as a vector bundle morphism from $F^{\check{}} \to M$ to $E^{\check{}} \to M$.*

Proof: These are all simple consequences of similar statements in section 3.1.2 and section 3.1.3. □

4.1.2. Sheaves of sections and morphisms

From this point onward, whenever we have a vector bundle that is denoted by a (possibly decorated) capital letter, the corresponding (similarly decorated) calligraphic letter will denote its sheaf of sections. Thus, $\mathscr{A}$, $\mathscr{E}$ and $\mathscr{E}^i$ shall for instance denote the sheaves of sections of the vector bundles $A \to M$, $E \to M$ and $E^i \to M$ respectively. More concretely, if we are given a smooth vector bundle $\pi\colon E \to M$, then $\mathscr{E}$ will thus denote the sheaf that assigns the space

$$\mathscr{E}(U) = \Gamma(U; E|_U)$$

to an open subset $U \subseteq M$. It comes with the restriction maps $r_{U',U}\colon \mathscr{E}(U) \to \mathscr{E}(U')$ for every inclusion $U' \subseteq U$ of open subsets of M that map $e \in \mathscr{E}(U)$ to $e|_{U'}$. We denote the sheaf of sections of the trivial bundle $\mathbf{R}_M \to M$ by $\mathscr{C}^\infty_M$ (or $\mathscr{C}^\omega_M$ in the analytic case) and the sheaf of vector fields, which we define as smooth (resp. analytic) sections of the tangent bundle $\mathrm{T}M \to M$, by $\mathscr{T}_M$.

We will view the sheaf $\mathscr{E}$ associated to a locally convex vector bundle $E \to M$ as a sheaf of locally convex $\mathscr{C}^\infty_M$-modules by endowing it with the topology described in the following proposition.

Proposition 4.1.8.
For any locally convex vector bundle $E \to M$ over a Fréchet manifold M, the spaces $\mathscr{E}(U)$ for $U \subseteq M$ open admit unique locally convex topologies such that $\mathscr{E}$ is a sheaf of locally convex $\mathscr{C}^\infty_M$-modules and every trivialisation chart $\Phi = (\tilde{\Phi}, \phi)\colon E|_U \to V \times T$ induces an isomorphism

$$\Phi_*\colon \mathscr{E}(U') \longrightarrow \mathrm{C}^\infty(\phi(U'), V), \qquad e \mapsto \tilde{\Phi} \circ e \circ \phi^{-1}$$

of topological vector spaces. If the bundle $E \to M$ is analytic, it restricts to an isomorphism between the subspaces of analytic local sections and analytic maps from $\phi(U')$ to V.

Because this proposition can be viewed as a special case of Proposition 4.1.13, we shall postpone its proof.

Base-preserving morphisms between vector bundles of Fréchet type can be characterised using their sheaves of sections and the topology described in Proposition 4.1.8.

Proposition 4.1.9.
Let M be a Fréchet manifold, let $E \to M$ and $F \to M$ be vector bundles of Fréchet type and let $\check{A}\colon E \to F$ be a function between their total spaces that covers id_M and is linear in the fibres of E (but not necessarily continuous). Then $\check{A}$ is smooth if and only if there exists a continuous linear operator

$$\hat{A}\colon \mathscr{E} \longrightarrow \mathscr{F}, \qquad \textit{with} \qquad \hat{A}(e)(x) = \check{A}(e(x)).$$

for every $e \in \mathscr{E}(U)$ with $U \subseteq M$ open and $x \in M$. It is analytic if and only if it maps analytic sections to analytic sections.

This describes a one-to-one correspondence between smooth vector bundle morphisms from E to F and morphisms $\hat{A}\colon \mathscr{E} \to \mathscr{F}$ of sheaves of locally convex $\mathscr{C}^\infty_M$-modules with the property that $\hat{A}(e)(x) = 0$ whenever $e(x) = 0$.

Proof: If M is a Fréchet domain and $E \to M$ is trivialisable, then the first assertion follows from Proposition 3.1.6 and the observation that continuity of $\hat{A}$ implies continuity of the restrictions $\check{A}|_{E_x}$ for every $x \in M$. The analytic version of the same statement uses Proposition 3.1.6 instead. For such bundles, the final assertion then follows from Corollary B.2.7.

The general case, where M is a Fréchet manifold and $E \to M$ is not trivialisable, can be derived by using that $\mathscr{E}(U)$ and $\mathscr{F}(U)$ for $U \subseteq M$ open are projective limits of the spaces $\mathscr{E}(U')$ and $\mathscr{F}(U')$ for open subsets $U' \subseteq U$ that *are* domains of trivialisation charts. ☐

The requirement that $\hat{A}(e)(x) = 0$ whenever $e(x)$ is essentially a reformulation of the fact that the definition of $\hat{A}$ is pointwise in nature, i.e. that it is induced by a function from E to F. One might hope that this is already implied by $\mathscr{C}^\infty_M$-linearity, but this is unfortunately not always the case. A counterexample can be constructed by choosing locally convex vector spaces T and V for which there exists a non-trivial continuous linear map $\alpha \in \mathrm{L}(T, V)$ for which $\alpha|_{T \check{\otimes} V} = 0$ and then defining $\hat{A}\colon \mathrm{C}^\infty(T, V) \to \mathrm{C}^\infty(T, V)$ by setting $\hat{A}(e) = \alpha \circ \mathrm{D}e$. This map is both continuous and $\mathrm{C}^\infty(M)$-linear, but it is clearly not a vector bundle morphism since it depends on the derivative of e.

The above counterexample cannot work if either T or V has the approximation property from Definition A.2.40. A locally convex vector space V has this property if the image of the inclusion map $V\check{} \otimes V \hookrightarrow \mathrm{L}(V,V)$ is dense (for the topology of precompact convergence). This is the case for all Hilbert spaces, all Banach spaces that admit a Schauder basis and all nuclear spaces. This includes the space of smooth sections of a finite-dimensional vector bundle, if it is endowed with the standard (nuclear) Fréchet topology.

The second half of the proof of Proposition 4.1.10, where the base is assumed to possess the approximation property, is partially based on the proof of Theorem 28.7 in [KM97b].

Proposition 4.1.10.
Let $E \to M$ and $F \to M$ be two locally convex vector bundles over a Fréchet manifold M whose fibres are either barrelled or metrisable. Assume moreover that either the fibres of E have the approximation property or that M is modelled on a space with this property. A continuous linear operator $\hat{\alpha}\colon \mathscr{E} \to \mathscr{F}$ comes from a vector bundle morphisms if and only if it is $\mathscr{C}^\infty_M$-linear.

Proof: Since this statement can be verified locally, we can assume without loss of generality that the bundles $E, F \to M$ are trivial and that M is a Fréchet domain. Assume therefore $M = X$ is a connected open subset of the Fréchet space T and that $E = V \times X \to X$ and $F = W \times X \to X$ are trivial bundles with respective fibres V and W. We will first assume that V has the approximation property, and then that T has this property. In both cases we will prove that the value of $\alpha(e)$ at $x \in X$ only depends on the value of e at this point.

First assume that V has the approximation property, i.e. that there exists a net $(A_\lambda)_{\lambda\in\Lambda}$ of elements of $V\check{} \otimes V \subseteq \mathrm{End}(V)$ that converges to the identity morphism $\mathrm{id}_V \in \mathrm{End}(V)$ for the topology of uniform convergence on precompact subsets. For any $\lambda \in \Lambda$, we can write the endomorphism $A_\lambda \in V\check{} \otimes V$ as a linear combination

$$A_\lambda = \sum_{j=1}^{N_\lambda} \xi_{\lambda,j} \otimes v_{\lambda,j}$$

of tensor products of forms $\xi_{\lambda,1}, \xi_{\lambda,2}, \ldots, \xi_{\lambda,N_\lambda} \in V\check{}$ and linearly independent elements $v_{\lambda,1}, v_{\lambda,2}, \ldots, v_{\lambda,N_\lambda} \in V$ with $N_\lambda \in \mathbf{N}_0$.

Now suppose we are given a local smooth section $e \in \mathscr{E}(U)$ that vanishes at a given point $x_0 \in U$. We will demonstrate that also $\alpha(e)$ vanishes at this point by considering the sections

$$A_\lambda \circ e = \sum_{j=1}^{N_\lambda} \xi_{\lambda,j}(e) \otimes v_{\lambda,j} \in \mathscr{C}_X^\infty(U) \otimes V \subseteq \mathscr{E}(U).$$

for $\lambda \in \Lambda$. The fact that $A_\lambda \circ e(x_0) = 0$ for every $\lambda \in \Lambda$, together with the linear independence of the elements $v_{\lambda,j}$ for $j \in \{1, 2, \ldots, N_\lambda\}$, tells us that each of the coefficients $\xi_{\lambda,j}(e) \in \mathscr{C}_X^\infty(U)$ must vanish at x_0. It follows that for any $\lambda \in \Lambda$,

$$\alpha(A_\lambda \circ e)(x_0) = \sum_{j=1}^{N_\lambda} \xi_{\lambda,j}(e)(x_0)\, \alpha(v_{\lambda,j}) = 0,$$

because we had assumed $\mathscr{C}_X^\infty$-linearity of α.

Since $(A_\lambda)_{\lambda\in\Lambda}$ converges to id_V and precomposition by e is a continuous operation, as we have seen in Proposition 3.1.8, the net $(A_\lambda \circ e)_{\lambda\in\Lambda}$ converges in $\mathscr{E}(U)$ to $\mathrm{id}_V \circ e = e$ and $\alpha(A_\lambda \circ e)$ therefore converges to $\alpha(e)$. Continuity of the evaluation map $\mathrm{ev}_{x_0} \colon \mathscr{E}(U) \to V$ subsequently tells us that also $\alpha(e)(x_0) = 0$. From this, we deduce that for any local section $e \in \mathscr{E}(U)$, the value of $\alpha(e)$ at a point $x \in U$ only depends on the value of e at this point.

Now assume that T has the approximation property, i.e. that there exists a net $(B_\lambda)_{\lambda\in\Lambda}$ in $T^\vee \otimes T$ that converges to the identity morphism id_T in $\mathrm{L}(T, T)$. We can write each morphism of B_λ as a linear combination

$$B_\lambda = \sum_{j=1}^{N_\lambda} \eta_{\lambda,j} \otimes \dot{x}_{\lambda,j}$$

of decomposable elements for $\eta_{\lambda,1}, \eta_{\lambda,2}, \ldots, \eta_{\lambda,N_\lambda} \in T^\vee$ and $\dot{x}_{\lambda,1}, \dot{x}_{\lambda,2}, \ldots, \dot{x}_{\lambda,N_\lambda} \in T$ with $N_\lambda \in \mathbf{N}_0$.

Let $e \in \mathscr{E}(U)$ be a local section defined on an open subset $U \subseteq X$ such that $e(x_0) = 0$ at a point $x_0 \in U$. Proposition B.1.9 tells us that we can choose a smooth function $\tilde{e} \colon U \to \mathrm{L}(T, V)$ such that $e(x) = \tilde{e}(x)(x - x_0)$ for every $x \in U$. Now consider the functions

$$e_\lambda \colon X \longrightarrow V, \qquad x \mapsto \tilde{e}(B_\lambda(x - x_0))$$

for $\lambda \in \Lambda$, which Proposition 3.1.8 and Corollary 3.1.9 tell us is smooth since both $\tilde{e}$ and $x \mapsto x - x_0$ are. Since Proposition 3.1.8 and Corollary 3.1.9 also tell us that precomposition by $\tilde{e}$ and evaluation on $x \mapsto x - x_0$ are continuous operations, we moreover deduce from these that the net $(e_\lambda)_{\lambda \in \Lambda}$ converges to $e \in \mathrm{C}^\infty(U, V)$.

By using the decomposition of B_λ, we see that e_λ can be expressed as

$$e_\lambda(x) = \tilde{e}(B_\lambda(x - x_0)) = \sum_{j=1}^{N_\lambda} \eta_{\lambda,j}(x - x_0)\, \tilde{e}(x)(\dot{x}_{\lambda,j}).$$

We observe that $x \mapsto \eta_{\lambda,j}(x - x_0)$ is a smooth real function that vanishes at x_0 and that $x \mapsto \tilde{e}(x)(\dot{x}_{\lambda,j})$ is a smooth as well. It thus follows from the $\mathscr{C}^\infty_M$-linearity of α that $\alpha(e_\lambda)(x_0) = 0$. We again conclude that the value of $\alpha(e)$ at a point $x \in U$ only depends on the value of e at this point.

We see that in both cases there exists a function $\check{\alpha}\colon E \to F$ between the total spaces of $E \to X$ and $F \to X$ such that $\alpha(e)(x) = (\check{\alpha})(e(x))$ for every $x \in X$ and every local section $e \in \mathscr{E}(U)$. Its restriction $\check{\alpha}|_{E_x} \colon E_x \to F_x$ is continuous because both the canonical inclusion map $V \hookrightarrow \mathrm{C}^\infty(X, V)$ and the evaluation map $\mathrm{ev}_x \colon \mathrm{C}^\infty(X, W) \to W$ are. That α is a vector bundle morphism now follows from either Proposition 3.1.6 or Lemma 3.1.7, depending on whether the fibres are assumed to be metrisable or barrelled. ☐

Although we did not state it, an analytic version of Proposition 4.1.10 holds and can be proven similarly. The same is true for the following corollary.

Corollary 4.1.11.
Let $E \to M$ and $F \to M$ be vector bundles of Fréchet type over a Fréchet manifold M and assume that either the fibres of E have the approximation property or that M is modelled on a space with this property. There is a one-to-one correspondence between base-preserving vector bundle morphisms from E to F and continuous $\mathscr{C}^\infty_M$-linear operators from $\mathscr{E}$ to $\mathscr{F}$ that relates $\alpha\colon E \to F$ to $\hat{\alpha}\colon \mathscr{E} \to \mathscr{F}$ if $\hat{\alpha}(e)(x) = \alpha_x(e_x)$ for every $e \in \mathscr{E}(U)$ and every $x \in U$ with $U \subseteq M$ open.

Proof: This follows from Proposition 4.1.10 and Proposition 4.1.9. ☐

Proposition 4.1.10 allows us to think of vector bundles whose base or whose fibres are modelled on Fréchet spaces with the approximation property purely as sheaves of topological modules. It moreover simplifies the definition of a smooth derivation on such bundles, as we will see in Proposition 4.2.10, and has a few other consequences.

We can also consider multilinear vector bundle morphisms. Any of the characterisations in the following proposition can be used to describe these.

Proposition 4.1.12.
Let M be a Fréchet manifold and let $E_1, E_2, \ldots, E_p \to M$ and $F \to M$ be vector bundles of Fréchet type. There are canonical one-to-one correspondences between each following objects:

(i) *smooth base-preserving maps*

$$B\colon E_1 \times_M \cdots \times_M E_p \longrightarrow F$$

that are linear when restricted to any of the bundles E_i with $i = 1, 2, \ldots, p$;

(ii) *continuous multilinear operators*

$$\hat{B}\colon \mathscr{E}_1 \times \cdots \times \mathscr{E}_p \longrightarrow \mathscr{F},$$

that are such that $\hat{B}(e_1, \ldots, e_p)(x) = 0$ whenever $e_i(x) = 0$ for some $i \in \{1, 2, \ldots, p\}$;

(iii) *smooth base-preserving vector bundle morphisms $B_\otimes\colon \bigotimes_{i=1}^p E_i$ to F.*

These are related by the requirement that

$$B(e_1(x), \ldots, e_p(x)) = \hat{B}(e_1, \ldots, e_p)(x) = B_\otimes(\textstyle\bigotimes_{i=1}^p e_i(x))$$

for all $x \in U$ with $U \subseteq M$ open and all local sections $e_i \in \mathscr{E}_i(U)$ for $i = 1, 2, \ldots, p$. If each of the bundles $E_i \to M$ is analytic, then analyticity of B and analyticity $B_\otimes$ are both equivalent to the condition that $\hat{B}$ map analytic sections to analytic sections.

Proof: This follows from an argument similar to that employed in Proposition 4.1.9, but it uses Lemma 3.1.21 instead of Proposition 3.1.6. □

Given two vector bundles $E \to M$ and $F \to M$ of Fréchet type and an open subset $U \subseteq M$, let $\mathscr{L}_{E,F}(U)$ denote the set of all vector bundle morphisms from $E|_U$ to $F|_U$. These form a sheaf of $\mathscr{C}^\infty_M$-modules, which we would like to topologise. To do this, we observe that any pair (Φ, Ψ) of trivialisation charts $\Phi = (\tilde{\Phi}, \phi)\colon E|_{U'} \to V \times T$ and $\Psi = (\tilde{\Psi}, \phi)\colon F|_{U'} \to W \times T$ determines a map

$$\Psi_*\Phi^*\colon \mathscr{L}_{E,F}(U) \longrightarrow \mathrm{C}^\infty(\phi(U \cap U'), \mathrm{L}(V, W)),$$

which is given by

$$\Psi_*\Phi^*(\alpha)(y)(v) = \tilde{\Psi}(\alpha(\Phi^{-1}(v, y))),$$

for $\alpha \in \mathscr{L}_{E,F}(U)$ and $(v, y) \in V \times T$. We wants these maps to be continuous for every such pair, and to be topological isomorphisms whenever $U = U'$.

Proposition 4.1.13.
Let $E \to M$ and $F \to M$ be locally convex vector bundles over a Fréchet manifold M. The spaces $\mathscr{L}_{E,F}(U)$, for $U \subseteq M$ open, admit unique locally convex topologies such that $\mathscr{L}_{E,F}$ is a sheaf of locally convex $\mathscr{C}^\infty_M$-modules and the map

$$\Psi_*\Phi^*\colon \mathscr{L}_{E,F}(U) \longrightarrow \mathrm{C}^\infty(\phi(U), \mathrm{L}(V, W)),$$

is a topological isomorphism for any two trivialisation charts $\Phi\colon E|_U \to V \times T$ and $\Psi\colon F|_U \to W \times T$ defined on U. If both $E \to M$ and $F \to M$ are analytic, it restricts to an isomorphism between the subspaces of analytic vector bundle morphisms and analytic maps from $\phi(U)$ to $\mathrm{L}(V, W)$.

Proof: Uniqueness of the described topologies is guaranteed because any open subset of M can be covered by domains of pairs of trivialisation charts and $\mathscr{L}_{E,F}$ is required to be sheaf of locally convex vector spaces. More specifically, these two facts together imply that the space $\mathscr{L}_{E,F}(U)$ for an arbitrary open subset $U \subseteq M$ carries the coarsest locally convex topology for which the map

$$\mathscr{L}_{E,F}(U) \longrightarrow \prod_{U' \subseteq U} \mathscr{L}_{E,F}(U'), \qquad \alpha \mapsto \big(\alpha|_{U'}\big)_{U'}$$

is continuous, where U' ranges over all open subsets of U that are the domain of a pair of trivialisation charts on E and F.

We shall need to verify that this is consistent with the requirement that $\Psi_*\Phi^*$ is a topological isomorphism whenever U itself is the domain of a pair (Φ, Ψ) of trivialisation charts. This amounts to showing any two such pairs of trivialisation charts defined over the same domain $U \subseteq M$ induce the same topology on the space $\mathscr{L}_{E,F}(U)$, and that restriction to a subset $U' \subseteq U$ is a continuous operation.

Suppose therefore that we are given two pairs (Φ, Ψ) and (Φ', Ψ') of trivialisation charts defined over a common domain $U \subseteq M$. We can then consider the map

$$(\Psi'_* \Phi'^*) \circ (\Psi_* \Phi^*)^{-1} \colon \mathrm{C}^\infty(\phi(U''), \mathrm{L}(V, W)) \longrightarrow \mathrm{C}^\infty(\phi'(U''), \mathrm{L}(V', W')),$$

which can also be expressed as $(\Psi' \circ \Psi^{-1})_* (\Phi' \circ \Phi^{-1})^*$. Continuity of this map now follows from Proposition 3.1.8 since both $(\Psi' \circ \Psi^{-1})$ and $(\Phi' \circ \Phi^{-1})$ are smooth vector bundle morphisms. By repeating this argument with the roles of (Φ, Ψ) and (Φ', Ψ') interchanged we deduce that $(\Psi'_* \Phi'^*) \circ (\Psi_* \Phi^*)^{-1}$ is an isomorphism of topological vector spaces. Every pair of trivialisation charts over U consequently induces the same topology on $\mathscr{L}_{E,F}(U)$.

Continuity of the restriction map $r_{U',U} \colon \mathscr{L}_{E,F}(U) \to \mathscr{L}_{E,F}(U')$ follows from the fact that the canonical maps $\mathrm{C}^\infty(\phi(U), \mathrm{L}(V, W))$ to $\mathrm{C}^\infty(\phi(U'), \mathrm{L}(V, W))$ are continuous (which is easy to verify), along with the fact that the restrictions $\Phi|_{U'}$ and $\Psi|_{U'}$ are also trivialisation charts.

If $U \subseteq M$ is the domain of a trivialisation chart, then continuity of the $\mathscr{C}^\infty_M(U)$-module structure on $\mathscr{L}_{E,F}(U)$ is a consequence of Corollary B.2.7. For an arbitrary open subset $U' \subseteq M$, it follows from the fact that the topologies on $\mathscr{C}^\infty_M(U')$ and $\mathscr{L}_{E,F}(U')$ are projective limits.

The analytic case is completely analogous and can be proven using results from section 3.1.3 and appendix B.1.3. □

4.1.3. Graded vector bundles

Many of the smooth vector bundles discussed in this chapter come with a **Z**-grading. A locally convex **Z**-graded vector bundle is almost, but not quite, the same thing as a collection $(E^n)_{n \in \mathbf{Z}}$ of ordinary vector bundles indexed by **Z**.

Definition 4.1.14.
A smooth **Z**-*graded locally convex vector bundle* over a Fréchet manifold M is a locally convex vector bundle $E = \bigoplus_{n \in \mathbf{Z}} E^n \to M$ whose fibres are **Z**-graded locally convex vector spaces and which admits an open cover $(U_\lambda)_{\lambda \in \Lambda}$ and

trivialisations of the form $\Phi_\lambda\colon E_{U_\lambda} \to V \times U_\lambda$ for some locally convex **Z**-graded vector space $V = \bigoplus_{n\in\mathbf{Z}} V^n$ such that $\Phi_\lambda(E_{U_\lambda}) \subseteq V \times U_\lambda$ for every $\lambda \in \Lambda$.

We say that the vector bundle $E = \bigoplus_{n\in\mathbf{Z}} E^n \to M$ is of Fréchet type, Banach type or finite-dimensional type whenever its fibres are **Z**-graded vector spaces of that type. Analytic locally convex vector bundles can be defined analogously.

If $E = \bigoplus_{n\in\mathbf{Z}} E^n \to M$ is a **Z**-graded locally convex vector bundle, then the homogeneous components $E^n \to M$ are of course locally convex vector bundles as well. Conversely, the following proposition allows us to construct a **Z**-graded vector bundle from a collection of ordinary vector bundles of Fréchet type.

Proposition 4.1.15.
*Let $(E^n)_{n\in\mathbf{Z}}$ be a sequence of smooth (or analytic) locally convex vector bundles over a Fréchet manifold M and assume that there exists an open cover $(U_\lambda)_{\lambda\in\Lambda}$ such that $E^n|_{U_\lambda}$ is trivial for every $\lambda \in \Lambda$ and every $n \in \mathbf{Z}$. There exists a smooth (resp. analytic) **Z**-graded locally convex vector bundle $E \to M$ that has E^n for $n \in \mathbf{Z}$ as its homogeneous components.*

Proof: This follows directly from Proposition 4.1.5, since the trivialisations constructed in that proposition are of the required type.

To demonstrate this assertion we need to construct trivialisations Φ_λ for $E \to M$ that restrict to $(\Phi^n_\lambda)_{\lambda\in\Lambda}$ for every $n \in \mathbf{Z}$. These trivialisations are necessarily given by $\Phi_\lambda(\sum_{n\in A} v_n) = \sum_{n\in A \Phi^n_\lambda(v_n)}$, which means that we only need to prove smoothness (resp. analyticity) of the map $\tilde{\Phi}_{\lambda,\kappa}\colon U \to \mathrm{L}(V,V)$ given by

$$\tilde{\Phi}_{\lambda,\kappa}(x)\big(\textstyle\sum_{n\in A} v_n\big) = \displaystyle\sum_{n\in A} \tilde{\Phi}_\lambda(x, \tilde{\Phi}_\kappa(x,v))$$

for $x \in U$ and $v_n \in V^n$ for all n in a finite subset $A \subseteq \mathbf{Z}$

We observe that $\tilde{\Phi}_{\lambda,\kappa}$ takes values in the subspace $\mathrm{L}(V,V)^0 \subseteq \mathrm{L}(V,V)$ of degree 0 endomorphisms of V and that this space isomorphic to the product $\prod_{n\in\mathbf{Z}} \mathrm{L}(V^n,V^n)$ due to Proposition A.2.25. Smoothness of $\tilde{\Phi}_{\lambda,\kappa}$ is thus equivalent to smoothness of $\tilde{\Phi}^n_{\lambda,\kappa}$ for all $n \in \mathbf{Z}$. □

The requirement that the bundles $E^n \to M$ for $n \in \mathbf{Z}$ admit local trivialisations with common domains is a bit of a technicality that is unlikely to come up in practice. It is not relevant for finite-dimensional smooth vector bundles, since these can be trivialised over any contractible neighbourhood, nor for infinite-dimensional vector bundles that arise as spaces of sections of finite-dimensional bundles.

Definition 4.1.16.
A homogeneous *morphism* from $E = \bigoplus_{n\in\mathbf{Z}} E^n \to M$ to $F = \bigoplus_{n\in\mathbf{Z}} F^n \to N$ of degree $d \in \mathbf{Z}$ is a vector bundle morphism $\alpha\colon E \to F$ such that $\alpha(E^n) \subseteq F^{n+d}$ for every $n \in \mathbf{Z}$.

Homogeneous morphisms of locally convex $\mathbf{Z}$-graded vector bundles of a given degree $d \in \mathbf{Z}$ can equivalently be characterised as sequences $(\alpha_n)_{n\in\mathbf{Z}}$ of morphisms $\alpha_n\colon E^n \to F^{n+d}$ between their homogeneous components. This is a consequence of Proposition 4.1.5.

We define the sheaf $\mathscr{E}$ of sections of a $\mathbf{Z}$-graded vector bundle $E = \bigoplus_{n\in\mathbf{Z}} E^n \to M$ as the sheaf of $\mathbf{Z}$-graded locally convex vector spaces given by

$$\mathscr{E}(U) = \bigoplus_{n\in\mathbf{Z}} \mathscr{E}^n(U).$$

for any open subset $U \subseteq M$, where $\mathscr{E}^n$ denotes the sheaf of sections of the homogeneous component $E^n \to M$. This is not a sheaf of ungraded locally convex vector spaces because a section that locally has finitely many homogeneous components might not globally have this property. It is, however, a sheaf of locally convex $\mathbf{Z}$-graded $\mathscr{C}^\infty_M$-modules.

We similarly refer to the sheaf $\mathscr{L}_{E,F}$ given by

$$\mathscr{L}_{E,F}(U) = \bigoplus_{d\in\mathbf{Z}} \mathscr{L}^d_{E,F}(U) \qquad \text{with} \qquad \mathscr{L}_{E,F}(U)^d = \prod_{n\in\mathbf{Z}} \mathscr{L}_{E^n,F^{n+d}}(U)$$

as the sheaf of morphisms from $E \to M$ to $F \to M$. It is a sheaf of locally convex $\mathbf{Z}$-graded $\mathscr{C}^\infty_M$-modules, but it is not generally the sheaf of sections of a $\mathbf{Z}$-graded vector bundle in the sense of Definition 4.1.1.

We can now make sense of the graded symmetric (co)algebra of any $\mathbf{Z}$-graded vector bundle.

Proposition 4.1.17.
Let $E = \bigoplus_{n\in\mathbf{Z}} E^n$ be a $\mathbf{Z}$-graded locally convex vector bundle with a trivialisation atlas $(\Phi_\lambda)_{\lambda\in\Lambda}$ and let

$$\bigodot(E) = \coprod_{x\in M} \bigoplus_{n\in\mathbf{Z}} \bigodot(E_x)^n \longrightarrow M,$$

denote its bundle of graded symmetric (co)algebras. It is a $\mathbf{Z}$-graded locally convex vector bundle with trivialisations

$$\bigoplus_{p\in\mathbf{N}_0} \bigoplus_{n_1\leq\ldots\leq n_p} \bigodot_{i=1}^{p} \Phi_\lambda^{n_i} \colon \bigodot(E)|_{U_\lambda} \longrightarrow \bigodot(V)$$

for $\lambda \in \Lambda$.

Proof: This proposition is essentially a combination of Proposition 4.1.4 and Proposition 4.1.5. It moreover uses that the graded symmetric tensor product $\bigodot_{i=1}^{p} V^{n_i}$ is a direct summand of $\bigotimes_{i=1}^{p} V^{n_i}$ for any sequence $n_1, n_2, \ldots, n_p \in \mathbf{Z}$ of degrees. □

We can also consider the topological dual of a $\mathbf{Z}$-graded vector bundle, provided its fibres are barrelled. Although the dual of a locally convex direct sum is a product, we shall view it as another locally convex direct sum to ensure that the fibres are still $\mathbf{Z}$-graded.

Definition 4.1.18.
Let $E = \bigoplus_{n\in\mathbf{Z}} \to M$ be a $\mathbf{Z}$-graded vector bundle of barrelled type. Its dual is the $\mathbf{Z}$-graded locally convex vector bundle $E^{\vee} = \bigoplus_{n\in\mathbf{Z}} (E^n)^{\vee} \to M$.

If $E \to M$ is a $\mathbf{Z}$-graded locally convex vector bundle of Fréchet type, then the corresponding bundle $\bigodot(E) \to M$ of graded symmetric (co)algebras has barrelled fibres. Its dual, $(\bigodot(E))^{\vee} \to M$, is therefore well-defined as a locally convex vector bundle. We denote its sheaf of sections by Ω_E and refer to it as the sheaf of graded symmetric forms on $E \to M$. It can be identified with the sheaf of degree zero vector bundle morphisms from $\bigodot(E) \to M$ to the trivial bundle $\mathbf{R}_M \to M$ (which is concentrated in degree 0).

We can in particular apply these construction to the desuspension $\mathrm{T}[1]M \to M$ of the tangent bundle of M. We denote the resulting sheaf by $\Omega_M = \Omega_{\mathrm{T}[1]M}$ and refer to its sections as differential forms on M. A homogeneous differential form $\alpha \in \Omega^d_M(M)$ of degree $d \in \mathbf{N}_0$ can equivalently be described as a degree 0 multilinear vector bundle morphism from $\mathrm{T}M$ to $\mathbf{R}_M$ of arity d which is symmetric if d is even and skew symmetric if d is odd.

4.2. Multiderivations

Now that infinite-dimensional vector bundles and their sheaves of sections have been explored in section 4.1, we can talk about smooth derivations and multiderivations. These are the objects that make up a curved L_∞-algebroid structure. The usual (finite-dimensional) definitions are modified slightly to suit our purposes, and a number of assertions that would normally be considered standard are reproven. We apply these definitions to describe linear connections and Lie algebroids in the context of Fréchet manifolds.

4.2.1. Derivations

A derivation on a smooth finite dimensional vector bundle $E \to M$ is usually defined as a linear map $\delta\colon \Gamma(M;E) \to \Gamma(M;E)$ for which there exists a vector field $X \in \mathfrak{X}(M)$ such that the Leibniz rule

$$\delta(f\, e) = f\, \delta(e) + (\mathscr{L}_X f)\, e \tag{4.2.1}$$

holds for every section $e \in \Gamma(M;E)$ and any function $f \in \mathrm{C}^\infty(M)$.

Although this definition is still meaningful in an infinite-dimensional context, it has several shortcomings that make it unsuitable for our purposes. We would expect δ to be a local operator (a morphism of sheaves), we would expect it to be a first order differential operator that has $X \otimes \mathrm{id}_E$ as its principal symbol, and we would expect δ to somehow respect the topology on the fibres of E. Because none of these properties follow from equation (4.2.1) in the general case, they will need to be imposed by hand.

In the definition below, $\mathrm{T}^{\mathrm{v}}_x e\colon \mathrm{T}_x M \to E_x$ denotes the vertical derivative of a local section $e \in \mathscr{E}(U)$ at a point $x \in U$ where $e(x) = 0$. This is the vertical component of the derivative $\mathrm{T}_x e\colon \mathrm{T}_x M \to \mathrm{T}_x E \simeq \mathrm{T}_x M \oplus \mathrm{T}_0 E_x$, viewed as a linear map from $\mathrm{T}_x M$ to E_x through the canonical identification $\mathrm{T}_0 E_x \simeq E_x$.

Definition 4.2.1.
A smooth *derivation* on a vector bundle $E \to M$ of Fréchet type consists of a morphism $\delta\colon \mathscr{E} \to \mathscr{E}$ of sheaves of topological vector spaces, along with a vector field $X \in \mathfrak{X}(M)$ such that

$$\delta(e)_x = \mathrm{T}^{\mathrm{v}}_x e(X(x)), \tag{4.2.2}$$

for any local section $e \in \mathscr{E}(U)$ and any point $x \in U$ where $e(x) = 0$.

Equation (4.2.1) and the linearity of δ can be used to show that for any two local sections $e_1, e_2 \in \mathscr{E}(U)$ and any point $x \in M$ where $e_1(x) = e_2(x)$,

$$\delta(e_1)(x) - \delta(e_2)(x) = (\mathrm{T}_x e_1 - \mathrm{T}_x e_2)(X_\delta(x)).$$

It thus tells us that the value of $\delta(e)$ at x only depends on the value and the derivative of e at x, and that the latter part of this dependence is described by the vector field X. We call X the *principal symbol* of the derivation and say that δ covers X.

While one can easily verify that any smooth derivation in the sense of Definition 4.2.1 satisfies the Leibniz rule (4.2.1), the converse implication only holds if further conditions are imposed on the topology of the fibres of $E \to M$. This is explored in Proposition 4.2.10 below.

Given a non-zero vector field $X \in \mathfrak{X}(M)$ and a point $x \in M$ such that $X(x) \neq 0$, one can always find a local section $e \in \mathscr{E}(U)$ that vanishes at x and is such that $\mathrm{T}^{\mathrm{v}}_x e(X(x)) \neq 0$ (by using the Hahn–Banach theorem), provided that $E \neq 0$. Unless $E = 0$, the vector field X in Definition 4.2.1 is therefore uniquely determined by the operator $\delta\colon \mathscr{E} \to \mathscr{E}$ through equation (4.2.2). If $E = 0$, on the other hand, then a smooth derivation on $E \to M$ is fully described by the underlying vector field, and the operator on the (trivial) sheaf $\mathscr{E}$ carries no information.

Anticipating the concept of a multiderivation, which will be discussed in section 4.2.3 below, we shall denote the space of derivations on a vector bundle $E \to M$ by $\mathrm{Der}^1(E)$. One can readily verify that smooth derivations form a sheaf of $\mathscr{C}^\infty_M$-modules, which we choose to denote by $\mathscr{D}^1_E\colon U \mapsto \mathrm{Der}^1(E|_U)$.

Lemma 4.2.2.
Let $E \to M$ be a smooth vector bundle of Fréchet type and let $A\colon E \to E$ be a

smooth vector bundle endomorphism. The associated operator

$$\hat{A}\colon \mathscr{E} \longrightarrow \mathscr{E}, \qquad e \mapsto A \circ e,$$

is a smooth derivation covering the trivial vector field $0 \in \mathfrak{X}(M)$*.*

Proof: Continuity of $\hat{A}(U)\colon \mathscr{E}(U) \to \mathscr{E}(U)$ for $U \subseteq M$ open is demonstrated in Proposition 4.1.9, and the condition that $\hat{A}(e)(x) = 0$ whenever $e(x) = 0$ holds by construction. □

Unless the distinction is important, we will implicitly identify the vector bundle morphism A and the associated morphism of sheaves, denoting the latter by A rather than $\hat{A}$.

Given a local trivialisation $\Phi\colon E|_U \to V \times U$ of $E \to M$ and a vector field $X \in \mathfrak{X}(U)$ on its domain, we can use the associated isomorphism $\Phi_*\colon \mathscr{E}(U) \xrightarrow{\sim} \mathrm{C}^\infty(U, V)$ to define a linear map

$$\mathscr{L}_X^\Phi := (\Phi_*)^{-1} \circ \mathscr{L}_X \circ \Phi_*\colon \mathscr{E}(U) \longrightarrow \mathscr{E}(U)$$

that associates to a local section of E its derivative along X with respect to the trivialisation Φ. Here, $\mathscr{L}_X\colon \mathrm{C}^\infty(U, V) \to \mathrm{C}^\infty(U, V)$ describes differentiation of V-valued functions on U by X, so that $\mathscr{L}_X v(x) = \mathrm{D}v(x)(X(x))$ for every $x \in U$. The next proposition shows that any smooth derivation can locally be expressed as the sum of a vector bundle morphism and an operator of this type.

Proposition 4.2.3.
Let $E \to M$ be a vector bundle of Fréchet type and let $\Phi\colon E|_U \to V \times U$ be a local trivialisation. For any smooth vector field $X \in \mathfrak{X}(U)$ and any vector bundle morphism $A\colon E|_U \to E|_U$, the operator $\delta = \mathscr{L}_X^\Phi + A$ is a smooth derivation on $E|_U$. Given Φ, any smooth derivation on $E|_U$ can be uniquely written in this form.

Proof: Let $X \in \mathfrak{X}(U)$ be a vector field on U and let $A\colon E|_U \to E|_U$ be a vector bundle morphism. The Lie derivative $\mathscr{L}_X$ can be written as a composition of two maps:

$$\mathscr{L}_X\colon \mathrm{C}^\infty(U, V) \xrightarrow{\mathrm{D}} \mathrm{C}^\infty(U, \mathrm{L}(T, V)) \xrightarrow{\mathrm{ev}_X} \mathrm{C}^\infty(U, V).$$

The first map map assigns to $v \in \mathrm{C}^\infty(U,V)$ its derivative $\mathrm{D}v \in \mathrm{C}^\infty(U,\mathrm{L}(T,V))$ and it is continuous due to Proposition B.2.5. The second map assigns to $B \in \mathrm{C}^\infty(U,\mathrm{L}(T,V))$ the composition $B \circ X \colon x \mapsto B(x)(X(x))$ and its continuity is a consequence of Corollary 3.1.9. Continuity of $\mathscr{L}_X$, and thus of $\mathscr{L}_X^\Phi$, follows.

If $e \in \mathscr{E}(U)$ and $x \in U$ are such that $e(x) = 0$, then $\mathscr{L}_X^\Phi e(x) = \mathrm{D}(\Phi_* e)(x)(X(x))$ coincides with the vertical derivative $\mathrm{T}_x^{\mathrm{v}} e(X(x))$. We thus deduce that $\mathscr{L}_X^\Phi$ is in fact a smooth derivation covering X. For any vector bundle morphism $A\colon E|_U \to E|_U$, $\mathscr{L}_X^\Phi + A$ is a smooth derivation covering the same vector field because the operator $A\colon \mathscr{E}|_U \to \mathscr{E}|_U$ is a smooth derivation covering $0 \in \mathfrak{X}(U)$.

To prove the converse statement, let $\delta\colon \mathscr{E}|_U \to \mathscr{E}|_U$ be a derivation covering $X \in \mathfrak{X}(U)$ and consider the derivative $\mathscr{L}_X^\Phi\colon \mathscr{E}|_U \to \mathscr{E}|_U$. Since both δ and $\mathscr{L}_X^\Phi$ are smooth derivations covering X, their difference, $\hat{A} := \delta - \mathscr{L}_X^\Phi$, is a derivation that covers the trivial vector field $0 \in \mathfrak{X}(U)$. Equation (4.2.2) now implies that for any $x \in U$, the value of $\hat{A}(e)$ at x is completely determined by the value of $e \in \mathscr{E}(U)$ at x, i.e. that there exists a map of sets $A\colon E|_U \to E|_U$ such that $\hat{A}(e)(x) = A(e(x))$ for every $x \in U$. It subsequently follows from Proposition 4.1.9 that A is in fact a smooth vector bundle morphism, which proves that $\delta = \mathscr{L}_X^\Phi + \hat{A}$ is of the desired form ☐

We recover the identity $\delta(f\,e) = f\,\delta(e) + X_\delta(f)e$ for $e \in \Gamma(M;E)$ and $f \in \mathrm{C}^\infty(M)$ from Proposition 4.2.3 because the derivative $\mathscr{L}_X$ is a derivation in this sense and A is $\mathrm{C}^\infty(M)$-linear. The proposition moreover implies the following.

Corollary 4.2.4.
The natural inclusion map from the sheaf $\mathscr{L}_{E,E}$ of endomorphisms of E into the sheaf $\mathscr{D}_E^1$ of derivations on E and the function that maps a derivation to its symbol together form an exact sequence

$$0 \longrightarrow \mathscr{L}_{E,E} \longrightarrow \mathscr{D}_E^1 \longrightarrow \mathscr{T}_M \longrightarrow 0 \tag{4.2.3}$$

of sheaves of $\mathscr{C}_M^\infty$-modules.

The sequence (4.2.3) can locally be split using a trivialisation $\Phi\colon E|_U \to V \times U$ and the corresponding operator $X \mapsto \mathscr{L}_X^\Phi$ from $\mathscr{T}_M|_U$ to $\mathscr{D}_E^1|_U$. This can be used to topologise the sheaf $\mathscr{D}_E^1$ in such a way that it is exact as a sequence of sheaves of locally convex $\mathscr{C}_M^\infty$-modules. A global splitting can be obtained if $E \to M$ (globally) admits a smooth connection.

If the fibres of E are finite-dimensional, then the sequence from Corollary 4.2.4 comes from a corresponding exact sequence of vector bundles

$$0 \longrightarrow \mathrm{End}(E) \longrightarrow D(E) \longrightarrow \mathrm{T}M \longrightarrow 0$$

where $D(E) \to M$ is a vector bundle whose space of sections is canonically isomorphic to $\mathrm{Der}(E)$. This is also more or less true in the general case, but the total spaces $\mathrm{End}(E)$ and $D(E)$ do not generally carry a smooth manifold structure.

The following lemma shows that the exact sequence from Corollary 4.2.4 can also be viewed as an exact sequence of sheaves of Lie algebras.

Lemma 4.2.5.
Let $E \to M$ be a vector bundle of Fréchet type and let δ_1 and δ_2 be smooth derivations on E covering X_1 and X_2 respectively. The commutator $[\delta_1, \delta_2]$ is a smooth derivation that covers $[X_1, X_2] \in \mathfrak{X}(M)$.

Proof: Since the statement that $[\delta_1, \delta_2]$ is a derivation can be verified locally, we can restrict ourselves to the domain of a trivialisation chart $\Phi\colon E|_U \to \varphi(U) \times V$ covering a chart $\varphi\colon U \to T$. For notational convenience, we will freely identify the spaces $\mathscr{E}(U)$ and $\mathrm{C}^\infty(U, V)$, as well as the spaces $\mathfrak{X}(U)$ and $\mathrm{C}^\infty(U, T)$. Due to Proposition 4.2.3, we can now write the two derivations as $\delta_i = \mathscr{L}_{X_i} + A_i$, with $i = 1, 2$, for two smooth maps $A_1, A_2\colon U \to \mathrm{L}(V, V)$.

We can work out the composition $\delta_1 \circ \delta_2$ using Proposition 3.1.8, which tells us that

$$\begin{aligned}
(\delta_1 \circ \delta_2)(e)(x) &= \mathscr{L}_{X_1}\big(\mathscr{L}_{X_2} e + A_2 \circ e\big)(x) + A_1(x)(\mathscr{L}_{X_2} e + A_2 \circ e(x)) \\
&= \mathrm{D}^2 e(x)(X_1(x), X_2(x)) + \mathrm{D}e(x)(\mathrm{D}X_1(x)(X_2(x))) \\
&\quad + (\mathrm{D}A_2(x)(X_1(x)))(e(x)) + A_2(x)(\mathrm{D}e(x)(X_1(x))) \\
&\quad\quad + A_1(x)(\mathrm{D}e(x)(X_2(x))) + A_1(x) \circ A_2(x) \circ e(x)
\end{aligned}$$

for any smooth map $e\colon U \to V$ and any $x \in \varphi(U)$. By subtracting the same expression with δ_1 and δ_2 interchanged we obtain

$$\begin{aligned}
[\delta_1, \delta_2](e)(x) &= \mathrm{D}e(x)(\mathrm{D}X_1(x)(X_2(x)) - \mathrm{D}X_2(x)(X_1(x))) \\
&\qquad + (\mathrm{D}A_2(x)(X_1(x)))(e(x)) - (\mathrm{D}B_2(x)(X_1(x)))(e(x)) \\
&\qquad\quad + A_1(x) \circ A_2(x) \circ e(x) - B_1(x) \circ B_2(x) \circ e(x) \\
&= \mathscr{L}_{[X_1,X_2]} e(x) + [A_1(x), A_2(x)](e(x)) \\
&\qquad + (\mathscr{L}_{X_1} A_2)(x)(e(x)) - (\mathscr{L}_{X_2} A_1)(x)(e(x)).
\end{aligned}$$

We observe that $[\delta_1, \delta_2]$ is of the form described in Proposition 4.2.3 for $X = [X_1, X_2]$ and $A = [A_1, A_2] + \mathscr{L}^{\Phi}_{X_1} A_2 - \mathscr{L}^{\Phi}_{X_2} A_1$ and that it is therefore a smooth derivation covering $[X_1, X_2]$. □

The proof of Lemma 4.2.5 is rather explicit because it uses the local description from Proposition 4.2.3. A significantly shorter argument is available when derivations can be defined using just the Leibniz rule (4.2.1).

Smooth derivations on a vector bundle $E \xrightarrow{\pi} M$ of Fréchet type can also be characterised as linear vector fields on its total space. A vector field $\tilde{X} \in \mathfrak{X}(E)$ is called linear if there exists a vector field $X \in \mathfrak{X}(M)$ on the base for which the diagram

$$\begin{array}{ccc}
E & \xrightarrow{\tilde{X}} & \mathrm{T}E \\
\big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \mathrm{T}\pi} \\
M & \xrightarrow{X} & \mathrm{T}M
\end{array}$$

commutes and X is a vector bundle morphism from $E \to M$ to $\mathrm{T}E \to \mathrm{T}M$.

Proposition 4.2.6.
Let $E \to M$ be a vector bundle of Fréchet type let $\tilde{X}$ be a linear vector field on E covering $X \in \mathfrak{X}(M)$. The operator $\delta_{\tilde{X}}\colon \mathscr{E} \to \mathscr{E}$ given by

$$\delta_{\tilde{X}}(e)(x) = \mathrm{T}_x e(X(x)) - \tilde{X}(e(x)) \in \mathrm{T}_{e(x)} E_x \simeq E_x$$

for $e \in \mathscr{E}(U)$ is a smooth derivation on $E \to M$ covering X. Every smooth derivation on this vector bundle is of this form.

Proof: We can assume without loss of generality that the bundle $E \to M$ is trivial with fibre V and that M is an open subset of a Fréchet space T. We will moreover implicitly identify elements of the spaces $\mathscr{E}(U)$ and $\mathrm{C}^\infty(U,V)$ for any open subset $U \subseteq M$, and will similarly write the vector field $\tilde{X}$ as a sum $\tilde{X} = X + \tilde{X}$ of the vector field $X \in \mathfrak{X}(M) \simeq \mathrm{C}^\infty(U,T)$ on the base and a vertical vector field $\tilde{X} \in \mathfrak{X}^{\mathrm{v}}(E) \simeq \mathrm{C}^\infty(V \times U, V)$. This vertical component is linear on the fibres and thus corresponds to a vector bundle endomorphism on $E \to M$.

In terms of this description, $\mathrm{T}_x e(X(x)) = \mathscr{L}_{X(x)} e + X(x)$ and $X^E(e(x)) = X(x) + \tilde{X}(x, e(x))$ for every $x \in M$, and thus

$$\delta_{\tilde{X}}(e)(x) = \mathscr{L}_{X(x)} e - \tilde{X}(x, e(x)),$$

for any local section $e \in \mathscr{E}(U) \simeq \mathrm{C}^\infty(U,V)$. We conclude that it is a smooth derivation because $\delta_{\tilde{X}}$ is thus precisely of the form described in Proposition 4.2.3. This argument can be reversed to construct a linear vector field from any smooth derivation. ☐

Although one can show that the commutator $[\delta_{\tilde{X}}, \delta_{\tilde{Y}}]$ of the derivations corresponding to linear vector fields $\tilde{X}$ and $\tilde{Y}$ is given by $\delta_{[\tilde{X},\tilde{Y}]}$, we will not perform the required computation here.

Example 4.2.7 (Lie group actions)**.**
Let G be a Fréchet Lie group, let $E \to M$ be a vector bundle of Fréchet type and assume we are provided a smooth right action $\alpha^E \colon E \times G \to E$ which covers an action $\alpha \colon M \times G \to M$ and is linear on the fibres of E. Differentiation of α^E at the identity element $\mathrm{e}_G \in G$ yields the infinitesimal action

$$\alpha^E_* \colon \mathrm{Lie}(G) \simeq \mathrm{T}_{\mathrm{e}_G} G \longrightarrow \mathfrak{X}(E), \qquad \alpha^E_*(a)(e_x) = \mathrm{T}_{(e(x),\mathrm{e}_G)} \alpha(a),$$

which one can readily verify produces linear vector fields. The derivation $\delta_{\alpha^E_*(a)}$ associated to a Lie algebra elements $a \in \mathrm{Lie}(G)$ is given by

$$\delta_{\alpha^E_*(X)}(e)(x) = \mathrm{T}_x e(\mathrm{T}_{(x,\mathrm{e}_G)} \alpha(X)) - \mathrm{T}_{(e(x),\mathrm{e}_G)} \alpha^E(X) \in \mathrm{T}_{e(x)} E_x \simeq E_x$$

for $e \in \mathscr{E}(U)$ and $x \in U$ with $U \subseteq M$ open. Moreover, the map $X \mapsto \delta_{\alpha^E_*(X)}$ from $\mathfrak{g}$ to $\mathrm{Der}^1(E)$ is a (continuous) Lie algebra morphism. ♢

Derivations on a graded vector bundle can be defined analogously.

Definition 4.2.8.
Let $E = \bigoplus_{n\in\mathbf{Z}} E^n \to M$ of Fréchet type which is non-zero. A homogeneous smooth derivation of degree d on $E \to M$ is a homogeneous morphism $\delta\colon \mathscr{E} \to \mathscr{E}$ of sheaves of topological graded vector spaces that of degree d for which there exists a vector field $\mathfrak{X} \in \mathfrak{X}(M)$ such that $\delta(e)_x = \mathrm{T}^{\mathrm{v}}_x e(X(x))$ for any homogeneous local section $e \in \mathscr{E}^n(U)$ of degree n and any point $x \in U$ where $e(x) = 0$.

A homogeneous derivation δ of degree 0 on a $\mathbf{Z}$-graded bundle $E \to M$ is thus equivalent to a collection of derivations $\delta_n\colon \mathscr{E}^n \to \mathscr{E}^n$ on the bundles $E^n \to M$ that cover the same vector field. Similarly, a derivation of degree $d \neq 0$ is equivalent to a collection of vector bundle morphisms $\delta_n\colon E^n \to E^{n+d}$. We denote the space of homogeneous smooth derivations of degree $d \in \mathbf{Z}$ on $E \to M$ by $\mathrm{Der}^1(E)^d$, and shall refer to the direct sum

$$\mathrm{Der}^1(E) := \bigoplus_{d\in\mathbf{Z}} \mathrm{Der}^1(E)^d$$

as the space of smooth derivations on $E \to M$. Note that if we forgot about the grading on $E \to M$, and viewed it as an ungraded locally convex vector bundle, this would be a product rather than a direct sum. The following lemma tells us that, because we chose to define it as a direct sum, this space comes with a graded Lie algebra structure.

Lemma 4.2.9.
Let δ_1 and δ_2 be homogeneous smooth derivations on a $\mathbf{Z}$-graded vector bundle $E \to M$ of Fréchet type covering X_1 and X_2 respectively. The graded commutator $[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - (-1)^{|\delta_1||\delta_2|}\delta_2 \circ \delta_1$ is a smooth derivation of degree $|\delta_1| + |\delta_2|$ covering $[X_1, X_2]$.

Apart from the fact that one has to take grading into account, the proof of this lemma is identical to that of Lemma 4.2.5, so we will not repeat it.

We conclude this section by briefly discussing when the Leibniz rule can be used to characterise smooth derivations. We recall from Definition A.2.40 that a locally convex vector space V has the approximation whenever the image of the inclusion map $V^{\check{}} \otimes V \hookrightarrow \mathrm{L}(V, V)$ is dense (for the topology of precompact convergence). It is possessed by many classes of commonly encountered locally convex vector spaces, which includes the space of smooth sections of any finite-dimensional vector bundle, if it is endowed with the standard Fréchet topology.

Proposition 4.2.10.
Let $E \to M$ be a vector bundle of Fréchet type and assume that either its fibres or its base is modelled on a space with the approximation property. Then a morphism $\delta\colon \mathscr{E} \to \mathscr{E}$ of sheaves of locally convex vector spaces is a smooth derivation covering the vector field $X \in \mathfrak{X}(M)$ if and only if it satisfies the Leibniz rule (4.2.1).

Proof: Since both conditions are local in nature, we may assume without loss of generality that the bundle is trivial. Assume therefore that $E = V \times M \to M$ is the trivial bundle over M with fibre V.

If a continuous operator $\delta\colon \mathscr{E} \to \mathscr{E}$ is such that the Leibniz rule (4.2.1) holds for a vector field $X \in \mathfrak{X}(M)$, then $\delta - \mathscr{L}_X\colon \mathscr{E} \to \mathscr{E}$ is both continuous and $\mathscr{C}^\infty_M$-linear. Proposition 4.1.10 therefore tells us that this operator is in fact a vector bundle morphism, from which we deduce through Proposition 4.2.3 that $\delta = \mathscr{L}_X + (\delta - \mathscr{L}_X)$ is a smooth derivation. □

4.2.2. Connections

We will make frequent use of linear connections when describing and comparing smooth multiderivations. The following definition is just one of several possible characterisations of such connections.

Definition 4.2.11.
A smooth *linear connection* on a **Z**-graded vector bundle $E \to M$ of Fréchet type is a morphism of sheaves of locally convex graded vector spaces,

$$\nabla\colon \mathscr{E} \longrightarrow \Omega^1(E), \qquad e \mapsto \nabla e,$$

of degree 1 such that $(\nabla e)_x = \mathrm{T}^{\mathrm{v}}_x e \in \mathrm{L}(\mathrm{T}_x[1]M, E_x)$ for any homogeneous local section $e \in \mathscr{E}(U)$ and any point $x \in U$ where $e(x) = 0$.

For any given open subset $U \subseteq M$, a smooth connection $\nabla\colon \mathscr{E} \to \Omega^1(\mathscr{E})$ on the bundle $E \to M$ induces a bilinear map

$$\nabla\colon \mathscr{T}_M(U) \times \mathscr{E}(U) \longrightarrow \mathscr{E}(U), \qquad (X, e) \mapsto \nabla_X e := (\nabla e) \circ X.$$

This map is continuous in its first argument since ∇e is a vector bundle morphism for any local section $e \in \mathscr{E}(U)$, and we will see in Lemma 4.2.12 that it is continuous in its second argument as well. Despite this, it is not generally (jointly) continuous as a bilinear map.

Lemma 4.2.12.

Let $E \to M$ be a $\mathbf{Z}$-graded vector bundle of Fréchet type and let $\nabla\colon \mathscr{E} \to \Omega^1(E)$ be a smooth linear connection. For any local vector field $X \in \mathscr{T}(U)$, the operator

$$\nabla_X\colon \mathscr{E}|_U \longrightarrow \mathscr{E}|_U, \qquad e \mapsto \nabla_X e := \nabla e \circ X$$

is a smooth derivation covering X.

Proof: Let $X \in \mathscr{T}(U)$ be a local vector field defined on $U \subseteq M$. The fact that ∇ is a linear connection directly tells us that it has the property that $(\nabla_X e)_x = (\mathrm{T}^{\mathrm{v}}_x e)\circ\uparrow$ for any local section $e \in \mathscr{E}(U')$ with $U' \subseteq U$. To show that ∇_X is a smooth derivation covering X, we thus only need to demonstrate that ∇_X is a continuous operator. Since $\mathscr{E}$ is a sheaf of locally convex vector spaces, it is sufficient to verify this property locally.

If U' is the domain of a trivialisation chart, then Corollary 3.1.9 tells us that evaluation on X is describes a continuous operator $\mathrm{ev}_X\colon \Omega^1(E)(U') \to \mathscr{E}(U')$. Because ∇ is continuous by assumption, continuity of $\nabla_X = \nabla \circ \mathrm{ev}_X$ follows. We conclude that ∇_X is a smooth derivation covering X because every open subset of M is covered by neighbourhoods of this type. ☐

Linear connections can be described more explicitly on the domain of a trivialisation chart using the following analogue of Proposition 4.2.3.

Proposition 4.2.13.

Let $E \to M$ be a vector bundle of Fréchet type and let $\Phi\colon E|_U \to V \times U$ be a local trivialisation. For any vector bundle morphism $A\colon \mathrm{T}[1]M|_U \otimes E|_U \to E|_U$, the operator

$$\nabla^{\Phi,A} = \mathscr{L}^{\Phi} + A\colon \mathscr{E}|_U \longrightarrow \Omega^1(E)|_U, \qquad \nabla^{\Phi,A}_X e = \mathscr{L}^{\Phi}_X e + A \circ (X \otimes e)$$

is a smooth connection on $E|_U$, and every connection on $E|_U$ is of this form.

Proof: Both $\mathscr{L}^\Phi$ and A are local operators and the condition that their sum is a smooth connection can be verified locally. We will therefore assume without loss of generality that the base is a Fréchet domain $M \subseteq T$ and that $E \to M$ is the trivial bundle $V \times M \to M$, and we will freely use the canonical identifications $\mathscr{E}(U) \simeq \mathrm{C}^\infty(U, V)$ and $\mathscr{T}_M(U) \simeq \mathrm{C}^\infty(U, T)$ for any open subset $U \subseteq M$.

A vector bundle morphism $A\colon \mathrm{T}U \otimes E|_U \to E|_U$ now corresponds to a smooth map $A\colon U \to \mathrm{L}(T \otimes V, V)$, in terms of which $\nabla^{\Phi,A}$ is given by

$$(\nabla^{\Phi,A}_X e)(x) = \mathrm{D}e(x)(X(x)) + A(x)(X(x) \otimes e(x))$$

for every $X \in \mathscr{T}_M(U')$, every $e \in \mathscr{E}(U')$ and every $x \in U'$ with $U' \subseteq U$ open.

As discussed in Proposition B.2.5, differentiation describes a continuous linear map $\mathrm{D}\colon \mathrm{C}^\infty(U', V) \to \mathrm{C}^\infty(U', \mathrm{L}(T, V))$. Since the spaces $\mathrm{L}(T \otimes V, V)$ and $\mathrm{L}(V, \mathrm{L}(T, V))$ are canonically isomorphic, Proposition 3.1.6 tells us that A also induces a continuous map $\hat{A}$ from $\mathrm{C}^\infty(U', V)$ to $\mathrm{C}^\infty(U, \mathrm{L}(T, V))$. Consequently, $\nabla^{\Phi,A} = \mathscr{L}^\Phi + \hat{A}$ is continuous as well.

In terms of the identification $\mathscr{E}(U') \simeq \mathrm{C}^\infty(T, V)$, the vertical derivative of a local section $e \in \mathscr{E}(U')$ that vanishes at $x \in U'$ is given by $\mathrm{T}^{\mathrm{v}}_x e = \mathrm{D}e(x)$, which coincides with $\mathscr{L}^{\Phi,0} e(x)$. We deduce that also $\nabla^{\Phi,A} e(x) = \mathrm{T}^{\mathrm{v}}_x e$ at this point since $\hat{A}(e)$ vanishes at any point where e has a zero. We deduce that $\nabla^{\Phi,A}$ is a smooth linear connection.

To show that any smooth connection on $E|_U$ is of this form, assume we are given a connection ∇ on E and consider the operator $\hat{A} = \nabla - \mathscr{L}^\Phi\colon \mathscr{E}|_U \to \Omega^1(E)|_U$. It is continuous and has the property that $\hat{A}(e)(x) = 0$ whenever e vanishes at x because $\nabla e(x)$ and $\mathscr{L}^\Phi e(x)$ coincide at such points. This tells us that there exists a function $A\colon \mathrm{T}U \otimes E|_U \to E|_U$ such that $\hat{A}(e)(X)(x) = A(X(x) \otimes e(x))$ for all $e \in \mathscr{E}(U')$, $X \in \mathscr{T}_M(U')$ with $U' \subseteq M$ open and any $x \in U'$. Proposition 4.1.9 tells us that A is a vector bundle morphism and, consequently, that ∇ is of the desired form.

We conclude that smooth connections on $E|_U$ are precisely those operators that are of the form $\mathscr{L}^\Phi + \hat{A}$ for some vector bundle morphism $A\colon \mathrm{T}[1]U \otimes E|_U \to E|_U$. □

Note that Proposition 4.2.13 in particular tells us that the operator $\mathscr{L}^\Phi$ is itself a connection on $E|_U$ and that the difference between any two connection on

$E \to M$ is a vector bundle morphism from $\mathrm{T}M \otimes E$ to E. From this we deduce that if $E \to M$ admits a global connection, then the set of connections on $E \to M$ is an affine space over the space of vector bundle morphisms from $\mathrm{T}M \otimes E$ to E. Unlike in the finite-dimensional case, however, the existence of a global connection on a given vector bundle $E \to M$ of Fréchet type is not guaranteed.

As in the finite-dimensional case, vector bundle connections can be pulled back along smooth maps to obtain connections on the corresponding pullback bundle.

Proposition 4.2.14 (Pullback connection)**.**
Let $\nabla\colon \mathscr{T}_M \otimes \mathscr{E} \to \mathscr{E}$ be a smooth connection on a vector bundle $E \to M$ of Fréchet type. For any smooth map $f : N \to M$ from another Fréchet manifold N to M, the pullback bundle $f^*E \to N$ admits a unique connection $f^*\nabla$ such that

$$(f^*\nabla)_{\dot{y}}(f^*e) = \nabla_{\mathrm{T}f(\dot{y})}e$$

for every local section $e \in \mathscr{E}(U)$ with $U \subseteq M$ open, and every tangent vector $\dot{y} \in \mathrm{T}_yN$ for $y \in f^{-1}(U)$.

Proof: Since the condition on $f^*\nabla$ is local in nature, we can assume without loss of generality that $E \to M$ is trivialisable. Let $\Phi\colon E \to V \times M$ be a trivialisation of and let $f^*\Phi\colon f^*E \to N \times V$ denote the induced trivialisation on $f^*E \to N$.

Proposition 4.2.13 now tells us that there exists a unique vector bundle morphisms $A\colon \mathrm{T}M \otimes E \to E$ such that $\nabla = \mathscr{L}^{\Phi} + A$. We claim that the pullback of ∇ is given by $f^*\nabla = \mathscr{L}^{f^*\Phi} + f^*A$, where $f^*A\colon \mathrm{T}N \otimes f^*E \to f^*E$ denotes the vector bundle morphism that is given by

$$(f^*A)_y(\dot{y}, e) = A_{f(y)}(\mathrm{T}_yf(\dot{y}), e) \in (f^*E)_y \simeq E_{f(y)}$$

for $y \in N$, $\dot{y} \in \mathrm{T}_yN$ and $e \in (f^*E)_y \simeq E_{f(y)}$.

Let $\nabla' = \mathscr{L}^{f^*\Phi} + B$ denote an arbitrary connection on f^*E and $U \subseteq M$ be an open subset, then

$$\nabla'_{\dot{y}}(f^*e) = \mathscr{L}^{\Phi}_{\dot{y}}(f^*e) + B(\dot{y}, e(y)) = \mathscr{L}^{\Phi}_{\mathrm{T}f(\dot{y})}e + B(\dot{y}, e(y))$$

for every section $e \in \mathscr{E}(U)$ and every tangent vector $\dot{y} \in \mathrm{T}_yU$ at a point $y \in U$, while

$$\nabla_{\mathrm{T}f(\dot{y})}e = \mathscr{L}^{\Phi}_{\mathrm{T}f(\dot{y})}e + A(\mathrm{T}f(\dot{y}), e(y)).$$

If U is sufficiently small, these two expressions coincide for every $e \in \mathscr{E}(U)$ and all $\dot{y} \in \mathrm{T}U$ if and only if $B = f^*A$. This tells us that $\mathscr{L}^{f^*\Phi} + f^*A$ satisfies the defining equation for the pull-back connection $f^*\nabla$ and that it is uniquely determined by this property. □

Remark 4.2.15.
To any smooth linear connection $\nabla\colon \mathscr{E} \to \Omega^1(E)$ on a vector bundle $E \to M$ of Fréchet type we can associate a sequence of continuous degree 1 operators

$$\mathscr{E} = \Omega^1(E) \xrightarrow{\mathrm{d}^\nabla} \Omega^2(E) \xrightarrow{\mathrm{d}^\nabla} \Omega^3(E) \xrightarrow{\mathrm{d}^\nabla} \Omega^4(E) \xrightarrow{\mathrm{d}^\nabla} \cdots$$

on the sheaves of E-valued forms. The $(p+1)$-st of these operators is given by

$$(\mathrm{d}^\nabla\alpha)(\dot{x}_{\{1,\ldots,p\}}) = \sum_{\{i\}\sqcup J=\{1,\ldots,p\}} \epsilon_{i,J}\,(\nabla_{\dot{x}_i}\alpha)(\dot{x}_J) - \sum_{\{i\}\sqcup\{j\}\sqcup K=\{1,\ldots,p\}} \tfrac{1}{2}\,\epsilon_{i,j,K}\,\alpha([\dot{x}_i,\dot{x}_j],\dot{x}_K).$$

for any form $\alpha \in \Omega^p(E)(U)$ defined on a neighbourhood $U \subseteq M$ of $x \in M$ and tangent vectors $\dot{x}_0, \dot{x}_1, \ldots, \dot{x}_p \in \mathrm{T}_x[1]M$. Existence and continuity of these operators can be verified by working out what they look like on a trivial bundle and subsequently applying the theory from section 3.1.2.

The operator $F^\nabla = \mathrm{d}^\nabla \circ \mathrm{d}^\nabla\colon \mathscr{E} \to \Omega^2(E)$ is the curvature of ∇, and it can be interpreted as vector bundle morphism from $\bigodot^2(\mathrm{T}[1]M) \otimes E$ to E. This is a special case of Corollary 4.2.25. △

4.2.3. Multiderivations of a given arity

Given a $\mathbf{Z}$-graded vector bundle $E \to M$ of Fréchet type, we can consider homogeneous graded symmetric multilinear operators on its sheaf of sections. We are interested in operators that are smooth derivations in each of their arguments.

Definition 4.2.16.
A homogeneous *multiderivation* of arity $p \in \mathbf{N}_0$ and degree $d \in \mathbf{Z}$ on a $\mathbf{Z}$-graded vector bundle $E \to M$ of Fréchet type consists of a morphism of presheaves and a graded symmetric multilinear vector bundle morphism,

$$\theta\colon \textstyle\bigodot^p(\mathscr{E}) \longrightarrow \mathscr{E} \qquad \text{and} \qquad \rho_\theta\colon \bigodot^{p-1}(E) \longrightarrow \mathrm{T}M,$$

both homogeneous of degree d, such that for any open subset $U \subseteq M$ and any sequence $e_1, e_2, \dots, e_{p-1} \in \mathscr{E}(U)$ of local sections,

$$\theta(e_1 \odot \cdots \odot e_{p-1} \odot \cdot)\colon \mathscr{E}|_U \longrightarrow \mathscr{E}|_U$$

is a smooth derivation on $E|_U$ covering $\rho_\theta(e_1 \odot \cdots \odot e_{p-1}) \in \mathscr{T}(U)$.

The vector bundle morphism ρ_θ is called the *principal symbol* or the *anchor* of the multiderivation. It is completely determined by the operator θ except when $E = 0$ and $(p, d) = (1, 0)$, in which case $\theta = 0$ and the pair (θ, ρ_θ) carries the same information as the vector field $\rho_\theta(1) \in \mathfrak{X}(M)$. One can show that the condition that ρ is a multilinear graded vector bundle morphism is redundant if $E \to M$ is of finite-dimensional type and E has rank at least 2.

For $p \in \mathbf{N}$ and $d \in \mathbf{Z}$, we denote the space of multiderivations of arity p and degree d on E by $\mathrm{Der}^p(E)^d$. Together, these form a bigraded vector space

$$\mathrm{Der}(E) := \bigoplus_{(p,d) \in \mathbf{N}_0 \times \mathbf{Z}} \mathrm{Der}^p(E)^d$$

indexed by arity $p \in \mathbf{N}_0$ and degree $d \in \mathbf{Z}$.

Since $\bigodot^{-1}(E) = 0$ and $\bigodot^0(\mathscr{E}) \simeq \mathbf{R}_M$, a multiderivation of arity 0 and degree d is equivalent to a global section of E^d. A multiderivation of arity 1 and degree d on E is a derivation on this graded vector bundle of the same degree, as defined in Definition 4.2.8: it is a (derivation) covering the vector field $\rho_\theta(1) \in \mathfrak{X}(M)$ on each of the bundles E^n for $n \in \mathbf{Z}$ if $d = 0$, and a degree d graded vector bundle morphism from E to itself if $d \neq 0$.

Note that the requirement that $\theta(e_1 \odot \cdots \odot e_{p-1} \odot \cdot)$ is a smooth derivation in particular implies that the operators

$$\theta(U)\colon \mathscr{E}^{n_1}(U) \times \cdots \times \mathscr{E}^{n_p}(U) \longrightarrow \mathscr{E}^{n_1 + \cdots + n_p + d}(U)$$

for $n_1, n_2, \dots, n_p \in \mathbf{Z}$ are continuous in each of their arguments separately. Although this implies joint continuity of $\theta(U)$ if $E \to M$ is of finite-dimensional type, this is not the case in general.

Locally, connections can be used to describe multiderivation more explicitly.

Proposition 4.2.17.
Let $E \to M$ be a $\mathbf{Z}$-graded vector bundle of Fréchet type let $\nabla\colon \mathscr{T}_U \otimes \mathscr{E}|_U \to \mathscr{E}|_U$ be a local connection. Given $p \in \mathbf{N}_0$, $d \in \mathbf{Z}$ and two vector bundle morphisms of degree d,

$$A\colon \bigodot^p(E)|_U \longrightarrow E|_U \qquad \text{and} \qquad \rho\colon \bigodot^{p-1}(E)|_U \longrightarrow \mathrm{T}M|_U,$$

the presheaf morphism $\theta = \nabla_\rho + A\colon \bigodot^p(\mathscr{E}|_U) \to \mathscr{E}|_U$ given by

$$\theta(e_1 \odot \cdots \odot e_p) = \sum_{I \sqcup \{j\} = \{1,\dots,p\}} \epsilon_{I,\{j\}} \nabla_{\rho(e_I)} e_j + A(e_1 \odot \ldots \odot e_p),$$

for $e_1, e_2, \ldots, e_p \in \mathscr{E}(U')$ with $U' \subseteq U$ open, is a multiderivation of arity p and degree d with anchor ρ. Conversely, any homogeneous multiderivation of a given arity on $E|_U$ is of this form.

Proof: Let $A\colon \bigodot^p(E)|_U \to E|_U$ and $\rho\colon \bigodot^{p-1}(E)|_U \to \mathrm{T}M|_U$ be smooth vector bundle morphisms of degree d and let $\theta = \nabla_\rho + A$ be defined as above. We need to show that $\theta(e_{\{1,\dots,p-1\}} \odot \cdot)\colon \mathscr{E}|_U \to \mathscr{E}|_U$ is a smooth derivation covering $\rho_{\{e_1,\dots,e_p\}} \in \mathscr{T}_M(U)$ for an arbitrary sequence of homogeneous local sections $e_1, e_2, \ldots, e_{p-1} \in \mathscr{E}(U')$ with $U' \subseteq U$ open.

Given such a sequence, we can write $\theta(e_{\{1,\dots,p-1\}} \odot \cdot)$ as $\nabla_X + B$ in terms of the vector field $X = \rho_{\{e_1,\dots,e_p\}}$ and another operator $B\colon \mathscr{E}|_{U'} \to \mathscr{E}|_{U'}$, which is given by

$$B(e) = \sum_{I \sqcup \{j\} = \{1,\dots,p-1\}} (-1)^{|e_j||e|} \epsilon_{I,\{j\}} \nabla_{\rho(e_I \odot e)} e_j + A(e_{\{1,\dots,p-1\}} \odot e),$$

for $e \in \mathscr{E}(U')$ homogeneous with $U'' \subseteq U'$ open.

We now observe that $\nabla_{\rho(e_I \odot \cdot)} e_j = \nabla e_j \circ \rho \circ (e_I \odot \cdot)$ for $j = 1, \ldots, p-1$ and $A(e_{\{1,\dots,p-1\}} \odot \cdot) = a \circ (e_{\{1,\dots,p-1\}} \odot \cdot)$ are smooth graded vector bundle morphisms since they can be expressed as compositions of such morphisms. Consequently, B is a smooth graded vector bundle morphism as well and Proposition 4.2.3 tells us that $\theta(e_{\{1,\dots,p-1\}} \odot \cdot) = \nabla_X + B$ is a smooth derivation covering X. This allows us to conclude that $\theta = \nabla_\rho + A$ is a smooth multiderivation that has ρ as its anchor.

For the converse statement, assume that we are given a smooth multiderivation $\theta\colon \bigodot^p(\mathscr{E}) \to \mathscr{E}$ of arity p and degree d covering $\rho\colon \bigodot^{p-1}(E) \to \mathrm{T}M$. Since both θ and ∇_ρ are multiderivations covering ρ, their difference $A = \theta - \nabla_\rho$ is a multiderivation covering the trivial anchor $0\colon \bigodot^{p-1}(E) \to \mathrm{T}M$. We conclude, through Proposition 4.1.12, that A is a vector bundle morphism and that $\theta = \nabla_\rho + A$ is consequently of the prescribed form. ☐

We obtain an exact sequence similar to the one described in Corollary 4.2.4.

Corollary 4.2.18.
The natural inclusion map from the sheaf $\mathscr{L}_{E,E}$ of p-linear vector bundle morphisms from E to itself into $\mathscr{D}^p_E$ and the anchor map $\rho\colon \theta \mapsto \rho_\theta$ together form an exact sequence

$$0 \longrightarrow \mathscr{L}_{\bigodot^p(E),E} \longrightarrow \mathscr{D}^p_E \longrightarrow \mathscr{L}_{\bigodot^{p-1}(E),\mathrm{T}M} \longrightarrow 0$$

of sheaves of $\mathscr{C}^\infty_M$-modules.

Given a local connection $\nabla\colon \mathscr{T}_U \otimes \mathscr{E}|_U \to \mathscr{E}|_U$, Proposition 4.2.17 allows us to write $\mathscr{D}^p_E(U)$ as a direct sum $\mathscr{L}_{\bigodot^p(E),E}(U) \oplus \mathscr{L}_{\bigodot^{p-1}(E),\mathrm{T}M}(U)$. This provides a local splitting

$$0 \longrightarrow \mathscr{L}_{\bigodot^p(E),E}|_U \longrightarrow \mathscr{D}^p_E|_U \overset{\nabla}{\underset{\longrightarrow}{\curvearrowleft}} \mathscr{L}_{\bigodot^{p-1}(E),\mathrm{T}M}|_U \longrightarrow 0$$

for the sequence from Corollary 4.2.18. In the finite-dimensional case, we can use this to interpret $\mathscr{D}^p_E$ as the sheaf of sections of another vector bundle over M.

We will define the graded commutator of two homogeneous multiderivations $\theta\colon \bigodot^p(\mathscr{E}) \to \mathscr{E}$ and $\theta\colon \bigodot^q(\mathscr{E}) \to \mathscr{E}$ of arity p and q respectively as the presheaf morphism $[\theta,\eta]\colon \bigodot^r(\mathscr{E}) \to \mathscr{E}$ for $r = p+q-1$, given by

$$[\theta,\eta](e_1,e_2,\ldots,e_r) = \sum_{I \sqcup J = \{1,\ldots,r\}} \epsilon_{I,J}\big(\theta(\eta(e_I)\odot e_J) - (-1)^{|\theta||\eta|}\eta(\theta(e_I)\odot e_J)\big),$$

for homogeneous $e_1, e_2, \ldots, e_r \in \mathscr{E}(U)$ with $U \subseteq M$ open.

Proposition 4.2.19.
Let $E \to M$ be a graded vector bundle of Fréchet type and let $\theta\colon \bigodot^p(\mathscr{E}) \to \mathscr{E}$ and $\eta\colon \bigodot^q(\mathscr{E}) \to \mathscr{E}$ be homogeneous smooth multiderivations of arity p and q respectively. The graded commutator $[\theta,\eta]\colon \bigodot^r(\mathscr{E}) \to \mathscr{E}$ is a homogeneous

multiderivation of arity $r = p + q - 1$ and degree $|\theta| + |\eta|$ whose anchor is given by

$$\rho_{[\theta,\eta]} = [\rho_\theta, \rho_\eta] + \rho_\theta \mathbin{\square} \eta - (-1)^{|\theta||\eta|} \rho_\eta \mathbin{\square} \theta.$$

Here $[\rho_\theta, \rho_\eta]\colon \bigodot^r(\mathscr{E}) \to \mathscr{E}$ is defined by the equation

$$[\rho_\theta, \rho_\eta](e_{\{1,\dots,r\}}) = \sum_{I \sqcup J = \{1,\dots,r\}} (-1)^{|e_I||\eta|} \epsilon_{I,J} [\rho_\theta(e_I), \rho_\eta(e_J)],$$

$\rho_\theta \mathbin{\square} \eta\colon \bigodot^r(\mathscr{E}) \to \mathscr{E}$ is given by

$$(\rho_\theta \mathbin{\square} \eta)(e_{\{1,\dots,r\}}) = \sum_{I \sqcup J = \{1,\dots,r\}} \epsilon_{I,J} \rho_\theta(\eta(e_I) \odot e_J)$$

and $\rho_\eta \mathbin{\square} \theta$ is defined similarly.

Proof: We can show that $[\theta, \eta]$ is a smooth multiderivation by repeatedly applying Lemma 4.2.5. Given a subset $I \subseteq \{1, 2, \dots, r\}$, let $\theta_{e_I}\colon \mathscr{E} \to \mathscr{E}$ and $\eta_{e_I}\colon \mathscr{E} \to \mathscr{E}$ be the operators given by $\theta_{e_I}(e_j) = \theta(e_I \odot e_j)$ and $\eta_{e_I}(e_j) = \eta(e_I \odot e_j)$. These are derivations covering $\rho_\theta(e_I)$ and $\rho_\eta(e_I)$ respectively.

To show that the graded commutator $[\theta, \eta]$ is a derivation in its last argument, we write out $[\theta, \eta](e_{\{1,\dots,r\}})$ as

$$\sum_{I \sqcup J = \{1,\dots,r-1\}} \begin{aligned} &\epsilon_{I,\{r\},J}\big(\theta(\eta(e_I \odot e_r) \odot e_J) - (-1)^{|\theta||\eta|}\eta(\theta(e_I \odot e_r) \odot e_J)\big) \\ &+ \epsilon_{I,J}\big(\theta(\eta(e_I) \odot e_J \odot e_r) - (-1)^{|\theta||\eta|}\eta(\theta(e_I) \odot e_J \odot e_r)\big). \end{aligned} \tag{4.2.4}$$

If we first investigate the first line, we see that can be rewritten as

$$\begin{aligned} \epsilon_{I,\{r\},J}\theta(\eta(e_I \odot e_r) \odot e_J) - (-1)^{|\theta||\eta|}\eta(\theta(e_J \odot e_r) \odot e_I) \\ &= (-1)^{|\eta||e_J|}\epsilon_{J,I}\left(\theta_{e_J} \circ \eta_{e_I} - (-1)^{|\theta_{e_J}||\eta_{e_I}|}\eta_{e_I} \circ \theta_{e_J}\right)(e_r) \\ &= (-1)^{|\eta||e_J|}\epsilon_{J,I}[\theta_{e_J}, \eta_{e_I}](e_r). \end{aligned}$$

Since both θ_{e_J} and η_{e_I} are smooth derivations, Lemma 4.2.5 tells us that this is a smooth derivation in e_r covering the vector field $(-1)^{|\eta||e_J|}\epsilon_{J,I}[\rho_\theta(e_J), \rho_\eta(e_I)]$.

The expressions on the second line of equation (4.2.4) are given by

$$\theta(\eta(e_I) \odot e_J \odot e_r) = \theta_{\eta(e_I) \odot e_J}(e_r)$$

and

$$\eta(\theta(e_I) \odot e_J \odot e_r) = \eta_{\theta(e_I)\odot e_J}(e_r)$$

respectively. These are smooth derivations in e_r covering $\rho(\eta(e_I) \odot e_J)$ and $\rho(\theta(e_J) \odot e_I)$ respectively.

By combining these observations, we can conclude that $[\theta, \eta](e_{\{1,\dots,r-1\}} \odot \bullet)$ is a smooth derivation covering $\rho_{[\theta,\eta]}(e_{\{1,\dots,r-1\}})$, with $\rho_{[\theta,\eta]}$ as described in the statement of this proposition.

What remains to be shown is that $\rho_{[\theta,\eta]} \colon \bigodot^{r-1}(\mathscr{E}) \to \mathscr{E}$ is a vector bundle morphism. By explicitly working out this operator on the domain of a trivialisation chart $\Phi \colon E|_U \to V \times U$ using Proposition 4.2.17, we obtain the identity

$$\rho_{[\theta,\eta]}(e_{\{1,\dots,p\}}) = \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J} \Big((\mathscr{L}^{\Phi}_{\rho_\theta(e_I)} \rho_\eta)(e_J) + \rho_\theta(A(e_I) \odot e_J) - (\mathscr{L}^{\Phi}_{\rho_\eta(e_I)} \rho_\theta)(e_J) - \rho_\eta(A(e_I) \odot e_J) \Big)$$

for $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$. Since every term in this sum is a smooth vector bundle morphism in each of the arguments $e_1, e_2, \dots, e_p$, the operator $\rho_{[\theta,\rho]}$ is as well. □

With respect to the bigrading on the space $\operatorname{Der}(E) = \bigoplus_{(p,d)\in \mathbf{N}_0 \times \mathbf{Z}} \operatorname{Der}^p(E)^d$ of multiderivations on E, the graded Lie bracket described in Proposition 4.2.19 has degree $(-1, 0)$: it is degree-preserving, but decreases arity by 1.

4.2.4. Lie algebroids

We can define Lie algebroid structures on vector bundles of Fréchet type by taking the usual definition for finite-dimensional vector bundles and replacing the Leibniz identity by the condition that the Lie bracket is a smooth derivation in the sense of Definition 4.2.1 in both of its arguments.

Definition 4.2.20.
Let M be a Fréchet manifold. A *Lie algebroid* $(A, [\![\bullet, \bullet]\!], \rho)$ of Fréchet type over M consists of:

1. a vector bundle $A \to M$ of Fréchet type whose sheaf of sections we denote by $\mathscr{A}$,

2. a morphism of presheaves

$$[\![\cdot,\cdot]\!]\colon \mathscr{A}\otimes\mathscr{A}\longrightarrow\mathscr{A}$$

such that $(\mathscr{A}(U),[\![\cdot,\cdot]\!])$ is a Lie algebra for every open subset $U\subseteq M$, and

3. a vector bundle morphism $\rho\colon A\to \mathrm{T}M$, called the *anchor map*, such that

$$[\![a,\cdot]\!]\colon \mathscr{A}|_U\longrightarrow\mathscr{A}|_U$$

is a smooth derivation covering $\rho(a)$ for any local section $a\in\mathscr{A}$.

Given a Lie algebroid $(A,[\![\cdot,\cdot]\!],\rho)$ of Fréchet type over M we can consider the $\mathbf{Z}$-graded vector bundle $A[1]\to M$ which is obtained from $A\to M$ by shifting all degrees down by 1. This bundle is concentrated in degree −1 and is given by $A[1]^{-1}=A$. It can be endowed with a canonical binary multiderivation θ by setting

$$\theta(\downarrow a,\downarrow b)=\downarrow[\![a,b]\!]$$

for $a,b\in\mathscr{A}(U)$ with $U\subseteq M$ open, where $\downarrow a$ denotes the local section of $A[1]$ corresponding to a and $\downarrow b$ is defined similarly. This multiderivation is homogeneous of degree 1 and it is self-commuting since

$$\tfrac{1}{2}[\theta,\theta](e_{\{1,2,3\}})=\theta(\theta(e_1,e_2),e_3)+\theta(\theta(e_2,e_3),e_1)+\theta(\theta(e_3,e_1),e_2)$$

is the Jacobiator of θ and thus vanishes by assumption. A binary multiderivation on $A[1]$ whose Jacobiator does not vanish but which does satisfy $\rho_{\mathrm{Jac}(\theta)}=0$ similarly corresponds to a pre-Lie algebroid.

Example 4.2.21 (Lie algebras).
A Fréchet Lie algebra is a Fréchet space $\mathfrak{g}$ endowed with a continuous Lie bracket $[\cdot,\cdot]\colon\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$. It can be viewed as a Fréchet Lie algebroid over a point. Conversely, any Fréchet Lie algebroid over a point is of this form, even though continuity of the Lie bracket was not explicitly imposed in Definition 4.2.23. ◊

Example 4.2.22 (Action algebroids).
An infinitesimal action of a Fréchet Lie algebra $(\mathfrak{g},[\cdot,\cdot])$ on a Fréchet manifold M is a continuous Lie algebra morphism $\alpha_*\colon\mathfrak{g}\to\mathfrak{X}(M)$ to the Lie algebra

$\mathfrak{X}(M) = \mathscr{T}_M(M)$ of vector fields on M. This induces a smooth vector bundle morphism

$$\rho\colon A \longrightarrow \mathrm{T}M, \qquad (X,x) \mapsto \alpha_*(X_x)(x)$$

from the trivial bundle $A = \mathfrak{g} \times M \to M$ to the tangent bundle of M. Smoothness of ρ can be derived from either Lemma 3.1.7 or from the proof of the implication (iii) ⇒ (ii) in Proposition 3.1.6.

The *action algebroid* $(\mathfrak{g} \ltimes M, \rho, [\![\cdot,\cdot]\!])$ corresponding to the aforementioned is the trivial bundle $A = \mathfrak{g} \times M \to M$, endowed with the anchor ρ described above and the Lie bracket $[\![\cdot,\cdot]\!]$ given by

$$[\![X,Y]\!]_x = [X_x, Y_x] + \mathscr{L}_{\rho(X_x)} Y - \mathscr{L}_{\rho(Y_x)} X$$

for $X, Y \in \mathscr{A}(U) \simeq \mathrm{C}^\infty(U,\mathfrak{g})$ with $U \subseteq M$ open. This is a smooth Lie algebroid structure because it is of the form described in Proposition 4.2.17 for the trivial connection on $\mathfrak{g} \times M \to M$, the anchor ρ and the constant bilinear vector bundle morphism $x \mapsto [\cdot,\cdot]$. ◊

Definition 4.2.23.
Let $A = (A, [\![\cdot,\cdot]\!], \rho) \to M$ be a Lie algebroid of Fréchet type and let $E \to M$ be a vector bundle of Fréchet type. An *A-connection* on a E is a morphism

$$\nabla\colon \mathscr{E} \longrightarrow \Omega^1_A(E), \qquad e \mapsto \nabla e$$

of sheaves of locally convex vector spaces such that $\nabla e(x) = \mathrm{T}^{\mathrm{v}}_x e \circ \rho_x$ for any local section $e \in \mathscr{E}(U)$ and any $x \in U$ where $e(x) = 0$. The curvature of ∇ is the form $F_\nabla \in \Omega^2_A(\mathrm{End}(E))$ given by

$$F_\nabla(a_1 \odot a_2)(e) = [\nabla_{a_1}, \nabla_{a_2}]e - \nabla_{[\![a_1,a_2]\!]} e,$$

for $a_1, a_2 \in \mathscr{A}(U)$ and $e \in \mathscr{E}(U)$ and (E,∇).

A *representation* of A is a vector bundle $E \to M$ endowed with an A-connection ∇ whose curvature vanishes identically.

Proposition 4.2.24.
Let $E \to M$ be a graded vector bundle of Fréchet type concentrated in degrees -1 and k for some $k \in \mathbf{Z}$ and let $\theta\colon \mathscr{E} \odot \mathscr{E} \to \mathscr{E}$ be a binary multiderivation on E. Denoting the restrictions of θ to $\mathscr{E}^{-1} \odot \mathscr{E}^{-1}$ and $\mathscr{E}^{-1} \otimes \mathscr{E}^k$ by $[\![\cdot,\cdot]\!]$ and ∇ respectively, we have

$$\theta = [\![\cdot,\cdot]\!] + \nabla \qquad \textit{and} \qquad [\theta,\theta] = \mathrm{Jac}([\![\cdot,\cdot]\!]) + F_\nabla,$$

where F_∇ is denotes the curvature from Definition 4.2.23 (which can be defined even if $\operatorname{Jac}([\![\cdot,\cdot]\!]) \neq 0$*).*

If $k \neq -1$, this describes a one-to-one correspondence between self-commuting binary multiderivations on $E = E^{-1} \oplus E^k$ and pairs $([\![\cdot,\cdot]\!], \nabla)$ consisting of a Lie algebroid structure on E^{-1} and an E^{-1}-representation on E^k.

Verifying Proposition 4.2.24 amounts to checking that the components of a binary multiderivation θ on $A \oplus E$ are of the type described in Definition 4.2.23 working out the commutator $[\theta, \theta]$.

Although we had claimed in Definition 4.2.23 that the curvature F_∇ is a smooth form, we had not yet established that this is the case.

Corollary 4.2.25.
Let $A = (A, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid and let $\nabla\colon \mathscr{A} \otimes \mathscr{E} \to \mathscr{E}$ be an A-connection. The curvature $F_\nabla\colon \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{E} \to \mathscr{E}$ of ∇ is a vector bundle morphism from $A \otimes A \otimes E$ to E.

Proof: If we pick some integer $k \in \mathbf{Z} \setminus \{-1\}$ and let $E = E^{-1} \oplus E^k$ be given by $E^{-1} = A$ and $E^k = E$, then Proposition 4.2.24 tells us that the curvature is given by $F_\nabla = [\nabla, \nabla] + 2[\nabla, [\![\cdot,\cdot]\!]]$. We know from Lemma 4.2.9 that this is a smooth multiderivation, and since its anchor vanishes due to homogeneity it must be a vector bundle morphism. ☐

Example 4.2.26 (Fibrewise linear Lie group actions)**.**
Let $\alpha\colon M \times G \to M$ be a smooth right action of a Fréchet Lie group G on a Fréchet manifold M, and let $(A, \rho, [\![\cdot,\cdot]\!])$ be the associated action algebroid. Suppose moreover that we are given a fibrewise linear action of G on a vector bundle $\pi\colon E \to M$ that covers α, i.e. a smooth map $\alpha^E\colon E \times G \to E$ such that $\alpha^E|_{E_x \times \{g\}}$ is a linear map from E_x to $E_{\alpha(x,g)}$ for every $x \in M$ and every $g \in G$.
We can associated a Lie algebroid representation to this action by endowing $E \to M$ with the A-connection

$$\nabla^E\colon \mathscr{E} \longrightarrow \Omega^1_A(E), \qquad \nabla_a e(x) = \delta_{\alpha^E_*(a_x)}(e) = \mathrm{T}_x e(\rho(a_x)) - \mathrm{T}_{(e(x), e_G)} \alpha^E(a_x),$$

where $\delta_{\alpha^E_*(a_x)}$ is the derivation described in Example 4.2.7. That $(a, e) \mapsto \nabla_a e$ is a smooth derivation in its first argument was essentially shown in this example. One can show that it is a smooth vector bundle by working out what $\nabla^E_a e$ looks like on the domain of a trivialisation chart. The curvature of ∇ vanishes because $X \mapsto \delta_{\alpha^E_*(X)}$ is a Lie algebra morphism, as noted in Example 4.2.7. ◊

4.2.5. Pluriderivations

Although we have thus far only considered multiderivations of a definite arity, we are also interested in formal linear combinations of multiderivations of different arities. To clearly distinguish the two concepts, we shall refer to such objects as pluriderivations.

Definition 4.2.27.
A homogeneous *pluriderivation* of degree d on a $\mathbf{Z}$-graded vector bundle $E \to M$ of Fréchet type is a sequence $(\theta, \rho) = (\theta_p, \rho_{p-1})_{p \in \mathbf{N}_0}$ of multiderivations of degree d indexed by arity.

We can think of a homogeneous pluriderivation (θ, ρ) as a pair consisting of a presheaf morphism and a graded vector bundle morphism

$$\theta\colon \bigodot(\mathscr{E}) \longrightarrow \mathscr{E}, \qquad \text{and} \qquad \rho\colon \bigodot(E) \longrightarrow \mathrm{T}M,$$

which are such that $\theta|_{\bigodot^p(\mathscr{E})} = \theta_p$ $\rho|_{\bigodot^p(E)} = \rho_p$ for all $p \in \mathbf{N}_0$ (note that $\rho_{-1} = 0$ by definition). They form a $\mathbf{Z}$-graded vector space

$$\widehat{\mathrm{Der}}(E) = \bigoplus_{d \in \mathbf{Z}} \widehat{\mathrm{Der}}(E)^d \qquad \text{with} \qquad \widehat{\mathrm{Der}}(E)^d = \prod_{p \in \mathbf{N}_0} \mathrm{Der}^p(E)^d,$$

which contains the space $\mathrm{Der}(E)$ from section 4.2.3 since this space is defined similarly, but uses a direct sum rather than a direct product.

Proposition 4.2.28.
Let $E \to M$ be a $\mathbf{Z}$-graded vector bundle of Fréchet type let $\nabla\colon \mathscr{T}_U \otimes \mathscr{E}|_U \to \mathscr{E}|_U$ be a local connection. Given $p \in \mathbf{N}_0$, $d \in \mathbf{Z}$ and two vector bundle morphisms of degree d,

$$A\colon \bigodot(E)|_U \longrightarrow E|_U \qquad \textit{and} \qquad \rho\colon \bigodot(E)|_U \longrightarrow \mathrm{T}M|_U,$$

the presheaf morphism $\theta = \nabla_\rho + A\colon \bigodot(\mathscr{E}|_U) \to \mathscr{E}|_U$ given by

$$\theta(e_1 \odot \cdots \odot e_p) = \sum_{I \sqcup \{j\} = \{1,\ldots,p\}} \epsilon_{I,\{j\}} \nabla_{\rho(e_I)} e_j + A(e_1 \odot \ldots \odot e_p),$$

for $e_1, e_2, \ldots, e_p \in \mathscr{E}(U')$ with $U' \subseteq U$ open, is a homogeneous pluriderivation with anchor ρ. Every homogeneous pluriderivation on $E|_U$ is of this form.

Proof: This is an immediate corollary to Proposition 4.2.17. ☐

The space $\widehat{\mathrm{Der}}(E)$ of smooth pluriderivations on $E \to M$ carries a natural graded Lie algebra structure. The graded commutator of two homogeneous pluriderivations $\theta = (\theta_p)_{p\in\mathbf{N}_0} \in \widehat{\mathrm{Der}}$ and $\eta = (\eta_q)_{q\in\mathbf{N}_0}$ is given by

$$[\theta,\eta] = \big([\theta,\eta]_r\big)_{r\in\mathbf{N}_0} \qquad \text{with} \qquad [\theta,\eta]_r = \sum_{p+q=r+1} [\theta_p,\eta_q].$$

in terms of the graded Lie algebra structure on $\mathrm{Der}(E)$. It is also characterised by the identity

$$[\theta,\eta](e_1 \odot \cdots \odot e_p) = \sum_{I\sqcup J=\{1,\ldots,p\}} \epsilon_{I,J}\big(\theta(\eta(e_I)\odot e_J) - (-1)^{|\theta||\eta|}\eta(\theta(e_I)\odot e_J)\big)$$

for homogeneous $e_1, e_2, \ldots, e_p \in \mathscr{E}(U)$ with $U \subseteq M$ open.

The following proposition is a plurilinear analogue of Proposition 4.2.29.

Proposition 4.2.29.

Let $E \to M$ be a graded vector bundle of Fréchet type and let $\theta\colon \bigodot(\mathscr{E}) \to \mathscr{E}$ and $\eta\colon \bigodot(\mathscr{E}) \to \mathscr{E}$ be homogeneous smooth pluriderivations. The graded commutator $[\theta,\eta]\colon \bigodot(\mathscr{E}) \to \mathscr{E}$ is a homogeneous pluriderivation of degree $|\theta|+|\eta|$ whose anchor is given by

$$\rho_{[\theta,\eta]} = [\rho_\theta,\rho_\eta] + \rho_\theta \,\square\, \eta - (-1)^{|\theta||\eta|}\rho_\eta \,\square\, \theta.$$

Here $[\rho_\theta,\rho_\eta]\colon \bigodot^r(\mathscr{E}) \to \mathscr{E}$ is defined by the equation

$$[\rho_\theta,\rho_\eta](e_{\{1,\ldots,r\}}) = \sum_{I\sqcup J=\{1,\ldots,r\}} (-1)^{|e_I||\eta|}\epsilon_{I,J}\,[\rho_\theta(e_I),\rho_\eta(e_J)],$$

$\rho_\theta \,\square\, \eta\colon \bigodot^r(\mathscr{E}) \to \mathscr{E}$ is given by

$$(\rho_\theta \,\square\, \eta)(e_{\{1,\ldots,r\}}) = \sum_{I\sqcup J=\{1,\ldots,r\}} \epsilon_{I,J}\,\rho_\theta(\eta(e_I)\odot e_J)$$

and $\rho_\eta \,\square\, \theta$ is defined similarly.

Proof: This follows from Proposition 4.2.29 because the arity r component of $[\theta,\eta]$ is the finite sum of the graded commutators $[\theta_p,\eta_q]$ for $p,q \in \mathbf{N}_0$ with $p+q=r$. ☐

4.3. L_∞-algebroids of Fréchet type

In this section, curved L_∞-algebroids of Fréchet type are finally introduced. Because we have taken some effort to first develop the theory of smooth multiderivations in section 4.2, their definition becomes rather straightforward: a curved split curved L_∞-algebroids is a **Z**-graded vector bundles whose spaces of sections come with curved L_∞-algebra structures that consists of smooth multiderivations. As in the finite-dimensional case, the definition of a morphism of curved L_∞-algebroids is a little more involved, as these are best characterised in terms of the dual description of the L_∞-algebroid structure as an (algebraic) derivation on the sheaf of graded symmetric multilinear forms on the underlying bundle.

Finite-dimensional L_∞-algebroids have been studied before and can be found in the literature under several different names. They initially only appeared in their dual description, as differential graded manifolds, in e.g. [Vor10] (as "non-linear" or "higher" Lie algebroids), [SSS12] and [Bru11] (in a $\mathbf{Z}_2$-graded setting). Lie n-algebroids, which are L_∞-algebroids whose underlying vector bundles are concentrated in degrees between $-n$ and -1, are studied in detail in [SZ17] and [BP13]. L_∞-algebroids have appeared in many other publications since then, such as [Lav16; LLS17], [PS18] and [Vit14]. The underlying vector bundle is usually assumed to be concentrated in negative degrees, and the term Lie ∞-algebroid is often used to refer to L_∞-algebroids with this property.

4.3.1. Split L_∞-algebroids of Fréchet type

Now that we have introduced smooth multiderivations and pluriderivations on graded vector bundles of Fréchet type, we can provide the definition of a curved split L_∞-algebroid. It is essentially a curved L_∞-algebra structure on the sheaf of sections of a **Z**-graded vector bundle whose brackets are all smooth multiderivations.

Definition 4.3.1.
A *curved split L_∞-algebroid* of Fréchet type over a Fréchet manifold M is a **Z**-graded vector bundle $E \to M$ of Fréchet type endowed with a degree 1 pluriderivation $(\theta, \rho) \in \widehat{\mathrm{Der}}(E)$ such that $[\theta, \theta] = 0$.

Curved split L_∞-algebroid structures on E are thus precisely the Maurer–Cartan elements for the graded Lie algebra $(\widehat{\mathrm{Der}}(E), [\,\cdot\,,\cdot\,])$. The word "split" in Definition 4.3.1 refers to either of the equivalent splittings

$$\bigodot(E) = \bigoplus_{p\in\mathbf{N}_0} \bigodot^p(E) \qquad \text{and} \qquad \bigodot(\mathscr{E}) = \bigoplus_{p\in\mathbf{N}_0} \bigodot^p(\mathscr{E}),$$

of the bundle $\bigodot(E) \to M$ and its sheaf of sections. This splitting is considered part of the structure.

Reformulated slightly, Definition 4.3.1 states that a curved split L_∞-algebroid structure consists of sequences $(\theta_p)_{p\in\mathbf{N}_0}$ and $(\rho_p)_{p\in\mathbf{N}_0}$ of graded symmetric *brackets* and *anchors*,

$$\theta_p\colon \bigodot^p(\mathscr{E}) \longrightarrow \mathscr{E} \qquad \text{and} \qquad \rho_p\colon \bigodot(E) \longrightarrow \mathrm{T}M$$

which are both homogeneous of degree 1. These are required to be such that $\theta_p(v_1,\dots,v_{p-1})$ is a smooth derivation covering $\rho_{p-1}(v_1,\dots,v_{p-1})$ for all local sections $v_1, v_2, \dots, v_{p-1} \in \mathscr{E}(U)$, and they should satisfy the Jacobi identities

$$\sum_{I\sqcup J=\{1,\dots,p\}} \epsilon_{I,J}\,\theta_{\#J+1}(\theta_{\#I}(v_I)\odot v_J) = 0$$

for $v_1, v_2, \dots, v_p \in \mathscr{E}(U)$ with $U \subseteq M$ open. The zeroth bracket, $\theta_0\colon \mathbf{R} \to \mathscr{E}^1$, is identified with the global section $\theta_0(1) \in \mathscr{E}^1(M)$ and is called the *curvature* of (E, θ, ρ). A *split L_∞-algebroid* is a curved split L_∞-algebroid for which $\theta_0 = 0$.

Even though the requirement that θ_p is a smooth multiderivation implies continuity of the maps $\theta_p(U)\colon \prod^p \mathscr{E}(U) \to \mathscr{E}(U)$ for $p \in \mathbf{N}_0$ in each of their arguments separately, these maps need not be jointly continuous if M is infinite dimensional. Consequently, $(\mathscr{E}(U), \theta(U))$ will not generally be a locally convex L_∞-algebra in the sense of Definition 2.3.2. Since they are smooth pluriderivations, curved split L_∞-algebroid structures on a graded vector bundle of Fréchet type can locally be described through an auxiliary connection.

Proposition 4.3.2.
Let $E \to M$ be a $\mathbf{Z}$-graded vector bundle of Fréchet type and let ∇ be a linear connection on $E|_U$ for some open subset $U \subseteq M$. Every curved split L_∞-algebroid (θ, ρ) structure on E is of the form

$$\nabla_\rho + \Theta\colon (e_1 \odot \cdots \odot e_p) \mapsto \sum_{I\sqcup\{j\}=\{1,\dots,p\}} \epsilon_{I,\{j\}} \nabla_{\rho(e_I)} e_j + \Theta(e_1 \odot \cdots \odot e_p)$$

for some graded vector bundle morphism $\Theta\colon \bigodot(E|_U) \to E|_U$ *of degree* 1.

Conversely, two homogeneous vector bundle morphisms $\Theta\colon \bigodot(E) \to E$ *and* $\rho\colon \bigodot(E) \to \mathrm{T}M$ *of degree* 1 *together determine a curved split* L_∞*-algebroid structure if and only if* $\rho \mathbin{\square} \Theta + T_{\nabla,\rho} = 0$ *and*

$$\mathrm{Jac}(\Theta) + \nabla_\rho \Theta + \tfrac{1}{2} F_\nabla \circ (\rho \otimes \rho) = 0, \tag{4.3.1}$$

where $\mathrm{Jac}(\Theta)$ *is the Jacobiator of* Θ*, which can be defined pointwise, and the tensors* $T_{\nabla,\rho}$*,* $\nabla_\rho\Theta$ *and* $F_\nabla \circ (\rho \otimes \rho)$ *are given by*

$$T_{\nabla,\rho} = \sum_{I\sqcup\{j\}\sqcup K} \epsilon_{I,\{j\},K}\, \rho(\nabla_{\rho(e_I)} e_j \odot e_K) - \sum_{I\sqcup J} \epsilon_{I,J}\, \tfrac{1}{2} [\rho(e_I), \rho(e_J)]$$

$$\nabla_\rho \Theta(e_{\{1,\dots,p\}}) = -\sum_{I\sqcup J} \epsilon_{I,J} \nabla_{\rho(e_I)} \Theta(e_J) + \sum_{I\sqcup\{j\}\sqcup K} \epsilon_{I,\{j\},K} \Theta(\nabla_{\rho(e_I)} e_j \odot e_K)$$

and

$$(F_\nabla \circ (\rho \otimes \rho))(e_{\{1,\dots,p\}}) = -\sum_{I\sqcup J\sqcup\{k\}} \epsilon_{I,J,\{k\}}\, F_\nabla(\rho(e_I), \rho(e_J))(e_k)$$

respectively, for homogeneous $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$ *with* $U \subseteq M$ *open.*

Proof: Since (θ, ρ) is a smooth pluriderivation, the first part of this proposition is a special case of Proposition 4.2.28.

Given a vector bundle morphism Θ as described above, the pluriderivation $\nabla_\rho + \Theta$ defines a curved L_∞-algebroid structure if and only if its Jacobiator vanishes. The second assertion can thus be demonstrated by working out this Jacobiator (which is a pluriderivation), as well as its anchor. We can read off from Proposition 4.2.19 that the anchor is given by

$$\rho_{\mathrm{Jac}(\nabla_\rho + \Theta)} = \rho \mathbin{\square} (\nabla_\rho + \Theta) + \tfrac{1}{2} [\rho, \rho] = \rho \mathbin{\square} \Theta + T_{\nabla,\rho},$$

and it necessarily vanishes when $\nabla_\rho + \Theta$ is a curved L_∞-algebroid structure. The Jacobiator $\mathrm{Jac}(\nabla_\rho + \Theta)$ itself is given by $\mathrm{Jac}(\Theta) + [\nabla_\rho, \Theta] + \frac{1}{2}[\nabla_\rho, \nabla_\rho]$, where

$$\begin{aligned}[\nabla_\rho, \Theta](e_{\{1,\dots,p\}}) = \sum_{I\sqcup J\sqcup\{k\}} \epsilon_{I,J,\{k\}}\, \nabla_{\rho(\Theta(e_I)\odot e_J)} e_k - \sum_{I\sqcup J} \epsilon_{I,J}\, \nabla_{\rho(e_I)}(\Theta(e_J)) \\ + \sum_{I\sqcup\{j\}\sqcup K} \epsilon_{I,\{j\},K}\, \Theta(\nabla_{\rho(e_I)} e_j \odot e_K)\end{aligned}$$

and

$$\tfrac{1}{2}[\nabla_\rho, \nabla_\rho](e_{\{1,\ldots,p\}}) = \sum_{I \sqcup \{j\} \sqcup J \sqcup \{k\}} \epsilon_{I,\{j\},K,\{l\}} \nabla_{\rho(\nabla_{\rho(e_I)} e_j \odot e_K)} e_l \\ - \sum_{I \sqcup J \sqcup k} \epsilon_{I,J,\{k\}} \nabla_{\rho(e_I)}(\nabla_{\rho(e_J)} e_k),$$

for homogeneous local sections $e_1, e_2, \ldots, e_p \in \mathscr{E}(U)$ with $U \subseteq M$ open. If we assume that $\rho \mathbin{\square} \Theta + T_{\nabla,\rho} = 0$, then the identity $\mathrm{Jac}(\nabla_\rho + \Theta) = 0$ reduces to equation (4.3.1). We conclude that $\nabla_\rho + \Theta$ is a curved L_∞-algebroid structure if and only if this equation holds and $\rho \mathbin{\square} \Theta + T_{\nabla,\rho} = 0$. ☐

We conclude this section by discussing a few simple examples of curved split L_∞-algebroids.

Example 4.3.3 (L_∞-algebras).
Any curved split L_∞-algebra structure $\ell\colon \bigodot(V) \to V$ on a $\mathbf{Z}$-graded vector space V of Fréchet type also describes a curved split L_∞-algebroid structure on the bundle $V \to \{*\}$, and every curved split L_∞-algebroid over a point is of this form. ◊

Example 4.3.4 (Lie algebroids).
We had seen in Section 4.2.4 that any Lie algebroid structure $([\![\cdot,\cdot]\!], \rho)$ on a vector bundle $A \to M$ of Fréchet type can be viewed as a binary multiderivation of degree 1 on the graded vector bundle $A[1] \to M$, which is concentrated in degree -1. The pluriderivation θ on E given by $\theta_2 = \downarrow[\![\uparrow\cdot, \uparrow\cdot]\!]$ and $\theta_p = 0$ for $p \neq 2$ is a curved split L_∞-algebroid structure since the only component of its Jacobiator that does not vanish due to homogeneity is the ternary component,

$$\mathrm{Jac}(\theta)_3\colon \textstyle\bigodot^3(\mathscr{E}) \longrightarrow \mathscr{E}, \qquad e_1 \odot e_2 \odot e_3 \mapsto \sum_{I \sqcup \{j\} = \{1,2,3\}} \epsilon_{I,j} \theta(\theta(e_I) \odot e_j),$$

whose vanishing is expressed by the ordinary Jacobi identity for $[\![\cdot,\cdot]\!]$. Every curved split L_∞-algebroid (E, θ, ρ) which is concentrated in degree -1 is of this form. ◊

Example 4.3.5 (Lie algebroid representations).
Let $(A, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid of Fréchet type over M, let $E \to M$ be a $\mathbf{Z}$-graded vector bundle and let $\nabla\colon \mathscr{T}_M \otimes \mathscr{E} \to \mathscr{E}$ be a flat Lie algebroid

connection on E. We can define a smooth binary multiderivation θ_2 on the $\mathbf{Z}$-graded vector bundle $F = A[1] \oplus E \to M$ by setting

$$\theta_2(a,b) = [\![a,b]\!], \qquad \theta_2(a,e_1) = \nabla_a e_1, \qquad \text{and} \qquad \theta_2(e_1 \odot e_2) = 0$$

for $a, b \in \mathscr{A}(U)$ and $e_1, e_2 \in \mathscr{E}(U)$ with $U \subseteq M$ open. The only potentially non-zero component of the Jacobiator $\mathrm{Jac}(\theta_2)$ is the ternary component, which is given by $\mathrm{Jac}_3(\theta) = \mathrm{Jac}([\![\cdot,\cdot]\!]) + F_\nabla$. It vanishes precisely because $[\![\cdot,\cdot]\!]$ is a Lie bracket and the connection ∇ is flat. ◊

Example 4.3.6 (Equivariant families of L_∞-algebras)**.**
One can add additional structure to the equivariant vector bundle $E \to M$ from Example 4.3.5 in the form of a homogeneous plurilinear vector bundle morphism $\alpha\colon \bigodot(E) \to E$ of degree 1. We may then consider the plurilinear operator $\theta = [\![\cdot,\cdot]\!] + \nabla + \alpha$ whose non-trivial components are given by

$$\theta_2(a,b) = [\![a,b]\!], \qquad \theta_2(a,e_1) = \nabla_a e_1, \qquad \text{and} \qquad \theta_2(e_{\{1,\dots,p\}}) = \alpha(e_{\{1,\dots,p\}})$$

for $a, b \in \mathscr{A}(U)$ and $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$ with $U \subseteq M$. This defines a curved split L_∞-algebroid structure on $F = A \oplus E$ if and only if (E_x, α_x) is a curved L_∞-algebra for every $x \in M$, and α is equivariant in the sense that

$$\nabla_a \alpha(e_{\{1,\dots,p\}}) = \sum_{i=1}^{p} \alpha\big(\textstyle\bigodot_{j<i} e_j \odot (\nabla_a e_i) \odot \bigodot_{j>i} e_j\big)$$

for $a \in \mathscr{A}(U)$ and $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$ with $U \subseteq M$ open. ◊

Example 4.3.7 (Representations up to homotopy)**.**
A more interesting class of examples is provided by representation up to homotopy. These were introduced in [Aba08; AC12] to study the cohomology of classifying spaces of Lie groupoids and to make sense of the adjoint representation of a Lie algebroid. We will not recall their definition here, but shall instead focus on the equivalent characterisation as a sequence $(\partial, \nabla, \omega_2, \omega_3, \dots)$ of multilinear operator, which can be found in Proposition 2.3.2 of [Aba08] or Proposition 3.2 of [AC12].

Let $(A, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over M and let $E \to M$ be a $\mathbf{Z}$-graded vector bundle, but of Fréchet type. A representation up to homotopy of $(A, [\![\cdot,\cdot]\!], \rho)$ on E can be described as a sequence $(\partial, \nabla, \omega_2, \omega_3, \dots)$ consisting of the following objects:

- a graded vector bundle morphism $\partial \colon E \to E$ of degree 1 such that $\partial \circ \partial = 0$;
- a Lie algebroid connection $\nabla \colon \mathscr{A} \otimes \mathscr{E} \to \mathscr{E}$ such that $\nabla\partial = 0$;
- a vector bundle morphism $\omega_2 \in \bigodot^2(A[1]) \otimes E \to E$ of degree 1 such that

$$\partial\,\omega_2(a,b,e) + \omega_2(a,b,\partial e) + F_\nabla(a,b)(e) = 0$$

  for any two three local sections $a, b \in \mathscr{A}(U)$ and $e \in \mathscr{E}(U)$; and
- a sequence $(\omega_p)_{p\geq 3}$ of vector bundle morphisms $\omega_p \in \bigodot^p(A[1]) \otimes E \to E$ of degree 1 satisfying a further equation,

$$\partial(\omega_p) + \mathrm{d}_\nabla \omega_{p-1} + \omega_2 \circ \omega_{p-2} + \omega_3 \circ \omega_{p-3} + \cdots + \omega_{p-2} \circ \omega_2 = 0,$$

  whose terms we will not define here.

These objects can be combined to form an operator $\theta = [\![\cdot,\cdot]\!] + \partial + \nabla + \sum_{p\geq 2} \omega_p$ from $\bigodot(\mathscr{A}[1] \oplus \mathscr{E})$ to $\mathscr{A}[1] \oplus \mathscr{E}$ similar to the operator θ from Example 4.3.6. This produces a curved L_∞-algebroid structure on $A[1] \oplus E \to M$ since each of the identities listed above corresponds to one of the components of the Jacobiator of θ, and every potentially non-trivial component is represented. $\lozenge$

When L_∞-algebroids are studied, it is often assumed that the underlying vector bundle $E \to M$ is finite-dimensional and negatively graded. A curved split L_∞-algebroid which on a $\mathbf{Z}$-graded vector bundle $E \to M$ which is concentrated in the negative degrees -1 up to $-n$ for some $n \in \mathbf{N}$ is called a Lie-n algebroid. Such L_∞-algebroids can be conveniently described as (positively graded) differential graded manifolds, as demonstrated in [SZ17] and [BP13]. Even though Definition 4.3.1 is meaningful for arbitrary $\mathbf{Z}$-graded vector bundles, we will often restrict our attention to bundles $E = \bigoplus_{n\in\mathbf{Z}} E^n \to M$ for which $E^0 = 0$.

4.3.2. The Maurer–Cartan locus

Since we mean to use curved L_∞-algebroids to describe deformation problems, we should describe how they might do so. In this section we will therefore describe analogues of the Maurer–Cartan locus and the concept of gauge equivalence from section 2.4 and introduce the deformation complex corresponding to a Maurer–Cartan element. How these are related to similar concepts for equivariant deformation problems and curved L_∞-algebras will be made explicit in chapter 5.

In this section, as well as in chapter 5, we will only consider smooth curved L_∞-algebroids (E,θ,ρ) that are concentrated away from degree 0, i.e. for which $E^0 = 0$. This restriction does not apply to any of the curved L_∞-algebroids considered in section 4.3.3 and section 4.3.4.

The role of the Maurer–Cartan map for an equivariant deformation problem and the Maurer–Cartan map of a convergent curved L_∞-algebra will be fulfilled by the curvature $\theta_0 \in \mathscr{E}^1(M)$. This leads to the following definition.

Definition 4.3.8.
The *Maurer–Cartan locus* of a curved L_∞-algebroid (E,θ,ρ) of Fréchet type over a Fréchet manifold M is the zero locus

$$\mathrm{MC}(E,\theta,\rho) = \left\{x \in M \;\middle|\; \theta_0(x) = 0\right\}$$

of the curvature $\theta_0 \in \mathscr{E}^1(M)$. It's elements are called *Maurer–Cartan elements*.

Gauge equivalence can also be defined in a way that is reminiscent of both Definition 2.4.6 and the notion of an A-path for a Lie algebroid $(A, [\![\cdot,\cdot]\!], \rho)$.

Definition 4.3.9.
A *homotopy* of Maurer–Cartan elements for a curved L_∞-algebroid (E,θ,ρ) of Fréchet type concentrated away from degree 0 is a smooth path $\Gamma\colon [0,1] \to E^{-1}$ which covers a path $\gamma = \pi \circ \Gamma\colon [0,1] \to \mathrm{MC}(E,\theta,\rho)$ and solves the differential equation

$$\gamma'(t) = \rho \circ \Gamma(t) \tag{4.3.2}$$

for $t \in [0,1]$. We say that two Maurer–Cartan elements $x_0, x_1 \in \mathrm{MC}(E,\theta,\rho)$ are *gauge equivalent* whenever there exists a homotopy (Γ,γ) with endpoints $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

We note that the requirement that $\gamma([0,1]) \subseteq \mathrm{MC}(E,\theta,\rho)$ is somewhat redundant in a finite-dimensional context, since this implied by equation (4.3.2) whenever $\gamma(0) \in \mathrm{MC}(V,\theta,\rho)$ in that case (but not in general). Moreover, if (θ,ρ) restricts to a Lie algebroid structure (θ_2,ρ_1) on $A = E^{-1}$, then the homotopies defined in Definition 4.3.9 are precisely the A-paths for this Lie algebroid that cover a path in $\mathrm{MC}(E,\theta,\rho)$.

As was the case for convergent L_∞-algebras, these homotopies can be reparametrised. Given a curve $\Gamma\colon [a,b] \to E^{-1}$ and a function $f\colon [c,d] \to [a,b]$, both smooth, with $a < b$ and $c < d$, the curve $t \mapsto f'(t)\Gamma(f(t))$ solves equation (4.3.2) for all $t \in [c,d]$ if and only Γ does for all $t \in f([a,b]) \subseteq [c,d]$. This makes it possible to reverse and concatenate homotopies. With this in mind, it becomes clear that gauge equivalence describes an equivalence relation on the Maurer–Cartan locus $\mathrm{MC}(E,\theta,\rho)$. We shall refer to the equivalence classes for this relation as the (gauge) *orbits* of this L_∞-algebroid and denote the orbit that contains a particular point $x \in \mathrm{MC}(E,\theta,\rho)$ by $[x]$. Given an open subset $U \subseteq M$, we denote the orbits of $(E|_U, \theta|_U, \rho|_U)$ containing $x \in \mathrm{MC}(E,\theta,\rho)$ by $[x]_U$.

One can readily verify that the structure maps of (E,θ,ρ) satisfy two additional identities,

$$\rho_1 \circ \theta_1|_{E^{-2}} = 0 \qquad \text{and} \qquad \mathrm{T}^{\mathrm{v}}\theta_0 \circ \rho_1$$

at any point $x \in M$ where θ_0 vanishes. Here $\mathrm{T}^{\mathrm{v}}_x\theta_0\colon \mathrm{T}_xM \to E^1_x$ denotes the vertical component of the derivative of θ_0 at x. The first identity follows from Proposition 4.2.29, and the second can be read off from Proposition 4.3.2 because the linear connection ∇ is by definition such that $\nabla\theta_0(x) = \mathrm{T}^{\mathrm{v}}\theta_0(x)$ whenever $\theta_0(x)$. These allow us to associate a differential complex to every Maurer–Cartan element.

Definition 4.3.10.

Let (E,θ,ρ) be a curved L_∞-algebroid (E,θ,ρ) of Fréchet type over M for which $E^0 = 0$. The *deformation complex* corresponding to a Maurer–Cartan element $x \in \mathrm{MC}(E,\theta,\rho)$ is the differential complex

$$\cdots \xrightarrow{d_x^{-3}=\theta_1} V_x^{-2} \xrightarrow{d_x^{-2}=\theta_1} V_x^{-1} \xrightarrow{d_x^{-1}=\rho_1} V^0 \xrightarrow{d_x^{0}=\mathrm{T}^{\mathrm{v}}_x\theta_0} V_x^{1} \xrightarrow{d_x^{1}=\theta_1} V_x^{2} \xrightarrow{d_x^{2}=\theta_1} \cdots$$

with $V_x^0 = \mathrm{T}_xM$ and $V_x^n = E_x^n$ for $n \in \mathbf{Z} \setminus \{0\}$.

A potentially more insightful way to think about gauge equivalence is in terms of the singular distribution determined by the images $\rho_1(E_x^{-1}) \subseteq \mathrm{T}_xM$ of the linear components of the anchor $\rho\colon \bigodot(E) \to \mathrm{T}M$. Its restriction to the Maurer–Cartan locus $\mathrm{MC}(E,\theta,\rho)$ can be made involutive in a precise sense.

Proposition 4.3.11.

Let (E,θ,ρ) be curved L_∞-algebroid of Fréchet type and consider the subsheaf

$\mathscr{F}_\theta \leq \mathscr{T}_M$ of the sheaf of vector fields on M which consist of all vector fields that are locally of the form

$$X = \rho_1 \circ e + A \circ \theta_0$$

for a local section $e \in \mathscr{E}^{-1}(U)$ and a local vector bundle morphism $A \in \mathscr{L}_{E^1,\mathrm{T}M}(U)$ with $U \subseteq M$ open. This sheaf is involutive in the sense that $[X,Y] \in \mathscr{F}(U)$ for all $X,Y \in \mathscr{F}_\theta(U)$ with $U \subseteq M$ open.

Proof: We need to verify that the commutator of two vector fields X and Y of the described form is again of this form. Since this is a local property, we can assume without loss of generality that $E \to M$ admits a global trivialisation chart $\Phi\colon E \to V \times T$ and use the associated flat linear connection $\nabla = \mathscr{L}^\Phi$ on E to write θ as $\theta = \nabla_\rho + \Theta$ for some vector bundle morphism $\Theta\colon \bigodot(E) \to E$.

We have three cases to consider:

1. If $X = \rho_1 \circ e$ and $Y = \rho_1 \circ e'$ for two local sections $e, e' \in \mathscr{E}^{-1}(U)$, then Lemma 4.3.25 tells us that that

   $$[X,Y] = \rho_1(\theta_2(e,e')) + \rho_2(\theta_1(e),e') - \rho_2(\theta_1(e'),e) + \rho_3(\theta_0,e,e')$$

   The two middle terms drop out because we had assumed that $E^0 = 0$, which leaves us with $[X,Y] = \rho_1(\theta(e,e')) + \rho_3(\theta_0,e,e')$. This vector field is of the prescribed form.

2. If $X = A \circ \theta_0$ and $Y = B \circ \theta_0$ for $A, B \in \mathscr{L}_{E,TM}(U)$, then

   $$[X,Y] = \big(\nabla(B \circ \theta_0)\big) \circ \rho_1 \circ A \circ \theta_0 - \big(\nabla(A \circ \theta_0)\big) \circ \rho_1 \circ B \circ \theta_0,$$

   which is an element $\mathscr{F}(U)$.

3. If $X = \rho_1 \circ e$ and $Y = A \circ \theta_0$ for $e \in \mathscr{E}^{-1}(U)$ and $A \in \mathscr{L}(E,\mathrm{T}M)(U)$, then

   $$[X,Y] = \big(\nabla_{\rho_1(e)} A\big) \circ \theta_0 + A \circ \nabla_{\rho_1(e)}\theta_0 - \nabla(\rho_1 \circ e) \circ \rho_1 \circ A \circ \theta_0.$$

   This vector field does not appear to be of the required form at first glance because of the term $A \circ \nabla_{\rho_1(e)}\theta_0$. Since $\theta = \nabla_\rho + \Theta$, we can however rewrite this term as

   $$\nabla_{\rho_1(e)}\theta_0 = -\theta_2(\theta_0,e) + \Theta_2(\theta_0,e) = \Theta_2(\theta_0,e)$$

by using the Jacobi identity $\theta_1 \circ \theta_1 + \theta_2(\theta_0, \cdot) = 0$ and the fact that $E^0 = 0$. Also this commutator is consequently of the form $C \circ \theta_0$ for some local vector bundle morphism $C \in \mathscr{L}_{E,\mathrm{T}M}(U)$.

Since the commutator $[X, Y]$ is a section of $\mathscr{F}$ in all three cases, the same is true for the commutator of two arbitrary sections. ☐

If the curved L_∞-algebroid (E, θ, ρ) is of finite-dimensional type and satisfies $E^0 = 0$, then Proposition 4.3.11 can be used to show that the singular distribution $\Delta \subseteq \mathrm{T}M$ spanned by $\mathscr{F}_\theta$ is integrable. The fibres of this distribution are given by

$$\Delta_x = \begin{cases} \rho_1(E_x^{-1}) & \text{if } \theta_0(x) = 0, \\ \mathrm{T}_x M & \text{if } \theta_0(x) \neq 0 \end{cases}$$

and each of its leaves is either an orbit for (E, θ, ρ) or a connected component of the complement of $\mathrm{MC}(E, \theta, \rho) \subseteq M$. We can go a little bit further than this, however, and state an analogue of Proposition 1.3.5 for curved L_∞-algebroids of finite-dimensional type.

Theorem 4.3.12.
Let (E, θ, ρ) be a curved L_∞-algebroid of finite-dimensional type over a finite-dimensional manifold M for which $E^0 = 0$ and let (V_{x_0}, d_{x_0}) be the deformation complex associated to a Maurer–Cartan element $x_0 \in \mathrm{MC}(E, \theta, \rho)$. There exist a decomposition $V_{x_0}^0 = B \oplus H \oplus B'$ with $B = \mathrm{im}(d_{x_0}^{-1})$ and $B \oplus H = \ker(d_{x_0}^0)$ and a smooth chart

$$\psi\colon U \subseteq M \longrightarrow V_{x_0}^0 = B \oplus H \oplus B',$$

defined on a neighbourhood $U \subseteq M$ of x_0, for which the following hold.

0. *The derivative $\mathrm{T}\psi(x_0)$ of ψ at x_0 is the identity map on $V_{x_0}^0$.*

1. *The Maurer–Cartan locus of $(E|_U, \theta|_U, \rho|_U)$ is given by*

$$\mathrm{MC}(E, \theta, \rho) \cap U = \psi^{-1}\big(B \times K \times \{0\}\big)$$

for some subset $K \subseteq H \simeq \mathrm{H}^0(V_{x_0}, d_{x_0})$. This subset contains 0, and it is a zero neighbourhood if $\mathrm{H}^1(V_{x_0}, d_{x_0}) = 0$.

2. *Each orbit of $(E|_U, \theta|_U, \rho|_U)$ is of the form*

$$[x] = \psi^{-1}\big(B \times A_x \times \{0\}\big)$$

for some immersed submanifold $A_x \subseteq H$. Specifically, $A_{x_0} = \{0\}$, and if $\mathrm{H}^{-1}(V_{x_0}, d_{x_0}) = 0$ then the neighbourhood U can be chosen such that A_x consists of a single point for every $x \in \mathrm{MC}(E, \theta, \rho) \cap U$.

Proof: The subsheaf $\mathscr{F}_\theta \subseteq \mathscr{T}_M$ is locally finitely generated by construction and its involutivity is demonstrated in Proposition 4.3.11. Integrability of the singular distribution that it spans therefore follows from Theorem 2.2 in [Her62], which tells us that the orbits of (E, θ, ρ) are submanifolds that are tangent to this distribution. A local model for the corresponding foliation is provided by e.g. Theorem 1 in [Ste74b] or Corollary 3.14 in [BLM16]. These both provide a chart

$$\phi\colon U \subseteq M \longrightarrow B \oplus (H \oplus B')$$

such that each leaf is of the form $\phi^{-1}(B \times \tilde{A})$ with $\tilde{A} \subseteq H \oplus B'$ and $\mathrm{T}\phi(x_0) = \mathrm{id}$.

We can locally decompose the vector bundle $E^1 \to M$ as a direct sum $F_1 \oplus F_2 \oplus F_3$ of three vector bundles such that $F_{1,x_0} = B$, $F_{2,x_0} = H$ and $F_{3,x_0} = B'$, and such that $\ker(\theta_{1,x}) \leq F_{1,x} \oplus F_{2,x}$ for every x near x_0. Let π_{F_1} denote the projection onto F_1 and consider the section $\pi_{F_1} \circ \theta_0$ of $F_1|_{\phi^{-1}(H \oplus B')}$. Since this section is transverse at x_0, its zero locus is a smooth submanifold with tangent space $H \oplus B' \cap \ker(d^0_{x_0}) = H$ at x_0. By straightening this submanifold we can obtain a new chart

$$\psi\colon U' \subseteq U \subseteq M \longrightarrow B \oplus H \oplus B'$$

such that $(\pi_{F_1} \circ \theta_0)^{-1}(0) \cap \phi^{-1}(H \oplus B') = \psi^{-1}(H)$, the leaves are still of the aforementioned form and $\mathrm{T}\psi(x_0) = \mathrm{id}$. It follows that $\mathrm{MC}(E, \theta, \rho) \cap U'$ is of the form $\psi^{-1}(B \times K \times \{0\})$ since this set is a union of leaves and is contained in the zero locus of $\pi_{F_1} \circ \theta_0$.

Since they consist of Maurer–Cartan elements, the orbits of $(E|_{U'}, \theta|_{U'}, \rho|_{U'})$ are each of the form $\psi^{-1}(B \times A \times \{0\})$ with $A \subseteq \mathrm{H}^0(V_{x_0}, d_{x_0})$. Because the leaves themselves are immersed submanifolds, these subsets are as well. The orbit through x_0 corresponds to $A_{x_0} = \{0\}$ because the tangent space of this orbit at x_0 is equal to B.

Every element x of $\psi^{-1}(H)$ satisfies $\theta_0(x) \in F_{2,x} \oplus F_{3,x}$ by construction and, since $\theta_1 \circ \theta_0 = 0$ and $\ker(\theta_1) \leq F_1 \oplus F_2$, it is also such that $\theta_0(x) \in F_{1,x} \oplus F_{2,x}$ for all x close to x_0. Consequently, $\theta_0(x) \in F_{2,x} \simeq \mathrm{H}^1(V_{x_0}, d_{x_0})$ for such x. It follows that if $\mathrm{H}^1(V_{x_0}, d_{x_0}) = 0$, every $x \in \psi^{-1}(H)$ close to x_0 is automatically Maurer–Cartan and that $K \subseteq \mathrm{H}^0(V_{x_0}, d_{x_0})$ is thus a zero neighbourhood.

Because $d_x^{-2}\colon E_X^{-2} \to E_x^{-1}$ and $d_x^{-1}\colon E_X^{-2} \to E_x^{-1}$ both vary smoothly with the base point $x \in M$, there exists a neighbourhood $U'' \subseteq M$ of x_0 such that

$$\dim\bigl(\mathrm{im}(d_{x_0}^{-2})\bigr) \le \dim\bigl(\mathrm{im}(d_x^{-2})\bigr) \qquad \text{and} \qquad \dim\bigl(\ker(d_x^{-1})\bigr) \le \dim\bigl(\ker(d_{x_0}^{-1})\bigr)$$

for all $x \in U''$. If $x \in \mathrm{MC}(E,\theta,\rho) \cap U''$, this in fact becomes a fourfold equality

$$\dim\bigl(\ker(d_{x_0}^{-2})\bigr) = \dim\bigl(\ker(d_x^{-2})\bigr) = \dim\bigl(\ker(d_x^{-1})\bigr) = \dim\bigl(\ker(d_{x_0}^{-1})\bigr)$$

because $\dim\bigl(\ker(d_{x_0}^{-2})\bigr) = \dim\bigl(\mathrm{im}(d_{x_0}^{-1})\bigr)$ and $\mathrm{im}\bigl(d_x^{-2}\bigr) \subseteq \ker\bigl(d_x^{-1}\bigr)$ for such x. This also implies that the dimension of the image $\mathrm{im}(d_x^{-1})$ is equal to that of $B = \mathrm{im}(d_{x_0}^{-1})$ for all such x. Since the leaf through x has $\mathrm{im}(d_x^{-1})$ as its tangent space, we deduce that A_x must be zero-dimensional for every $x \in \mathrm{MC}(E,\theta,\rho) \cap U''$. □

Remark 4.3.13.

The addition of vector fields of the form $A \circ \theta_0$ in Proposition 4.3.11 may seem a bit arbitrary at first glance. To make sense of it, we should think of the Maurer–Cartan locus as the C^∞-scheme cut out by θ_0. The following is only meaningful in a finite-dimensional context, but also serves as a motivation for the general case.

We can declare that the space of "smooth functions" on $\mathrm{MC}(E,\theta,\rho)$ should consist of equivalence classes of functions on M, where two functions are identified whenever their difference is an element of the ideal of functions of the form $A \circ \theta_0$ for a linear form $\xi\colon E^1 \to \mathbf{R}_M$. Derivations of such functions can then be described as elements of the quotient of the space vector fields on M that preserve this ideal by the submodule which consist of those vector fields that are of the form $A \circ \theta_0$ for a vector bundle morphism : $E^1 \to \mathrm{T}M$. Now, Proposition 4.3.11 simply states that vector field that can be represented as $\rho_1 \circ e$ for some smooth section $e \in \mathscr{E}^{-1}(M)$ form an involutive submodule. △

Remark 4.3.14.

Theorem 4.3.12 also holds for curved L_∞-algebroids of Banach type if one assumes that the kernels $\ker(d_{x_0}^n) \subseteq V^n$ and the images $\mathrm{im}(d_{x_0}^{n-1}) \subseteq \ker(d_{x_0}^n)$ are (closed) complemented subspaces for $n = -1, 0, 1$. The proof becomes significantly more involved, however, because being locally finitely generated is no longer a viable condition for integrability. One will instead have to use an alternative characterisation of integrability, such as Theorem 1 from [CS76]. We can verify that the conditions of this theorem are met by (1) describing the curved L_∞-algebroids as a holonomic family of curved L_∞-algebras using

Theorem 5.1.1, then (2) integrating homotopies to isomorphisms between the corresponding L_∞-algebras, (3) showing that these isomorphism describe the flow of vector fields that are tangent to the distribution. △

4.3.3. The Chevalley–Eilenberg algebra

Given a curved L_∞-algebroid (E, θ, ρ) of Fréchet type over M, we can consider the corresponding sheaf of graded symmetric multilinear forms on the underlying $\mathbf{Z}$-graded vector bundle $E \to M$. This sheaf carries a natural graded commutative algebra structure, as well as a differential that is induced by the L_∞-algebroid structure (θ, ρ). It is referred to as the Chevalley–Eilenberg algebra of the curved L_∞-algebroid, since its definition is analogous to that of the Chevalley–Eilenberg algebra of a Lie algebra (or Lie algebroid). The Chevalley–Eilenberg algebra is closely related to the interpretation of a curved L_∞-algebroid as a differential graded manifold, which we have alluded to before.

For finite-dimensional L_∞-algebroids whose underlying graded vector bundle is concentrated in negative degrees, the relation between the definition in terms of brackets and the description as a differential graded manifold is discussed in varying levels of detail in [SZ17], [Bru11], [BP13] and [Lav16]. The $\mathbf{Z}$-graded case is discussed in [Vit14] as a special case of the more general notion of a strong homotopy Lie-Rinehart algebra.

A graded symmetric p-form of degree d on the graded vector bundle $E \to M$ of Fréchet type is a graded symmetric multilinear vector bundle morphism $\varphi\colon \bigodot^p(E) \to \mathbf{R}_M$ of degree d from E to the trivial line bundle over M (which is concentrated in degree 0). A 0-form of degree 0 is thus just a smooth function on M and a 1-form of degree d is simply a vector bundle morphism from E^{-d} to $\mathbf{R}_M$. A general p-form of a given degree $d \in \mathbf{Z}$ can be described as a collection of vector bundle morphisms

$$\varphi^{n_1,\dots,n_p}\colon \textstyle\bigodot_{i=1}^p E^{n_i} \longrightarrow \mathbf{R}_M$$

indexed by (non-decreasing) sequences of degrees $n_1 \le n_2 \le \dots \le n_p \in \mathbf{Z}$ with the property that $\sum_{i=1}^p n_i = -d$. We denote the sheaf of such forms by $\Omega^d_{E,p}$.

A homogeneous plurilinear form on E can similarly be defined as either a homogeneous vector bundle morphism from $\bigodot(E)$ to $\mathbf{R}_M$ or as a sequence $(\varphi_p)_{p\in\mathbf{N}_0}$ of graded symmetric forms $\varphi_p\colon \bigodot^p(E) \to \mathbf{R}_M$ of the same degree indexed by arity. The sheaf of such forms is given by $\Omega_E = \bigoplus_{d\in\mathbf{Z}} \Omega_E^d$ with $\Omega_E^d = \prod_{p\in\mathbf{N}_0} \Omega_{E,p}^d$. This sheaf comes with a canonical graded commutative algebra structure, as the following lemma shows.

Lemma 4.3.15.
For any $\mathbf{Z}$*-graded vector bundle* $E \to M$ *of Fréchet type and any open subset* $U \subseteq M$*, the locally convex vector space* $\Omega_E(U)$ *admits a topological graded commutative unital algebra structure which is given by* $\varphi \cdot \psi = (\varphi \otimes \psi) \circ \Delta$*, or*

$$(\varphi \cdot \psi)(x)(e_{\{1,\dots,p\}}) = \sum_{I\sqcup J=\{1,\dots,p\}} (-1)^{|\psi||e_I|} \epsilon_{I,J}\, \varphi(x)(e_I)\, \psi(x)(e_J)$$

at a point $x \in M$ *for any two homogeneous forms* $\varphi, \psi \in \Omega_E(U)$ *defined on a neighbourhood* U *of* x*, and arbitrary homogeneous elements* $e_1, e_2, \dots, e_p \in E_x$ *of the corresponding fibre.*

Proof: Since Ω_E is a sheaf of locally convex vector spaces, it is sufficient to demonstrate these properties on arbitrarily small neighbourhoods. We shall therefore assume that M is an open subset of a Fréchet space T and that $E \to M$ is the trivial bundle with fibre V. Then $\Omega_E(U)$ for $U \subseteq M$ open is isomorphic to the product

$$\prod_{n_1\le\cdots\le n_p} \mathrm{C}^\infty\big(U, \big(\textstyle\bigodot_{i=1}^p E^{n_i}, \mathbf{R}\big)^\vee\big)$$

over non-decreasing sequences of integers $n_1, n_2, \dots, n_p \in \mathbf{Z}$. Since this is a product, it is sufficient to demonstrate continuity of the composition of the multiplication map with the projection onto $\mathrm{C}^\infty(U, \mathrm{L}(\bigodot_{i=1}^p E^{n_i}, \mathbf{R}))$ for an arbitrary such sequence.

The multiplication map described in this lemma is defined pointwise. Its restriction to $\mathrm{L}\big(\bigodot_{i=1}^p V^{n_i}\big)$ and $\mathrm{L}\big(\bigodot_{i=p+1}^r V^{n_i}\big)$ factors as

$$\big(\textstyle\bigodot_{i=1}^p V^{n_i}\big)^\vee \times \big(\bigodot_{i=p+1}^r V^{n_i}\big)^\vee \xrightarrow{\chi} \big(\bigodot_{i=1}^p V^{n_i} \otimes \bigodot_{i=p+1}^r V^{n_i}\big)^\vee \xrightarrow{\Delta^\vee} \big(\bigodot_{i=1}^r V^{n_i}\big)^\vee,$$

where χ is the tensor product map from Definition A.2.35 and $\Delta^\vee$ is the transpose of the coalgebra structure Δ on $\bigodot(V)$. The first map is continuous by definition, and continuity of the second follows from Proposition A.2.30. It now follows from Proposition A.2.18 that the operation described above defines a continuous map from $\bigodot(V)^\vee \times \bigodot(V)^\vee$ to $\bigodot(V)^\vee$, and Lemma B.2.4 subsequently tells us that it is also continuous as bilinear map from $\mathrm{C}^\infty(U, \bigodot(V)^\vee)$ to itself. □

We observe that for any open subset $U \subseteq M$, the smooth functions on U with values in $\mathbf{R}$ form a unital subalgebra $\mathscr{C}^\infty_M(U) \leq \Omega_E(U)$. The induced $\mathscr{C}^\infty_M$-module structure on Ω_E is, as one might expect, such that $(f \cdot \psi)(x) = f(x)\psi(x)$ for any $x \in X$ and any two sections $f \in \mathscr{C}^\infty_M(U)$ and $\psi \in \Omega_E(U)$ defined on a neighbourhood $U \subseteq M$ of x. We can thus view Ω_E as a sheaf of locally convex $\mathscr{C}^\infty_M$-algebras.

Definition 4.3.16.
A derivation on Ω_E of degree d is a continuous operator $Q\colon \Omega_E \to \Omega_E$ which is homogeneous of degree d and is such that

$$Q(\varphi \cdot \psi) = Q(\varphi) \cdot \psi + (-1)^{|Q||\varphi|}\varphi \cdot Q(\psi)$$

for any two homogeneous forms $\varphi, \psi \in \Omega_E(U)$ with $U \subseteq M$ open.

Even though we refer to such operators as derivations, they need not necessarily be smooth derivations in the sense of Definition 4.2.1. One can readily verify that the graded commutator $[Q, Q'] = Q \circ Q' - (-1)^{|Q||Q'|}Q' \circ Q$ of two derivations on Ω_E is again a derivation, which means that the space of all such derivations is a graded Lie algebra.

Proposition 4.3.17.
Given a $\mathbf{Z}$-graded vector bundle $E \to M$ of Fréchet type and let $(\theta, \rho_\theta) \in \widehat{\mathrm{Der}}(E)$ be a smooth homogeneous pluriderivation on E, then the operator $Q_\theta \colon \Omega_E \to \Omega_E$ given by

$$(-1)^{|\theta||\varphi|}\langle Q_\theta(\varphi), e_{\{1,\dots,p\}}\rangle = \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J} \mathscr{L}_{\rho(e_I)}\langle \varphi, e_J\rangle - \langle \varphi, \bar{\theta}(e_{\{1,\dots,p\}})\rangle, \tag{4.3.3}$$

for $\varphi \in \Omega_E(U)$ with $U \subseteq M$ open and homogeneous $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$, is a derivation on Ω_E of degree $|\theta|$.

The map the map $\theta \mapsto Q_\theta$ that assigns the operator Q_θ to a homogeneous pluriderivation $\theta \in \widehat{\mathrm{Der}}(E)$ is an injective graded Lie algebra morphism from $\widehat{\mathrm{Der}}(E)$ to the graded Lie algebra of homogeneous derivations on Ω_E in the sense of Definition 4.3.16.

Proof: Let (θ, ρ) be a smooth homogeneous pluriderivation on the graded vector bundle $E \to M$. We first need to show that the operator Q_θ described in this proposition is well-defined, since this is a priori unclear from its implicit definition. To do this, we can assume without loss of generality that the bundle $E \to M$ is trivial and that $\theta = \nabla_\rho^{\mathrm{triv}} + A$ for some vector bundle morphism $A\colon \bigodot(E) \to E$. All terms involving derivatives of the sections on the right-hand side of the defining equation for Q_θ now cancel out, and we are left with

$$(-1)^{|\theta||\varphi|}\big\langle Q_\theta(\varphi), e_{\{1,\dots,p\}}\big\rangle = \sum_{I\sqcup J=\{1,\dots,p\}} \epsilon_{I,J}\big(\langle \mathscr{L}^{\Phi}_{\rho(e_I)}\varphi, e_J\rangle - \langle \varphi, A(e_I)\odot e_J\rangle\big).$$

At a point $x \in U$, this only depends on the values $e_i(x)$ for $i = 1, 2, \dots, p$, and it describes a smooth function for every sequence $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$ of homogeneous sections. Proposition 3.2.1 consequently tells us that $Q_\theta(\varphi)$ is a smooth graded symmetric plurilinear vector bundle morphism from E to $\mathbf{R}_M$.

To prove that Q_θ is a derivation, we can apply $Q_\theta(\varphi\cdot\psi)$ to homogeneous local sections of E and expand the resulting expression. We thus obtain

$$\begin{aligned}
&(-1)^{|\theta|(|\varphi|+|\psi|)}\big\langle Q_\theta(\varphi\cdot\psi), e_{\{1,\dots,p\}}\big\rangle\\
&\quad= \sum_{I\sqcup J=\{1,\dots,p\}} \epsilon_{I,J}\mathscr{L}_{\rho(e_I)}\langle\varphi\cdot\psi, e_J\rangle - \big\langle \varphi\cdot\psi, \bar\theta(e_{\{1,\dots,p\}})\big\rangle\\
&\quad= \sum_{I\sqcup J\sqcup K=\{1,\dots,p\}} (-1)^{|\psi||e_I|}\epsilon_{I,J,K}\Big(\big(\mathscr{L}_{\rho(e_I)}\langle\varphi, e_J\rangle\big)\langle\psi, e_K\rangle\\
&\qquad\qquad + \langle\varphi, e_J\rangle\big(\mathscr{L}_{\rho(e_I)}\langle\psi, e_K\rangle\big)\Big)\\
&\qquad - \sum_{I\sqcup J=\{1,\dots,p\}} \epsilon_{I,J}\Big((-1)^{|\psi|(|e_I|+|\theta|)}\langle\varphi, \bar\theta(e_I)\rangle\,\langle\psi, e_J\rangle\\
&\qquad\qquad + (-1)^{(|\psi|+|\theta|)|e_I|}\langle\varphi, e_I\rangle\,\langle\psi, \bar\theta(e_J)\rangle\Big)\\
&\quad= \sum_{I\sqcup J=\{1,\dots,p\}} \epsilon_{I,J}(-1)^{(|\psi|+|\theta|)|e_I|+|\psi|}\langle\varphi, e_I\rangle\,\langle Q_\theta(\psi), e_J\rangle\\
&\qquad\qquad + (-1)^{|\varphi|+|\psi|(|e_I|+|\theta|)}\langle Q_\theta(\varphi), e_I\rangle\,\langle\psi, e_J\rangle\\
&\quad= (-1)^{|\theta|(|\varphi|+|\psi|)}\big\langle Q_\theta(\varphi)\cdot\psi + (-1)^{|\theta||\varphi|}\varphi\cdot Q_\theta(\psi), e_{\{1,\dots,p\}}\big\rangle
\end{aligned}$$

for arbitrary homogeneous local sections $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$ with $U \subseteq M$ open.

To show that $\theta \mapsto Q_\theta$ is a graded Lie algebra morphism, we can similarly work out and compare $[Q_\theta, Q_\eta]$ and $Q_{[\theta,\eta]}$ for two homogeneous pluriderivations

(θ,ρ_θ) and $(\eta,\rho_\eta)\in\widehat{\mathrm{Der}}(E)$. The former is given by

$$\begin{aligned}\pm\big\langle[Q_\theta,Q_\eta](\varphi),e_{\{1,\dots,p\}}\big\rangle &= \sum_{I\sqcup J\sqcup K}\epsilon_{I,J,K}\,[\mathscr{L}_{\rho_\theta(e_I)},\mathscr{L}_{\rho_\eta(e_J)}]\langle\varphi,e_K\rangle\\ &\quad-\sum_{I\sqcup J}\epsilon_{I,J}\left(\mathscr{L}_{\rho_\eta(\bar\theta(e_I))}-(-1)^{|\theta||\eta|}\mathscr{L}_{\rho_\theta(\bar\eta(e_I))}\right)\langle\varphi,e_J\rangle\\ &\quad+\sum_{I\sqcup J}\epsilon_{I,J}\,\langle\varphi,[\eta,\theta](e_I)\odot e_J\rangle,\end{aligned}$$

where $\pm$ denotes the sign $(-1)^{|\eta||\varphi|+(|\varphi|+|\eta|)|\theta|}$, and the latter is given by

$$\begin{aligned}(-1)^{|[\theta,\eta]||\varphi|}\langle Q_{[\theta,\eta]}(\varphi),e_{\{1,\dots,p\}}\rangle &= \sum_{I\sqcup J}\epsilon_{I,J}\,\mathscr{L}_{\rho_{[\theta,\eta]}(e_I)}\langle\varphi,e_J\rangle\\ &\quad-\langle\varphi,\overline{[\theta,\eta]}_{\mathrm{mor}}(e_{\{1,\dots,p\}})\rangle\end{aligned}$$

for a homogeneous form $\varphi\in\Omega_E(U)$ and homogeneous $e_1,e_2,\dots,e_p\in\mathscr{E}(U)$ with $U\subseteq M$ open. The anchor $\rho_{[\theta,\eta]}$ was worked out in Proposition 4.2.19. By comparing these two expressions, we deduce that $[Q_\theta,Q_\eta]$ and $Q_{[\theta,\eta]}$ are in fact equal.

To show that the assignment $\theta\mapsto Q_\theta$ is injective, suppose that (θ,ρ) is a homogeneous smooth pluriderivation with the property that $Q_\theta=0$. By applying Q_θ to a rank 0 form $f\in\mathscr{C}^\infty_M(U)\subseteq\Omega_E(U)$ with $U\subseteq M$ open, we find that

$$\mathscr{L}_{\rho(e_{\{1,\dots,p\}})}f=\langle Q_\theta(f),e_{\{1,\dots,p\}}\rangle=0$$

for all sections $e_1,e_2,\dots,e_p\in\mathscr{E}(U)$. Since this holds for any smooth function $f\in\mathscr{C}^\infty_M(U)$, it follows that the anchor ρ is necessarily trivial. Using this, we can derive that

$$\langle\varphi,\theta(e_{\{1,\dots,p\}})\rangle=-(-1)^{|\theta||\varphi|}\langle Q_\theta(\varphi),e_{\{1,\dots,p\}}\rangle=0$$

for any homogeneous rank 1 form $\varphi\in\Omega_{E,1}(U)$ and $e_1,e_2,\dots,e_p\in\mathscr{E}(U)$. We therefore conclude that $\theta=0$ whenever $Q_\theta=0$, and that the map $\theta\mapsto Q_\theta$ is thus indeed injective. □

Proposition 4.3.17 tells us that every smooth pluriderivation on a **Z**-graded vector bundle $E\to M$ of Fréchet type is uniquely determined by the corresponding derivation $Q_\theta\colon\Omega_E\to\Omega_E$. It also leads to the following observation.

Corollary 4.3.18.
A degree 1 *pluriderivation* (θ, ρ) *on a* **Z***-graded vector bundle* $E \to M$ *is a curved* L_∞*-algebroid structure if and only if* $Q_\theta \circ Q_\theta = 0$.

Proof: Proposition 4.3.17 tells us that $Q_\theta \circ Q_\theta = \frac{1}{2}[Q_\theta, Q_\theta]$ is equal to $\frac{1}{2}Q_{[\theta,\theta]}$, which vanishes if and only if $[\theta, \theta] = 0$ due to the injectivity of the assigment $\theta' \mapsto Q_{\theta'}$. Consequently, Q_θ squares to zero precisely when θ satisfies the Jacobi identities. □

If (E, θ, ρ) is a curved split L_∞-algebroid of Fréchet type, then (Ω_E, Q_θ) is a differential graded commutative algebra called the *Chevalley–Eilenberg algebra*. The differential Q_θ is referred to as the *Chevalley–Eilenberg differential*.

Example 4.3.19 (Manifolds).
Any smooth manifold M can be viewed as a curved L_∞-algebroid by endowing it with the trivial vector bundle $E = 0$. Its Chevalley–Eilenberg algebra is simply the sheaf $\Omega_E = \mathscr{C}^\infty_M$ of smooth functions on M, which is concentrated in degree 0. It is endowed with the usual commutative unital algebra structure and the trivial differential $Q = 0$. ◊

Example 4.3.20 (The tangent bundle).
If $E = \mathrm{T}[1]M \to M$ is the tangent bundle of M, shifted down so that it is concentrated in degree -1, then $\Omega_{\mathrm{T}[1]M} = \Omega_M(\mathbf{R})$ is just the sheaf of differential forms on M. The algebra structures on $(\Omega_{\mathrm{T}[1]M}, \cdot)$ and $(\Omega_M(\mathbf{R}), \wedge)$ are opposite in the sense that $\varphi \cdot \psi = \psi \wedge \varphi$ for any two homogeneous forms $\varphi, \psi \in \Omega_{\mathrm{T}[1]M}(U)$ with a common domain $U \subseteq M$. Also the differential on $\Omega_{\mathrm{T}[1]M}$ differs from the usual exterior derivative on Ω_M by a sign: it is given by $Q_{[\cdot,\cdot]}(\varphi) = (-1)^{|\varphi|}\mathrm{d}\varphi$.

Despite this discrepancy in signs, the differential graded commutative algebras $(\Omega_M, \wedge, \mathrm{d})$ and $(\Omega_{\mathrm{T}[1]M}, \cdot, Q_{[\cdot,\cdot]})$ are isomorphic. An explicit isomorphism between the two is given by the operator

$$\Omega_{\mathrm{T}[1]M} \longrightarrow \Omega_M(\mathbf{R}), \qquad \varphi \mapsto (-1)^{\frac{1}{2}|\varphi|(|\varphi|-1)}\varphi,$$

which sends a product $\varphi_1 \varphi_2 \cdots \varphi_p$ of rank 1 forms $\varphi_1, \varphi_2, \ldots, \varphi_p \in \Omega_{\mathrm{T}[1]M,1}$ to the opposite product $\varphi_p \varphi_{p-1} \cdots \varphi_1 = (-1)^{\frac{1}{2}|\varphi|(|\varphi|-1)}\varphi_1 \varphi_2 \cdots \varphi_p$. ◊

Example 4.3.21 (Lie algebroids).
Given a Lie algebroid $(A, [\![\cdot,\cdot]\!], \rho)$, we can consider the associated split L_∞-algebroid $(A[1], \downarrow\circ[\![\uparrow\cdot,\uparrow\cdot]\!], \rho\circ\uparrow)$ from Example 4.3.4. Its Chevalley–Eilenberg algebra, as we have just defined it, is canonically isomorphic to the usual Chevalley–Eilenberg algebra for Lie algebroids. The isomorphism is similar to the morphism described in Example 4.3.20. ◊

The following proposition shows that graded symmetric multilinear vector bundle morphisms induce morphisms between the associated sheaves of algebras.

Proposition 4.3.22.
Let $E \to M$ and $F \to N$ be two $\mathbf{Z}$-graded vector bundles of Fréchet type and let $(\alpha, \phi)\colon \bigodot(E) \to F$ be a morphism of graded vector bundles of degree 0 such that $\alpha(1) = 0$. The operator $\alpha^*\colon \Omega_F \to \Omega_E$ given by $\alpha^*\varphi = \varphi \diamond \alpha$, or

$$\langle \alpha^*(\varphi), e_{\{1,\dots,p\}}\rangle(x) = \sum_{I_1,\dots,I_m} \epsilon_{I_1,\dots,I_m} \left\langle \varphi(f(x)), \textstyle\bigodot_{j=1}^m \alpha(e_{I_j}(x))\right\rangle$$

for $\varphi \in \Omega_F(U)$, homogeneous $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$ with $U \subseteq M$ open and $x \in U$, is a morphism of graded commutative algebras.

Proof: Smoothness of the composition $\varphi \diamond \alpha$ was shown in Proposition 3.2.11. That α^* is an algebra morphism follows from the fact that $\varphi \cdot \psi = (\varphi \otimes \psi) \circ \Delta$, since this tells us that

$$\alpha^*(\varphi \cdot \psi) = (\varphi \otimes \psi) \circ \Delta \circ \bar{\alpha}_{\mathrm{mor}}$$

and

$$(\alpha^*\varphi) \cdot (\alpha^*\psi) = (\varphi \otimes \psi) \circ (\bar{\alpha}_{\mathrm{mor}} \otimes \bar{\alpha}_{\mathrm{mor}}) \circ \Delta$$

for any two forms $\varphi, \psi \in \Omega_E(U)$ with $U \subseteq M$ open. These two expressions coincide because $\bar{\alpha}_{\mathrm{mor}}$ is a coalgebra morphism. □

Remark 4.3.23.
A finite-dimensional graded manifold is a locally ringed space $(M, \mathscr{O})$ which is locally isomorphic to (U, Ω_E) for some domain $U \subseteq \mathbf{R}^n$ with $n \in \mathbf{N}_0$ and a (trivial) $\mathbf{Z}$-graded vector bundle $E \to M$ of finite-dimensional type for which $E^0 = 0$. A homogeneous derivation $Q\colon \mathscr{O} \to \mathscr{O}$ is called a homological vector field if $[Q, Q] = 0$, and a graded manifold endowed with a homological vector field of degree 1 is called a *differential graded manifold*.

It follows from the preceding discussion that the triple (M, Ω_E, Q_θ) associated to any curved L_∞-algebroid (E, θ, ρ) of finite-dimensional type over a manifold M is a differential graded manifold. Every non-negatively graded differential graded manifold (by which we mean that $\mathscr{O}$ is non-negatively graded) is in fact of this form for some negatively graded curved L_∞-algebroid (E, θ, ρ). This is known as *Batchelor's theorem*, and it was first proven for $\mathbf{Z}_2$-graded manifolds in [Bat79]. A proof for positively graded differential graded manifolds can for instance be found in [BP13], but its $\mathbf{Z}$-graded analogue does not hold.

△

4.3.4. Morphisms of L_∞-algebroids

Before discussing general morphism, we will first describe morphisms between curved L_∞-algebroids over a fixed base manifold that cover the identity map on that manifold. These are relatively easy to define and help motivate the general definition. We then describe morphisms that cover arbitrary maps and conclude by defining what it means for a morphisms to be splitting preserving.

Base-preserving L_∞-algebroid morphisms

Base-preserving morphisms are relatively simple to describe, since they are simply plurilinear maps of vector bundles that are compatible with the curved L_∞-algebra structure their spaces of sections.

Definition 4.3.24.
Let (E, θ, ρ) and (E', θ', ρ') be two curved split L_∞-algebroids of Fréchet type over the same base manifold M. A base preserving *morphism* from (E, θ, ρ) to (E', θ', ρ') is a base-preserving graded vector bundle morphism $\alpha\colon \bigodot(E) \to E'$ of degree 0 such that $\alpha(1) = 0$ with the following properties:

1. **compatibility with the brackets:** it satisfies $\theta' \diamond \alpha = \alpha \mathbin{\square} \theta$, i.e.

$$\sum_{I_1,\ldots,I_m} \epsilon_{I_1,\ldots,I_m} \tfrac{1}{m!}\, \theta'\big(\textstyle\bigodot_{j=1}^{m} \alpha(e_{I_j})\big) = \sum_{I \sqcup J = \{1,\ldots,p\}} \epsilon_{I,J}\, \alpha(\theta(e_I) \odot e_J)$$

for all homogeneous $e_1, e_2, \ldots, e_p \in \mathscr{E}(U)$ with $U \subseteq M$ open;

2. **compatibility with the anchors:** it satisfies $\rho' \diamond \alpha = \rho$, i.e.

$$\sum_{I_1,\dots,I_m} \epsilon_{I_1,\dots,I_m} \tfrac{1}{m!}\, \rho'\big(\textstyle\bigodot_{j=1}^m \alpha(e_{I_j})\big) = \rho(e_{\{1,\dots,p\}})$$

for all homogeneous $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$ with $U \subseteq M$ open.

Given two base-preserving curved split L_∞-algebroids $\alpha\colon (E,\theta,\rho) \to (E',\theta',\rho')$ and $\beta\colon (E',\theta',\rho') \to (E'',\theta'',\rho'')$ with a common base M, one can readily verify that also the composition $\beta \diamond \alpha\colon E \to E$ is a morphism of curved split L_∞-algebroids from (E,θ,ρ) to (E'',θ'',ρ'').

If we recall the definition of a curved L_∞-algebra morphism, Definition 2.2.8, we see that the first property in Definition 4.3.24 simply states that the induced map

$$\alpha_*\colon \bigodot(\mathscr{E}(U)) \longrightarrow \mathscr{F}(U), \qquad e_{\{1,\dots,p\}} \mapsto \alpha \circ e_{\{1,\dots,p\}}$$

is a morphism of curved L_∞-algebras from $(\mathscr{E}(U),\theta)$ to $(\mathscr{F}(U),\theta')$ for every open subset $U \subseteq M$. The second property is automatically satisfied at every point where the linear component $\alpha_1\colon E \to E'$ is non-trivial, which is in particular true whenever α is invertible.

Lemma 4.3.25.
For any curved split L_∞-algebroid (E,θ,ρ) of Fréchet type with base M, the map $\downarrow\!\rho\colon \bigodot(E) \to \mathrm{T}[1]M$ is a base-preserving morphism of curved L_∞-algebroids to the shifted tangent bundle $(\mathrm{T}[1]M, [\cdot,\cdot], \uparrow)$.

Proof: The first condition in Definition 4.3.24 follows from Proposition 4.2.19. Since $[\theta,\theta] = 0$, this proposition tells us that $\rho_{[\theta,\theta]} = [\rho_\theta,\rho_\theta] + 2\rho_\theta \,\square\, \theta$ vanishes, and that therefore

$$\sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J}\, 2\rho(\theta(e_I) \odot e_J) = \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J}\, [\rho(e_I),\rho(e_J)]$$

for homogeneous $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$. Compatibility of $\downarrow\!\rho$ with the anchors ρ and id_{TM} is tautological, as it is expressed by the equation $\mathrm{id}_{TM} \circ \rho = \rho$. □

Given any base-preserving morphism α of curved L_∞-algebroids from (E,θ,ρ) to (E',θ',ρ'), Lemma 4.3.25 provides a diagram of curved L_∞-algebroid morphisms,

$$\begin{array}{ccc} (E,\theta,\rho) & \xrightarrow{\alpha} & (E',\theta',\rho') \\ & \searrow{\scriptstyle\downarrow\rho} \quad {\scriptstyle\downarrow\rho'}\swarrow & \\ & (\mathrm{T}[1]M,[\cdot,\cdot],\mathrm{id}_{TM}). & \end{array}$$

Commutativity of this diagram is precisely the second property from Definition 4.3.24.

Definition 4.3.24 can be reformulated rather concisely in terms of the dual description of a curved L_∞-algebroid through its Chevalley–Eilenberg algebra. We note that every homogeneous graded vector bundle morphism $\alpha\colon \bigodot(E)\to F$ induces a morphism

$$\alpha^*\colon \Omega_F \longrightarrow \Omega_E, \qquad \varphi\mapsto\varphi\diamond\alpha$$

between the corresponding sheaves of plurilinear forms. It is not difficult to see that this operator is in fact a morphism of sheaves of associative algebras, since

$$\alpha^*(\varphi\cdot\psi)=(\varphi\otimes\psi)\circ\Delta\circ\bar\alpha_{\mathrm{mor}}=(\varphi\otimes\psi)\circ(\bar\alpha_{\mathrm{mor}}\otimes\bar\alpha_{\mathrm{mor}})\circ\Delta=\alpha^*\varphi\cdot\alpha^*\psi$$

for any two forms $\varphi,\psi\in\Omega_F(U)$ defined on an open subset $U\subseteq N$.

Proposition 4.3.26.
Let (E,θ,ρ) and (F,θ',ρ') be two curved split L_∞-algebroids over M and let $\alpha\colon \bigodot(E)\to F$ be a degree 0 graded vector bundle morphism covering id_M. Then α is a morphism of curved L_∞-algebroids if and only if $\alpha^*\circ Q_{\theta'}=Q_\theta\circ\alpha^*$.

Proof: Let $\varphi\in\Omega_F(U)$ be a homogeneous plurilinear form defined on $U\subseteq M$ and let $e_1,e_2,\dots,e_p\in\mathscr{E}(U)$ denote homogeneous local sections with the same domain. If we apply $\alpha^*\circ Q_{\theta'}$ and $Q_\theta\circ\alpha^*$ to these, we obtain the expressions

$$\begin{aligned}(-1)^{|\varphi|}\langle\alpha^*\circ Q_{\theta'}(\varphi),e_{\{1,\dots,p\}}\rangle=\sum_{I\sqcup J=\{1,\dots,p\}}&\epsilon_{I,J}\,\mathscr{L}_{\rho'(\bar\alpha_{\mathrm{mor}}(e_I))}\langle\varphi,\bar\alpha_{\mathrm{mor}}(e_J)\rangle,\\ &+\langle\varphi,\bar\theta'_{\mathrm{cod}}\circ\bar\alpha_{\mathrm{mor}}(e_{\{1,\dots,p\}})\rangle\end{aligned}$$

and

$$(-1)^{|\varphi|}\langle Q_\theta \circ \alpha^*(\varphi), e_{\{1,\dots,p\}}\rangle = \sum_{I\sqcup J=\{1,\dots,p\}} \epsilon_{I,J}\, \mathscr{L}_{\rho(e_I)}\langle \varphi, \bar{\alpha}_{\mathrm{mor}}(e_J)\rangle + \langle \varphi, \bar{\alpha}_{\mathrm{mor}} \circ \bar{\theta}_{\mathrm{cod}}(e_{\{1,\dots,p\}})\rangle$$

respectively. For $\varphi \in \mathrm{C}^\infty(U) \subseteq \Omega_F(U)$, the identity $\alpha^* \circ Q_{\theta'}(\varphi) = Q_\theta \circ \alpha^*(\varphi)$ is thus equivalent to the equation

$$\mathscr{L}_{\rho'(\bar{\alpha}_{\mathrm{mor}}(e_{\{1,\dots,p\}}))}\varphi = \mathscr{L}_{\rho(e_{\{1,\dots,p\}})}\varphi.$$

Since a tangent vector is determined by the corresponding derivation, this holds for all $\varphi \in \mathrm{C}^\infty(U)$ with $U \subseteq M$ open if and only if $\rho'(x) \diamond \alpha(x) = \rho(x)$ for all $x \in U$.

Once we know that $\rho' \diamond \alpha = \rho$, the identity $Q_{\theta'} \circ \alpha^* = \alpha^* \circ Q_\theta$ reduces to

$$\langle \varphi, \bar{\theta}'_{\mathrm{cod}} \circ \bar{\alpha}_{\mathrm{mor}}(e_{\{1,\dots,p\}})\rangle = \langle \varphi, \bar{\alpha}_{\mathrm{mor}} \circ \bar{\theta}_{\mathrm{cod}}(e_{\{1,\dots,p\}})\rangle$$

for $\varphi \in \Omega_F(U)$ and $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$ with $U \subseteq M$ open. We conclude that the identity $\alpha^* \circ Q_{\theta'} = Q_\theta \circ \alpha^*$ holds if and only if both of the equations $\rho' \diamond \alpha = \rho$ and $\alpha \mathbin{\scriptstyle\square} \theta = \theta' \diamond \alpha$ are satisfied. □

This equivalent characterisation of curved L_∞-algebroid morphisms will be the basis for the general definition.

Morphisms covering smooth maps

While Definition 4.3.24 works well for morphisms of curved split L_∞-algebroids covering diffeomorphisms, it cannot easily be adapted to describe curved split L_∞-algebroid morphisms that cover a general smooth map. The problem is that a general vector bundle morphism from $\bigodot(E) \to M$ to $F \to N$ does not necessarily induce a map between the corresponding spaces of sections, which makes compatibility with the brackets difficult to state.

A simple way to circumvent this problem is by using the dual characterisation of curved split L_∞-algebroid morphisms from Proposition 4.3.26. This leads us to the following definition.

Definition 4.3.27.

Let (E, θ, ρ) and (F, θ', ρ') be two curved split L_∞-algebroids of Fréchet type over M and N respectively and let $f : M \to N$ be a smooth map. A *morphism* of curved L_∞-algebroids from (E, θ, ρ) to (F, θ', ρ') covering f is a graded vector bundle morphism $(\alpha, f)\colon \bigodot(E) \to F$ of degree 0 such that $\alpha^* \circ Q_{\theta'} = Q_\theta \circ \alpha^*$.

Given two morphisms $(\alpha, f)\colon (E, \theta, \rho) \to (F, \theta', \rho')$ and $(\beta, g)\colon (F, \theta', \rho') \to (G, \theta'', \rho'')$ between curved split L_∞-algebroids, we define their composition as the vector bundle morphism $(\beta \diamond \alpha, g \circ f)\colon \bigodot(E) \to G$ that covers $g \circ f$ and is given by $(\beta \diamond \alpha)_x = \beta_{f(x)} \diamond \alpha_x$ for $x \in M$. In other words, it is such that

$$(\beta \diamond \alpha)_x(e_1, \ldots, e_p) = \sum_{I_1,\ldots,I_m} \epsilon_{I_1,\ldots,I_m} \tfrac{1}{m!}\, \beta_{f(x)}\big(\textstyle\bigodot_{j=1}^m \alpha_x(e_{I_j})\big) \in G_{g(f(x))}$$

for all $x \in M$ and all homogeneous $e_1, e_2, \ldots, e_m \in E_x$. This composition is also characterised by the property that $(\beta \diamond \alpha)^* = \alpha^* \circ \beta^*$, and it is thus itself a morphism of curved L_∞-algebroids from (E, θ, ρ) to (G, θ'', ρ''). That Definition 4.3.27 and Definition 4.3.24 agree for morphisms that cover the identity map id_M on a manifold M was demonstrated in Proposition 4.3.26 above.

We will see two equivalent formulations of this definition in Proposition 4.3.30 and Corollary 4.3.32 below. Before we get to those, however, we shall verify that any morphism of curved L_∞-algebroids is compatible with the associated anchor maps.

Lemma 4.3.28.

Let (E, θ, ρ) and (F, θ', ρ') be two curved split L_∞-algebroids of Fréchet type over M and N respectively. The identity $\rho' \diamond \alpha = \mathrm{T}f \circ \rho$ holds for every morphism (α, f) of curved L_∞-algebroids from (E, θ, ρ) to (F, θ', ρ').

Proof: Let $U' \subseteq N$ be an open subset and let $\varphi \in \mathrm{C}^\infty(U') \subseteq \Omega_F(U')$ be a smooth function on U'. We will show that $\rho' \diamond \alpha = \mathrm{T}f \circ \rho$ by applying both sides of the equation $Q_\theta \circ \alpha^*(\varphi) = \alpha^* \circ Q_{\theta'}(\varphi)$ to a sequence of homogeneous local sections $e_1, e_2, \ldots, e_p \in \mathscr{E}(U)$ with $U \subseteq f^{-1}(U')$ and comparing the resulting expressions.

For the left-hand side of this equation, inserting $e_1, e_2, \ldots, e_p \in \mathscr{E}(U)$ produces the expression.

$$\begin{aligned}\langle Q_\theta \circ \alpha^*(\varphi), e_{\{1,\ldots,p\}}\rangle &= \sum_{I \sqcup J = \{1,\ldots,p\}} \epsilon_{I,J}\, \mathscr{L}_{\rho(e_I)} \langle \alpha^* \varphi, e_J\rangle - \langle \alpha^*\varphi, \bar{\theta}_{\mathrm{cod}}(e_{\{1,\ldots,p\}})\rangle \\ &= \sum_{I \sqcup J = \{1,\ldots,p\}} \epsilon_{I,J}\, \mathscr{L}_{\rho(e_I)} \langle f^*\varphi, \bar{\alpha}^!_{\mathrm{mor}}(e_J)\rangle - \langle f^*\varphi, \bar{\alpha}^!_{\mathrm{mor}} \circ \bar{\theta}_{\mathrm{cod}}(e_{\{1,\ldots,p\}})\rangle \\ &= \mathscr{L}_{\rho(e_{\{1,\ldots,p\}})} f^*\varphi = \mathscr{L}_{\mathrm{T}f \circ \rho(e_{\{1,\ldots,p\}})}\varphi,\end{aligned}$$

where $\alpha^!$ denotes the canonical vector bundle morphism from $\bigodot(E)$ to the pullback bundle f^*F associated with α.

If we similarly apply $\alpha^* \circ Q_{\theta'}(\varphi)$ to these local sections, we obtain

$$\langle \alpha^*(\varphi) \circ Q_{\theta'}(\varphi), e_{\{1,\ldots,p\}}\rangle = \left\langle f^* Q_{\theta'}(\varphi), \bar{\alpha}^!_{\mathrm{mor}}(e_{\{1,\ldots,p\}})\right\rangle = \mathscr{L}_{\rho'(\bar{\alpha}_{\mathrm{mor}}(e_{\{1,\ldots,p\}}))}\varphi.$$

To obtain this last expression, we have used that $Q_{\theta'}$ is, more or less by definition, such that $Q_{\theta'}(\varphi) = \mathrm{d}\varphi \circ \rho'$ for $\varphi \in \mathrm{C}^\infty(U')$. Since the function φ and the local sections $e_1, e_2, \ldots, e_p$ were arbitrary, we may now conclude that $\mathrm{T}f \circ \rho = \rho' \diamond \alpha$ whenever $Q_\theta \circ \alpha^* = \alpha^* \circ Q_{\theta'}$. □

We infer from Lemma 4.3.28 that every morphism $(\alpha, f)\colon (E, \theta, \rho) \to (F, \theta', \rho')$ of curved split L_∞-algebroids is part of a commutative diagram of morphisms

$$\begin{array}{ccccc} (E,\theta,\rho_\theta) & \xrightarrow{(\rho_\theta, \mathrm{id}_M)} & (\mathrm{T}[1]M, [\cdot,\cdot], \mathrm{id}_{\mathrm{T}M}) & \xrightarrow{(0,\mathrm{id}_M)} & (M,0,0) \\ \downarrow{\scriptstyle (\alpha,f)} & & \downarrow{\scriptstyle (\mathrm{T}f,f)} & & \downarrow{\scriptstyle (0,f)} \\ (F,\theta',\rho_{\theta'}) & \xrightarrow{(\rho\theta', \mathrm{id}_N)} & (\mathrm{T}[1]N, [\cdot,\cdot], \mathrm{id}_{\mathrm{T}N}) & \xrightarrow{(0,\mathrm{id}_N)} & (N,0,0) \end{array}$$

of curved split L_∞-algebroids. Here $(M, 0, 0)$ and $(M, 0, 0)$ denote the trivial L_∞-algebroid structures on the trivial vector bundles $M \to M$ and $N \to N$ of rank 0. The duals of the vector bundle morphism α and the anchors ρ and ρ' are similarly part of a commutative diagram of morphisms of differential graded commutative unital algebras,

$$\begin{array}{ccccc}
(\Omega_E(U), Q_\theta) & \xleftarrow{\rho_\theta^*} & (\Omega_{\mathrm{T}[1]M}(U), Q_{[\cdot,\cdot]}) & \hookleftarrow & (\mathscr{C}_M^\infty(U), 0), \\
\uparrow \alpha^* & & \uparrow \mathrm{T}f^* & & \uparrow f^* \\
(\Omega_F(U'), Q_{\theta'}) & \xleftarrow{\rho_{\theta'}^*} & (\Omega_{\mathrm{T}[1]N}(U'), Q_{[\cdot,\cdot]}) & \hookleftarrow & (\mathscr{C}_N^\infty(U'), 0)
\end{array}$$

for any two open subsets $U' \subseteq N$ and $U \subseteq f^{-1}(U')$.

Example 4.3.29 (Lie algebroid morphisms).
Morphisms of Lie algebroids can be defined in several equivalent ways. One possible definition states that a morphism from $(A, [\![\cdot,\cdot]\!], \rho)$ to $(A', [\![\cdot,\cdot]\!]', \rho')$ is as a vector bundle morphism (α, f) from A to A' with the property that induces map $\alpha^*\colon \Omega_{A'} \to \Omega_A$ on the sheaves of alternating forms on A and A' commutes with the Chevalley–Eilenberg differentials d_A and $\mathrm{d}_{A'}$.

Despite the discrepancy in sign conventions, this definition is equivalent to Definition 4.3.27 if one interprets these Lie algebroids as curved split L_∞-algebroids as described Example 4.3.4 and uses the isomorphism from Example 4.3.21 to relate the usual Chevalley–Eilenberg algebras to those defined in section 4.3.3. Note that any morphism of curved L_∞-algebroids between curved split L_∞-algebroids concentrated in degree -1 necessarily only has a linear component due to homogeneity. ◊

While Definition 4.3.27 is certainly clean, it may not always be practical to work with. We would like to state that a plurilinear graded vector bundle morphism $\alpha\colon \bigodot(E) \to F$ is a morphism of curved L_∞-algebroids from (E, θ, ρ) to (E', θ', ρ') whenever $\alpha \mathbin{\square} \theta = \theta' \diamond \alpha$, but the right-hand side of this equation is not generally well-defined because α does not map sections of $\bigodot(E) \to M$ to local sections of $F \to N$ unless f is a local diffeomorphism.

The plurilinear vector bundle morphism $\alpha\colon \bigodot(E) \to F$ does factor through the pull-back bundle $f^*F \to M$. By this we mean that there exists a unique base preserving graded vector bundle morphism $\alpha^!\colon \bigodot(E) \to F$ for which the diagram

$$\begin{array}{ccccc} & & \overset{\alpha}{\frown} & & \\ \bigodot(E) & \xrightarrow[\alpha^!]{} & f^*F & \xrightarrow[\mathrm{id}_F]{} & F \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{\mathrm{id}_M} & M & \xrightarrow{f} & N \end{array}$$

is commutative, and that therefore $(\alpha, f) = (\mathrm{id}_F, f) \circ (\alpha^!, \mathrm{id}_M)$. If a smooth pluriderivation $\theta' \in \widehat{\mathrm{Der}}(F)$ is expressed as $\theta' = \nabla_{\rho'} + A'$ in terms of a smooth connection ∇ on F and a form $A'\colon \bigodot(F) \to F$, both A' and ∇ can be pulled back to f^*F separately.

Proposition 4.3.30.
Let $(E, \theta\rho)$ and (F, θ', ρ') be curved split L_∞-algebroids of Fréchet type over M and N respectively and let $(\alpha, f)\colon \bigodot(E) \to F$ be a smooth vector bundle morphism of degree 0. Given an open subset $U' \subseteq N$ and a connection ∇ on $F|_{U'}$ such that $\theta' = \nabla_{\rho'} + A'$, the restriction $\alpha|_{f^{-1}(U')}$ is a morphism of curved L_∞-algebroids if and only if $\rho' \diamond \alpha|_U = \mathrm{T}f \circ \rho|_U$ and

$$\big(\alpha^! \mathbin{\square} \theta - (f^*A') \diamond \alpha^!\big)(e_{\{1,\dots,p\}}) = \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J} (f^*\nabla)_{\rho(e_I)}\big(\alpha^!(e_J)\big) \qquad (4.3.4)$$

for all local sections $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$ with $U \subseteq f^{-1}(U')$ open.

Proof: Since the proposed conditions are local in nature, we can assume without loss of generality that N is a Fréchet domain and that the bundle $F \to N$ is trivial. Let F therefore be of the form $F = W \times N \to N$ for some **Z**-graded Fréchet space W and some Fréchet domain N. We will freely identify the spaces $\mathscr{F}(U')$ and $\mathrm{C}^\infty(U', W)$ for any open subset $U' \subseteq N$. We shall moreover assume that the condition $\rho' \diamond \alpha = \mathrm{T}f \circ \rho$ is satisfied since it is part of the description proposed in this proposition and Lemma 4.3.28 tells us that it also holds whenever (α, f) is a morphism of curved L_∞-algebroids.

Now (α, f) is a morphism of curved L_∞-algebroids if and only if $\alpha^* \circ Q_{\theta'} = Q_\theta \circ \alpha^*$, or

$$(-1)^{|\varphi|} \langle Q_\theta \circ \alpha^*(\varphi), e_{\{1,\dots,p\}} \rangle(x) = (-1)^{|\varphi|} \langle \alpha^* \circ Q_{\theta'}(\varphi), e_{\{1,\dots,p\}} \rangle(x) \qquad (4.3.5)$$

for every form $\alpha \in \Omega_F(U')$, all homogeneous sections $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$ and every point $x \in U$ with $U' \subseteq N$ and $U \subseteq f^{-1}(U')$ open. The global factor

$(-1)^{|\varphi|}$ has been included for convenience. To compare this condition to equation (4.3.4), we will first work out both sides of this equation separately.

If we work out the left-hand side of equation (4.3.5), we see that it is given by

$$\sum_{I\sqcup J=\{1,\ldots,p\}} \epsilon_{I,J}\, \mathscr{L}_{\rho(e_I)(x)}\langle f^*\varphi, \bar{\alpha}^!_{\mathrm{mor}} e_J\rangle - \langle \varphi(f(x)), \bar{\alpha}_{\mathrm{mor}}(x)\circ \bar{\theta}_{\mathrm{cod}}(e_{\{1,\ldots,p\}})(x)\rangle,$$

which can be worked out further using Lemma 4.3.31 below and a few results from section 3.1.2 to obtain the expression

$$\begin{aligned}\sum_{I\sqcup J=\{1,\ldots,p\}} &\epsilon_{I,J}\left\langle \mathscr{L}_{\mathrm{T}f(x)\rho(e_I)(x)}\varphi, \bar{\alpha}_{\mathrm{mor}}(x)(e_J(x))\right\rangle \\ &+ \sum_{I\sqcup J\sqcup K=\{1,\ldots,p\}} \epsilon_{I,J,K}\left\langle \varphi(f(x)), \mathscr{L}_{\rho(e_I)(x)}\big(\alpha^!(e_J)\big)\odot \bar{\alpha}(e_K(x))\right\rangle \\ &\qquad - \left\langle \varphi(f(x)), \bar{\alpha}_{\mathrm{mor}}(x)\circ\bar{\theta}_{\mathrm{cod}}(e_{\{1,\ldots,p\}})(x)\right\rangle.\end{aligned}$$

Since N is a Fréchet domain and $F\to N$ is trivial by assumption, we now express the L_∞-algebroid structure on this bundle as $\theta' = \mathscr{L}_{\rho'} + A'$ for some smooth map $A'\colon N\to \mathrm{L}(\bigodot(W),W)$. By using this to work out the right-hand side of equation (4.3.3) and employing a few results from section 3.1.2 we obtain

$$\begin{aligned}(-1)^{|\varphi'|}\langle Q_{\theta'}(\varphi'), e'_{\{1,\ldots,p\}}\rangle &= -\sum_{I\sqcup\{j\}\sqcup K} \epsilon_{I,\{j\},K}\langle \varphi', \mathscr{L}_{\rho(e'_I)} e'_j \odot e'_K\rangle \\ &\qquad + \sum_{I\sqcup J}\epsilon_{I,J}\left(\mathscr{L}_{\rho(e'_I)}\langle\varphi', e'_J\rangle - \langle \varphi', A'(e'_I)\odot e'_J\rangle\right) \\ &= \sum_{I\sqcup J=\{1,\ldots,p\}} \epsilon_{I,J}\Big(\langle \mathscr{L}_{\rho(e'_I)}\varphi, e'_J\rangle - \langle \varphi, A'(e'_I)\odot e'_J\rangle\Big)\end{aligned}$$

for any homogeneous form $\varphi'\in\Omega_F(U')$ and $e'_1, e'_2, \ldots, e'_p\in\mathscr{F}(U')$ with $U'\subseteq N$ open. Consequently, the right-hand side of equation (4.3.5) is equal to

$$\begin{aligned}-\sum_{I\sqcup J=\{1,\ldots,p\}} \epsilon_{I,J}\Big(&\left\langle \varphi(f(x)), A'(f(x))(\bar{\alpha}_{\mathrm{mor}}(x)(e_I(x)))\odot \bar{\alpha}_{\mathrm{mor}}(x)(e_J(x))\right\rangle \\ &+ \left\langle \mathscr{L}_{\rho'(\bar{\alpha}_{\mathrm{mor}}(x)(e_I(x)))}\varphi, \bar{\alpha}_{\mathrm{mor}}(x)(e_J(x))\right\rangle\Big),\end{aligned}$$

where we have used that $\Delta\circ\bar{\alpha}_{\mathrm{mor}} = (\bar{\alpha}_{\mathrm{mor}}\otimes\bar{\alpha}_{\mathrm{mor}})\circ\Delta$.

If we compare this to the expression we had obtained for the left-hand side of equation (4.3.5), we see that $Q_\theta \circ \alpha^* = \alpha^* \circ Q_{\theta'}$ if and only if

$$\begin{aligned}\sum_{I\sqcup J=\{1,\dots,p\}} \epsilon_{I,J}\,\alpha(x)\big(\alpha(x)(e_I(x))\big) \odot \bar{\alpha}_{\mathrm{mor}}(x)(e_J(x)) \\ = \sum_{I\sqcup J\sqcup K=\{1,\dots,p\}} \epsilon_{I,J,K}\,\mathscr{L}_{\rho(e_I)(x)}\big(\alpha^!(e_J)\big) \odot \bar{\alpha}(e_K(x)) \\ + \sum_{I\sqcup J=\{1,\dots,p\}} \epsilon_{I,J}\,A'(f(x))(\bar{\alpha}_{\mathrm{mor}}(x)(e_I(x))) \odot \bar{\alpha}_{\mathrm{mor}}(x)(e_J(x))\end{aligned}$$

for all $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$ and all $x \in U$ with $U \subseteq M$ open. This allows us to conclude that $(\alpha, f)\colon \bigodot(E) \to F$ is a morphism of curved L_∞-algebroids if and only if equation (4.3.4) holds for the trivial connection $\nabla^{\mathrm{triv}} = \mathscr{L}$.

It is not difficult to show that the validity of equation (4.3.4) is independent of the chosen connection, as both sides transform in the same way if a different pair (∇, A') is chosen. $\square$

The difference between the left- and right-hand side of equation (4.3.4) is a vector bundle morphism

$$R_{(\alpha,f)} = \alpha^! \mathbin{\scriptstyle\square} \theta - f^*A'_{\nabla'} \diamond \alpha^! - (f^*\nabla)_\rho \diamond \alpha^! \colon \bigodot(E) \longrightarrow f^*F,$$

provided that $\mathrm{T}f \circ \rho = \rho' \diamond \alpha^!$. This tensor measures the failure of (α, f) to be a morphism of curved L_∞-algebroids and it is independent of the chosen connection. The value of $R_{(\alpha,f)}$ at $x \in M$ can alternatively be expressed as

$$\begin{aligned}R_{(\alpha,f)}(x)(e_1, \dots, e_p) = \sum_{I\sqcup J=\{1,\dots,p\}} \epsilon_{I,J}\,\alpha(\theta(e_I) \odot e_J)(x) \\ - \sum_{I_1,\dots,I_m} \epsilon_{I_1,\dots,I_m}\,\tfrac{1}{m!}\,\theta'\big(\textstyle\bigodot_{j=1}^m a_{I_j}\big)(x) \\ - \sum_{I\sqcup J=\{1,\dots,p\}} \epsilon_{I,J}\,\mathrm{T}^{\mathrm{v}}_x(\alpha^!(e_J) - f^*a_J)(\rho(e_I))\end{aligned}$$

if $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$ are homogeneous local sections defined on an open neighbourhood $U \subseteq M$ of x and $a_I \in \mathscr{F}(U')$ for $I \subseteq \{1, \dots, p\}$ are arbitrary local sections which are defined on an open subset $U' \subseteq N$ that contains $f(x)$ and have the property that $a_I(f(x)) = \alpha(e_I(x))$. The final term accounts for the fact that the sections $\alpha^!(e_J)$ and f^*a_J of $f^*F \to M$ for $J \subseteq \{1,\dots,p\}$ could have different derivatives at x.

The following lemma was used in the proof of Proposition 4.3.30.

Lemma 4.3.31.
Let $V g$ and W be $\mathbf{Z}$-graded vector spaces of Fréchet type and let $X \subseteq V^0$ be a Fréchet domain. Given smooth maps $\alpha\colon X \to \mathrm{L}(\bigodot(V), W)$ and $e_1, e_2, \dots, e_p \colon X \to V$, Given the hypotheses of Proposition 4.3.30,

$$\mathscr{L}_{\dot{x}}(\bar{\alpha}_{\mathrm{mor}}(e_{\{1,\dots,p\}})) = \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J} \mathscr{L}_{\dot{x}} \alpha^!(e_I) \odot \bar{\alpha}^!_{\mathrm{mor}}(e_J),$$

for any $\dot{x} \in \mathrm{T}_x M$ and homogeneous $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$ with $U \subseteq M$ open.

Proof: Since $\bar{\alpha}_{\mathrm{mor}}$ is described as a sum of graded symmetric tensor products in equation (3.2.6), we can use Proposition 3.1.11 to compute the derivative $\mathscr{L}_{\dot{x}} \bar{\alpha}_{\mathrm{mor}}(e_{\{1,\dots,p\}})$. After some reordering of terms and indices, this yields

$$\begin{aligned} &\sum_{I_0,\dots,I_m} \tfrac{1}{(m+1)!} \epsilon_{I_0,\dots,I_m} \sum_{i=0}^{m} \textstyle\bigodot_{j<i} \alpha(e_{I_j}) \odot \mathscr{L}_{\dot{x}} \alpha(e_{I_i}) \odot \textstyle\bigodot_{j>i} \alpha(e_{I_j}), \\ &\qquad = \sum_{I_0,\dots,I_m} \tfrac{1}{m!} \epsilon_{I_0,\dots,I_m} \mathscr{L}_{\dot{x}} \alpha(e_{I_0}) \odot \textstyle\bigodot_{j\neq 1}^{m} \alpha(e_{I_j}), \end{aligned} \tag{4.3.6}$$

which is equal to the desired expression. ☐

We will need a slight reformulation of Proposition 4.3.30 that involves choosing smooth connections on both the domain and the codomain of the curved L_∞-algebroid morphism.

Corollary 4.3.32.
Let $(E, \theta\rho)$ and (F, θ', ρ') be curved split L_∞-algebroids of Fréchet type over M and N respectively and let $(\alpha, f)\colon \bigodot(E) \to F$ be a smooth vector bundle morphism of degree 0. Given open subsets $U' \subseteq N$ and $U \subseteq f^{-1}(U')$ and two connections ∇ and ∇' on $E|_U$ and $F|_{U'}$ respectively such that $\theta|_U = \nabla_\rho + A$ and

$\theta'|_{U'} = \nabla'_{\rho'} + A'$, the restriction $\alpha|_U$ is a morphism of curved L_∞-algebroids if and only if $\rho' \diamond \alpha|_U = \mathrm{T}f \circ \rho|_U$ and

$$\big(\alpha^! \mathbin{\square} A - (f^*A') \diamond \alpha^!\big)(e_{\{1,\dots,p\}}) = \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J}\, (f^*\nabla')_{\rho(e_I)}\big(\alpha^!(e_J)\big) - \sum_{I \sqcup \{j\} \sqcup K = \{1,\dots,p\}} \epsilon_{I,\{j\},K}\, \alpha^!\big(\nabla_{\rho(e_I)} e_j \odot e_K\big)$$

for all local sections $e_1, e_2, \dots, e_p \in \mathscr{E}(\tilde{U})$ with $\tilde{U} \subseteq U$ open.

Proof: This follows from Proposition 4.3.30 and the fact that

$$(\alpha^! \mathbin{\square} \theta)(e_{\{1,\dots,p\}}) = (\alpha^! \mathbin{\square} A)(e_{\{1,\dots,p\}}) + \sum_{I \sqcup \{j\} \sqcup K = \{1,\dots,p\}} \epsilon_{I,\{j\},K}\, \alpha^!\big(\nabla_{\rho(e_I)} e_j \odot e_K\big)$$

for all homogenoeus local sections $e_1, e_2, \dots, e_p \in \mathscr{E}(\tilde{U})$. □

If $E = V \times M \to M$ and $F = W \times N \to N$ are both trivial vector bundles, then we can identify the space of plurilinear graded vector bundle morphisms from the former to the latter with the space $\mathrm{C}^\infty(M, \mathrm{L}(\bigodot(V), W)) \times \mathrm{C}^\infty(M, N)$. A plurilinear graded vector bundle morphism $(\alpha, f)\colon \bigodot(E) \to F$ is now a morphism of curved L_∞-algebroids from (E, θ, ρ) to (F, θ', ρ') if and only if

$$\big(\alpha^! \mathbin{\square} A - (f^*A') \diamond \alpha^!\big)(e_{\{1,\dots,p\}}) = \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J}\, \big(\mathscr{L}_{\rho(e_I)}\alpha\big)(e_J),$$

for all homogeneous $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$ with $U \subseteq M$ open. This follows from Proposition 4.3.30 by choosing the trivial connection on both sides.

Remark 4.3.33.
We had seen in Remark 4.3.23 that (M, Ω_E, Q_θ) is a differential graded manifold whenever (E, θ, ρ) is a curved L_∞-algebroid of finite-dimensional type. If $(\alpha, f)\colon \bigodot(E) \to F$ is a morphism from (E, θ, ρ) to another such curved L_∞-algebroid (F, θ', ρ'), then (α^*, f) is a morphism of locally ringed spaces and thus describes a morphism between the corresponding differential graded manifolds. For finite-dimensional negatively graded L_∞-algebroids every morphism between the corresponding (non-negatively graded) differential graded manifolds can be described in this way [BP13]. △

Splitting preserving morphisms

Up to this point, we have been careful to only talk about curved split L_∞-algebroids, but we have referred to morphisms between them as morphisms of curved L_∞-algebroids without referencing the splitting. This is because curved split L_∞-algebroids, as they were defined in Definition 4.3.1, come with a splitting that is not preserved by these morphisms.

Given a $\mathbf{Z}$-graded vector bundle $E \to M$ of Fréchet type, the corresponding bundle $\bigodot(E) \to M$ of graded symmetric coalgebras comes with a canonical increasing filtration

$$E = \bigodot^{\leq 0}(E) \subseteq \bigodot^{\leq 1}(E) \subseteq \bigodot^{\leq 2}(E) \subseteq \cdots \subseteq \bigodot(E) = \bigcup_{p \in \mathbf{Z}} \bigodot^{\leq p}(E)$$

by subbundles $\bigodot^{\leq p}(E) = \bigoplus_{q=0}^{p} \bigodot^{p}(E)$. This filtration is completely determined by the canonical coalgebra structure on the fibres of $\bigodot(E) \to M$ and it is preserved by plurilinear vector bundle morphisms in the sense that every such morphism $\alpha\colon \bigodot(E) \to F$ satisfies $\bar{\alpha}_{\mathrm{mor}}(\bigodot^{\leq p}(E)) \subseteq \bigodot^{\leq q}(F)$ for all $p \in \mathbf{N}_0$.

The splitting that is implicitly referenced in Definition 4.3.1 is the decomposition

$$\bigodot(E) = \bigoplus_{p \in \mathbf{N}_0} \bigodot^{p}(E),$$

which splits the aforementioned filtration. Unlike the filtration, it is not preserved by the family $\bar{\alpha}_{\mathrm{mor}}$ of coalgebra morphisms associated to a plurilinear vector bundle morphism α unless the components α_p for $p \neq 1$ all vanish, i.e. if α is an ordinary morphism of $\mathbf{Z}$-graded vector bundles. This leads to the following definition.

Definition 4.3.34.
A *splitting preserving* morphism of curved split L_∞-algebroids from (E, θ, ρ) to (E', θ', ρ') is a graded vector bundle morphism $\alpha\colon E \to E'$ of degree 0 such that $\alpha^* \circ Q_{\theta'} = Q_\theta \circ \alpha^*$.

The Chevalley–Eilenberg algebra (Ω_E, Q_θ) of a $\mathbf{Z}$-graded vector bundle $E \to M$ comes with a decreasing filtration

$$\Omega_E = \Omega_{E,\geq 0} \supseteq \Omega_{E,\geq 1} \supseteq \Omega_{E,\geq 2} \supseteq \cdots$$

which is dual to the increasing filtration on $\bigodot(E) \to M$. This filtration is similarly respected by the transpose α^* of a plurilinear map $\alpha\colon E \to F$, while the product decomposition

$$\Omega_E = \prod_{p \in \mathbf{N}_0} \Omega_{E,p}$$

is only preserved when α is linear.

Example 4.3.35.

If M and N are smooth Fréchet manifolds, then their tangent bundles carry canonical Lie algebroid structures, which in turn give rise to canonical L_∞-algebroids $(\mathrm{T}[1]M, \downarrow[\uparrow\cdot, \uparrow\cdot], \downarrow)$ and $(\mathrm{T}[1]N, \downarrow[\uparrow\cdot, \uparrow\cdot], \downarrow)$. Any smooth map $f : M \to N$ induces a splitting preserving morphism $(\mathrm{T}f, f)$ from the former to the latter. ◊

Example 4.3.36.

A morphism of Lie algebroids from $(E, [\![\cdot, \cdot]\!])$ to $(F, [\![\cdot, \cdot]\!])$ is often defined as a vector bundle morphism $\alpha\colon E \to F$ that induces a morphism between the corresponding Chevalley–Eilenberg algebras $(\Omega_F, \wedge, d_F)$ and $(\Omega_E, \wedge, d_E)$. These consequently induce splitting preserving morphisms between the associated L_∞-algebroids.

Every curved L_∞-algebroid morphism between L_∞-algebroids of Fréchet type that are concentrated in degree -1 is splitting preserving due to homogeneity, so Lie algebroids in fact form a full subcategory of the category of curved split L_∞-algebroids. ◊

Example 4.3.37.

The homotopies from Definition 4.3.9 can be described as curved L_∞-algebroid morphisms. A homotopy Γ for a curved L_∞-algebroid (E, θ, ρ) of Fréchet type with $E^0 = 0$ corresponds to a morphism $(\Gamma\, \mathrm{d}t \circ \uparrow, \gamma)$ from the shifted tangent bundle $\mathrm{T}[1]I$ of the unit interval $[0, 1]$, endowed with its canonical L_∞-algebroid structure, to (E, θ, ρ). Any degree 0 plurilinear vector bundle morphism from $\mathrm{T}[1]I$ to another $\mathbf{Z}$-graded vector bundle $E \to M$ is automatically splitting preserving, and one can show that it is a morphism of curved L_∞-algebroids if and only if $\gamma' = \rho \circ \Gamma$ and $\theta_0 \circ \gamma = 0$. ◊

Example 4.3.38.

One can subsequently define a homotopy between two homotopies (Γ_0, γ_0) and (Γ_1, γ_1) of Maurer–Cartan elements as a curved L_∞-algebroid morphisms (H, η) from the shifted tangent bundle $\mathrm{T}[1]I^2$ of the unit square $I^2 = [0, 1] \times [0, 1]$

to (E,θ,ρ). This morphism should additionally be such that $s \mapsto \eta(0,s)$ and $s \mapsto \eta(1,s)$ are constant and $\Gamma_s = H \circ f_s$ for $s = 0,1$ if f_s denotes the canonical inclusion map $I \to I \times \{s\} \subseteq I^2$. This describes an equivalence relation on the set of homotopies of Maurer–Cartan elements.

Because homotopies that are related by reparametrisation are homotopic, concatenation of homotopies can be used to define a groupoid structure on the set of equivalence classes of homotopies. If (E,θ,ρ) arises as the L_∞-algebroid of an finite-dimensional integrable Lie algebroid, this object is its integrating Lie groupoid [CF03]. In general, however, this groupoid does not come with a smooth structure. $\lozenge$

CHAPTER 5.

Correspondences

From the start, our objective has been to demonstrate that every equivariant deformation problem of Fréchet type is controlled by an L_∞-algebra. The principal objects in this statement, equivariant deformation problems and topological L_∞-algebras, were presented in chapter 1 and chapter 2 respectively. We have then introduced two intermediate objects: holonomic families of curved L_∞-algebras were defined in chapter 3, and curved L_∞-algebroids were the subject of chapter 4. Now that all of the pieces are in place, we are ready to demonstrate how these classes of objects are related.

We will first of all demonstrate how any curved L_∞-algebroid of Fréchet type can locally be recast into a family of curved L_∞-algebras of Fréchet type and, conversely, how any such family gives rise to a curved L_∞-algebroid structure. Also morphisms will be discussed, as it turns out that morphisms of curved L_∞-algebroids correspond to fibre-preserving morphisms of the corresponding holonomic families.

To relate equivariant deformation problems of Fréchet type to L_∞-algebras, we will show how such deformation problems can be interpreted as smooth curved L_∞-algebroids and then apply the aforementioned procedure. The resulting curved L_∞-algebras can to some extent be said to control the original deformation problem. In an analytic setting this means precisely that its Maurer–Cartan elements correspond to integrable structures and gauge equivalence of these elements implies equivalence of these structures. In general, the only thing we can recover from these L_∞-algebras is an algebraic description of formal families of deformations up to formal gauge equivalence.

Section 5.1 is dedicated to the relation between the categories of holonomic families of curved L_∞-algebras and curved L_∞-algebroids of Fréchet type. Equivariant deformation problems, and their description in terms of the aforementioned objects, are considered in section 5.2.

5.1. L_∞-algebroids as holonomic families

The aim of this section is to demonstrate that locally, there is an equivalence between curved L_∞-algebroids of Fréchet type and holonomic families of curved L_∞-algebras of Fréchet type. To turn such L_∞-algebroids into holonomic families, we will need to choose a trivialisation chart and consider the formal power series expansion of the relevant structure maps. Although this translation involves the choice of a trivialisation chart on the underlying bundle, we will see that the isomorphism class of the resulting family does not depend on this choice. We will moreover see that morphisms of curved L_∞-algebroids can similarly be described as fibre-preserving morphisms of the corresponding families.

5.1.1. Infinite jets of curved L_∞-algebroids

We recall from section 4.2.5 that a curved L_∞-algebroid structure on a $\mathbf{Z}$-graded vector bundle $E \to M$ of Fréchet type for which $E^0 = 0$ is a Maurer–Cartan element for the graded Lie algebra $\widehat{\mathrm{Der}}(E)$. Holonomic families of curved L_∞-algebras on a $\mathbf{Z}$-graded vector space V of Fréchet type parametrised by a domain $X \subseteq V^0$, on the other hand, are Maurer–Cartan elements of the graded Lie algebra $\mathrm{C}^\infty_{\mathrm{hol}}\big(\phi(U), \mathrm{L}(\bigodot(V), V)\big)$ from Proposition 3.3.3. To relate these two notions we need to choose a local trivialisation for the vector bundle $E \to M$.

Given a $\mathbf{Z}$-graded vector bundle $E \to M$ of Fréchet type with $E^0 = 0$ and a local trivialisation chart $\Phi = (\widetilde{\Phi}, \phi)\colon E|_U \to V^{\neq 0} \times V^0$, let $\nabla^\Phi = \mathscr{L}^\Phi$ denote the associated flat connection. By Proposition 4.2.28, every homogeneous pluriderivation on $E|_U \to U$ is then of the form $\theta = \nabla^\Phi_\rho + A$ for two homogeneous vector bundle morphisms

$$\rho\colon \bigodot(E)|_U \longrightarrow \mathrm{T}U \qquad \text{and} \qquad A\colon \bigodot(E)|_U \longrightarrow E|_U$$

of the same degree.

Through the trivialisation chart Φ, these can be described as smooth maps, $\tilde{\rho}_\Phi$ and $\tilde{A}_\Phi$, from $\phi(U)$ to the subspaces $\mathrm{L}(\bigodot(V^{\neq 0}), V^0)$ and $\mathrm{L}(\bigodot(V^{\neq 0}), V^{\neq 0})$ of $\mathrm{L}(\bigodot(V^{\neq 0}), V)$. We define these maps to be such that

$$\widetilde{\Phi} \circ A_x(e_{\{1,\dots,p\}}) = \tilde{A}_\Phi(\phi(x))\big(\textstyle\bigodot_{i=1}^p \widetilde{\Phi}_x(e_i)\big)$$

and

$$\mathrm{D}\phi \circ \rho_x(e_{\{1,\dots,p\}}) = \tilde{\rho}_\Phi(\phi(x))\big(\textstyle\bigodot_{i=1}^p \widetilde{\Phi}_x(e_i)\big)$$

for every $x \in U$ and all $e_1, e_2 \dots, e_p \in E_x$. The infinite jets of A and ρ are similarly represented by two maps, $\mathrm{j}^\infty_\Phi \rho$ and $\mathrm{j}^\infty_\Phi A$, from $\phi(U)$ to $\mathrm{L}(\bigodot(V), V)$, which are given by

$$\mathrm{j}^\infty_\Phi \rho(u)(\dot{u}_{\{1,\dots,q\}} \odot v_{\{1,\dots,p\}}) = \mathrm{D}^q(\tilde{\rho}_\Phi)(u)(\dot{u}_{\{1,\dots,q\}})(v_{\{1,\dots,p\}})$$

and

$$\mathrm{j}^\infty_\Phi A(u)(\dot{u}_{\{1,\dots,q\}} \odot v_{\{1,\dots,p\}}) = \mathrm{D}^q(\tilde{A}_\Phi)(u)(\dot{u}_{\{1,\dots,q\}})(v_{\{1,\dots,p\}})$$

for $u \in \phi(U)$, $\dot{u}_1, \dot{u}_2, \dots, \dot{u}_q \in V^0$ and $v_1, v_2, \dots, v_p \in V^{\neq 0}$.

Theorem 5.1.1 (L_∞-algebroids $\Leftrightarrow$ holonomic families of L_∞-algebras)**.** *Let $E \to M$ be a $\mathbf{Z}$-graded vector bundle of Fréchet type for which $E^0 = 0$ and let $\Phi\colon E|_U \to V^{\neq 0} \times V^0$ be a local trivialisation chart for this bundle. The map*

$$\mathrm{j}^\infty_\Phi \colon \widehat{\mathrm{Der}}(E|_U) \longrightarrow \mathrm{C}^\infty_{\mathrm{hol}}\big(\phi(U), \mathrm{L}(\textstyle\bigodot(V), V)\big), \qquad \nabla^\Phi_\rho + A \mapsto \mathrm{j}^\infty_\Phi A - \mathrm{j}^\infty_\Phi \rho$$

is an isomorphism of graded Lie algebras.

It restricts to a one-to-one correspondence between curved split L_∞-algebroid structures on $E|_U$ and holonomic families of curved L_∞-algebra structures on V parametrised by $\phi(U) \subseteq V^0$.

While the relation described in Theorem 5.1.1 explicitly depends on the choice of a trivialisation chart, we will see in Corollary 5.1.9 that the isomorphism class of the holonomic family of curved L_∞-algebras $\mathrm{j}^\infty_\Phi \theta$ is independent of the chosen trivialisation chart Φ (but does of course depend on its domain).

Although the proof of Theorem 5.1.1 is not complicated, it involves a number of lengthy computations. These computations make up Lemma 5.1.2 and Lemma 5.1.3 below, and the proof is concluded on page 202. To prove that j^∞_Φ is a morphism of graded Lie algebras, we need to show that $\mathrm{j}^\infty_\Phi[\theta, \eta]$ is equal to $[\mathrm{j}^\infty_\Phi\theta, \mathrm{j}^\infty_\Phi\eta]$ for any two homogeneous pluriderivations θ and η on $E \to M$. The first lemma considers just the components of the restriction of $\mathrm{j}^\infty_\Phi[\theta,\eta](u)|_{\bigodot(V^{\neq 0})}$ that are determined by the anchor of $[\theta,\eta]$.

Lemma 5.1.2.
Given two homogeneous pluriderivations θ and η on $E \to M$,

$$\mathrm{j}^\infty_\Phi[\theta,\eta](u)(v_{\{1,\dots,p\}}) = [\mathrm{j}^\infty_\Phi\theta(u), \mathrm{j}^\infty_\Phi\eta(u)](v_{\{1,\dots,p\}})$$

for every $u \in U$ and all $v_1, v_2, \dots, v_p \in V^{\neq 0}$ such that $\sum_{i=1}^p |v_i| = -|\theta| - |\eta|$.

Proof: We can assume without loss of generality that M is an open subset of a Fréchet space, that the bundle $E \to M$ is trivial and that the trivialisation chart Φ is tautological. Let V therefore be a $\mathbf{Z}$-graded vector space of Fréchet type and let $U \subseteq V^0$ be an open subset such that $E = V^{\neq 0} \times U \to U$, and let $\Phi = \mathrm{id}_E : E \to V^{\neq 0} \times U$. We shall suppress Φ in our notation from this point onward.

Let $\theta = \nabla_\rho + A$ and $\eta = \nabla_{\rho'} + A'$ be homogeneous pluriderivations on $E \to U$, and let $v_1, v_2, \dots, v_p \in V^{\neq 0}$ be such that $\sum_{i=1}^p |v_i| = -|\theta| - |\eta|$. If we denote the corresponding constant sections of E by $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$, so that $e_i(u) = v_i$ for every $u \in U$ and $i = 1, 2, \dots, p$, then

$$\mathrm{j}^\infty_u[\theta,\eta](v_{\{1,\dots,p\}}) = -\rho_{[\theta,\eta]}(e_{\{1,\dots,p\}})(u)$$

for all $u \in U$.

Proposition 4.2.19 tells us that the left-hand side of this equation is given by

$$\rho_{[\theta,\eta]}(e_{\{1,\dots,p\}}) = \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J}\big(\rho(\eta(e_I)\odot e_J) - (-1)^{|\theta||\eta|}\rho'(\theta(e_I)\odot e_J) + (-1)^{|v_I||\eta|}[\rho(e_I), \rho'(e_J)]\big).$$

When working out $\rho(\eta(\theta_J)\odot e_I)$, $\rho'(\theta(\eta_I)\odot e_J)$ and $[\rho(e_I), \rho'(e_J)]$ for a given decomposition $I \sqcup J = \{1,\dots,p\}$ there are two cases to consider. Since $|e_{\{1,\dots,p\}}| = -|\theta| - |\eta|$, the decomposition is such that either $|e_I| = -|\theta|$ and $|e_J| = -|\eta|$, or $|e_I| \neq -|\theta|$ and $|e_J| \neq -|\eta|$.

(a) If $|e_I| = -|\theta|$ and $|e_J| = -|\eta|$, then both $\rho(\theta(\eta_I) \odot e_J)$ and $\rho(\eta(\theta_I) \odot e_J)$ vanish due to homogeneity. The third term, however, is given by

$$\begin{aligned}[\rho(e_I), \rho'(e_J)](u) &= \mathscr{L}_{\rho(e_I)(u)}(\rho'(e_J)) - \mathscr{L}_{\rho'(e_I)(u)}(\rho(e_J)) \\ &= \mathscr{L}_{\rho(e_I)(u)}(\rho')(e_J) - \mathscr{L}_{\rho'(e_I)(u)}(\rho)(e_J), \\ &= \mathrm{j}^\infty_u \eta(\mathrm{j}^\infty_u \theta(v_I))(v_J) - \mathrm{j}^\infty_u \theta(\mathrm{j}^\infty_u \eta(v_J))(v_I).\end{aligned}$$

Therefore,

$$\begin{aligned}(-1)^{|v_I||\eta|}\epsilon_{I,J}[\rho(e_I), \rho'(e_J)](u) = (-1)^{|\theta||\eta|}\epsilon_{I,J}\, &\mathrm{j}^\infty_u \eta(\mathrm{j}^\infty_u \theta(v_I))(v_J) \\ -\epsilon_{J,I}\, &\mathrm{j}^\infty_u \theta(\mathrm{j}^\infty_u \eta(v_J))(v_I).\end{aligned}$$

(b) If $|e_I| \neq -|\theta|$ and $|e_J| \neq -|\eta|$, then $[\rho(e_I), \rho'(e_J)]$ since both $\rho(e_I)$ and $\rho'(e_J)$ are trivial, while

$$\begin{aligned}\rho(\eta(e_I) \odot e_J)(u) &= \rho(A'(e_I) \odot e_J)(u), \\ &= -\mathrm{j}^\infty_u \theta\big(\mathrm{j}^\infty_u \eta(v_I) \odot v_J\big)\end{aligned}$$

and, similarly,

$$\rho'(\theta(e_J) \odot e_I)(u) = -\mathrm{j}^\infty_u \eta\big(\mathrm{j}^\infty_u \theta(v_J) \odot v_I\big).$$

Consequently,

$$\begin{aligned}&\epsilon_{I,J}\, \rho(\eta(e_I) \odot e_J) - (-1)^{|\theta||\eta|}\epsilon_{J,I}\, \rho'(\theta(e_J) \odot e_I) \\ &\qquad = (-1)^{|\theta||\eta|}\epsilon_{I,J}\, \mathrm{j}^\infty_u \eta(\mathrm{j}^\infty_u \theta(v_I))(v_J) - \epsilon_{J,I}\, \mathrm{j}^\infty_u \theta(\mathrm{j}^\infty_u \eta(v_J))(v_I).\end{aligned}$$

By combining these expressions, we deduce that

$$\begin{aligned}\mathrm{j}^\infty_u[\theta, \eta](v_{\{1,\dots,p\}}) &= \sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{J,I}\, \mathrm{j}^\infty_u \theta(\mathrm{j}^\infty_u \eta(v_J))(v_I) \\ &\qquad\qquad - (-1)^{|\theta||\eta|}\epsilon_{I,J}\, \mathrm{j}^\infty_u \eta(\mathrm{j}^\infty_u \theta(v_I))(v_J) \\ &= [\mathrm{j}^\infty_u \theta, \mathrm{j}^\infty_u \eta](v_{\{1,\dots,p\}}).\end{aligned}$$

for all homogeneous $v_1, v_2, \dots, v_p \in V^{\neq 0}$ such that $\sum_{i=1}^p |v_i| = -|\theta| - |\eta|$ and every $u \in U$. □

The second lemma deals with remaining components of $\mathrm{j}^\infty_\Phi[\theta,\eta](u)|_{\bigodot(V^{\neq 0})}$.

Lemma 5.1.3.
Given two homogeneous pluriderivations θ and η on $E \to M$,

$$\mathrm{j}^\infty_\Phi[\theta,\eta](u)(v_{\{1,\ldots,p\}}) = [\mathrm{j}^\infty_\Phi\theta, \mathrm{j}^\infty_\Phi\eta](u)(v_{\{1,\ldots,p\}})$$

for every $u \in U$ and all $v_1, v_2, \ldots, v_p \in V^{\neq 0}$ such that $\sum_{i=1}^p |v_i| \neq -|\theta| - |\eta|$.

Proof: We again assume, without loss of generality, that $E = V^{\neq 0} \times U \to U$ for some $\mathbf{Z}$-graded vector space V of Fréchet type and an open subset $U \subseteq V^0$. Let the trivialisation Φ moreover be equal to the identity map $\mathrm{id}_E\colon E \to V^{\neq 0} \times U$, as in the proof of Lemma 5.1.2.

Let $\theta = \nabla_{\rho_\theta} + A_\theta$ and $\eta = \nabla_{\rho_\eta} + A_\eta$ be homogeneous pluriderivations on $E \to U$, and let $v_1, v_2, \ldots, v_p \in V^{\neq 0}$ be such that $\sum_{i=1}^p |v_i| \neq -|\theta| - |\eta|$. If we denote the corresponding constant sections of E by $e_1, e_2, \ldots, e_p \in \mathscr{E}(U)$, so that $e_i(u) = v_i$ for every $u \in U$ and $i = 1, 2, \ldots, p$, then

$$\mathrm{j}^\infty_u[\theta,\eta](v_{\{1,\ldots,p\}}) = [\theta,\eta](e_{\{1,\ldots,p\}})(u)$$

for every $u \in U$, which is equal to

$$\sum_{I \sqcup J = \{1,\ldots,p\}} \epsilon_{I,J}\, \theta(\eta(e_I) \odot e_J)(u) - (-1)^{|\theta||\eta|} \epsilon_{J,I}\, \eta(\theta(e_J) \odot e_I)(u).$$

This sum can then be split up into three parts: it is equal to $S_1(u) + S_2(u) + S_3(u)$ for

$$S_1 = \sum_{\substack{|e_I| \neq -|\eta| \\ |e_J| \neq -|\theta|}} \epsilon_{I,J}\, \theta(\eta(e_I) \odot e_J) - (-1)^{|\theta||\eta|} \epsilon_{J,I}\, \eta(\theta(e_J) \odot e_I),$$

$$S_2 = \sum_{\substack{|e_I| \neq -|\eta| \\ |e_J| = -|\theta|}} \epsilon_{I,J}\, \theta(\eta(e_I) \odot e_J) - (-1)^{|\theta||\eta|} \epsilon_{J,I}\, \eta(\theta(e_J) \odot e_I)$$

and

$$S_3 = \sum_{\substack{|e_I| = -|\eta| \\ |e_J| \neq -|\theta|}} \epsilon_{I,J}\, \theta(\eta(e_I) \odot e_J) - (-1)^{|\theta||\eta|} \epsilon_{J,I}\, \eta(\theta(e_J) \odot e_I).$$

We will work these out separately.

S_1: If both $|e_I| \neq -|\eta|$ and $|e_J| \neq -|\theta|$, then $\rho_\eta(e_I) = 0$ and $\rho_\theta(e_J) = 0$. This tells us that

$$\theta(\eta(e_I) \odot e_J)(u) = A_\theta(A_\eta(e_I) \odot e_J) = \mathrm{j}^\infty_u \theta\big(\mathrm{j}^\infty_u \eta(v_I) \odot v_J\big),$$

and that $\eta(\theta(e_J) \odot e_I)(u)$ can be expressed similarly.

Consequently,

$$S_1(u) = \sum_{\substack{|e_I| \neq -|\eta| \\ |e_J| \neq -|\theta|}} \epsilon_{I,J}\, \mathrm{j}^\infty_u \theta\big(\mathrm{j}^\infty_u \eta(v_I) \odot v_J\big) - (-1)^{|\theta||\eta|} \epsilon_{J,I}\, \mathrm{j}^\infty_u \eta\big(\mathrm{j}^\infty_u \theta(v_J) \odot v_I\big).$$

S_2: If $|e_J| = -|\theta|$ and $|e_I| \neq -|\eta|$, then $\rho_\eta(e_I) = 0$, but $\rho_\theta(e_J)$ might be non-zero. Therefore, $\epsilon_{I,J}\, \theta(\eta(e_I) \odot e_J)$ is equal to

$$\begin{aligned}\epsilon_{I,J} \big(A_\theta(A_\eta(e_I) \odot e_J) + (-1)^{|e_J|(|e_I|+|\eta|)} \mathscr{L}_{\rho_\theta(e_J)}(A_\eta \circ e_I)\big) \\ = \epsilon_{I,J} A_\theta(A_\eta(e_I) \odot e_J) + \epsilon_{J,I} (-1)^{|\theta||\eta|} \big(\mathscr{L}_{\rho_\theta(e_J)} A_\eta\big) \circ e_I,\end{aligned}$$

which, when evaluated at a point $u \in U$, can be re-expressed as

$$\epsilon_{I,J}\, \mathrm{j}^\infty_u \theta\big(\mathrm{j}^\infty_u \eta(v_I) \odot v_J\big) - (-1)^{|\theta||\eta|} \epsilon_{J,I}\, \mathrm{j}^\infty_u \eta\big(\mathrm{j}^\infty_u \theta(v_J) \odot v_I\big).$$

On the other hand, $\eta(\theta(e_J) \odot e_I) = 0$ because $|e_J| = -|\theta|$, which tells us that

$$S_2(u) = \sum_{\substack{|e_I| \neq -|\eta| \\ |e_J| = -|\theta|}} \epsilon_{I,J}\, \mathrm{j}^\infty_u \theta\big(\mathrm{j}^\infty_u \eta(v_I) \odot v_J\big) - (-1)^{|\theta||\eta|} \epsilon_{J,I}\, \mathrm{j}^\infty_u \eta\big(\mathrm{j}^\infty_u \theta(v_J) \odot v_I\big).$$

S_3: If $|e_I| = -|\eta|$ and $|e_J| \neq -|\theta|$, then the situation is completely analogous to the previous case. Therefore,

$$S_3(u) = \sum_{\substack{|e_I| = -|\eta| \\ |e_J| \neq -|\theta|}} \epsilon_{I,J}\, \mathrm{j}^\infty_u \theta\big(\mathrm{j}^\infty_u \eta(v_I) \odot v_J\big) - (-1)^{|\theta||\eta|} \epsilon_{J,I}\, \mathrm{j}^\infty_u \eta\big(\mathrm{j}^\infty_u \theta(v_J) \odot v_I\big).$$

By combining the obtained expressions for $S_1(u)$, $S_2(u)$ and $S_3(u)$, we now deduce that

$$\begin{aligned} \mathrm{j}^\infty[\theta,\eta](u)(v_{\{1,\dots,p\}}) &= S_1(u)+S_2(u)+S_3(u) \\ &= \sum_{I\sqcup J=\{1,\dots,p\}} \begin{aligned}[t] &\epsilon_{I,J}\, \mathrm{j}^\infty_u\theta\big(\mathrm{j}^\infty_u\eta(v_I)\odot v_J\big) \\ &-(-1)^{|\theta||\eta|}\epsilon_{I,J}\, \mathrm{j}^\infty_u\theta\big(\mathrm{j}^\infty_u\eta(v_I)\odot v_J\big)\end{aligned} \\ &= [\mathrm{j}^\infty_u\theta, \mathrm{j}^\infty_u\eta](v_{\{1,\dots,p\}}) \end{aligned}$$

for all homogeneous $v_1, v_2, \dots, v_p \in V^{\neq 0}$ such that $\sum_{i=1}^p |v_i| \neq -|\theta| - |\eta|$ and every $u \in U$. ☐

With these lemmas at our disposal, the proof of Theorem 5.1.1 becomes relatively straightforward.

Proof of Theorem 5.1.1: Every pluriderivation on $E|_U$ is of the form $\nabla_\rho + A$ for a unique vector bundle morphism $A - \rho\colon \bigodot(E) \to F \oplus \mathrm{T}U$, which corresponds to a unique smooth map $\tilde{A}_\Phi - \tilde{\rho}_\Phi$ from $\phi(U)$ to $\mathrm{L}(\bigodot(V^{\neq 0}), V)$ since Φ describes an isomorphism from $F \oplus \mathrm{T}U \to U$ to $V \times \phi(U) \to \phi(U)$. Similarly, every holonomic map from $\phi(U)$ to $\mathrm{L}(\bigodot(V), V)$ is of the form $\mathrm{j}^\infty\tilde{\mathfrak{l}}$ for a unique map $\tilde{\mathfrak{l}}$ from $\phi(U)$ to $\mathrm{L}(\bigodot(V^{\neq 0}), V)$. Bijectivity of j^∞_Φ, as a map from $\widehat{\mathrm{Der}}(E|_U)$ to $\mathrm{C}^\infty_{\mathrm{hol}}\big(U, \mathrm{L}(\bigodot(V), V)\big)$, now follows from the observation that it relates a pluriderivations and a holonomic maps precisely when they correspond to the same element of $\mathrm{C}^\infty(\phi(U), \mathrm{L}(\bigodot(V^{\neq 0}), V))$.

We can use Lemma 5.1.2 and Lemma 5.1.3 to show that j^∞_Φ is a morphism of graded Lie algebras. For any two homogeneous pluriderivations θ and η on $E \to M$, these lemmas together tell us that the restrictions

$$\mathrm{j}^\infty_\Phi[\theta,\eta](u)|_{\bigodot(V^{\neq 0})} \qquad \text{and} \qquad [\mathrm{j}^\infty\theta, \mathrm{j}^\infty\eta](u)|_{\bigodot(V^{\neq 0})}$$

coincide for every $u \in U$. This implies that $\mathrm{j}^\infty_\Phi[\theta,\eta] = [\mathrm{j}^\infty_\Phi\theta, \mathrm{j}^\infty_\Phi\eta]$ because both $\mathrm{j}^\infty_\Phi[\theta,\eta]$ and $[\mathrm{j}^\infty_\Phi\theta, \mathrm{j}^\infty_\Phi\eta]$ are holonomic, and thus determined by these restrictions. Since it is bijective, we conclude that j^∞_Φ is an isomorphism of graded Lie algebras.

Because curved split L_∞-algebroid structures on $E|_U$ are Maurer–Cartan elements for $\widehat{\mathrm{Der}}(E|_U)$ and holonomic families of curved L_∞-algebras are Maurer–Cartan elements for $\mathrm{C}^\infty_{\mathrm{hol}}\big(U, \mathrm{L}(\bigodot(V), V)\big)$, the isomorphism j^∞_Φ provides a one-to-one correspondence between objects of these respective types. ☐

All manifolds, bundles and maps in this section have been assumed smooth, but analyticity could have been assumed instead. The following proposition provides an analytic analogue of Theorem 5.1.1.

Proposition 5.1.4.
Let $E \to M$ be an analytic vector bundle of Fréchet type for which $E^0 = 0$ and let $\Phi = (\tilde{\Phi}, \phi)\colon E|_U \to V^{\neq 0} \times V^0$ be an analytic local trivialisation chart. A smooth curved L_∞-algebroid structure $\theta \in \mathrm{MC}(\widehat{\mathrm{Der}}(E|_U), [\,\cdot\,,\,\cdot\,])$ is analytic if and only if $\mathrm{j}^\infty_\Phi \theta\colon \phi(U) \to \mathrm{L}(\bigodot(V), V)$ is.

If θ is analytic and $(V, \ell) = (V, \mathrm{j}^\infty_\Phi(\phi(x_0)))$ is the L_∞-algebra corresponding to a fixed reference point $x_0 \in U$, then (V, ℓ) is convergent and

$$(V, \ell^y) = (V, \mathrm{j}^\infty_\Phi(\phi(x_0) + y))$$

for all y in a zero neighbourhood $U' \subseteq V^0$.

Proof: The proof of Theorem 5.1.1 can be repeated for analytic pluriderivations and analytic families of linear maps without major modifications to show that analytic curved L_∞-algebroid structures are locally in one-to-one correspondence with analytic holonomic families of curved L_∞-algebras. References to section 3.1.2 and appendix B.1.1 should be replaced by similar references to section 3.1.3 and appendix B.1.3 respectively.

Once analyticity of the holonomic family $\mathrm{j}^\infty_\Phi \theta$ is established, the final assertion follows directly from Proposition 3.3.2. □

Unlike in the smooth case, we can directly relate zeros of $\theta_0\colon M \to E^1$ to Maurer–Cartan elements when analyticity is assumed. Similarly, homotopies in the sense of Definition 2.4.6 locally correspond to homotopies in the sense of Definition 4.3.9.

Corollary 5.1.5.
Given the assumptions from Proposition 5.1.4, there exists an open neighbourhood $U' \subseteq M$ of x_0 such that ϕ restricts to a bijection

$$\phi\colon \mathrm{MC}(E, \theta, \rho) \cap U' \longrightarrow \mathrm{MC}(V, \ell) \cap \phi(U')$$

between a neighbourhood of x_0 in the zero locus of θ_0 and a zero neighbourhood in the Maurer–Cartan locus of (V, ℓ). It also induces a bijection

$$\Phi_*\colon \Gamma \mapsto \phi \circ \pi \circ \Gamma + \tilde{\Phi}_{-1} \circ \Gamma\, \mathrm{d}t$$

between the set of homotopies of Maurer–Cartan elements for $(E|_{U'}, \theta|_{U'}, \rho|_{U'})$ *and the set of homotopies* $\gamma + \eta\,\mathrm{d}t$ *for* $(V, \mathrm{j}^\infty_\Phi(0))$ *for which* $\gamma([0,1]) \subseteq \phi(U')$.

Proof: Proposition 5.1.4 tells us that the curvature of $(V, \ell^{\phi(x)})$ with $x \in U'$ is given by $\ell_0^{\phi(x)} = \theta_0(x)$. Consequently, $\phi(x)$ is a Maurer–Cartan element if and only if $\theta_0(x)$ vanishes.

The map Φ_* is a bijection from $\mathrm{C}^\infty([0,1], E^{-1}|_U)$ to $\mathrm{C}^\infty([0,1], V^0 \times V^{-1}\,\mathrm{d}t)$ because Φ is a trivialisation chart. The only thing that thus remains to be shown is that $\pi_* \circ \Gamma'(t) = \rho \circ \Gamma(t)$ if and only if $\gamma + \eta\,\mathrm{d}t = \Phi_*\Gamma$ solves the equation $\gamma'(t) + \ell_1^{\gamma(t)}(\eta(t)) = 0$ for $t \in [0,1]$. This follows from the fact that the left-hand side of the latter equation can be expressed as

$$\gamma'(t) + \ell_1^{\gamma(t)}(\eta(t)) = \mathrm{D}\phi(\pi(\Gamma(t)))\big(\pi_* \circ \Gamma'(t) - \rho \circ \Gamma(t)\big)$$

because $\ell_1^y \circ \Phi_{-1}(v) = -\mathrm{D}\phi(\phi^{-1}(y)) \circ \rho(v)$ for all $y \in \phi(U)$ and $v \in E_{\phi^{-1}(y)}$. □

5.1.2. Infinite jets of morphisms

We will now describe how the correspondence between curved split L_∞-algebroids and holonomic families of curved L_∞-algebras described in Theorem 5.1.1 extends to morphisms. We will show that curved L_∞-algebroid morphisms correspond to fibre-preserving (resp. fibrewise linear) morphisms of holonomic families of curved L_∞-algebras, and that splitting preserving morphisms of curved L_∞-algebroids correspond to fibrewise linear morphisms of holonomic families of curved L_∞-algebras.

Let $E \to M$ and $F \to N$ be $\mathbf{Z}$-graded vector bundles of Fréchet type for which $E^0 = 0$ and $F^0 = 0$, and assume that we are given two local trivialisation charts $\Phi(\widetilde{\Phi}, \phi)\colon E|_U \to V^{\neq 0} \times V^0$ and $\Phi' = (\widetilde{\Phi}', \phi')\colon F|_{U'} \to W^{\neq 0} \times W^0$. In terms of these charts, a graded vector bundle morphism (α, f) from $\bigodot(E)|_U$ to $F|_{U'}$ is described by smooth maps $\tilde{f}_{\Phi',\Phi}\colon \phi(U) \to W^0$ and $\tilde{\alpha}_{\Phi',\Phi}\colon \phi(U) \to \mathrm{L}(\bigodot(V^{\neq 0}), W^{\neq 0})$, which are such that

$$\phi' \circ f(x) = \tilde{f}_{\Phi',\Phi} \circ \phi(x)$$

and

$$\widetilde{\Phi}' \circ \alpha_x(e_{\{1,\dots,p\}}) = \tilde{\alpha}_{\Phi',\Phi}(\phi(x))\big(\textstyle\bigodot_{i=1}^p \Phi_x(e_i)\big)$$

for every $x \in U$ and all $e_1, e_2, \ldots, e_p \in E_x$. Since both $\tilde{f}_{\Phi',\Phi}$ and $\tilde{\alpha}_{\Phi',\Phi}$ take values in subspaces of $\mathrm{L}(\bigodot(V^{\neq 0}), W)$, we can consider their sum, $\tilde{f}_{\Phi',\Phi} + \tilde{\alpha}_{\Phi',\Phi}$ as a map from $\phi(U)$ to $\mathrm{L}(\bigodot(V^{\neq 0}), W)$. We denote the set of homogeneous vector bundle morphisms of degree 0 from $\bigodot(E)|_U \to U$ to $F|U' \to U'$ by $\mathrm{Mor}(\bigodot(E|_U), F|_{U'})$.

Proposition 5.1.6.
Let (E, θ, ρ) and (F, θ', ρ') be two curved split L_∞-algebroids of Fréchet type and let $\Phi\colon E|_U \to V^{\neq 0} \times V^0$ and $\Phi'\colon F|_{U'} \to W^{\neq 0} \times W^0$ be local trivialisation charts. The map $\mathrm{j}^\infty_{\Phi',\Phi}$ from $\mathrm{Mor}(\bigodot(E)|_U, F|_{U'})$ to $\mathrm{C}^\infty_{\mathrm{hol}}(\phi(U), \mathrm{L}(\bigodot(V), W))$ given by

$$\mathrm{j}^\infty_{\Phi',\Phi}(\alpha, f) = \mathrm{j}^\infty(\tilde{\alpha}_{\Phi',\Phi} + \tilde{f}_{\Phi',\Phi})$$

restricts to a one-to-one correspondence between curved L_∞-algebroid morphisms from $(E|_U, \theta|_U, \rho|_U)$ to $(F|_{U'}, \theta'|_{U'}, \rho'|_{U'})$ and fibre-preserving morphisms of holonomic families of curved L_∞-algebras from $\mathrm{j}^\infty_\Phi \theta$ to $\mathrm{j}^\infty_{\Phi'} \theta'$.

Proof: We can assume without loss of generality that both $E \to M$ and $F \to N$ are trivial bundles over Fréchet domains and that the chosen trivialisation charts are both tautological.

We thus assume that the graded vector bundles supporting (θ, ρ) and (θ', ρ') are of the form $E = V^{\neq 0} \times U \to U$ and $F = W^{\neq 0} \times U' \to U'$ for two $\mathbf{Z}$-graded vector spaces $V = V^{\neq 0} \oplus V^0$ and $W = W^{\neq 0} \oplus W^0$ of Fréchet type and open subsets $U \subseteq V^0$ and $U' \subseteq W^0$. We denote the corresponding holonomic families of curved L_∞-algebras, with respect to the tautological trivialisations $\Phi = \mathrm{id}_E$ and $\Phi' = \mathrm{id}_F$, by $\mathfrak{l} = \mathrm{j}^\infty_\Phi(\theta, \rho)$ and $\mathfrak{l}' = \mathrm{j}^\infty_{\Phi'}(\theta', \rho')$. Let $A\colon U \to \mathrm{L}(\bigodot(E), E)$ and $A'\colon U' \to \mathrm{L}(\bigodot(F), F)$ moreover be such that $\theta = \nabla^\Phi_\rho + A$ and $\theta' = \nabla^{\Phi'}_{\rho'} + A'$.

We recall from Proposition 3.3.8 that a holonomic family of plurilinear maps from V to W parametrised by $U \subseteq V^0$ is fibre-preserving precisely when it is of the form $\mathrm{j}^\infty(\mathfrak{g} + \mathfrak{f}_0)$ for smooth maps $\mathfrak{f}_0\colon U \to W$ and $\mathfrak{g}\colon U \to \mathrm{L}(\bigodot(V^{\neq 0}), W^{\neq 0})$. By construction, $\mathrm{j}^\infty_{\Phi',\Phi}$ thus describes a one-to-one correspondence between vector bundle morphisms from $\bigodot(E)$ to F and such fibre-preserving holonomic families of plurilinear maps. We do still need to verify that compatibility of $\mathrm{j}^\infty_{\Phi',\Phi}(\alpha, f)$ with $\mathrm{j}^\infty_\Phi(\theta, \rho)$ and $\mathrm{j}^\infty_{\Phi'}(\theta', \rho')$ is equivalent to compatibility of (α, f) with (θ, ρ) and (θ', ρ').

Let $(\alpha, f)\colon \bigodot(E) \to F$ be a homogeneous smooth vector bundle morphism and let $\mathfrak{f} = \mathrm{j}^\infty_{\Phi',\Phi}(\alpha, f)$ denote the corresponding holonomic family of plurilinear maps. Then $\mathfrak{f}$ is a morphism of holonomic families of curved L_∞-algebras if and only if it satisfies the equation $\mathfrak{f} \,\square\, \mathfrak{l} = \mathfrak{l}' \diamond \mathfrak{f}$, or

$$\sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J}\, \mathfrak{f}(u)\big(\mathfrak{l}(u)(v_I) \odot v_J\big) = \sum_{I_1,\dots,I_m} \epsilon_{I_1,\dots,I_m} \tfrac{1}{m!}\, \mathfrak{l}'(\mathfrak{f}_0(u))\big(\textstyle\bigodot_{j=1}^m \mathfrak{f}(u)(v_{I_j})\big) \quad (5.1.1)$$

for every $u \in U$ and all homogeneous $v_1, v_2, \dots, v_p \in V$. Since both $\mathfrak{l}' \diamond \mathfrak{f}$ and $\mathfrak{f} \,\square\, \mathfrak{l}$ are holonomic due to Proposition 3.2.11 and Proposition 3.2.7, it is sufficient for this equation to hold for homogeneous elements of non-zero degree.

Let $v_1, v_2, \dots, v_p \in V^{\neq 0}$ be homogeneous elements of V of non-zero degree. When distinguish two cases when examining equation (5.1.1).

(a) If $|v_{\{1,\dots,p\}}| = -1$, then many of the terms in equation (5.1.1) vanish because $\mathfrak{f}$ is fibre preserving. This is demonstrated in Proposition 3.3.8, from which we can deduce that equation (5.1.1) then reduces to

$$\mathfrak{f}(u)\big(\mathfrak{l}(u)(v_{\{1,\dots,p\}})\big) = \sum_{|v_{I_1}|,\dots,|v_{I_m}| \neq 0} \epsilon_{I_1,\dots,I_m} \tfrac{1}{m!}\, \mathfrak{l}'(\mathfrak{f}_0(u))\big(\textstyle\bigodot_{j=1}^m \mathfrak{f}(u)(v_{I_j})\big).$$

If we use holonomicity of $\mathfrak{f}$ to replace the left-hand side of this equation by $\mathrm{D}\mathfrak{f}_0(u)\big(\mathfrak{l}(u)(v_{\{1,\dots,p\}})\big)$, we see that it states precisely that

$$-\mathrm{T}\mathfrak{f}_0(u)(\rho_u(v_{\{1,\dots,p\}})) = -\rho'_{\mathfrak{f}_0(u)} \diamond \alpha_u(v_{\{1,\dots,p\}})$$

for all $u \in U$ and all homogeneous $v_1, v_2, \dots, v_p \in E_u \simeq V^{\neq 0}$. Note that both sides of this equation vanish due to homogeneity whenever $|v_{\{1,\dots,p\}}| \neq -1$.

(b) If $|v_{\{1,\dots,p\}}| \neq -1$, all of the terms in equation (5.1.1) are potentially non-zero. The equation then reads

$$\sum_{\substack{I \sqcup J = \{1,\dots,p\} \\ |v_I| = -1}} \epsilon_{I,J} \mathrm{D}\mathfrak{f}(u)\big(\mathfrak{l}(u)(v_I)\big)(v_J) + \sum_{\substack{I \sqcup J = \{1,\dots,p\} \\ |v_I| \neq -1}} \epsilon_{I,J} \mathfrak{f}(u)(\mathfrak{l}(u)(v_I) \odot v_J)$$
$$= \sum_{|v_{I_1}|,\dots,|v_{I_m}| \neq 0} \epsilon_{I_1,\dots,I_m} \tfrac{1}{m!}\, \mathfrak{l}'(u)\big(\textstyle\bigodot_{j=1}^m \mathfrak{f}(v_{I_j})\big).$$

In terms of the vector bundle morphisms A, A' and α, this becomes

$$\sum_{I \sqcup J = \{1,\dots,p\}} \epsilon_{I,J} \left(\mathscr{L}_{\rho(v_I)} \alpha^!_{\mathfrak{f}}\right)(u)(v_J) + \alpha_{\mathfrak{f}}(\mathfrak{l}_0(u)) \,\square\, A_{\mathfrak{l}}(u)(v_{\{1,\dots,p\}})$$
$$= A_{\mathfrak{l}'}(\mathfrak{f}_0(u)) \diamond \alpha_{\mathfrak{f}}(u)(v_{\{1,\dots,p\}})$$

for $u \in U$ and homogeneous $v_1, v_2, \dots, v_p \in V^{\neq 0}$. Note that both sides of this equation vanish if $|v_{\{1,\dots,p\}}| = -1$ due to homogeneity.

When combined, these two cases tell us that the hypotheses of Corollary 4.3.32 are satisfied (or the trivial connections) precisely when $\mathfrak{f} \,\square\, \mathfrak{l} = \mathfrak{l}' \diamond \mathfrak{f}$. Therefore, $\mathrm{j}^\infty_{\Phi',\Phi}(\alpha, f)$ is a morphism of holonomic families of curved L_∞-algebras if and only if (α, f) is a morphism of curved L_∞-algebroids. □

Proposition 5.1.7.
The map $\mathrm{j}^\infty_{\Phi',\Phi}$ described in Proposition 5.1.6 restricts to a one-to-one correspondence between splitting preserving morphisms of curved L_∞-algebroids and fibrewise linear morphisms of holonomic families of curved L_∞-algebras.

Proof: This follows from the characterisation of fibrewise linear morphisms of holonomic families provided by Proposition 3.3.10, since a morphism of curved L_∞-algebroids is splitting preserving precisely when it is described by a pair of maps $\tilde{f} : U \to U'$ and $\tilde{\alpha} : U \to \mathrm{L}(V, W)$. □

As one might expect, the correspondence described in Proposition 5.1.6 is compatible with the composition of morphisms.

Proposition 5.1.8.
Let $(\alpha, f) \colon (E, \theta, \rho) \to (F, \theta', \rho')$ and $(\beta, g) \colon (F, \theta', \rho') \to (G, \theta'', \rho'')$ be two morphisms of curved L_∞-algebroids of Fréchet type, then

$$\mathrm{j}^\infty_{\Phi'',\Phi}\big((\beta, g) \diamond (\alpha, f)\big) = \mathrm{j}^\infty_{\Phi'',\Phi'}(\beta, g) \diamond \mathrm{j}^\infty_{\Phi',\Phi}(\alpha, f).$$

for any three local trivialisation charts Φ, Φ' and Φ'' on $E|_U$, $F|_{U'}$ and $G|_{U''}$ respectively such that $f(U) \subseteq U'$ and $g(U') \subseteq U''$.

Moreover, $\mathrm{j}^\infty_{\Phi,\Phi}(\mathrm{id}_E, \mathrm{id}_M)$ is the identity morphism on $(\phi(U), V, \mathrm{j}^\infty_{\Phi',\Phi}\theta)$.

Proof: As in the proof Proposition 5.1.8, we shall assume without loss of generality that all three graded vector bundles are trivial over open subsets of Fréchet spaces and that this choice.

Both $\mathrm{j}^\infty_{\Phi'',\Phi}\big((\beta, g) \diamond (\alpha, f)\big)$ and $\mathrm{j}^\infty_{\Phi'',\Phi'}(\beta, g) \diamond \mathrm{j}^\infty_{\Phi',\Phi}(\alpha, f)$ are holonomic families of plurilinear maps due to Proposition 3.2.11. To show that they are equal it thus suffices to show that they coincide when restricted to $\bigodot(V^{\neq 0}) \subseteq \bigodot(V)$. We observe that

$$\mathrm{j}^{\infty}_{\Phi'',\Phi}\big((\beta,g)\diamond(\alpha,f)\big)(u)(1) = \mathrm{j}^{\infty}_{\Phi'',\Phi'}(\beta,g)(u')\diamond \mathrm{j}^{\infty}_{\Phi',\Phi}(\alpha,f)(u)(1),$$

for any $u \in \phi(U) = U$ and $u' = f(u) \in \phi'(U') = U'$, since both sides are equal to $g \circ f(u)$. A simple computation moreover shows that

$$\mathrm{j}^{\infty}_{\Phi'',\Phi}\big((\beta,g)\diamond(\alpha,f)\big)(u)(v_{\{1,\dots,p\}}) = \mathrm{j}^{\infty}_{\Phi'',\Phi'}(\beta,g)(u')\diamond \mathrm{j}^{\infty}_{\Phi',\Phi}(\alpha,f)(u)(v_{\{1,\dots,p\}}),$$

for u and u' as before and $v_1, v_2, \dots, v_p \in V^{\neq 0}$ with $p \geq 1$, since both sides of this equation are equal to $\beta_{u'} \diamond \alpha_u(v_{\{1,\dots,p\}})$.

With respect to the trivialisation chart Φ, the identity morphism $(\mathrm{id}_E, \mathrm{id}_M)$ is represented by two maps, $(\widetilde{\mathrm{id}}_M)_{\Phi,\Phi}$ and $(\widetilde{\mathrm{id}}_E)_{\Phi,\Phi}(u)$, that are given by $(\widetilde{\mathrm{id}}_M)_{\Phi,\Phi}(u) = u$ and $(\widetilde{\mathrm{id}}_E)_{\Phi,\Phi}(u) = \mathrm{id}_{V^{\neq 0}}$ for $u \in U$. Consequently, $\mathrm{j}^{\infty}_{\Phi,\Phi}(\mathrm{id}_E, \mathrm{id}_M) = \mathrm{id}_U + \mathrm{id}_V$ is the identity morphism. ☐

We had claimed that the holonomic family of curved L_∞-algebras constructed in Theorem 5.1.1 is independent of the chosen trivialisation chart (up to isomorphism). This is an immediate consequence of Proposition 5.1.6 and Proposition 5.1.8.

Corollary 5.1.9.
For any curved split L_∞-algebroid (E, θ, ρ) of Fréchet type and two local trivialisation charts $\Phi\colon E|_U \to V^{\neq 0} \times V^0$ and $\Phi'\colon E|_U \to W^{\neq 0} \times W^0$ on the same open subset $U \subseteq M$, the map

$$\mathrm{j}^{\infty}_{\Phi',\Phi}(\mathrm{id}_E, \mathrm{id}_M)\colon (\phi(U), V, \mathrm{j}^{\infty}_{\Phi}\theta) \longrightarrow (\phi'(U), W, \mathrm{j}^{\infty}_{\Phi'}\theta)$$

is a fibrewise linear isomorphism of holonomic families of curved L_∞-algebras.

Proof: It follows immediately from Proposition 5.1.6 that $\mathrm{j}^{\infty}_{\Phi',\Phi}(\mathrm{id}_E, \mathrm{id}_M)$ and $\mathrm{j}^{\infty}_{\Phi,\Phi'}(\mathrm{id}_E, \mathrm{id}_M)$ are both morphisms of holonomic families of curved L_∞-algebras. We can work out from Proposition 5.1.8 that these are mutually inverse. ☐

Remark 5.1.10.
Theorem 5.1.1 tells us that every curved split L_∞-algebroid can locally be described as a holonomic family of curved L_∞-algebras, while Proposition 5.1.6 and Proposition 5.1.8 tell us that splitting-preserving morphisms correspond

to fibrewise linear morphisms of such holonomic families in a functorial fashion. This allow us to obtain a curved split L_∞-algebroid by gluing together holonomic families of curved L_∞-algebras using fibrewise linear morphisms.

△

5.1.3. Equivalence of categories

Section 5.1.1 and section 5.1.2 can be summarised rather concisely in the language of categories.

To do this, we shall consider the category of curved L_∞-algebroids of Fréchet type endowed with a global trivialisation chart. An object in this category is a pair $((E,\theta,\rho),\Phi)$ that consist of a curved L_∞-algebroid (E,θ,ρ) of Fréchet type and a trivialisation chart $\Phi\colon E \to V^{\neq 0} \times V^0$. We define a morphism between two such objects as a morphism between the underlying curved L_∞-algebroids, and require no compatibility with the trivialisation charts Φ and Φ'. Consequently, any two objects $((E,\theta,\rho),\Phi)$ and $((E,\theta,\rho),\Phi')$ in this category that only differ by a choice of trivialisation chart are canonically isomorphic through the identity morphism id_E for (E,θ,ρ).

Theorem 5.1.11.
The category of curved split L_∞-algebroids of Fréchet type endowed with global trivialisation charts is equivalent to the category of holonomic families of curved L_∞-algebras of Fréchet type with fibre-preserving morphisms.

Proof: Consider the infinite jet functor which we have effectively studied in section 5.1.1 and Section 5.1.2. It assigns to every pair $((E,\theta,\rho),\Phi)$ consisting of a curved L_∞-algebroid (E,θ,ρ) and a trivialisation chart $\Phi = (\widetilde{\Phi},\phi)\colon E \to V^{\neq 0} \times V^0$, the holonomic family

$$\mathrm{j}^\infty_\Phi(\theta,\rho)\colon \mathrm{im}(\phi) \longrightarrow \mathrm{L}(\bigodot(V),V)$$

described by Theorem 5.1.1, and to a morphism (α,f) from $((E,\theta,\rho),\Phi)$ to $((F,\theta',\rho'),\Phi')$ it similarly associates the morphism of holonomic families

$$\mathrm{j}^\infty_{\Phi',\Phi}(\alpha,f)\colon \mathrm{im}(\phi) \longrightarrow \mathrm{L}(\bigodot(V),W)$$

described by Proposition 5.1.6. Functoriality of this construction was in demonstrated Proposition 5.1.8, and we have seen in Proposition 5.1.6 that j^∞ is bijective on morphisms, i.e. that it is both full and faithful.

Conversely, we can associate to every holonomic family $\mathfrak{l}\colon U \to \mathrm{L}(\bigodot(V), V)$ of curved L_∞-algebras, a trivial $\mathbf{Z}$-graded vector bundle $E_\mathfrak{l} = V^{\neq 0} \times U \to U$ for which $E^0 = 0$. Theorem 5.1.1 tells us that this bundle admits a unique curved L_∞-algebroid structure $\theta_\mathfrak{l}$ such that $\mathrm{j}^\infty_{\mathrm{id}}(\theta_\mathfrak{l}) = \mathfrak{l}$, which establishes surjectivity of j^∞. Since j^∞ is fully faithful and essentially surjective, it describes a weak equivalence of categories.

A weak inverse to j^∞ is given by the functor Θ that assigns to a holonomic family $\mathfrak{l}$ of curved L_∞-algebras the curved L_∞-algebroid $(E_\mathfrak{l}, \theta_\mathfrak{l}, \rho_{\theta_\mathfrak{l}})$ described above, and which assigns to a morphism $\mathfrak{f}\colon \mathfrak{l} \to \mathfrak{l}'$ of such holonomic families the unique morphism $\Theta(\mathfrak{f})$ from $\Theta(\mathfrak{l})$ to $\Theta(\mathfrak{l}')$ with the property that $\mathrm{j}^\infty(\Theta(\mathfrak{f})) = \mathfrak{f}$. This functor is by construction such that $\mathrm{j}^\infty \circ \Theta = \mathrm{id}$. A simple natural isomorphism relating the other composition, $\Theta \circ \mathrm{j}^\infty$, to the identity on its domain is the one that sends a pair $((E, \theta, \rho), \Phi)$ to the trivialisation chart $\Phi\colon E \xrightarrow{\sim} V^{\neq 0} \times U$, viewed as a L_∞-algebroid morphism from (E, θ, ρ) to the L_∞-algebroid structure $\Theta(\mathrm{j}^\infty_\Phi \theta)$ on the trivial bundle $V^{\neq 0} \times U \to U$. □

We observe that under the equivalence described in the proof of Theorem 5.1.11, splitting-preserving morphisms of curved L_∞-algebroids correspond to fibrewise linear morphisms of holonomic families of L_∞-algebras, due to Proposition 5.1.7.

We can forget about the choice of trivialisation chart and talk about the category of curved L_∞-algebroids of Fréchet type that admit such charts. This category is weakly equivalent to the category of curved L_∞-algebroids of Fréchet type endowed with a trivialisation chart because the forgetful functor from the latter to the former is surjective and fully faithful. This directly leads to the following corollary.

Corollary 5.1.12.
The category of curved L_∞-algebroids that admit global trivialisation charts is weakly equivalent to the category of holonomic families of curved L_∞-algebras with fibre-preserving morphisms.

In this sense, curved L_∞-algebroids and holonomic families of curved L_∞-algebras are locally equivalent.

Remark 5.1.13.

The proof of Theorem 5.1.11 could arguably have been concluded upon observing that j^∞ is fully faithful and essentially surjective, since a weak inverse to any such functor is ostensibly provided by the axiom of choice. The word "weakly" could then similarly be removed of Corollary 5.1.12. Since none of the categories discussed in this section is described as a set, however, whether or not the axiom of choice can be applied in this circumstance depends on one's foundational approach to category theory. △

5.2. Equivariant deformation problems

In this section, we will describe how an equivariant deformation problem of Fréchet type can be viewed as a special type of curved split L_∞-algebroid. As such, it will give rise to holonomic families of curved L_∞-algebras whose members can be said to formally control the deformations of the corresponding structures. In an analytic setting, these L_∞-algebras become slightly more meaningful since their Maurer–Cartan elements then correspond to (integrable) structure and gauge equivalence of these elements then implies gauge equivalence of the structures they represent.

5.2.1. Deformation problems as L_∞-algebroids

We will first demonstrate how every equivariant deformation problem can be viewed as a special type of curved L_∞-algebroid.

Although equivariant deformation problems were discussed in detail in chapter 1, we shall briefly recall their definition here. An equivariant deformation problem $(M \xrightarrow{\mu} E^1 \xrightarrow{\beta} E^2) \overset{\alpha}{\circlearrowleft} G$ of Fréchet type with (optional) Bianchi identity consists of the following objects:

- a Fréchet manifold M,
- a Fréchet Lie group G acting on M from the right by $\alpha\colon M \times G \to M$,
- two equivariant vector bundles $E^1 \to M$ and $E^2 \to M$ of Fréchet type,
- an equivariant section $\mu\colon M \to E^1$ of $E^1 \to M$, and

- an equivariant vector bundle morphism $\beta\colon E^1 \to E^2$ such that $\beta \circ \mu = 0$.

When we write $(M \xrightarrow{\mu} E^1 \xrightarrow{\beta} E^2) \overset{\alpha}{\circlearrowleft} G$, the letter α is understood to refer not only to the right action of G on M explicitly described above, but also to the implicit right actions of G on the equivariant vector bundles $E^1 \to M$ and $E^2 \to M$. It is thus an abbreviation for the triple $(\alpha, \alpha^1, \alpha^2)$ of right actions

$$\alpha\colon M \times G \longrightarrow M \qquad \text{and} \qquad \alpha^n\colon E^n \times G \longrightarrow E^n$$

for $n = 1, 2$.

The inclusion of the vector bundle E^2 and the Bianchi map $\beta\colon E^1 \to E^2$ is optional in the sense that one can always choose to set $E^2 = 0$ and (thus necessarily) $\beta = 0$. The Bianchi identity $\beta \circ \mu = 0$ is then automatically satisfied for any $\mu \in \mathscr{E}^1(M)$ and the unique (equivariant) vector bundle morphism $\beta = 0$ from E^1 to E^2.

Infinitesimal actions

The right actions α and α^n for $n = 1, 2$ can be differentiated to obtain Lie algebra morphisms

$$\alpha_*\colon \mathrm{Lie}(G) \longrightarrow \mathfrak{X}(M) \qquad \text{and} \qquad \alpha^n_*\colon \mathrm{Lie}(G) \longrightarrow \mathrm{Der}(E^n)$$

from $\mathrm{Lie}(G)$ to the Lie algebra of vector fields on M and the Lie algebras of smooth derivations on E^1 and E^2 respectively. The first is given by

$$\alpha_*(X)(x) = \mathrm{T}_{(x, \mathrm{e}_G)} \alpha(X),$$

for $X \in \mathrm{Lie}(G)$ and $x \in M$, and the second by

$$\alpha^n_*(X)(e)(x) = \mathrm{T}_x e\big(\alpha_*(X)(x)\big) - \mathrm{T}_{(e(x), \mathrm{e}_G)} \alpha^n(X) \in \mathrm{T}^{\mathrm{v}} E^n_{e(x)} \simeq E^n_x$$

for any Lie algebra element $X \in \mathrm{Lie}(G)$ and any local section $e \in \mathscr{E}^n(U)$ defined on an open subset $U \subseteq M$ of $x \in M$. These actions were also considered in Example 4.2.7 and Example 4.2.26.

We shall consider the action algebroid corresponding to the Lie algebra action described above, which we will denote by $(E^{-1}, [\![\cdot,\cdot]\!], \rho)$. It was defined in

Example 4.2.22 and consists of the trivial bundle $E^{-1} = \mathrm{Lie}(G) \times M \to M$, the anchor

$$\rho\colon \mathrm{Lie}(G) \times M \longrightarrow \mathrm{T}M, \qquad (X, x) \mapsto \alpha_*(X)(x) = \mathrm{T}_{(x,\mathrm{e}_G)}\alpha(X),$$

which describes the infinitesimal action of $\mathrm{Lie}(G)$, and a Lie bracket which is determined by the property that $[\![a, b]\!](x) = [a(x), b(x)]$ for any two constant sections $a, b \in \mathscr{E}^{-1}(M) \simeq \mathrm{C}^\infty(M, \mathrm{Lie}(G))$. The Lie bracket of two arbitrary local sections $a_1, a_2 \in \mathscr{E}^{-1}(U)$ with $U \subseteq M$ open is thus given by

$$[\![a_1, a_2]\!](x) = [a_1(x), a_2(x)] + \mathscr{L}_{\rho(a_1(x))}a_2 - \mathscr{L}_{\rho(a_2(x))}a_1$$

for $x \in U$.

The infinitesimal actions of $\mathrm{Lie}(G)$ on E^1 and E^2 similarly give rise to two representations (E^1, ∇^1) and (E^2, ∇^2) of the action the Lie algebroid $(E^{-1}, [\![\cdot, \cdot]\!], \rho)$. These representations are such that $\nabla^n_a = \alpha^n_*(a)$ for any constant section $a \in \mathscr{E}^{-1}(M) \simeq \mathrm{C}^\infty(M, \mathrm{Lie}(G))$, so that

$$\nabla^n_a e(x) = \mathrm{T}_x e(\rho(a(x))) - \mathrm{T}_{(e(x),\mathrm{e}_G)}\alpha^n(a(x))$$

for two arbitrary local sections $a \in \mathscr{A}(U)$ and $e \in \mathscr{E}^n(U)$ with $U \subseteq M$ open, for $x \in U$ and for $n = 1, 2$. Since μ and β were equivariant for the original action of G by assumption, they are also equivariant for this infinitesimal action.

This discussion is summarised by the following proposition.

Proposition 5.2.1.
Let $(M \xrightarrow{\mu} E^1 \xrightarrow{\beta} E^2) \overset{\alpha}{\circlearrowleft} G$ be an equivariant deformation problem of Fréchet type with (optional) Bianchi identity and let E^{-1} denote the trivial vector bundle $E^{-1} = \mathrm{Lie}(G) \times M \to M$. If ρ, $[\![\cdot, \cdot]\!]$ and ∇^n for $n = 1, 2$ are as described above, then:

1. *$(E^{-1}, [\![\cdot, \cdot]\!], \rho)$ is a Lie algebroid:*
   For any local section $a \in \mathscr{E}^{-1}(U)$ with $U \subseteq M$ open, $[\![a, \cdot]\!]\colon \mathscr{E}^{-1}|_U \to \mathscr{E}^{-1}|_U$ is a smooth derivation covering $\rho(a)$, ρ is a vector bundle morphism, and

$$\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(a, b, c) := [\![[\![a, b]\!], c]\!] + [\![[\![b, c]\!], a]\!] + [\![[\![c, a]\!], b]\!] = 0$$

   for any three local sections $a, b, c \in \mathscr{E}^{-1}(U)$ with $U \subseteq M$ open.

2. *(E^1, ∇^1) and (E^2, ∇^2) are representations of $(E^{-1}, [\![\cdot,\cdot]\!], \rho)$:*
   The operators ∇^n for $n = 1, 2$ are vector bundle morphisms in their first arguments, $\nabla^n_a \colon \mathscr{E}^n|_U \to \mathscr{E}^n|_U$ is a smooth derivation covering $\rho(a)$ for every local section $a \in \mathscr{E}^{-1}(U)$ with $U \subseteq M$ open, and

   $$F_{\nabla^n}(a, b) := [\nabla^n_a, \nabla^n_b] - \nabla^n_{[\![a,b]\!]} = 0$$

   for any two local sections $a, b \in \mathscr{E}^{-1}(U)$ with $U \subseteq M$ open and $n = 1, 2$.
3. *The section $\mu \colon \mathscr{E}^1(M)$ is infinitesimally equivariant:*

   $$\nabla^1 \mu = 0.$$

4. *The vector bundle morphism $\beta \colon E^1 \to E^2$ is infinitesimally equivariant:*

   $$\nabla^{2,1}_a \beta := \nabla^2_a \circ \beta - \beta \circ \nabla^1_a = 0$$

   for any local section $a \in \mathscr{E}^{-1}(U)$ with $U \subseteq M$ open.
5. *The Bianchi identity holds:*

   $$\beta \circ \mu = 0.$$

Proof: Each of the above identities follows from an explicit computation. □

The associated L_∞-algebroid

The section $\mu \in \mathscr{E}^1(M)$, the vector bundle morphism $\beta \colon E^1 \to E^2$ and the operators

$$[\![\cdot,\cdot]\!] \colon \mathscr{E}^{-1} \odot \mathscr{E}^{-1} \longrightarrow \mathscr{E}^{-1} \qquad \text{and} \qquad \nabla^n \colon \mathscr{E}^{-1} \otimes \mathscr{E}^n \longrightarrow \mathscr{E}^n$$

for $n = 1, 2$ can be combined to form a curved split L_∞-algebroid.

Proposition 5.2.2 (Equivariant deformation problems ⇒ L_∞-algebroids)**.** *Let $(M \xrightarrow{\mu} E^1 \xrightarrow{\beta} E^2) \circlearrowleft^{\alpha} G$ be an equivariant deformation problem of Fréchet type with (optional) Bianchi identity and let $(E^{-1}, [\![\cdot,\cdot]\!], \rho)$ and (E^n, ∇^n) for $n = 1, 2$ be as in Proposition 5.2.1, then*

$$\theta_{\alpha,\mu,\beta} = [\![\cdot,\cdot]\!] + \mu + \beta + \nabla^1 + \nabla^2,$$

defines a curved split L_∞-algebroid structure on E with anchor ρ.

Proof: To prove that (E, θ, ρ) is a curved split L_∞-algebroid we first need to show that the components of θ are smooth multiderivations of degree 1 whose symbols are determined by ρ. Homogeneity of θ follows from its construction, so we only need to verify that

$$\theta(e_{\{1,\dots,p\}} \odot \,\cdot\,)\colon \mathscr{E}|_U \longrightarrow \mathscr{E}|_U$$

is a smooth derivation covering $\rho(e_{\{1,\dots,p\}})$ for any sequence of homogeneous local sections $e_1, e_2, \dots, e_p \in \mathscr{E}(U)$ with $U \subseteq M$ open.

Since it takes no arguments, there is effectively nothing to verify for the nullary component $\theta_0 = \mu \in \mathscr{E}^1 \simeq \operatorname{Der}^0(E)^1$, and the unary component $\theta_1 = \beta$ is a smooth derivation covering $\rho(1) = 0$ precisely because β is a vector bundle morphism. For the binary component $\theta_2 = [\![\,\cdot\,,\,\cdot\,]\!] + \nabla^1 + \nabla^2$ we need to verify that

$$\theta_2(a \odot \,\cdot\,) = [\![a, \,\cdot\,]\!] + \nabla^1_a + \nabla^2_a\colon \mathscr{E} \longrightarrow \mathscr{E}$$

is a smooth derivation covering $\rho(a) \in \mathfrak{X}(M)$ for every $a \in \mathscr{E}^{-1}(U)$ with $U \subseteq M$ open, and that

$$\theta_2(e \odot \,\cdot\,) = (-1)^n\, \nabla^n e\colon \mathscr{E}^{-1} \longrightarrow \mathscr{E}^n$$

is a vector bundle morphism for every $e \in \mathscr{E}^n(U)$ with $n = 1, 2$ with $U \subseteq M$ open. Both of these conditions hold because $([\![\,\cdot\,,\,\cdot\,]\!], \rho)$ is a Lie algebroid structure and ∇^1 and ∇^2 are compatible Lie algebroid connections. Since θ has no components of arity 3 or greater, we conclude that it is a smooth pluriderivation of degree 1 on E and that $\rho\colon \bigodot(E) \to \mathrm{T}M$ is its anchor.

The Jacobiator of θ can be expressed in terms of the graded Lie bracket on $\widehat{\operatorname{Der}}(E)$ as $\operatorname{Jac}(\theta) = \frac{1}{2}[\theta, \theta]$. If we expand θ in terms of the components $[\![\,\cdot\,,\,\cdot\,]\!]$, μ, β and $\nabla = \nabla^1 + \nabla^2$, we thus the expression

$$\tfrac{1}{2}\big[[\![\,\cdot\,,\,\cdot\,]\!], [\![\,\cdot\,,\,\cdot\,]\!]\big] + [\beta, \mu] + [\nabla, \mu] + [\nabla, \beta] + \Big(\big[\nabla, [\![\,\cdot\,,\,\cdot\,]\!]\big] + \tfrac{1}{2}[\nabla, \nabla]\Big).$$

for $\operatorname{Jac}(\theta)$. These five terms are the restrictions of $\operatorname{Jac}(\theta)$ to $\bigodot^3(\mathscr{E}^{-1})$, $\bigodot^0(\mathscr{E})$, $\mathscr{E}^{-1}$, $\mathscr{E}^{-1} \odot (\mathscr{E}^1 \oplus \mathscr{E}^2)$ and $\bigodot^2(\mathscr{E}^{-1}) \odot (\mathscr{E}^1 \oplus \mathscr{E}^2)$ respectively. For $(\mathscr{E}, \theta)$ to be a curved L_∞-algebra, all five of these components should vanish.

Each of these graded commutators are relatively simple to work out, provided some care is taken when determining the appropriate signs. The first expression is simply the Jacobiator of $[\![\,\cdot\,,\,\cdot\,]\!]$, since

$$\tfrac{1}{2}\big[[\![\,\cdot\,,\,\cdot\,]\!], [\![\,\cdot\,,\,\cdot\,]\!]\big](a, b, c) = [\![[\![a, b]\!], c]\!] + [\![[\![b, c]\!], a]\!] + [\![[\![c, a]\!], b]\!]$$

for all $a, b, c \in \mathscr{E}^{-1}(U)$ with $U \subseteq M$ open. Going from left to right, the remaining terms are given by

$$[\beta, \mu] = \beta \circ \mu,$$
$$[\nabla, \mu](a) = -\nabla_a \mu,$$
$$[\nabla, \beta](a, e) = -\nabla_a(\beta(e)) + \beta(\nabla_a e)$$

for $a \in \mathscr{E}^{-1}(U)$ and $e \in \mathscr{E}^1(U)$ with $U \subseteq M$ open, and

$$\big([\nabla, [\![\cdot, \cdot]\!]] + \tfrac{1}{2}[\nabla, \nabla]\big)(a \odot b \odot e) = \nabla_{[\![a,b]\!]} e - \nabla_a \nabla_b e + \nabla_b \nabla_a e.$$

for $a, b \in \mathscr{E}^{-1}(U)$ and $e \in \mathscr{E}^1(U) \oplus \mathscr{E}^2(U)$ with $U \subseteq M$ open.

All in all, we deduce that the Jacobiator of θ is given by

$$\mathrm{Jac}(\theta) = \mathrm{Jac}([\![\cdot, \cdot]\!]) + \beta \circ \mu - \nabla\mu - \nabla\beta - F_\nabla,$$

where $F_\nabla = F_{\nabla^1} + F_{\nabla^2}$ denotes the combined curvature of $\nabla = \nabla^1 + \nabla^2$. Since these six terms live in different components of $\widehat{\mathrm{Der}}(E)$, the Jacobiator of θ vanishes if and only if $[\![\cdot, \cdot]\!]$ is a Lie bracket, the composition $\beta \circ \mu$ vanishes, μ and β are equivariant and the curvature of ∇ is zero. This allows us to conclude that (E, θ, ρ) is indeed a curved split L_∞-algebroid. ☐

We observe that each of components of the Jacobiator of θ corresponds to one of the five identities featured in Proposition 5.2.1. The arguments in this proof can thus be reversed to show that any curved split L_∞-algebroid structure (θ, ρ) on $E = E^{-1} \oplus E^1 \oplus E^2$ which has no components of arity greater than 2 is of the form

$$\theta = [\![\cdot, \cdot]\!] + \mu + \beta + \nabla^1 + \nabla^2$$

where $([\![\cdot, \cdot]\!], \rho)$ is a Lie algebroid structure on E^{-1}, ∇^1 and ∇^2 are flat Lie algebroid connections on E^1 and E^2, μ is an equivariant section of E^1 and β is an equivariant vector bundle morphism from E^1 to E^2 with the property that $\beta \circ \mu = 0$. This does not necessarily mean that it comes from an equivariant deformation problem as defined in Definition 1.1.5 however, since not every Lie algebroid is an action algebroid and not every (Fréchet) Lie algebra integrates to a Lie group.

If G is a connected regular Fréchet Lie group, then the original deformation problem is still fully described by the curved L_∞-algebroid provided by Proposition 5.2.2 in the following sense.

Proposition 5.2.3.
Let $(M \xrightarrow{\mu} E^1 \xrightarrow{\beta} E^2) \overset{\alpha}{\circlearrowleft} G$ be an equivariant deformation problem of Fréchet type and let (E, θ, ρ) be its associated curved split L_∞-algebroid, then

$$\mathrm{MC}(E, \theta, \rho) = \mu^{-1}(0).$$

If the Lie group G is regular (in the sense of Definition B.3.2), then two Maurer–Cartan elements $x, y \in \mathrm{MC}(E, \theta, \rho)$ are homotopic if and only if there exists a group element $g \in G_0$ in the identity component of G such that $y = x \cdot g$.

Proof: The Maurer–Cartan locus of (E, θ, ρ) is defined as the zero locus of the curvature $\theta_0 \in \mathscr{E}^1(M)$, which is equal to μ by construction. Therefore, $\mathrm{MC}(E, \theta, \rho) = \mu^{-1}(0)$.

If G is a regular Fréchet Lie group, then Corollary B.3.4 tells us that two points $x, y \in M$ are in the same orbit for the action of G_0 if and only if there exist two smooth paths $\eta \colon [0,1] \to \mathrm{Lie}(G)$ and $\gamma \colon [0,1] \to M$ such that $\gamma(0) = x$, $\gamma(1) = y$ and

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = \alpha_*(\eta(t))(\gamma(t))$$

for all $t \in [0,1]$. Since $\alpha_*(\eta(t))(\gamma(t)) = \rho(\eta(t), \gamma(t))$ for all $t \in [0,1]$, this equation states precisely that $(\gamma, \eta) \colon [0,1] \to E^{-1}$ is a homotopy with endpoints x and y, as defined in Definition 4.3.9. ☐

5.2.2. The associated curved L_∞-algebra

We wish to associate a curved L_∞-algebra to an arbitrary equivariant deformation problem. Because we have just seen in section 5.2.1 that equivariant deformation problems can be described as curved L_∞-algebroids, we need only apply the theory from section 5.1 to do this. This will provide a holonomic family of L_∞-algebras corresponding to different reference points.

Suppose we are given an equivariant deformation problem $(M \xrightarrow{\mu} E^1 \xrightarrow{\beta} E^2) \overset{\alpha}{\circlearrowleft} G$ of Fréchet type and let (E, θ, ρ) denote the corresponding curved L_∞-algebroid. Assume moreover that we are given a trivialisation chart $\Phi = (\tilde{\Phi}, \varphi) \colon E|_U \to V^{\neq 0} \times V^0$ centred at $x_0 \in U \subseteq M$. We recall from section 5.1.1 that the associated holonomic family of curved L_∞-algebras was the family

$$\mathrm{j}^\infty_\Phi \theta = \mathrm{j}^\infty_\Phi A_{\theta,\Phi} - \mathrm{j}^\infty_\Phi \rho \colon \phi(U) \longrightarrow \mathrm{L}(\bigodot(V), V),$$

where $A_{\theta,\Phi}\colon \bigodot(E)|_U \to \mathrm{T}U$ is a vector bundle morphism with the property that $\theta|_U = \mathscr{L}_\rho^\Phi + A_{\theta,\Phi}$, and the infinite jets $\mathrm{j}_\Phi^\infty A_{\theta,\Phi}$ and $\mathrm{j}_\Phi^\infty \rho$ are as defined in Theorem 5.1.1. Since the bundle $E^{-1} = \mathrm{Lie}(G)\times M \to M$ is trivial by construction, we will assume that it is comes with its canonical trivialisation.

We can explicitly describe the curved L_∞-algebra $(V, \mathrm{j}_\Phi^\infty\theta(0))$ in terms of the original data, but the resulting expressions are not particularly appealing or enlightening. Below, we shall use the notation

$$f(C,y) \approx \sum_{p\in\mathbf{N}_0} f_p(C,y)$$

to say that a smooth function $y\mapsto f(C,y)$ from V^0 to another locally convex vector space has $\sum_{p\in\mathbf{N}_0} f_p(C,\cdot)$ as its formal power series around 0.

Theorem 5.2.4 (Equivariant deformation problems ⇒ L_∞-algebras)**.** *Let $(M\xrightarrow{\mu}E^1\xrightarrow{\beta}E^2)\overset{\alpha}{\circlearrowleft}G$ be an equivariant deformation problem of Fréchet type. Let $V^{-1} = \mathrm{Lie}(G)$ and let $\Phi_1 = (\widetilde{\Phi}_1,\varphi)\colon E^1|_U \to V^1\times V^0$ and $(\widetilde{\Phi}_2,\varphi)\colon E^2|_U \to V^2\times V^0$ be two trivialisation charts that cover the same chart $\varphi\colon U\subseteq M\to V^0$ centred at $x_0\in M$. The graded vector space*

$$V^{-1}\oplus V^0\oplus V^1\oplus V^2 \simeq \mathrm{Lie}(G)\oplus \mathrm{T}_{x_0}M\oplus E^1_{x_0}\oplus E^2_{x_0}$$

admits a topological curved L_∞-algebra structure $\ell\colon \bigodot(V)\to V$ which is determined by the following properties.

(i) *The restriction of ℓ to $\bigodot(V^0)$ is such that*

$$\widetilde{\Phi}_1\circ\mu\circ\varphi^{-1}(y) \approx \sum_{p\in\mathbf{N}_0}\frac{1}{p!}\,\ell_p\big(\textstyle\bigodot^p y\big). \tag{5.2.1}$$

(ii) *The restriction of ℓ to $\bigodot(V^0)\odot V^1$ is such that*

$$\widetilde{\Phi}_2\circ\beta\circ\Phi_1^{-1}(v,y) \approx \sum_{p\in\mathbf{N}_0}\frac{1}{p!}\,\ell_{p+1}\big(\textstyle\bigodot^p y\odot v\big) \tag{5.2.2}$$

for $v\in V^1$.

(iii) *The restriction of ℓ to $\bigodot(V^0)\odot V^{-1}\odot V^{-1}$ is given by*

$$[a_1,a_2] \approx \sum_{p\in\mathbf{N}_0}\frac{1}{p!}\,\ell_{p+2}\big(\textstyle\bigodot^p y\odot a_1\odot a_2\big) \tag{5.2.3}$$

for all $a_1,a_2\in V^{-1} = \mathrm{Lie}(G)$.

(iv) *The restriction of ℓ to $\bigodot(V^0)\odot V^{-1}$ is such that*

$$\mathrm{d}\varphi\circ \mathrm{D}\alpha(\varphi^{-1}(y),\mathrm{e}_G)(a)\approx\sum_{p\in\mathbf{N}_0}-\frac{1}{p!}\,\ell_{p+1}\big(\textstyle\bigodot^p y\odot a\big) \tag{5.2.4}$$

for $a\in V^{-1}=\mathrm{Lie}(G)$.

(v) *The restriction of ℓ to $\bigodot(V^0)\odot V^i\odot V^{-1}$ with $i=1,2$ is such that*

$$\mathrm{d}\widetilde{\Phi}_i\circ \mathrm{D}\alpha_i(\Phi_i^{-1}(w,y),\mathrm{e}_G)(a)\approx\sum_{p\in\mathbf{N}_0}\frac{1}{p!}\,\ell_{p+1}\big(\textstyle\bigodot^p y\odot a\odot w\big) \tag{5.2.5}$$

for all $a\in V^{-1}=\mathrm{Lie}(G)$ and all $w\in V^i$.

(vi) *All remaining coefficients of $\ell\colon \bigodot(V)\to V$ vanish.*

This curved L_∞-algebra is uncurved if and only if $\mu(x_0)=0$ and it coincides with the curved L_∞-algebra $(V,\mathrm{j}^\infty_\Phi\theta(0))$ described above. Its isomorphism class is independent of the chosen trivialisation chart.

Proof: We have already seen that $\mathrm{j}^\infty_\Phi\theta\colon\varphi(U)\to\mathrm{L}(\bigodot(V),V)$ is a smooth family of curved L_∞-algebra in Remark 5.1.13. Independence of the L_∞-algebra structures $\mathrm{j}^\infty_\Phi\theta_{\alpha,\mu,\beta}(\phi(x))$ on the chosen trivialisation chart, up to isomorphism, was stated and proven in Corollary 5.1.9.

Proving that this L_∞-algebra is given by the equations listed above amounts to performing a series of explicit computations. These rely on the observations that if $\theta=\mathscr{L}^\Phi_\rho+A_{\theta,\Phi}$, then

(i) $A_{\theta,\Phi}(x)(1)=\mu(x)$ for any $x\in U$;

(ii) $A_{\theta,\Phi}(x)(e(x))=\beta(e)(x)$ for any $x\in U$ and any section $e\in\mathscr{E}^1(U)$;

(iii) $A_{\theta,\Phi}(x)(a(x),b(x))=[a(x),b(x)]$ for any any $x\in U$ and any two constant sections $a,b\in\mathscr{E}^{-1}(U)$;

(iv) $\rho(x)(a(x))=\mathrm{D}\alpha(x,\mathrm{e}_G)(a)$ for any $x\in U$ and any section $a\in\mathscr{E}^{-1}(U)$;

(v) $A_{\theta,\Phi}(x)(a(x)\odot e(x))=-\mathrm{D}\alpha^n(e(x),\mathrm{e}_G)(a)$ for any $x\in U$, any section $a\in\mathscr{E}^{-1}(U)$ and any section $e\in\mathscr{E}^n(U)$ such that $\tilde{\Phi}_n\circ e$ is constant $(n=1,2)$.

(vi) all other components of $A_{\theta,\Phi}$ and ρ are trivial. ☐

We have taken an equivariant deformation problem, along with a point in its base, and associated a very explicit curved L_∞-algebra to it. This in itself is not very meaningful, since we have said nothing about whether or how this curved L_∞-algebra can tell us anything about the deformations of the corresponding structure. Since it is obtained from the original deformation problem by taking power series expansions, it can be viewed as the infinite jet of this problem.

In an analytic setting, the fact that the L_∞-algebra described by Theorem 5.2.4 is convergent allows us to consider its Maurer–Cartan locus and compare it to a neighbourhood of the reference structure in the space of structures $\mu^{-1}(0) \subseteq M$. These will of course coincide by construction, and something similar will generally hold for the individual orbits if the these are defined appropriately.

Theorem 5.2.5 uses the following definitions:

- Given a convergent curved L_∞-algebra (V, ℓ) of Fréchet type and an open subset $U \subseteq D_\ell \subseteq V^0$ of its domain of convergence, we call two Maurer–Cartan elements $x, y \in \mathrm{MC}(V, \ell) \cap U$ gauge equivalent *within* U whenever there exists a smooth homotopy $\gamma + \eta\, \mathrm{d}t$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\gamma(t) \in U$ for every $t \in [0, 1]$.
- Given an equivariant deformation problem $(M \xrightarrow{\mu} E^1 \xrightarrow{\beta} E^2) \overset{\alpha}{\circlearrowleft} G$ and an open subset $U \subseteq M$, we say that two are structures $x, y \in \mu^{-1}(0)$ are equivalent *within* U whenever there exists a path $\gamma\colon [0, 1] \to G$ such that $\gamma(0) = \mathrm{e}_G$, $x \cdot \gamma(1) = y$ and $x \cdot \gamma(t) \in U$ for every $t \in [0, 1]$.

Given this set-up, the following theorem becomes nearly tautological given what we already know.

Theorem 5.2.5.
Let $(M \xrightarrow{\mu} E^1 \xrightarrow{\beta} E^2) \overset{\alpha}{\circlearrowleft} G$ be an analytic equivariant deformation problem of Fréchet type and assume that the Lie group G is regular. Let $x_0 \in M$ be a fixed reference point, let $\Phi = (\tilde{\Phi}, \varphi)\colon E|_U \to V^{\neq 0} \times V^0$ be an analytic trivialisation chart centred at x_0 and consider the associated curved L_∞-algebra (V, ℓ) described in Theorem 5.2.4.

There exists a neighbourhood $U' \subseteq M$ of x_0 such that the chart φ restricts to a bijection

$$\varphi\colon \mu^{-1}(0) \cap U' \xrightarrow{\sim} \mathrm{MC}(V, \ell) \cap \varphi(U')$$

and two structures $x, y \in \mu^{-1}(0) \cap U'$ are equivalent within U' precisely when $\varphi(x)$ and $\varphi(y)$ are gauge equivalent within $\varphi(U')$.

Proof: That Maurer–Cartan elements for (V,ℓ) are in one-to-one correspondence with zeros of θ_0 was stated in Corollary 5.1.5, and Proposition 5.2.3 tells us that these correspond to zeros of μ. Corollary 5.1.5 also asserts that homotopies of Maurer–Cartan elements within $\phi(U) \subseteq V^0$ correspond homotopies for $(E|_{U'}, \theta|_{U'}, \rho|_{U'})$. By following the proof of Proposition 5.2.3, it can then be shown that this is equivalent to the existence of a path in G as described above. $\square$

Appendix A.

Functional analysis

This thesis uses a large number of results from functional analysis that many of its readers are unlikely to be familiar with. Most of these can be found in some form spread out over text books such as [Bou87],[Jar81], [Köt69; Köt79] or [Trè67] or [Sch66], but are not always easy to find. The aim of this appendix is therefore to provide a summary of relevant definitions and theorems that are used throughout the other chapters. Proofs of a few results that were not found in the literature, but are likely known, have also been included.

All definitions and assertions in this appendix are valid for both real and complex vector spaces, as long as all vector spaces are consistently defined over the same field. Whenever an explicit reference to this base field is necessary, we will denote it by $\mathbf{K}$ rather than either $\mathbf{R}$ or $\mathbf{C}$ (even though we are mostly interested in real vector spaces). All topological spaces, which includes topological vector spaces, are assumed to be Hausdorff unless otherwise stated.

A.1. Locally convex vector spaces

A.1.1. Definitions

A *topological vector space* is a vector space V endowed with a topology for which the maps $m\colon \mathbf{K} \times V \to V$ and $a\colon V \times V \to V$ describing scalar multiplication and addition are both continuous. Morphisms of topological vector spaces are continuous linear maps, and we denote the set of morphism from V to W by $\mathrm{L}(V, W)$.

Unless specifically stated otherwise, all topological vector spaces considered in this thesis are assumed to be Hausdorff. Any non-Hausdorff topological vector space V can be made Hausdorff by taking the quotient V / N by the closure $N = \overline{\{0\}} \subseteq V$ of its origin.

Definition A.1.1.
A subset $A \subseteq V$ of a topological vector space is

- a *zero neighbourhood* if its $0 \in V$ is contained in its interior;
- *bounded* if for every zero neighbourhood $N \subseteq V$ there exists a $\lambda \in \mathbf{K}$ such that $A \subseteq \lambda N$;
- *precompact* or *totally bounded* if for every zero neighbourhood $N \subseteq V$ there exists a finite subset $F \subseteq V$ such that $A \subseteq F + N$;
- *convex* if $t\,v + (1-t)\,w \in A$ for all $t \in [0,1]$ whenever $v, w \in A$;
- *balanced* if $\lambda A \subseteq A$ for every $\lambda \in \mathbf{K}$ with $|\lambda| < 1$;
- *absolutely convex* if it is both balanced and convex;
- *absorbing* if for any point $v \in V$ there exists a $r > 0$ such that $\lambda v \in A$ for every $\lambda \in \mathbf{K}$ with $|\lambda| \leq r$;
- a *barrel* if it is absolutely convex, absorbing and closed.

It should be noted that the term “precompact” is somewhat ambiguous, since it is often used as a synonym for “relatively compact”. For subsets of complete locally convex vector spaces, total boundedness (or precompactness) and relative compactness are equivalent (cf. Proposition A.2.3).

Proposition A.1.2.
The closure $\bar{A}$ of a subset $A \subseteq V$ with one of the properties described in Definition A.1.1 has the same property. The same is true for the absolutely convex hull

$$\mathrm{hull}(A) = \Big\{ \textstyle\sum_{i=1}^{n} \lambda_i\, v_i \;\Big|\; n \in \mathbf{N}_0,\ v_i \in A \text{ and } \sum_{i=1}^{n} |\lambda_i| \leq 1 \Big\} \subseteq V,$$

which is the smallest absolutely convex subset of V containing A, as well as for its closure $\overline{\mathrm{hull}(A)} \subseteq V$.

One can show that every zero neighbourhood is absorbing, and that the closure of its absolutely convex hull is consequently a barrel.

Locally convex vector spaces arguably form the most important class of topological vector spaces.

Definition A.1.3.
A *locally convex vector space* is a topological vector space V whose topology admits a base consisting of convex subsets.

Since addition is continuous, any vector space topology on V is completely determined by its collection of zero neighbourhoods. Local convexity is thus equivalent to the existence of a basis of zero neighbourhoods that are convex. In fact, every locally convex vector space admits a basis of zero neighbourhoods that consists of open absolutely convex, absorbing subsets or, alternatively, of barrels.

Locally convex vector spaces are often described as vector spaces endowed with a (separated) collections of seminorms. A seminorm on a vector space V is a function $p\colon V \to [0,\infty)$ such that

$$p(v+w) \le p(v) + p(w) \qquad \text{and} \qquad p(\lambda v) = |\lambda|\, p(v)$$

for all $v, w \in V$ and all $\lambda \in \mathbf{K}$. A collection $\mathscr{P}$ of such seminorms is separated if for every $v \in V$ there exists at least one $p \in \mathscr{P}$ such that $p(v) > 0$. The vector space topology generated by a separated collection $\mathscr{P}$ of seminorms is the coarsest vector space topology on V for which each $p \in \mathscr{P}$ is continuous. A basis of zero neighbourhoods for it is given by the collection $\{N_{p,\epsilon} \mid p \in \mathscr{P}, \epsilon > 0\}$ of subsets

$$N_{p,\epsilon} = \{v \in V \mid p(v) < \epsilon\} \subseteq V$$

for $p \in \mathscr{P}$ and $\epsilon > 0$. Since these subsets are (absolutely) convex, the resulting topology is always locally convex. Conversely, every locally convex topology is generated by the collection $\mathscr{P}(V)$ of all continuous seminorms on this space.

There are many classes of topological vector spaces that have interesting or useful properties. Some of the terms in Definition A.1.4 will be explained later.

Definition A.1.4.
A topological vector space V is:

- *normable* if its topology is generated by a single norm;

- *metrisable* if it is metrisable as a topological space;
- *complete* if every Cauchy net (or every Cauchy filter) in V is convergent;
- *Banach* if it is complete and normable;
- *Fréchet* if it is locally convex, complete and metrisable;
- *barrelled* if every barrel in V is a zero neighbourhood;
- *reflexive* if it is isomorphic to its strong bidual;
- *Montel* if it is barrelled and has the *Heine–Borel property*: every closed and bounded subset of V is compact.

Completeness is discussed in Definition A.2.1, reflexivity in Definition A.2.34. Finite-dimensional vector spaces have all of the properties described in Definition A.1.4.

Banach spaces are often defined as complete vector spaces that come endowed with a topology-inducing norm, rather than as a topological vector space whose topology *can* be described in this way. What we call a Banach space is thus sometimes referred to as a Banachable space, but this distinction is rarely relevant.

Some of the properties listed in Definition A.1.4 are related.

Proposition A.1.5.

- *Every Banach space is also a Fréchet space.*
- *Every Fréchet space is barrelled.*
- *Every locally convex Montel space is complete.*
- *Every barrelled nuclear space is Montel.*
- *Every normable Montel space is finite dimensional.*
- *Every reflexive space is barrelled.*
- *Every reflexive metrisable locally convex vector space is Fréchet.*
- *The strong dual of a Montel space is Montel.*

Metrisability of a locally convex vector space can be characterised in several equivalent ways.

Proposition A.1.6.
The following statements are equivalent for any locally convex vector space V:

(i) *V is metrisable;*

(ii) *V is first countable;*

(iii) *the topology on V is generated by a countable subset $\mathscr{P}' \subseteq \mathscr{P}(V)$;*

(iv) *the topology on V is generated by a metric $d\colon V \times V \to [0, \infty)$ which is translation invariant.*

Some very important property of Fréchet spaces is spelled out by the open mapping theorem and its corollaries.

Theorem A.1.7 (Open mapping theorem)**.**
Let V and W be Fréchet spaces and let $f : V \to W$ be a continuous surjective linear map, then f is open.

Most relevant to us will be the following corollary.

Corollary A.1.8.
Every continuous linear bijection $f : V \to W$ of Fréchet spaces is an isomorphism of topological vector spaces.

A.2. Operations on locally convex vector spaces

In this section we will describe several ways to use collections of locally convex vector spaces to construct new ones. We also state a number of important and well-known theorems concerning the resulting objects, and prove a few that should certainly be known but not as readily available in the literature.

The constructions that we consider are completions, products, direct sums, linear mapping spaces, topological duals, tensor products and finally multilinear mapping spaces. While the spaces resulting from all of these constructions are unambiguously defined as sets, a few can be topologised in (many) different ways. The choices that have been made in this regard were very much made with the intended application in mind.

A.2.1. The completion

To define completeness for locally convex vector spaces that are not necessarily metrisable we will need to work with either nets or filters.

A net in V is a map $\lambda \mapsto v_\lambda$ from a directed set $(\Lambda, \leq)$ to V. The notation $(v_\lambda)_{\lambda \in \Lambda}$ is generally used, since sequences are just nets parametrised by $\mathbf{N}$. It is said to converge to an element $v \in V$ if one can find an index $\lambda_0 \in \Lambda$ for every neighbourhood U of v such that $v_\lambda \in U$ whenever $\lambda \geq \lambda_0$. It is called Cauchy if one can find an index $\lambda_0 \in \Lambda$ for every zero neighbourhood N such that $v_\mu - v_\nu \in U$ for all $\mu, \mathscr{P} \geq \lambda_0$. Every convergent net is Cauchy.

A filter basis on V is a non-empty collection $\mathscr{F}$ of subsets of V such that $\varnothing \notin \mathscr{F}$ and such that for all $A, B \in \mathscr{F}$ there exists a $C \in \mathscr{F}$ such that $C \subseteq A \cap B$. A filter basis $\mathscr{F}$ converges to $v \in V$ if every neighbourhood of V is an element of $\mathscr{F}$, and it is Cauchy if for every zero neighbourhood $N \subseteq V$ there exists a set $A \in \mathscr{F}$ such that $v - w \in N$ for all $v, w \in A$. Every convergent filter basis is Cauchy and the set of all neighbourhoods of a given point in V is a filter basis converging to this point.

Definition A.2.1.
A topological vector space V is *complete* if every Cauchy net (or equivalently, every Cauchy filter basis) on V is convergent.

For metrisable vector spaces, completeness can be characterised using Cauchy sequences instead of nets or filters.

The completion of a topological vector space V is morphism $i\colon V \to \overline{V}$ from V to a complete vector space $\overline{V}$ that is determined up to a unique isomorphism by the universal property that every morphism $f : V \to W$ to another complete vector space W factors through it. Every topological vector space admits a completion and it can be characterised, in more mundane terms, as follows.

Definition A.2.2.
The *completion* of a topological vector space V is a complete topological vector space $\overline{V}$ along with a linear topological embedding $i\colon V \hookrightarrow \overline{V}$ such that $i(V) \subseteq \overline{V}$ is dense.

Precompactness of subsets of V and of subsets of $\overline{V}$ are strongly related.

Proposition A.2.3.
For any subset $A \subseteq V$ of a locally convex vector space V, the following statements are equivalent:

(i) *A is precompact as a subset of V;*

(ii) *A is precompact as a subset of $\overline{V}$;*

(iii) *the closure of A in $\overline{V}$ is compact.*

It is not true that every compact subset of $\overline{V}$ arises as the closure of a subset of V, but if V is metrisable we can get close to this statement. The following lemma corresponds to Theorem 9.4.2 in [Jar81] and Corollary 1 on page IV.25 of [Bou87].

Lemma A.2.4.
Let V be a metrisable locally convex vector space and let $K \subseteq \overline{V}$ be a precompact subset, then there exists a null sequence $(x_n)_{n\in\mathbf{N}}$ in V such that K is contained in the completion of $\mathrm{hull}(\{x_n \mid n \in \mathbf{N}\}) \subseteq V$.

Note that the points of any convergent sequence $(x_n)_{n\in\mathbf{N}}$ in V form a precompact subset $\{x_n \mid n \in \mathbf{N}\} \subseteq V$, and that precompactness of the closed convex hull described in Lemma A.2.4 thus follows from Proposition A.1.2.

Corollary A.2.5.
Let V be a metrisable locally convex vector space, then a subset $A \subseteq \overline{V}$ is precompact if and only if it is contained in the completion of a precompact subset $K \subseteq V$.

It is possible to define the integrals of a continuous function from a compact interval to an arbitrary locally convex vector space if one allows it to take values in the completion of that space. There are multiple ways to define such integrals, and the corresponding notion of integrability, but the details of these constructions are largely inconsequential to us since they all apply to continuous maps and yield the same values.

Proposition A.2.6.
For every locally convex vector space V and all $a, b \in \mathbf{R}$, integration describes a linear map

$$\int_a^b : \mathrm{C}^0([a,b],V) \longrightarrow \overline{V}, \qquad f \mapsto \int_a^b f = \int_a^b f(t)\,\mathrm{d}t,$$

where $[a,b] = \mathrm{hull}(\{a,b\})$ *denotes the smallest closed interval containing both* a *and* b*. It has the following properties:*

1. *if* $f \in \mathrm{C}^0([a,b],\mathbf{R})$*, then* $\int_a^b f$ *coincides with the usual (Lebesgue or Riemann) integral of* f *from* a *to* b*;*
2. *if* $A\colon V \to W$ *is a continuous linear map, then* $\int_a^b A\circ f = A\big(\int_a^b f\big)$ *for every* $f \in \mathrm{C}^0([a,b],V)$*;*
3. *if* $f \in \mathrm{C}^0(\mathrm{hull}(\{a,b,c\}),V)$*, then* $\int_a^c f = \int_a^b f + \int_b^c f$*;*
4. *the mean value theorem holds: if* $f \in \mathrm{C}^0([a,b],V)$*, then* $\int_a^b f$ *is an element of the closure of* $(b-a)\,\mathrm{hull}(f([a,b]))$ *in* $\overline{V}$*;*
5. *if* $f \in \mathrm{C}^0([a,b],V)$*, then* $F\colon t \mapsto \int_a^t f$ *is a differentiable function from* $[a,b]$ *to* $\overline{V}$ *with* $F(a)=0$ *and* $F'(t) = f(t)$ *for all* $t \in [a,b]$*;*
6. *it is continuous for the compact-open topology on* $\mathrm{C}^0([a,b],V)$*.*

Integration of continuous functions is uniquely characterised by properties 1 and 2, as well as by properties 3 and 4 or by just property 5. A detailed discussion about integrals with values in locally convex vector spaces can for instance be found in Appendix A.2 of [Kel74].

A.2.2. Products and projective limits

Let $(V_i)_{i\in I}$ be a collection of locally convex vector spaces indexed by $i \in I$ and let $V = \prod_{i\in I} V_i$ denote their (algebraic) direct product.

Definition A.2.7.
The *product topology* on $V = \prod_{i\in I} V_i$ is the coarsest topology for which each of the canonical projection maps $\pi_i\colon V \twoheadrightarrow V_i$ is continuous.

The topology described in Definition A.2.7 is just the usual topology on the product of topological spaces. When endowed with this topology, the product $\prod_{i\in I} V_i$ is a locally convex vector space.

Given a finite subset $A \subseteq I$ and zero neighbourhoods N_i for V_i with $i \in A$, the inverse image of the canonical projection maps $\pi_A\colon \prod_{i\in I} V_i \to \prod_{i\in A} V_i$ is a zero neighbourhood for $\prod_{i\in I} V_i$. Such neighbourhoods form a basis of zero neighbourhoods for the product topology. If $\mathscr{P}_i$ is a generating collection of seminorms for V_i for every $i \in I$, then a generating collection of seminorms is given by the maps

$$\max_{i \in A} p_i\colon \prod_{i \in I} V_i \longrightarrow [0, \infty), \qquad (v_i)_{i \in I} \mapsto \max_{i \in A} p_i(v_i)$$

where A ranges over the finite subsets of I and $(p_i)_{i \in A} \in \prod_{i \in A} \mathscr{P}_i$.

Proposition A.2.8 (Universal property).
If $f_i\colon W \to V_i$ is a morphism of locally convex vector spaces for every $i \in I$ and let $V = \prod_{i \in I}$. There exists exactly one continuous linear map $f\colon W \to V$ such that $\pi_i \circ f = f_i$ for all $i \in I$.

The locally convex vector space V and the morphisms $\pi_i\colon V \to V_i$ are uniquely determined by this property up to a unique isomorphism.

We deduce from Proposition A.2.8 that for any collection $(V_i)_{i \in I}$ of locally convex vector spaces and any other locally convex vector space W, the map

$$\mathrm{L}\big(W, \textstyle\prod_{i \in I} V_i\big) \longrightarrow \displaystyle\prod_{i \in I} \mathrm{L}(W, V_i), \qquad f \mapsto (\pi_i \circ f)_{i \in I}$$

is a linear bijection. We will see in Proposition A.2.25 that this map is itself an isomorphism of locally convex vector spaces if its domain and codomain are topologised appropriately. While there exists a canonical inclusion map

$$\bigoplus_{i \in I} \mathrm{L}(V_i, W) \hookrightarrow \mathrm{L}\big(\textstyle\prod_{i \in I} V_i, W\big), \qquad (f_i)_{i \in A} \mapsto \displaystyle\sum_{i \in A} f_i,$$

the spaces $\bigoplus_{i \in I} \mathrm{L}(V_i, W)$ and $\mathrm{L}\big(\prod_{i \in I} V_i, W\big)$ are generally not even algebraically isomorphic unless W is finite-dimensional.

A subset $B \subseteq \prod_{i \in I} V_i$ is bounded (or precompact) if and only if $\pi_i(B) \subseteq V_i$ is bounded (resp. precompact) for every $i \in I$.

Definition A.2.9.
A (closed) linear subspace $W \subseteq V$ of a locally convex vector space V is complemented if there exists another (closed) subspace $W' \subseteq V$ such that $V = W \times W'$ as a topological vector space.

The requirement that the spaces W and W' both be closed in Definition A.2.9 is optional since this also follows from the condition that $V = W \times W'$.

A projective system of locally convex vector spaces indexed by a directed set $(I, \leq)$ consists of a collection $(V_i)_{i \in I}$ of locally convex vector spaces and a collection $(g_{i,j})_{i \leq j}$ of morphism $g_{i,j} \colon V_j \to V_i$ of locally convex vector spaces for $i \leq j \in I$ with the following properties:

1. $g_{i,i} = \mathrm{id}_{V_i}$ for every $i \in I$, and
2. $g_{i,k} = g_{i,j} \circ g_{j,k}$ for all $i, j, k \in I$ with $i \leq j \leq k$.

Definition A.2.10.
The *projective limit* of a projective system $\big((V_i)_{i \in I}, (g_{i,j})_{i \leq j}\big)$ is the (closed) subspace

$$\varprojlim_{i \in I} V_i = \left\{ (v_i)_{i \in I} \;\middle|\; v_i = g_{i,j}(v_j) \text{ whenever } i \leq j \right\} \subseteq \prod_{i \in I} V_i.$$

It comes with the canonical maps $\pi_i \colon \varprojlim_{j \in I} V_j \to V_i$ for $i \in I$.

Note that if the relation $\leq$ on I is such that $i \not\leq j$ for all $i, j \in I$, then $\varprojlim_{i \in I} V_i$ is simply equal to the product $\prod_{i \in I} V_i$.

The projective limit is determined up to isomorphism by the following universal property.

Proposition A.2.11 (Universal property)**.**
Let $\big((V_i)_{i \in I}, (g_{i,j})_{i \leq j}\big)$ be a projective system of locally convex vector spaces and let $V = \varprojlim_{i \in I} V_i$. For any collection $(f_i)_{i \in I}$ of morphisms $f_i \colon V_i \to W$ of locally convex vector spaces such that $f_j = f_i \circ g_{i,j}$ whenever $i \leq j$, there exists a unique morphism $f \colon V \to W$ such that $f_i = f \circ \pi_i$ for every $i \in I$.

The locally convex vector space V and the morphisms $\pi_i \colon V \to V_i$ for $i \in I$ are uniquely determined by this property up to a unique isomorphism.

Many properties of locally convex vector spaces are inherited by their products and projective limits.

Proposition A.2.12 (Heredity)**.**
Let V either denote the product $\prod_{i \in I} V_i$ of locally convex vector spaces V_i for $i \in I$, or the projective limit $\varprojlim_{i \in I} V_i$ of a projective system $\big((V_i)_{i \in I}, (g_{i,j})_{i \leq j}\big)$.

(i) *If each of the spaces V_i is complete, then so is V.*

(ii) *If I is finite and each of the spaces V_i is normable, then so is V.*

(iii) *If I is countable and each of the spaces V_i is metrisable, then so is V.*

(iv) *If each of the spaces V_i is barrelled, then so is V.*

(v) *If I is countable and each of the spaces V_i is Fréchet, then so is V.*

(vi) *If each of the spaces V_i is reflexive, then so is V.*

(vii) *If each of the spaces V_i is nuclear, then so is V.*

(viii) *If each of the spaces V has the approximation property, then so does V.*

(ix) *If each of the spaces V_i is Montel, then so is V.*

A.2.3. Locally convex direct sums

Let $(V_i)_{i\in I}$ be a collection of locally convex vector spaces indexed by $i \in I$ and let $V = \bigoplus_{i\in I} V_i$ denote their (algebraic) direct sum.

Definition A.2.13.
The *locally convex direct sum* topology on $V = \bigoplus_{i\in I} V_i$ is the finest locally convex topology for which each of the canonical inclusion maps $\iota_i \colon V_i \hookrightarrow V$ is continuous.

Unless specifically stated otherwise, the direct sum $\bigoplus_{i\in I} V_i$ of locally convex vector spaces V_i will always come with the locally convex direct sum topology. Instead of this topology, the finest vector space topology for which the inclusions $\nu_i \hookrightarrow V$ are continuous is sometimes used, e.g. in [Jar81] and [Trè67]. This topology is sometimes called the box topology and it not generally locally convex.

The locally convex direct sum topology is uniquely characterised by the following universal property.

Proposition A.2.14 (Universal property)**.**
Let $f_i \colon V_i \to W$ be a morphism of locally convex vector spaces for every $i \in I$ and let $V = \bigoplus_{i\in I} V_i$. There exists a unique continuous linear map $f \colon V \to W$ such that $f \circ \iota_i = f_i$ for every $i \in I$.

The locally convex vector space V and the morphism $\iota_i \colon V_i \to V$ are uniquely determined by this property up to a unique isomorphism.

We deduce from Proposition A.2.14 that for any collection $(V_i)_{i\in I}$ of locally convex vector spaces and any other locally convex vector space W, the canonical map

$$\mathrm{L}\big(\textstyle\bigoplus_{i\in I} V_i, W\big) \longrightarrow \prod_{i\in I} \mathrm{L}(V_i, W), \qquad f \mapsto (f|_{V_i})_{i\in I}$$

is a linear bijection. We will see in Proposition A.2.25 this map is in fact itself an isomorphism of locally convex vector spaces if its domain and codomain are topologised appropriately. Unless V is finite-dimensional, the spaces $\mathrm{L}(V, \bigoplus_{i\in I} W_i)$ and $\bigoplus_{i\in I} \mathrm{L}(V, W_i)$ are not generally isomorphic (even algebraically).

While general inductive limits exist in the category of possibly non-Hausdorff locally convex vector spaces, these will not be discussed here.

Given a collection $(N_i)_{i\in I}$ zero neighbourhoods $N_i \subseteq V_i$, the absolutely convex hull of their union,

$$\mathrm{hull}\big(\textstyle\bigcup_{i\in I} N_i\big) = \big\{ \sum_{i\in A} \lambda_i\, v_i \;\big|\; A \subseteq I \text{ finite, } v_i \in N_i \text{ and } \sum_{i\in A} |\lambda_i| \le 1 \,\big\}$$

is a zero neighbourhood for $\bigoplus_{i\in I} V_i$. Such neighbourhoods form a basis of zero neighbourhoods for the locally convex direct sum topology.

A seminorm on $\bigoplus_{i\in I} V_i$ is continuous if and only if its restriction to V_i is continuous for every $i \in I$. Given a collection $(p_i)_{i\in I} \in \prod_{i\in I} \mathscr{P}(V_i)$ of seminorms on each of the spaces V_i, we consider the seminorm

$$\sum_{i\in I} p_i \colon \bigoplus_{i\in I} V_i \longrightarrow [0, \infty), \qquad \sum_{i\in A} v_i \mapsto \sum_{i\in A} p_i(v_i).$$

If $\mathscr{P}_i$ is a generating collection of seminorms for V_i for $i \in I$, then the topology on $\bigoplus_{i\in I} V_i$ is generated by the seminorms $\sum_{i\in I} p_i$ for all $(p_i)_{i\in I} \in \prod_{i\in I} \mathscr{P}_i$.

A subset $B \subseteq \bigoplus_{i\in I} V_i$ is bounded (or precompact) if and only if there exists a finite subset $A \subseteq I$ and bounded (resp. precompact) subsets $B_i \subseteq V_i$ for each $i \in A$ such that $B \subseteq \mathrm{hull}\big(\bigcup_{i\in A} B_i\big)$.

Many properties of locally convex vector spaces are inherited by their locally convex direct sums and inductive limits.

Proposition A.2.15 (Heredity)**.**
Let $(V_i)_{i\in I}$ be a collection of locally convex vector spaces and let $V = \bigoplus_{i\in I} V_i$

(i) *If each of the spaces V_i is complete, then so is V.*

(ii) *If each of the spaces V_i is metrisable and nontrivial, then V is metrisable if and only if I is finite.*

(iii) *If I is finite and each of the spaces V_i is normable, then so is V.*

(iv) *If each of the spaces V_i is Fréchet and I is finite, then also V is Fréchet.*

(v) *If each of the spaces V_i is barrelled, then so is V.*

(vi) *If each of the spaces V_i is reflexive, then so is V.*

(vii) *If each of the spaces V_i is nuclear and I is countable, then also V is nuclear.*

(viii) *If each of the spaces V has the approximation property, then so does V.*

(ix) *If each of the spaces V_i is Montel, then so is V.*

Proposition A.2.16.
For any collection $(V_i)_{i \in I}$ of locally convex vector spaces, the canonical inclusion map

$$\bigoplus_{i \in I} V_i \hookrightarrow \prod_{i \in I} V_i$$

is continuous. It is an isomorphism of topological vector spaces if I is finite.

We will need the following proposition in several places in this document to prove the continuity of certain maps between locally convex vector direct sums.

Proposition A.2.17.
Let $f\colon \bigoplus_{i \in I} V_i \to \bigoplus_{j \in J} W_j$ be a linear map between locally convex direct sums such that there exists a finite subset $A_i \subseteq J$ for every $i \in I$ such that $f(V_i) \subseteq \bigoplus_{j \in A_i} W_j$. Then f is continuous if and only if the composition

$$f_{j,i} = \pi_j \circ f \circ \iota_i \colon V_i \longrightarrow W_j,$$

of f with the canonical maps $\iota_i\colon V_i \hookrightarrow \bigoplus_{i' \in I} V_{i'}$ and $\pi_j\colon \bigoplus_{j' \in J} W_{j'} \twoheadrightarrow W_j$ is continuous for every $i \in I$ and every $j \in J$.

Proof: Because of the universal property from Proposition A.2.14, f is continuous if and only if the restriction $f|_{V_i} = f \circ \iota_i$ is continuous for every $i \in I$. Since $f(V_i)$ is contained in $\bigoplus_{j \in A_i} W_j \simeq \prod_{j \in A_i} W_j$, the universal property from Proposition A.2.8 now tells us that $f \circ \iota_i$ is continuous if and only if $\pi_j \circ f \circ \iota_i$ is continuous for every $j \in J$. □

Proposition A.2.17 has an analogue for products, which is just as simple to prove.

Proposition A.2.18.
Let $f\colon \prod_{i\in I} V_i \to \prod_{j\in J} W_j$ *be a linear map between locally convex direct sums such that for every* $j \in J$ *there exists a finite subset* $A_j \subseteq I$ *such that* $\pi_j \circ f$ *factors through the projection map* $\pi_{A_j}\colon \prod_{i\in I} \to \prod_{i\in A_j} V_i$*. Then* f *is continuous if and only if the composition*

$$f_{j,i} = \pi_j \circ f|_{V_i}$$

is continuous for every $i \in I$ *and every* $j \in J$*.*

Proof: Due to the universal property from Proposition A.2.8, f is continuous if and only if the composition $\pi_j \circ f$ is continuous for every $j \in I$, which is equivalent to continuity to the restriction to $\prod_{i\in A_j} V_i$. Since this product is finite it is isomorphic to the corresponding locally convex direct sum, this restriction is continuous if and only if the restriction of $\pi_j \circ f|_{V_i}$ is continuousfor all $i \in A_j$ (as well as all $i \notin A_j$ since $f|_{V_i} = 0$ for those). ☐

A.2.4. Spaces of morphisms

Given two locally convex vector spaces V and W, there are many inequivalent ways to topologise the space $\mathrm{L}(V,W)$ of morphism from the former to the latter. We will exclusively be working with the topology of precompact convergence subsets, which is defined below, but shall also briefly mention the topology of simple convergence and the topology of bounded convergence.

Definition A.2.19.
Let V and W be locally convex vector spaces and let $\mathfrak{S}$ be a collection of bounded subsets that covers V. The *topology of uniform convergence* on sets in $\mathfrak{S}$ is the vector space topology on $\mathrm{L}(V,W)$ for which the subsets

$$\mathscr{N}(A,N) = \{\alpha \in \mathrm{L}(V,W) \mid \alpha(A) \subseteq N\} \subseteq \mathrm{L}(V,W),$$

for $A \in \mathfrak{S}$ and zero neighbourhoods $N \subseteq V$, form a basis of zero neighbourhoods. When endowed with this topology, we denote the space of continuous linear maps from V to W by $\mathrm{L}_{\mathfrak{S}}(V,W)$.

Some additional conditions are usually imposed on the collection $\mathfrak{S}$ in Definition A.2.19, but these are not required to obtain a locally convex topology. Note that the topology of uniform convergence on sets of $\mathfrak{S}$ does not change if one replaces $\mathfrak{S}$ by either the set of closures of elements of $\mathfrak{S}$ or by the set of absolutely convex hulls of elements of $\mathfrak{S}$. The topology of uniform convergence on sets of $\mathfrak{S}$ is locally convex, and it is generated by seminorms of the form

$$p_B \colon \alpha \mapsto \sup_{v \in B} p(\alpha(v))$$

for $p \in \mathscr{P}(W)$ and $B \in \mathfrak{S}$.

Definition A.2.20.
The topology of uniform convergence on sets in $\mathfrak{S}$ is called:

- the topology of *simple convergence* if $\mathfrak{S}$ is the collection $\mathfrak{S}_{\mathrm{s}}(V)$ of all finite subsets of V;
- the topology of *precompact convergence* if $\mathfrak{S}$ is the collection $\mathfrak{S}_{\mathrm{pc}}(V)$ of all precompact subsets of V;
- The topology of *bounded convergence* if $\mathfrak{S}$ is collection $\mathfrak{S}_{\mathrm{b}}(V)$ of all bounded subsets of V.

When endowed with one of these topologies, we denote the space of morphisms from V to W by $\mathrm{L}_{\mathrm{s}}(V,W)$, $\mathrm{L}_{\mathrm{pc}}(V,W)$ or $\mathrm{L}_{\mathrm{b}}(V,W)$ respectively.

Since every finite subset is precompact and every precompact subset is bounded, these topologies are presented in order of increasing granularity. Unless otherwise indicated, we will always endow the space $\mathrm{L}(V,W)$ with the topology of precompact convergence.

Any morphism $\alpha \colon V \to W$ of locally convex vector spaces can be extended uniquely to a continuous linear map $\bar{\alpha} \colon \overline{V} \to \overline{W}$ between their completions. Algebraically, the spaces $\mathrm{L}(V,W)$ and $\mathrm{L}(\overline{V},W)$ are therefore isomorphic whenever W is complete.

Proposition A.2.21.
For any metrisable locally convex vector space V and any complete locally convex vector space W, the restriction maps

$$\mathrm{L}_{\mathrm{pc}}(\overline{V},W) \longrightarrow \mathrm{L}_{\mathrm{pc}}(V,W), \qquad \textit{and} \qquad \mathrm{L}_{\mathrm{b}}(\overline{V},W) \longrightarrow \mathrm{L}_{\mathrm{b}}(V,W),$$

are isomorphisms of topological vector spaces.

The following proposition combines Corollaries 2 and 4 on pages 344 and 345 of [Trè67] and uses that the the preceding proposition.

Proposition A.2.22.
Let V and W be locally convex vector spaces. If V is metrisable and W is complete, then both $\mathrm{L}_{\mathrm{b}}(V, W)$ and $\mathrm{L}_{\mathrm{pc}}(V, W)$ are complete.

Proposition A.2.23.
Let V, W and Z be three locally convex vector spaces and consider the composition map

$$\circ\colon \mathrm{L}(W, Z) \times \mathrm{L}(V, W) \longrightarrow \mathrm{L}(V, Z), \qquad (\beta, \alpha) \mapsto \beta \circ \alpha.$$

This map is (separately) continuous in both of its arguments if all three spaces of morphisms are endowed with the same topology from Definition A.2.20.

Proposition A.2.23 follows simply from the fact that continuous linear maps map finite, precompact and bounded subsets to subsets of the same type and the pre-image of an zero neighbourhood is a zero neighbourhood.

Corollary A.2.24.
Let V and W be locally convex vector spaces and consider the evaluation map

$$\mathrm{ev}\colon \mathrm{L}(V, W) \times V \longrightarrow W, \qquad (\alpha, v) \mapsto \alpha(v)$$

This map is separately continuous in each of its arguments separately if $\mathrm{L}(V, W)$ is endowed with any of the topologies from Definition A.2.20.

The space $\mathrm{L}(V, W)$ admits no vector space topology for which the corresponding evaluation map is jointly continuous unless either V is normable or $W = 0$.

Proposition A.2.25.
Let $(V_i)_{i \in I}$ be a collection of locally convex vector spaces and let W be another locally convex vector space. The canonical bijections

$$\mathrm{L}_{\mathrm{pc}}\big(W, \textstyle\prod_{i \in I} V_i\big) \longrightarrow \displaystyle\prod_{i \in I} \mathrm{L}_{\mathrm{pc}}(W, V_i), \qquad f \mapsto (\pi_i \circ f)_{i \in I}$$

and

$$\mathrm{L}_{\mathrm{pc}}\big(\textstyle\bigoplus_{i \in I} V_i, W\big) \longrightarrow \displaystyle\prod_{i \in I} \mathrm{L}_{\mathrm{pc}}(V_i, W), \qquad f \mapsto (f \circ \iota_i)_{i \in I}$$

are isomorphisms of topological vector spaces. The same is true for the topologies of simple and bounded convergence.

Note that due to Proposition A.2.16, the product of a finite number of locally convex vector spaces is identical to the locally convex sum of these spaces.

The author is not aware of a reference for the following lemma, but its proof is relatively straightforward.

Lemma A.2.26.
Let V and W be locally convex vector spaces, then

$$K_{\mathrm{L}}(K_V) = \{\alpha(v) \mid \alpha \in K_{\mathrm{L}}, v \in K_V\} \subseteq W$$

is precompact for any two precompact subsets $K_V \subseteq V$ and $K_{\mathrm{L}} \subseteq \mathrm{L}_{\mathrm{pc}}(V, W)$.

Proof: We need to show that there exists a finite subset $F_W^N \subseteq W$ for every absolutely convex zero neighbourhood $N \subseteq W$ such that $f(K_X)(K_V) \subseteq F_W^N + N$. Let $N \subseteq W$ be an absolutely convex zero neighbourhood.

Because $K_{\mathrm{L}} \subseteq \mathrm{L}_{\mathrm{pc}}(V, W)$ is precompact by assumption, we can choose a finite subset $F_{\mathrm{L}}^N \subseteq \mathrm{L}(V, W)$ such that $K_{\mathrm{L}} \subseteq F_{\mathrm{L}}^N + \mathscr{N}(K_V, \frac{1}{2}N)$. For every $\alpha \in \mathrm{L}(V, W)$, precompactness of $\alpha(K_V) \subseteq W$ moreover allows us to choose finite subsets F_W^α such that $\alpha(K_V) \subseteq F_W^\alpha + \frac{1}{2}N$. Let $F_W^N = \bigcup_{\alpha \in F_{\mathrm{L}}^N} F_W^\alpha \subseteq W$, and note that this set is finite.

For every pair $(\alpha, v) \in K_{\mathrm{L}} \times K_V$ we can now choose a morphism $\beta \in F_{\mathrm{L}}^N$ such that $\alpha \in \beta + \mathscr{N}(K_V, \frac{1}{2}N)$ and thus $\alpha(v) \in \beta(v) + \frac{1}{2}N$. Since $\beta(v) \in F_W^\beta + \frac{1}{2}N$, it now follows that $\alpha(v) \in F_W^N + N$. We deduce that $K_{\mathrm{L}}(K_V) \subseteq F_W^N + N$. ☐

We will occasionally need the Uniform boundedness principle and Banach–Steinhaus theorem, which can be derived from it.

Theorem A.2.27 (Uniform boundedness principle)**.**
Let V be a barrelled locally convex vector space and let W be an arbitrary locally convex vector space, then every bounded subset $H \subseteq \mathrm{L}_{\mathrm{s}}(V, W)$ is equicontinuous.

Theorem A.2.28 (Banach–Steinhaus theorem)**.**
Let V be a barrelled space and let W be a locally convex vector space. Let $(\alpha_n)_{n \in \mathbf{N}}$ be a sequence of continuous linear maps $\alpha_n \in \mathrm{L}(V, W)$ and let $\alpha \colon V \to W$ be such that $\alpha_n(v) \to \alpha(v)$ for every $v \in V$. Then $\alpha \in \mathrm{L}(V, W)$ and $\alpha_n \to \alpha$ as $n \to \infty$ for the topology of precompact convergence.

The Banach–Steinhaus theorem can be used to prove that any continuous map from a metrisable space X to $\mathrm{L}_{\mathrm{s}}(V, W)$ is also continuous as a map to $\mathrm{L}_{\mathrm{pc}}(V, W)$ whenever V is barrelled.

Corollary A.2.29.
Let V and W be locally convex vector spaces and assume that V is barrelled and let X be a metrisable space. Let $\alpha\colon X \setminus \{x_0\} \to \mathrm{L}(V, W)$ be a function defined on the complement of a point $x_0 \in X$ such that $\alpha(x)(v)$ converges to some element $\alpha_0(v) \in W$ as $x \to x_0$ for every $v \in V$. Then $\alpha_0\colon V \to W$ is a continuous linear map and $\lim_{x \to x_0} \alpha(x) = \alpha_0$ with respect to the topology of uniform convergence of precompact subsets of V.

A.2.5. Topological duals

The *topological dual* of a locally convex vector space V is the space $V^{\vee} = \mathrm{L}(V, \mathbf{K})$ of continuous linear $\mathbf{K}$-valued linear maps on V. Depending on how this space is topologised, we denote it by either

$$V^{\vee}_{\mathrm{s}} = \mathrm{L}_{\mathrm{s}}(V, \mathbf{K}), \qquad V^{\vee}_{\mathrm{pc}} = \mathrm{L}_{\mathrm{pc}}(V, \mathbf{K}), \qquad \text{or} \qquad V^{\vee}_{\mathrm{b}} = \mathrm{L}_{\mathrm{b}}(V, \mathbf{K}).$$

The space $V^{\vee}_{\mathrm{b}}$ and $V^{\vee}_{\mathrm{s}}$ are usually called the *strong dual* and the *weak dual* of V respectively. We will usually work with the precompact dual $V^{\vee}_{\mathrm{pc}}$, however.

Proposition A.2.30.
Let $\alpha\colon V \to W$ be a morphism of locally convex vector spaces, then its transpose

$$\alpha^{\vee}\colon W^{\vee} \longrightarrow V^{\vee}, \qquad \xi \mapsto \xi \circ \alpha$$

is a continuous map if both topological duals are endowed with the topology of simple, precompact or bounded convergence.

Transposition describes a continuous linear map from $\mathrm{L}_{\mathrm{pc}}(V, W)$ to $\mathrm{L}_{\mathrm{pc}}(W^{\vee}_{\mathrm{pc}}, V^{\vee}_{\mathrm{pc}})$ if W is barrelled.

Proof: Continuity of the transpose $\alpha^{\vee}$ is a special case of Proposition A.2.23, but continuity of the operation $\alpha \mapsto \alpha^{\vee}$ is a little more involved. The spaces $V^{\vee}$, $W^{\vee}$, $\mathrm{L}(V, W)$ and $\mathrm{L}(W^{\vee}, V^{\vee})$ are all endowed with the topology of precompact convergence below.

Subsets of the form $\mathscr{N}(K_{W^\vee}, \mathscr{N}(K_V)) \subseteq \mathrm{L}(W^\vee, V^\vee)$ for precompact subsets $K_V \subseteq V$ and $K_{W^\vee} \subseteq W^\vee$ form a basis of zero neighbourhoods for the topology of precompact convergence on $\mathrm{L}(W^\vee, V^\vee)$. We need to show that any such neighbourhood contains the image of a zero neighbourhood in $\mathrm{L}(V, W)$.

Let K_V and $K_{W^\vee}$ be as described above and consider the subset

$$N_W := \tfrac{1}{2} \bigcap_{\xi \in K_{W^\vee}} \xi^{-1}([-1,1]) \subseteq W,$$

which we claim is a zero neighbourhood. Since W is barrelled, we can show that N_W is open by demonstrating that it is a barrel. It is clearly closed and absolutely convex, so we only need to show that it is absorbing.

Let $w \in W$ be an arbitrary point and let $U_{W^\vee} = \{\xi \in W^\vee \mid |\xi(w)| < 1\}$. This is a non-empty open subset of $W^\vee$, so the fact that $K_{W^\vee}$ is precompact informs us that we can find a finite subset $A \subseteq W^\vee$ such that $K_{W^\vee} \subseteq A + U_{W^\vee}$. It follows that $K_{W^\vee} \subseteq (2\lambda + 1) U_{W^\vee}$ with $\lambda = \max_{\xi \in A} |\xi(w)|$, and consequently that $w \in (4\lambda + 2) N_W$. Since the point $w \in W$ was arbitrary, we deduce that N_W is absorbing and that it is therefore a zero neighbourhood.

Now let α be an element of the zero neighbourhood $\mathscr{N}(K_V, N_W) \subseteq \mathrm{L}(V, W)$ determined by the subsets described above. The transpose $\alpha^\vee$ is such that for any $\xi \in K_{W^\vee}$ and any $v \in K_V$, $\alpha^\vee(\xi)(v) = \xi(\alpha(v))$ is an element of $\xi(N_W)$, which is a subset of $(-1, 1)$ because of how N_W was constructed. From this, we can conclude that $\alpha^\vee$ is an element of $\mathscr{N}(K_{W^\vee}, \mathscr{N}(K_V))$ and that the assignment $\alpha \mapsto \alpha^\vee$ is in fact continuous. ☐

Proposition A.2.31 (Hahn–Banach theorem)**.**
Let p be a seminorm on a vector space V and let $W \subseteq V$ be a linear subspace. Any linear form $\xi \colon F \to \mathbf{K}$ for which $|\xi| \leq p|_F$ can be extended to a linear form $\tilde{\xi} \colon V \to \mathbf{K}$ such that $|\tilde{\xi}| \leq p$.

Corollary A.2.32.
Let V be a locally convex vector space and let $W \subseteq V$ be a linear subspace. Every continuous linear form $\xi \in W^\vee$ extends to a continuous linear form $\tilde{\xi} \in W^\vee$.

Corollary A.2.33.
For any locally convex vector space V and any $v \in V \setminus \{0\}$ there exists a form $\xi \in V^\vee$ such that $\xi(v) = 1$.

Every element $v \in V$ of a locally convex vector space V induces a form $\mathrm{ev}_v \colon \xi \mapsto \xi(v)$ on its topological dual. This functional is continuous for all three of the aforementioned topologies due to Corollary A.2.24. Corollary A.2.33 moreover tells us that the map

$$\mathrm{ev}_{\bullet} \colon V \longrightarrow (V^{\vee})^{\vee}, \qquad v \mapsto \mathrm{ev}_v$$

is injective if $V^{\vee}$ is endowed with any of the three topologies described above.

Definition A.2.34.
The locally convex vector space V is *reflexive* if the map

$$\mathrm{ev}_{\bullet} \colon V \longrightarrow (V^{\vee}_{\mathrm{b}})^{\vee}_{\mathrm{b}}, \qquad v \mapsto (\mathrm{ev}_v \colon \xi \mapsto \xi(v))$$

from V to its bidual is an isomorphism of topological vector spaces.

We call V semi-reflexive if the map $\mathrm{ev}_{\bullet}$ from Definition A.2.34 is merely a bijection, rather than a isomorphism.

A.2.6. The projective tensor product

We are interested in the tensor product $\bigotimes_{i=1}^n V_i = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ of locally convex vector spaces $V_1, V_2, \dots, V_n$. Although there are many natural ways to topologise this space, the following is most suitable for our purposes.

Definition A.2.35.
The *projective topology* topology on the tensor product $\bigotimes_{i=1}^n V_i$ of locally convex vector spaces $V_1, V_2, \dots, V_n$ is the coarsest locally convex topology for which the multilinear map

$$\chi \colon \prod_{i=1}^n V_i \longrightarrow \bigotimes_{i=1}^n V_i, \qquad (v_i)_{i=1}^n \mapsto \bigotimes_{i=1}^n v_i = v_1 \otimes \cdots \otimes v_n$$

is continuous.

When endowed with this topology, the space $\bigotimes_{i=1}^n V_i$ is called the *projective tensor product*, and it is often denoted by $V_1 \otimes_\pi V_2 \otimes_\pi \cdots \otimes_\pi V_n$ to distinguish it from other topological tensor products. The tensor product of a collection zero spaces, $\bigotimes_{i \in \varnothing} V$, is simply the base field $\mathbf{K}$ and it comes with its usual topology.

Given zero neighbourhoods $N_i \subseteq V_i$ for $i = 1, 2, \dots, n$, one may obtain a zero neighbourhood in $\bigotimes_{i=1}^n V_i$ by taking the absolutely convex hull

$$\bigotimes_{i=1}^n N_i = \mathrm{hull}\bigl(\bigl\{\bigotimes_{i=1}^n v_i \bigm| v_i \in N_i \text{ for } i = 1, 2, \dots, n\bigr\}\bigr)$$

of the image of $\chi(\prod_{i=1}^n N_i) \subseteq \bigotimes_{i=1}^n V_i$. The collection of such sets is a basis of zero neighbourhoods for the projective tensor product.

It is also possible to construct a continuous seminorm $\bigotimes_{i=1}^n p_i$ on $\bigotimes_{i=1}^n V_i$ from a sequence $(p_i)_{i=1}^n$ of continuous seminorms $p_i \in \mathscr{P}(V_i)$. This seminorm is given by

$$\bigl(\bigotimes_{i=1}^n p_i\bigr)(z) = \inf\bigl\{\textstyle\sum_{j=1}^r p_1(v_{j,1}) \cdots p_n(v_{j,n}) \bigm| z = \sum_{j=1}^r v_{j,1} \otimes \cdots \otimes v_{j,n}\bigr\}$$

where the infimum is over all ways in which z can be written as a linear combination $z = \sum_{j=1}^r v_{j,1} \otimes \cdots \otimes v_{j,n}$ of decomposable elements. The projective topology on $\bigotimes_{i=1}^n V_I$ is generated by these seminorms.

More relevant to us, however, is the fact that the projective tensor product has the following universal property.

Proposition A.2.36 (Universal property)**.**
Let $(V_a)_{a \in A}$ be a finite collection of locally convex vector spaces and define $Z = \bigotimes_{a \in A} V_a$. Any multilinear map $f\colon \prod_{a \in A} V_a \to W$ of locally convex vector spaces is continuous if and only if there exists a continuous linear map $g\colon \bigotimes_{a \in A} V_a$ such that $f = g \circ \chi$.

The locally convex vector space Z and the continuous map $\chi\colon \prod_{a \in A} V_a \to Z$ are uniquely determined by this property up to a unique isomorphism.

For a sequence $V_1, V_2, \dots, V_n$ of locally convex vector spaces, let $\mathrm{L}_n\bigl(\prod_{i=1}^n V_i, W\bigr)$ denote the set of continuous multilinear maps from $\prod_{i=1}^n V_i$ to another locally convex vector space W. Proposition A.2.36 implies that there exists a canonical bijection

$$\mathrm{L}_n\bigl(\prod_{i=1}^n V_i, W\bigr) \xrightarrow{\sim} \mathrm{L}\bigl(\bigotimes_{i=1}^n V_i, W\bigr), \qquad \alpha \mapsto \alpha \circ \chi$$

between this space and the space of continuous linear maps from $\bigotimes_{i=1}^n V_i$ to W.

Given bounded (or precompact) subsets $B_i \subseteq V_i$ for $i = 1, 2, \dots, n$, the image $\bigotimes_{i=1}^n B_i$ and its absolutely convex hull in $\bigotimes_{i=1}^n V_i$ are again bounded (resp. precompact). The converse statement is generally false. For precompact subsets of metrisable vector spaces we do however have the following proposition.

Proposition A.2.37.
If each of the spaces $V_1, V_2, \dots, V_n$ *is metrisable, then a subset* $K \subseteq \bigotimes_{i=1}^n V_i$ *is precompact if and only if there exist precompact subsets* $K_i \subseteq V_i$ *for* $i = 1, 2, \dots, n$ *such that* $K \subseteq \overline{\mathrm{hull}(\bigotimes_{i=1}^n K_i)}$.

Morphisms of locally convex vector spaces induce morphisms between their tensor products. The following proposition is a corollary to Proposition A.2.37.

Proposition A.2.38.
Let $V_1, V_2, \dots, V_n$ *and* $W_1, W_2, \dots, W_n$ *be locally convex vector spaces and let* $f_i \colon V_i \to W_i$ *be a morphism of topological vector spaces for* $i = 1, 2, \dots, n$*. The induced map*

$$\bigotimes_{i=1}^n \alpha_i \colon \bigotimes_{i=1}^n V_i \longrightarrow \bigotimes_{j=1}^n W_j, \qquad \bigotimes_{i=1}^n v_i \mapsto \bigotimes_{i=1}^n \alpha_i(v_i)$$

is continuous. If V_i *is metrisable for* $i = 1, 2, \dots, n$*, then the multilinear map*

$$\otimes \colon \prod_{i=1}^n \mathrm{L}_{\mathrm{pc}}(V_i, W_i) \longrightarrow \mathrm{L}_{\mathrm{pc}}\big(\bigotimes_{i=1}^n V_i, \bigotimes_{j=1}^n W_j\big), \qquad (\alpha_i)_{i\in I} \mapsto \bigotimes_{i=1}^n \alpha_i$$

is continuous as well.

The universal property from Proposition A.2.36 moreover tells us that the operator $\otimes$ from can be viewed as a continuous linear map from $\bigotimes_{i=1}^n \mathrm{L}_{\mathrm{pc}}(V_i, W_i)$ to $\mathrm{L}_{\mathrm{pc}}\big(\bigotimes_{i=1}^n V_i, \bigotimes_{j=1}^n W_j\big)$. The following corollary can be thought of as a special case, but can also be proven by simpler means.

Corollary A.2.39.
For any two locally convex vector spaces V *and* W*, the canonical inclusion map*

$$V^{\vee}_{\mathrm{pc}} \otimes W \lhook\joinrel\longrightarrow \mathrm{L}_{\mathrm{pc}}(V, W) \qquad \xi \otimes w \mapsto (v \mapsto \xi(v)\, w)$$

is a continuous linear embedding.

Unlike Proposition A.2.38, Corollary A.2.39 remains true if the topology of bounded or simple convergence is used instead of the topology of precompact convergence. Elements of the image of $V^{\vee} \otimes W$ in $\mathrm{L}(V, W)$ are the so-called finite rank operators. While they form a dense subset for the topology of simple convergence, the same is not generally true for other topologies of uniform convergence.

Definition A.2.40.
A locally convex vector space V has the *approximation property* if one of the following equivalent conditions hold:

(i) $V^\vee \otimes V$ is dense in $\mathrm{L}_{\mathrm{pc}}(V, V)$;

(ii) $V^\vee \otimes W$ is dense in $\mathrm{L}_{\mathrm{pc}}(V, W)$ for any locally convex vector space W;

(iii) $\mathrm{id}_V \in \mathrm{L}(V, V)$ is the limit of a net in $V^\vee \otimes V \subseteq \mathrm{L}(V, V)$.

The approximation property is highly desirable because it allows one approximate arbitrary continuous operators on a space by operators of finite-rank. All Banach spaces that admit a Schauder basis have the approximation property, as do all Hilbert spaces and all nuclear spaces. This in particular includes spaces of smooth sections of finite-dimensional vector bundles.

Given a complete locally convex vector space V, $V^\vee \otimes V \subseteq \mathrm{L}_{\mathrm{b}}(V, V)$ is dense if and only if V has the approximation property and the Heine–Borel property (which states that every bounded subset is precompact).

Proposition A.2.41.
If each of the spaces $V_1, V_2, \ldots, V_n$ is metrisable, then every element z of the tensor product $\bigotimes_{i=1}^n V_i$ (or its completion) can be be written as the limit of a series

$$z = \sum_{n \in \mathbf{N}} \lambda_n \, v_n^{(1)} \otimes v_n^{(2)} \otimes \cdots \otimes v_n^{(n)}$$

for some absolutely summable sequence $(\lambda_n)_{n \in \mathbf{N}} \in \mathbf{K}^{\mathbf{N}}$ and bounded sequences $(x_n^{(i)})_{n \in \mathbf{N}}$ in V_i for $i = 1, 2, \ldots, n$.

Remark A.2.42.
When considering complete locally convex vector spaces such as Banach spaces or Fréchet spaces, it makes sense to consider the *completed tensor product*. This is the completion

$$\overline{\bigotimes}_{i=1}^n V_i = V_1 \mathbin{\bar\otimes} V_2 \mathbin{\bar\otimes} \cdots \mathbin{\bar\otimes} V_n := \overline{\bigotimes_{i=1}^n V_i}.$$

of the projective tensor product as it was defined in Definition A.2.35, and it is often simply referred the projective tensor product.

Since every continuous linear map can be extended to the completion of its domain, the spaces, there is a canonical bijection between the spaces $\mathrm{L}_{\mathrm{pc}}\big(\bigotimes_{i=1}^n V_i, W\big)$ and $\mathrm{L}_{\mathrm{pc}}\big(\overline{\bigotimes}_{i=1}^n V_i, W\big)$. This is an isomorphism of topological vector spaces if V_i is metrisable for $i = 1, 2, \dots, n$ due to Proposition A.2.21. Within the category of complete locally convex vector spaces, the completed projective tensor product is also uniquely determined by the universal property described in Proposition A.2.36. △

Proposition A.2.43 (Heredity).
Let $V = \bigotimes_{i=1}^n V_i$ denote the projective tensor product of the locally convex vector spaces $V_1, V_2, \dots, V_n$, and let $\overline{V} = \overline{\bigotimes}_{i=1}^n V_i$ denote its completion.

(i) *V is complete if and only if either at most one one of the spaces V_i for $i = 1, 2, \dots, n$ is finite dimensional or at least one of them is trivial.*

(ii) *If each of the spaces V_i is normable, then so is V.*

(iii) *If each of the spaces V_i is metrisable, then so is V.*

(iv) *If each of the spaces V_i is both metrisable and barrelled, then so is V.*

(v) *If I is countable and each of the spaces V_i is Fréchet, then so is $\overline{V}$.*

(vi) *If each of the spaces V_i is nuclear, then so is V.*

(vii) *If each of the spaces V_i is metrisable and has the approximation property, then so does V.*

(viii)

On the n-fold tensor product $\bigotimes^n V$ of a locally convex vector space V, we may define the symmetrisation operator

$$s\colon \bigotimes^n V \longrightarrow \bigotimes^n V, \qquad \bigotimes_{i=1}^n v_i \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \bigotimes_{j=1}^n v_{\sigma(j)}, \tag{A.2.1}$$

and the anti-symmetrisation operator

$$a\colon \bigotimes^n V \longrightarrow \bigotimes^n V, \qquad \bigotimes_{i=1}^n v_i \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \bigotimes_{j=1}^n v_{\sigma(j)}.$$

Both s and a are continuous projection maps and thus induce decompositions

$$\bigotimes^n V = \ker(s) \oplus \bigodot^n V \qquad \text{or} \qquad \bigotimes^n V = \ker(a) \oplus \bigwedge^n V,$$

where $\bigodot^n V = \operatorname{im}(s) \simeq \bigotimes^n V \,/\, \ker(s)$ and $\bigwedge^n V = \operatorname{im}(a) \simeq \bigotimes^n V \,/\, \ker(a)$.

The space $\bigodot^n V$ is called the n-fold *symmetric tensor product* of V, and $\bigwedge^n V$ is called the n-th *exterior power* of V. These spaces have universal properties similar to the one described in Proposition A.2.36.

Proposition A.2.44 (Universal property).
Any continuous multilinear map $f: \prod^n V \to W$ of locally convex vector spaces that is either symmetric or alternating factors through

$$\chi_s = s \circ \chi : \prod^n V \longrightarrow \bigodot^n V \qquad \text{or} \qquad \chi_a = a \circ \chi : \prod^n V \longrightarrow \bigwedge^n V$$

respectively.

The spaces $\bigodot^n V$ and $\bigwedge^n V$ and the continuous multilinear maps χ_s and χ_a are uniquely determined by this property up to a unique isomorphism.

A.2.7. Multilinear maps

We are also interested in continuous multilinear maps between locally convex vector spaces. These are continuous maps

$$\alpha \colon \prod_{i=1}^n V_i = V_1 \times V_2 \times \cdots \times V_n \longrightarrow W$$

from the product of locally convex vector spaces $V_1, V_2, \ldots, V_n$ to another such space W that are linear in each argument separately. We denote the set of such maps by $\mathrm{L}(V_1, V_2, \ldots, V_n; W)$.

It is sometimes sufficient to verify that a multilinear map is continuous in each of its arguments separately. The following lemma for instance applies if each of the spaces V_i is Fréchet. The bilinear case corresponds to Theorem 11.6 from [Jar81], where its proof was left as an exercise to the reader.

Lemma A.2.45.
Let $V_1, V_2, \ldots, V_n$ be a barrelled metrisable locally convex vector spaces and let W be a locally convex vector space. For any $n \in \mathbf{N}$, every multilinear $\alpha \colon \prod_{i=1}^n V_i \to W$ that is continuous in each of its arguments separately is (jointly) continuous.

Proof: Assume that the conclusion of this lemma is false and let $n \in \mathbf{N}$ be the smallest positive integer for which it fails. Then $n \geq 2$ since the case $n = 1$ is tautological.

Let $\alpha\colon \prod_{i=1}^n V_i \to W$ be a counter-example, i.e. a multilinear map that is separately continuous but not (jointly) continuous. Then there exists a zero neighbourhood U_W and sequences $(v_{i,j})_{j\in\mathbf{N}}$ that converge to $0 \in V_i$ for $i = 1, 2, \ldots, n$ such that $\alpha(v_{1,j}, \ldots, v_{n,j}) \notin U_W$ for every $j \in \mathbf{N}$. We can assume without loss of generality that U_W is convex, balanced and closed.

Because α is continuous and linear in its first argument, the inverse images $\alpha(v_{1,j}, \ldots, y_{n-1,j}, \cdot)^{-1}(U_W)$ are again convex, balanced and closed for every $j \in \mathbf{N}$, as is the intersection $U_{V_n} := \bigcap_{j=1}^{\infty} \alpha(v_{1,j}, \ldots, v_{n-1,j}, \cdot)^{-1}(U_W)$ of V. We claim that it is also absorbing, which would make it a barrel.

Let $u \in V_n$, then the multilinear map $\alpha(\cdot, \ldots, \cdot, u)\colon \prod_{i=1}^{n-1} V_i \to W$ is (jointly) continuous because we had chosen the smallest n which the lemma fails. Consequently, $\alpha(v_{1,j}, \ldots, v_{n-1,j}, u)$ converges to 0 as $j \to \infty$ and we can choose an $j_0 \in \mathbf{N}$ such that $\alpha(v_{1,j}, \ldots, v_{n-1,j}, u) \in U_W$ whenever $j > j_0$. If we moreover pick some number $\lambda \geq 1$ such that $\alpha(v_{1,j}, \ldots, v_{n-1,j}, u) \in \lambda\, U_W$ for $j = 1, 2, \ldots, j_0$, then $\alpha(v_{1,j}, \ldots, v_{n-1,j}, u) \in \lambda\, U_W$, and thus $u \in \lambda\, U_{V_n}$, for every $j \in \mathbf{N}$. In this manner, we can find a $\lambda > 0$ for any $u \in V_n$ such that $u \in \lambda\, V_n$, so U_{V_n} is indeed absorbing.

Since V_n is a barrelled space and $U_{V_n} \subseteq V_n$ is a barrel, the latter is in fact a zero neighbourhood. There thus exists an $j_1 \in \mathbf{N}$ such that $v_{n,j} \in U_{V_n}$ for every $j > j_1$, which implies that $\alpha(v_{1,j}, \ldots, v_{n,j}) \in U_W$ for all such j. We have arrived at a contradiction since the neighbourhood U_W and the sequences $(v_{i,j})_{j\in\mathbf{N}}$ for $i = 1, 2, \ldots, n$ had been chosen precisely such that $\alpha(v_{1,j}, \ldots, v_{n,j}) \notin U_W$ for every $j \in \mathbf{N}$.

We conclude that this lemma must hold for every $n \in \mathbf{N}$. ☐

The space $\mathrm{L}(V_1, \ldots, V_n, W)$ of multilinear maps can be topologised in the same way as spaces morphisms using topologies of uniform convergence. We shall only consider the topology of precompact convergence, which has the subsets

$$\mathscr{N}\big(\textstyle\prod_{i=1}^n K_i, N\big) = \big\{\alpha \in \mathrm{L}(V_1, \ldots, V_n; W) \;\big|\; \alpha\big(\prod_{i=1}^n K_i\big) \subseteq N\big\}$$

for precompact $K_i \subseteq V_i$ and zero neighbourhoods $N \subseteq W$ as a basis of zero neighbourhoods.

We recall from Proposition A.2.36 that the projective tensor product is characterised by the fact that every continuous multilinear map factors through it. The following proposition is obtained by combining this universal property with the characterisation of precompact subsets of tensor products from Corollary A.2.5.

Proposition A.2.46.
If $V_1, V_2, \dots, V_n$ are metrisable locally convex vector spaces and W is an arbitrary locally convex vector space, then the canonical bijection

$$\mathrm{L}_{\mathrm{pc}}(V_1, \dots, V_n; W) \xrightarrow{\sim} \mathrm{L}_{\mathrm{pc}}\big(\textstyle\bigotimes_{i=1}^{n} V_i, W\big), \qquad \alpha \mapsto \alpha \circ \chi$$

is an isomorphisms of topological vector spaces.

We will often identify the spaces $\mathrm{L}_{\mathrm{pc}}(V_1, \dots, V_n; W)$ and $\mathrm{L}_{\mathrm{pc}}\big(\bigotimes_{i=1}^{n} V_i, W\big)$, and use the term "multilinear map" to refer to either a multilinear map or to the linear map on the corresponding tensor product.

The decompositions $\bigotimes^n V = \ker(s) \oplus \bigodot^n V$ and $\bigotimes^n V = \ker(a) \oplus \bigwedge^n V$ of the n-fold tensor product of a locally convex vector space V induce a similar decomposition

$$\mathrm{L}_{\mathrm{pc}}(\textstyle\bigotimes^n V, W) = \mathrm{L}_{\mathrm{pc}}(\ker(s), W) \oplus \mathrm{L}_{\mathrm{pc}}(\bigodot^n V, W)$$

and

$$\mathrm{L}_{\mathrm{pc}}(\textstyle\bigotimes^n V, W) = \mathrm{L}_{\mathrm{pc}}(\ker(a), W) \oplus \mathrm{L}_{\mathrm{pc}}(\bigwedge^n V, W)$$

of the space of n-linear maps from V to W (cf. Proposition A.2.25 and the remark just below it). The spaces $\mathrm{L}_{\mathrm{pc}}(\bigodot^n V, W)$ and $\mathrm{L}_{\mathrm{pc}}(\bigwedge^n V, W)$ of symmetric and alternating n-linear maps are topologised accordingly.

Proposition A.2.47.
Let $V_1, V_2, \dots, V_n$ and W be locally convex vector space and let $p, q \in \mathbf{N}_0$ be such that $p + q = n$, then there exists a canonical embedding

$$\begin{aligned} \mathrm{L}_{\mathrm{pc}}(V_1, \dots, V_n; W) &\hookrightarrow \mathrm{L}_{\mathrm{pc}}(V_1, \dots, V_p; \mathrm{L}_{\mathrm{pc}}(V_{p+1}, \dots, V_n; W)), \\ \alpha &\mapsto \big((v_1, \dots, v_p) \mapsto \alpha(v_1, \dots, v_p, \cdot, \dots, \cdot)\big) \end{aligned}$$

and the same is true for the topologies of simple and bounded convergence.

These maps are isomorphisms of topological vector spaces for the topologies of precompact and bounded convergence whenever each of the spaces V_i is metrisable, and also for the topologies of simple convergence if these spaces are moreover barrelled.

The following corollary combines Proposition A.2.46 and Proposition A.2.47.

Corollary A.2.48.
Let V and W be locally convex vector space and define the spaces $\mathrm{L}_k(V,W)$ for $k \in \mathbf{N}_0$ recursively as

$$\mathrm{L}_l(V,W) = \mathrm{L}(V, \mathrm{L}_{l-1}(V,W)) \qquad \textit{and} \qquad \mathrm{L}_0(V,W) = W.$$

for $l \in \mathbf{N}$. If V is metrisable, then the spaces

$$\mathrm{L}^k_{\mathrm{pc}}(V,W), \qquad \mathrm{L}_{\mathrm{pc}}(\underbrace{V,\dots,V}_{k \text{ copies}};W), \qquad \textit{and} \qquad \mathrm{L}_{\mathrm{pc}}\Big(\bigotimes^k V, W\Big)$$

are canonically isomorphic

APPENDIX B.

Differential calculus

In addition to functional analysis, we will also need some amount of differential calculus, mainly on Fréchet spaces, although we will consider maps whose codomains are arbitrary locally convex vector spaces. The overall message is that many of the definitions and results that one usually employs in the context of finite-dimensional geometry continue to work, but that some amount of care is required.

Not everything that works for finite-dimensional or Banach spaces can be generalised to Fréchet spaces, however. Most notable amongst these are probably the inverse function theorem and the existence and uniqueness theorem for ordinary differential equations, and many results that rely on either of these are consequently false as well.

The for us relevant notions of (continuous) differentiability and analyticity are discussed in appendix B.1, and the spaces of such maps are topologised in appendix B.2. Finally, appendix B.3 contains a brief discussion of regular Fréchet Lie groups.

B.1. Smoothness and analyticity

The purpose of this appendix is to provide definitions for smoothness and analyticity for functions from a Fréchet space to a locally convex vector space, since such maps are used throughout this thesis. We will additionally briefly consider continuous polynomials, as well as jets and formal power series of smooth (or analytic) functions.

B.1.1. Differentiability and smoothness

Although there are many inequivalent ways to define what it means for such maps to be k times continuously differentiable for any $k \in \mathbf{N}$, most give rise to the same notion of smoothness because all maps that are of class C^{k+1} in one sense are also of class C^k in the others. Since we are mostly interested in smooth calculus, the choice of one notion of differentiability over another therefore turns out to be largely inconsequential and we can use the definition that is most convenient for our purposes. An overview of some of these notions of continuous differentiability, as well as their relations, can be found in [Kel74]. The notion of k-fold continuous differentiability described here corresponds to what is called "differentiability of class $\mathrm{C}^k_{\mathrm{pk}}$ in the aforementioned reference. It is equivalent to the definition used in [Ham82] whenever the domain is metrisable (cf. Proposition B.1.4 below).

Throughout this appendix we will be considering maps that are parametrised by a *Fréchet domain*, by which we mean an open subset $X \subseteq T$ of some Fréchet space T. We say that a function $f : X \to V$ to a locally convex vector space V is *weakly differentiable* at a point $x \in X$ if there exists a continuous linear map $\mathrm{D}f(x) \in \mathrm{L}(T, V)$ such that

$$\lim_{t\to 0} \tfrac{1}{t}\big(f(x + t\,\dot{x}) - f(x)\big) = \mathrm{D}f(x)(\dot{x}),$$

in which case $\mathrm{D}f(x)$ is called the derivative of f at x.

Throughout this document, we shall endow the space $\mathrm{L}(T, V)$ of continuous linear maps from T to V with the topology of precompact convergence from Definition A.2.20. We therefore say that f is continuously differentiable, or of class C^1, if it is weakly differentiable and the map $\mathrm{D}f : X \to \mathrm{L}(T, V)$ is continuous for this topology. In [Kel74], f would be said to be of class $\mathrm{C}^1_{\mathrm{pk}}$, and it is shown there that this is equivalent to the existence this is equivalent to the existence of a continuous map $\delta f : X \times T \to V$ such that $\mathrm{D}f(x)(v) = \delta f(x, v)$ for all $(x, v) \in X \times T$.

We define the spaces $\mathrm{L}^k(T,V)$ recursively by starting with $\mathrm{L}^0(T,V) = V$ and setting $\mathrm{L}^{l+1} := \mathrm{L}(T,\mathrm{L}^l(T,V))$ for every $l \in \mathbf{N}_0$. Whenever T is metrisable, Corollary A.2.48 provides canonical isomorphisms

$$\mathrm{L}^k(T,V) \simeq \mathrm{L}(\underbrace{T,\ldots,T}_{k \text{ copies}},V) \simeq \mathrm{L}\Big(\bigotimes^k T;V\Big)$$

of topological vector spaces between this space, the space $\mathrm{L}(T,\ldots,T,V)$ of continuous multilinear maps and the space $\mathrm{L}\big(\bigotimes^k T,V\big)$ of continuous linear maps from the k-fold (projective) tensor product of T.

We say that f is of class C^2 whenever it is continuously differentiable and its derivative, $\mathrm{D}f\colon X \to \mathrm{L}(T,V)$, is continuously differentiable as well. We then refer to $\mathrm{D}^2 f := \mathrm{D}(\mathrm{D}f)\colon X \to \mathrm{L}^2(T,V)$ as its second order derivative. Higher order derivatives can be defined by repeating this process.

Definition B.1.1.
Let T be a metrisable locally convex vector space, let V be a locally convex vector space and let $X \subseteq T$ be an open subset. A continuous map $f\colon X \to V$ is of class C^k for some $k \in \mathbf{N}_0$ if there exist (unique) continuous maps $\mathrm{D}^l f\colon X \to \mathrm{L}^l(T,V)$ for $l = 0,1,\ldots,k$ such that $\mathrm{D}^0 f = f$ and

$$\lim_{t\to 0} \tfrac{1}{t}\left(\mathrm{D}^{l-1} f(x + t\,\dot{x}) - \mathrm{D}^{l-1} f(x)\right) = \mathrm{D}^l f(x)(\dot{x}).$$

for all $x \in X$, $\dot{x} \in T$ and $l = 1,2,\ldots,k$. We say that f is *smooth*, or of class C^∞, whenever it is of class C^k for every $k \in \mathbf{N}_0$.

Below, we state a number of important results for such maps. Versions of each of these lemmas can be found in [Kel74]

Proposition B.1.2.
If $f\colon X \subseteq T \to V$ is of class C^k for some $k \in \mathbf{N}_0 \cup \{\infty\}$, then the derivatives $\mathrm{D}^l f\colon X \to \mathrm{L}^l(T,V)$ for $l \in \mathbf{N}_0$ with $l \leq k$ are symmetric in the sense that

$$\mathrm{D}^l f(x)(\dot{x}_1,\dot{x}_2,\ldots,\dot{x}_l) = \mathrm{D}^l f(x)(\dot{x}_{\sigma(1)},\dot{x}_{\sigma(2)},\ldots,\dot{x}_{\sigma(l)})$$

for any permutation $\sigma \in S_l$.

Proposition B.1.2 is an immediate consequence of e.g. Theorem 2.4.0 from [Kel74].

We call a map $f : X \times Y \to V$ continuously partially differentiable if both $f|_{X\times\{y\}}$ and $f|_{\{x\}\times Y}$ are continuously differentiable for every $x \in X$ and every $y \in Y$. The following proposition essentially states that continuous partial differentiability implies continuous differentiability.

Proposition B.1.3.
Let S and T be metrisable locally convex vector spaces, let V be a locally convex vector space and let $X \subseteq S$ and $Y \subseteq T$ be open subsets. Let $f : X \times Y \to V$ be a continuous function with the following properties for a fixed $k \in \mathbf{N}_0 \cup \{\infty\}$:

1. *the restriction $f|_{X\times\{y\}}$ is of class C^k for every $y \in Y$;*
2. *for any $p \in \mathbf{N}_0$ such that $p \leq k$ and every $x \in X$, the partial derivatives $\mathrm{D}_1^p f$ are such that the restriction*
$$\mathrm{D}_1^p f|_{\{x\}\times Y} : \{x\} \times Y \longrightarrow \mathrm{L}^p(S, V)$$
is of class C^{k-p}.
3. *for any pair $(p, q) \in \mathbf{N}_0 \times \mathbf{N}_0$ such that $p + q \leq l$, the map*
$$\mathrm{D}_2^q \mathrm{D}_1^p f : X \times Y \longrightarrow \mathrm{L}^q(T, \mathrm{L}^p(S, V)) \subseteq \mathrm{L}^{p+q}(S \oplus T, V)$$
is continuous.

Then f is of class C^k and its derivatives are given by
$$\mathrm{D}^{p+q} f(x, y)(\dot{x}_{\{1,\dots,p\}} \odot \dot{y}_{\{1,\dots,q\}}) = (\mathrm{D}_2^q \mathrm{D}_1^p f)(x, y)(\dot{x}_{\{1,\dots,p\}})(\dot{y}_{\{1,\dots,q\}})$$
for all $x, y \in X \times Y$, all $\dot{x}_1, \dot{x}_2, \dots, x_p \in S$ and $\dot{y}_1, \dot{y}_2, \dots, \dot{y}_q \in T$ for which $p + q \leq k$.

Proposition B.1.3 can be proven using the same methods one might apply in the finite-dimensional case, which in particular includes the mean value theorem from Proposition A.2.6.

The characterisation of continuous differentiability provided by the next proposition is often used as a definition. This is for instance the how continuous differentiability is defined in [Ham82].

Proposition B.1.4.
Let T be a metrisable locally convex vector space, let V be a locally convex vector

space and let $X \subseteq T$ be an open subset. Given some $k \in \mathbf{N}_0 \cup \{\infty\}$, a continuous map $f : X \subseteq T \to V$ is of class C^k if and only if there exists a continuous map

$$\delta^p f : X \times \prod^p T \longrightarrow V$$

for every $p \in \mathbf{N}_0$ with $p \leq k$, where $\delta^0 f = f$, such that

$$\delta^p f(x; \dot{x}_1, \dots, \dot{x}_p) = \lim_{t\to 0} \tfrac{1}{t}\big(\delta^{k-1} f(x+t\,\dot{x}_1)(\dot{x}_2, \dots, \dot{x}_p) - \delta^{k-1} f(x)(\dot{x}_2, \dots, \dot{x}_p)\big)$$

for all $x \in X$ and all $\dot{x}_1, \dots, \dot{x}_p \in T$. The derivatives $\mathrm{D}^p f$ and $\delta^p f$ are related by

$$\delta^p f(x; \dot{x}_1, \dots, \dot{x}_p) = \mathrm{D}^p f(x)(\dot{x}_1, \dots, \dot{x}_p)$$

for $x \in X$ and $\dot{x}_1, \dot{x}_2, \dots, \dot{x}_p \in T$.

As they should, the usual differentiation rules are valid in the present context.

Proposition B.1.5 (Sum rule).
Let T be a metrisable locally convex vector space and let V be an arbitrary locally convex vector space. If $f : X \to V$ and $g : X \to V$ are both of class C^k for some $k \in \mathbf{N}_0 \cup \{\infty\}$, then so is $f + g$. Its derivatives are given by $\mathrm{D}^p(f+g) = \mathrm{D}^p f + \mathrm{D}^p g$.

The chain rule can be proven by applying Corollary 1.3.4 from [Kel74] repeatedly, and one obtains expression for the derivatives through diligent bookkeeping.

Proposition B.1.6 (Chain rule).
Let T and V be metrisable locally convex vector spaces, let Z be a locally convex vector space and let $X \subseteq T$ and $Y \subseteq V$ be open subsets. Given two maps $\alpha \colon X \to V$ and $\beta \colon Y \to Z$ of class C^k for some $k \in \mathbf{N}_0 \cup \{\infty\}$ that are such that $\alpha(X) \subseteq Y$, also the composition $\beta \circ \alpha \colon X \to Z$ is of class C^k. Its derivatives are given by

$$\mathrm{D}^p(g \circ f)(x)(\dot{x}_1, \dots, \dot{x}_p) = \sum_{I_1,\dots,I_m} \tfrac{1}{m!}\,\mathrm{D}^m g(f(x))\big(\textstyle\bigodot_{j=1}^m \mathrm{D}^{\# I_j} f(x)(\dot{x}_{I_j})\big)$$

for all $x \in X$ and all $\dot{x}_1, \dot{x}_2, \dots, \dot{x}_p \in T$ with $p \in \mathbf{N}_0$. Here, summation is over all $m \in \mathbf{N}_0$ and all ordered partitions of $\{1, 2, \dots, p\}$ into non-empty pairwise disjoint subsets $I_1, I_2, \dots, I_m$.

Proposition B.1.7 below is a generalisation of the usual product rule for real-valued functions that uses the projective tensor product, which is described in Definition A.2.35.

Proposition B.1.7 (Product rule).
Let $X \subseteq T$ be an open subset of a metrisable locally convex vector space T, and let $V_1, V_2, \dots, V_q$ be locally convex vector spaces. The tensor product

$$e_i \otimes \cdots \otimes e_q : X \longrightarrow \bigotimes_{i=1}^q V_i, \qquad x \mapsto e_1(x) \otimes \cdots \otimes e_q(x)$$

of maps $e_i \in \mathrm{C}^k(X, V_i)$ for $i = 1, 2, \dots, q$ and $k \in \mathbf{N}_0 \cup \{\infty\}$ is of class C^k and its derivatives are given by

$$\mathrm{D}^p(e_1 \otimes \cdots \otimes e_q)(x)(\dot{x}_1, \dots, \dot{x}_p) = \sum_{I_1, \dots, I_q} \bigotimes_{i=1}^q \mathrm{D}^{\# I_i} e_i(x)(\dot{x}_{I_i})$$

for $x \in X$ and $\dot{x}_1, \dot{x}_2, \dots, \dot{x}_p \in T$ with $p \in \mathbf{N}_0$ such that $p \le k$.

Proof: We can assume without loss of generality that $q = 2$ since the general case can be derived from this case by a simple recursive argument.

Let $e_1 : X \to V_1$ and $e_2 : X \to V_2$ be continuous and let $N \subseteq V_1 \otimes V_2$ be a zero neighbourhood. We can then pick absolutely convex zero neighbourhoods $N_1 \subseteq V_1$ and $N_2 \subseteq V_2$ such that $\mathrm{hull}(N_1 \otimes N_2) \subseteq \frac{1}{3} N$ and choose $\kappa, \lambda \ge 1$ such that $e_1(x) \in \kappa N_1$ and $e_2(x) \in \lambda N_2$. Now choose a neighbourhood $U \subseteq X$ of x such that $e_1(y) \in e_1(x) + \lambda^{-1} N_1$ and $e_2(y) \in e_2(x) + \kappa^{-1} N_2$ for all $y \in U$. Then $(e_1 \otimes e_2)(y) - (e_1 \otimes e_2)(x)$ can be written as the sum of

$$e_1(x) \otimes (e_2(y) - e_2(x)) \in (\kappa N_1) \otimes (\kappa^{-1} N_2) \subseteq N_1 \otimes N_2,$$
$$(e_1(y) - e_1(x)) \otimes e_2(x) \in (\lambda^{-1} N_1) \otimes (\lambda N_2) \subseteq N_1 \otimes N_2$$

and

$$(e_1(y) - e_1(x)) \otimes (e_2(y) - e_2(x)) \in (\kappa^{-1} N_1) \otimes (\lambda^{-1} N_2) \subseteq N_1 \otimes N_2$$

and is therefore an element of $3\,\mathrm{hull}(N_1 \otimes N_2) \subseteq N$. This shows that $e_1 \otimes e_2$ is continuous whenever e_1 and e_2 are.

It follows from a very similar argument that $e_1 \otimes e_2$ is weakly differentiable with derivative $\mathrm{D}e_1 \otimes e_2 + e_1 \otimes \mathrm{D}e_2 : X \to \mathrm{L}(T, V_1 \otimes V_2)$ whenever e_1 and e_2 are differentiable. The general case follows from an inductive argument using Proposition A.2.38. ☐

One can prove, using essentially the same arguments as in the finite-dimensional case, that integrals of continuously differentiable functions are continuously differentiable.

Lemma B.1.8.
Let $X \subseteq T$ be a Fréchet domain, let V be a complete locally convex vector space and assume we are given a map $f : [0,1] \times X \to V$ which is of class C^k for some $k \in \mathbf{N}_0 \cup \{\infty\}$. The function

$$F : X \longrightarrow V, \qquad x \mapsto \int_0^1 f(t,x)\,\mathrm{d}t$$

is of class C^k, and its derivatives are given by $\mathrm{D}^p F(x) = \int_0^1 \mathrm{D}^p f(t,x)\,\mathrm{d}t \in \mathrm{L}(\bigodot^p(T), V)$ for every $x \in X$ and any $p \in \mathbf{N}_0$ such that $p \leq k$.

Proof: We will first show that F is continuous and then demonstrate differentiability whenever f is of class C^1. The general case then follows from an inductive argument.

Let $x \in X$ be arbitrary and pick an absolutely convex zero neighbourhood $N_V \subseteq V$. The continuity of f and the compactness of the interval $[0,1]$ allow us to choose a neighbourhood $U_X \subseteq X$ such that $f(t,y) \in f(t,x) + \frac{1}{2} N_Y$ for every $t \in [0,1]$ and every $y \in U_X$. Due to the mean value theorem from part 4 of Proposition A.2.6, this in turn implies that

$$F(y) - F(x) = \int_0^1 f(t,y) - f(t,x)\,\mathrm{d}t \in \overline{\tfrac{1}{2} N_Y} \subseteq N_Y$$

for every $y \in U_Y$. Continuity of F follows.

Assume that f is of class C^1. To show that F is weakly differentiable in the direction $\dot{x} \in T$ at a point $x \in X$, we use the fact that f is continuously differentiable to write

$$\begin{aligned} F(x + t\,\dot{x}) - F(x) &= \int_0^1 f(s, x + t\,\dot{x}) - f(s,x)\,\mathrm{d}s \\ &= \int_0^1 \int_0^1 \mathrm{D}f(s, x + r\,t\,h)(t\,\dot{x})\,\mathrm{d}r\,\mathrm{d}s. \end{aligned}$$

Now let $N_V \subseteq V$ be an arbitrary zero neighbourhood. Because $\mathrm{D}f(s,x)(\dot{x})$ depends continuously on $(s,x) \in [0,1] \times X$ and $[0,1]$ is compact, we can now choose a convex neighbourhood U_X of $x \in X$ such that $\mathrm{D}f(s,y)(\dot{x}) \in$

$\mathrm{D}f(s,x)(\dot{x}) + \frac{1}{2}N_V$ for all $y \in U_X$ and all $s \in [0,1]$. If again follows from the mean value theorem that for these choices and for any $t \in \mathbf{R}$ such that $x + t\,\dot{x} \in U_X$,

$$\tfrac{1}{t}\big(F(x+t\,\dot{x}) - F(x)\big) - \int_0^1 \mathrm{D}f(s,x)(\dot{x})\,\mathrm{d}s$$
$$= \int_0^1 \int_0^1 \mathrm{D}f(s,x+t\,\dot{x})(\dot{x}) - \mathrm{D}f(s,x)(\dot{x})\,\mathrm{d}r\,\mathrm{d}s$$

is an element of $\overline{\frac{1}{2}N_V} \subseteq N_V$. We deduce that F is weakly differentiable and that, due to part 2 of Proposition A.2.6, its derivative is given by $\mathrm{D}F(x) = \int_0^1 \mathrm{D}f(s,x)\,\mathrm{d}s$.

Continuity of $\mathrm{D}F$ can be demonstrated by essentially repeating the argument used to prove the continuity of F, and one can then go on to prove that $\mathrm{D}F$ is in fact differentiable whenever f is of class C^2. The general case for $k > 1$ follows by induction. □

With Lemma B.1.8, we can now prove an infinite-dimensional analogue of Hadamard's lemma for smooth functions.

Proposition B.1.9.
Let $X \subseteq T$ be a convex Fréchet domain and let V be a locally convex vector space. For any smooth map $f : X \to V$ and any $x_0 \in X$ there exists a smooth function $F : X \to \mathrm{L}(T, \overline{V})$ such that

$$f(x) = f(x_0) + F(x)(x - x_0)$$

for all $x \in X$

Proof: Let f and x_0 be as described above. The fundamental theorem of calculus, in the form of part 5 of Proposition A.2.6, tells us that because f is continuously differentiable it is of the aforementioned form for the function

$$F : X \longrightarrow \overline{\mathrm{L}(T,V)}, \qquad x \mapsto \int_0^1 \mathrm{D}f(x_0 + t\,(x - x_0))\,\mathrm{d}t.$$

We can deduce from from Proposition A.2.22 that $\mathrm{L}(T,\overline{V})$ is the completion of $\mathrm{L}(T,V)$, and Lemma B.1.8 tells us that F is smooth. □

All it really takes to be able to be able to define smooth manifolds modelled on Fréchet spaces is the chain rule from Proposition B.1.6. Although they are defined in exactly the same way as in the finite-dimensional case, we will recall the definition for the sake of completeness.

A continuous chart on a topological space M of Fréchet type is a homeomorphism map $\varphi\colon U \to \varphi(U) \subseteq V$ from an open subset $U \subseteq M$ onto a Fréchet domain $\varphi(U) \subseteq V$. A collection $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ of such charts $\varphi_\lambda\colon U_\lambda \to V_\lambda$ whose domains cover M is called an atlas and we call this atlas smooth whenever the transition functions

$$\varphi_{\kappa,\lambda} \coloneqq \varphi_\lambda \circ \varphi_\kappa\colon \varphi_\kappa(U_\kappa \cap U_\lambda) \longrightarrow \varphi_\lambda(U_\kappa \cap U_\lambda)$$

are smooth for any two indices $\kappa, \lambda \in \Lambda$. We call two smooth atlases (smoothly) equivalent whenever their union is a smooth atlas as well, and we refer to an equivalence class of smooth atlases as a smooth structure.

Definition B.1.10.
A (smooth) *Fréchet manifold* is a Hausdorff space endowed with a smooth structure.

A morphism of Fréchet manifolds from $(M, [(\varphi_\lambda)_{\lambda\in\Lambda}])$ to $(N, [(\psi_\kappa)_{\kappa\in K}])$ is a function $f : M \to N$ such that the composition

$$\varphi_\lambda \circ f \circ \psi_\kappa^{-1}\colon \psi_\kappa(U_\kappa \cap U_\lambda) \longrightarrow \varphi_\lambda(U_\kappa \cap U_\lambda)$$

is smooth for all $\kappa \in K$ and all $\lambda \in \Lambda$. This is independent of the atlases chosen to represent the smooth structures on M and N. We will usually just refer to such morphism as smooth maps and call $f : M \to N$ a diffeomorphism if it is a bijection and its inverse, $f^{-1}\colon N \to M$, is smooth as well. Also smooth vector bundles and fibre bundles of Fréchet can be defined exactly as they would in the finite-dimensional case.

All finite-dimensional manifolds considered in this thesis will be assumed to be second countable. Since the types of Fréchet spaces we are interested in are separable (and therefore also second countable), it is actually reasonable to impose the same condition on Fréchet manifolds as well.

B.1.2. Polynomials and jets

Polynomials on locally convex vector spaces can be characterised using symmetric multilinear maps. This topic is discussed in detail in [BS71b].

Definition B.1.11.
Let V and W be locally convex vector spaces. A continuous function $f_k \colon V \to W$ is a *homogeneous polynomial* of degree $k \in \mathbf{N}$ if there exists a continuous symmetric multilinear map $\bar{f} \colon \bigotimes^k V \to W$ such that $f(x) = \frac{1}{k!}\bar{f}_k(x, \dots, x)$ for all $x \in V$.

The factor $\frac{1}{k!}$ in Definition B.1.11 was included for compatibility with existing conventions for L_∞-algebras.

Given a homogeneous polynomial f_k, the corresponding multilinear map $\frac{1}{k!}\bar{f}_k$ is referred to as the *polarisation* of f_k. It is uniquely determined by f_k, and it can be obtained through one of several polarisation identities, such as

$$2^k \bar{f}_k(v_1 \otimes \cdots \otimes v_k) = \sum_{\epsilon_1,\dots,\epsilon_k \in \{\pm 1\}} \epsilon_1 \cdots \epsilon_k f_k\big(\epsilon_1 v_1 + \cdots + \epsilon_k v_k\big).$$

For any two locally convex vector spaces V and W this describes a one-to-one correspondence $f_k \longleftrightarrow \bar{f}_k$ between homogeneous polynomials of degree k and symmetric continuous k-multilinear maps from the former to the latter. We will henceforth use these two concepts interchangeably and will generally use the same symbol to denote both the function homogeneous polynomial $f \colon V \to W$ and its polarisation.

It was not necessary to require continuity of both the function f and its polarisation $\bar{f}$ in Definition B.1.11 since continuity of the former is equivalent to continuity of the latter. We denote the space of homogeneous polynomials of degree k on V with values in W by $\mathrm{Poly}_k(V, W) \subseteq \mathrm{C}^0(V, W)$ and note that it is canonically isomorphic to $\mathrm{L}(\bigodot^k(V), W)$ whenever V is metrisable due to Proposition A.2.44.

Definition B.1.12.
Let V and W be locally convex vector spaces. A continuous *polynomial* of degree at most k with $k \in \mathbf{N}_0$ from V to W is a function $f \colon V \to W$ of the form $f = f_0 + f_1 + \cdots + f_k$ with $f_i \in \mathrm{Poly}_i(V, W)$ for $i = 0, 1, 2, \dots, k$.

We denote the space of all continuous polynomials from V to W of degree at most k by $\mathrm{Poly}_{\leq k}(V,W) \simeq \prod_{p=0}^{k} \mathrm{Poly}_p(V,W)$ and endow it with the corresponding product topology. Note that Proposition A.2.44 and Proposition A.2.25 tells us that whenever V is metrisable,

$$\mathrm{Poly}_{\leq k}(V,W) \simeq \prod_{i=1}^{k} \mathrm{L}\big(\bigodot^p(V), W\big) \simeq \mathrm{L}\big(\bigoplus_{p=0}^{k} \bigodot^p(V), W\big)$$

as topological vector spaces. We denote the space of all polynomials from V to W by $\mathrm{Poly}(V,W) = \bigcup_{p\in\mathbf{N}} \mathrm{Poly}_{\leq p}(V,W)$ and endow it with the subspace topology for the inclusion into $\prod_{p=0}^{k} \mathrm{Poly}_p(V,W)$.

The sum $f+g$ of two continuous polynomials $f \in \mathrm{Poly}_{\leq k}(V,W)$ and $g \in \mathrm{Poly}_{\leq l}(V,W)$ of degree at most k and l respectively is a continuous polynomial of degree at most $\max(k,l)$. Its homogeneous components are polarised by $\overline{(f+g)}_p = f_p + g_p$.

The product $f \otimes g$ of two continuous polynomials $f \in \mathrm{Poly}_{\leq l}(V,W)$ and $g \in \mathrm{Poly}_{\leq l}(V,Z)$ of degree at most k and l respectively is a continuous polynomial of degree at most $k+l$ with values in $W \otimes Z$. Its is polarised by the multilinear maps $\overline{(f\otimes g)}_p = \sum_{r+s=p} \binom{p}{r}(f_r \otimes g_s) \circ s$ for $p \in \mathbf{N}_0$, where s denotes the symmetrisation operator from $\bigotimes^r V \otimes \bigotimes^s V$ to $\bigotimes^p V$.

The composition $h = g \circ f$ of two continuous polynomials $f = \sum_{i=1}^{k} f_i \colon V \to W$ and $g = \sum_{i=1}^{l} \colon W \to W$ of degree at most k and l respectively is a continuous polynomial of degree at most kl. Its homogeneous component h_p of degree p is polarised by the multilinear form $\bar{h}_p \colon \bigotimes^p V \to Z$ given by

$$\bar{h}_p(v_1, v_2, \dots, v_p) = \sum_{I_1,\dots,I_m} \tfrac{1}{m!}\, \bar{g}_m\big(\bar{f}_{\#I_1}(v_{I_1}), \dots, \bar{f}_{\#I_m}(v_{I_m})\big)$$

where summation is over all partitions of $\{1,2,\dots,p\}$ by non-empty subsets and v_{I_j} denotes the tensor product $\bigotimes_{i\in I_m} v_i$.

Proposition B.1.13.
Any continuous polynomial $f = \sum_{i=0}^{k} f_i \colon V \to W$ from a Fréchet space to a locally convex vector space W is smooth and its derivatives at $x \in V$ are given by

$$\mathrm{D}^p f(x)(X_1, \dots, X_p) = \sum_{i=0}^{k-p} \tfrac{1}{i!}\, \bar{f}_{p+i}\big((\bigotimes^i x) \otimes X_1 \otimes \cdots \otimes X_p\big)$$

for $p = 0, 1, \dots, k$ *and* $X_1, X_2, \dots, X_p \in V$, *and by* $\mathrm{D}^p f = 0$ *if* $p > k$.

We can now provide a definition for a formal power series on V with values in W. The following definition differs slightly from Definition 5.1 in [BS71a], where homogeneous continuous polynomials were not required to be continuous. What we call a formal power series is called a formal series with continuous terms there.

Definition B.1.14.
Let V be a Fréchet space and let W be a locally convex vector space. A *formal power series* on V with values in W is a formal sum $f = \sum_{k=0}^{\infty} f_k$ of homogeneous continuous polynomials $f_k \in \mathrm{Poly}_k(V, W)$.

The space of formal power series is the product $\prod_{p=0}^{k} \mathrm{Poly}_k(V, W)$ and is topologised as such. If V is metrisable, it is consequently isomorphic to

$$\prod_{p \in \mathbf{N}_0} \mathrm{L}\big(\bigodot^p(V), W\big) \simeq \mathrm{L}\big(\bigoplus_{p \in \mathbf{N}_0} \bigodot^p(V), W\big)$$

as a topological vector space. The composition $g \circ f$ of two formal power series makes sense if the zeroth component $f_0 \in \mathrm{L}(\bigodot^0(V), W) \simeq W$ is trivial.

Example B.1.15 (Taylor series)**.**
Let $f : U \subseteq V \to W$ be a smooth map from a Fréchet domain $U \subseteq V$ to another locally convex vector space W. The *Taylor series* of f at x is the formal power series

$$\sum_{k \in \mathbf{N}_0} f_{x,k} \qquad \text{with } \bar{f}_{x,k} = \mathrm{D}^k f(x).$$

We will write $f(x + h) \approx \sum_{k \geq 0} f_k(x)(h)$ to indicate that $\sum_{k \in \mathbf{N}_0} f_k$ is the Taylor series of f at x. ◊

Very closely related to formal power series is the notion of a jet.

Definition B.1.16 (Jets)**.**
Let $X \subseteq T$ be a Fréchet domain, let V be a locally convex vector space and let $k \in \mathbf{N}_0$. A *k-jet* from X to V at $x \in X$ is an equivalence class

$$\mathrm{j}_x^k f = [f] \in \mathrm{J}_x^k(X, V) = \mathrm{C}^k(X, V) \big/ \sim$$

of the set of maps of class C^k from X to V, where $f \sim g$ whenever $\mathrm{D}^p f(x) = \mathrm{D}^p g(x)$ for $p = 0, 1, \ldots, k$. An ∞-*jet* is an element

$$\tilde{f}_x \in \mathrm{J}^\infty_x(V, W) = \varprojlim_p \mathrm{J}^k_x(V, W)$$

of the limit of the projective system which consists of the canonical projection maps $\pi_{l,k} \colon \mathrm{J}^k_x(V, W) \to \mathrm{J}^l_x(V, W)$ with $l \le k \in \mathbf{N}_0$.

For finite k, every k-jet from X to V at an a point $x \in X$ can be uniquely represented as the jet of a polynomial of degree at most k. We thus obtain a canonical bijection

$$\mathrm{Poly}_{\le k}(T, V) \xrightarrow{\sim} \mathrm{J}^k(T, V) \qquad f \mapsto \mathrm{j}^k_x f,$$

which we declare to be a topological isomorphism. We endow the space of ∞-jets with the limit topology for the projective system defining it, so that

$$\mathrm{J}^\infty_x(X, V) \simeq \prod_{p=0}^{k} \mathrm{L}\big(\textstyle\bigodot^p(T), V\big) \simeq \mathrm{L}\big(\bigoplus_{p=0}^k \bigodot^p(T), V\big)$$

as a topological vector space. We will generally identify these spaces and their elements.

Theorem B.1.17 (Borel's Theorem).
If T is finite dimensional and V is a Fréchet space, then the map

$$\mathrm{j}^\infty_x \colon \mathrm{C}^\infty(X, V) \longrightarrow \mathrm{J}^\infty_x(X, V), \qquad f \mapsto \mathrm{j}^\infty_x f$$

is surjective for any open subset $X \subseteq T$ and $x \in X$.

Borel's theorem is generally false if T is infinite-dimensional or if V is replaced by a locally convex vector space that not metrisable or not complete.

Given a function $f : X \to V$ of class C^k from a Fréchet domain $X \subseteq T$ to a locally convex vector space V, the k-jet of f is the function

$$\mathrm{j}^k f : X \longrightarrow \mathrm{L}\big(\textstyle\bigoplus_{p=0}^k \bigodot^p(T), V\big). \qquad x \mapsto \mathrm{j}^k_x f$$

which assigns to every $x \in X$ the k-jet $\mathrm{j}^k_x \in \mathrm{J}^k_x(X, V) \simeq \mathrm{L}\big(\bigoplus_{p=0}^k \bigodot^p(T), V\big)$. This function is smooth for any $k \in \mathbf{N}_0 \cup \{\infty\}$ if f assumed to be smooth.

Definition B.1.18.
A continuous function $\tilde{f} : X \to \mathrm{J}^k(T, V)$ is *holonomic* if it is of the form $\mathrm{j}^k f$ for some $f \in \mathrm{C}^k(T, V)$.

B.1.3. Analyticity

Let $D \subseteq T$ be a Fréchet domain and let V be a locally convex vector space. We say that the formal power series $\sum_{k=0}^{\infty} f_k$ converges on D if the sequence $(F_k(x))_{k\in\mathbf{N}_0}$ of partial sums $F_k(x) = \sum_{p=0}^{k} f_p(x)$ of partial sums converges for every $x \in D$. In this case, we identify the series it with its limit $f : X \to T$. The largest open subset $D \subseteq T$ on which f converges is called the *domain of convergence* of f.

Proposition B.1.19.
The domain of convergence of a formal power series $\sum_{k=0}^{\infty} f_k$ from a Fréchet space T to a locally convex vector space V is either empty or a zero neighbourhood.

In light of Proposition B.1.20 below, an equivalent definition would have been obtained if convergence of partial sums had been replaced by either absolute convergence or unconditional convergence. This proposition is part of Proposition 5.3 in [BS71a].

Proposition B.1.20 (Absolute convergence).
Let $D \subseteq T$ be a Fréchet domain and let $f = \sum_{n\in\mathbf{N}_0} f_n$ be a formal power series with values in a locally convex vector space V that converges on D. For any continuous seminorm q on V, the series $\sum_{n=0}^{\infty} q(f_n(x))$ then converges in $\mathbf{R}$.

A function is analytic if it can locally be described as the limit of convergent power series.

Definition B.1.21.
Let $X \subseteq T$ be a Fréchet domain and let V be a locally convex vector space. A continuous function $f : X \to V$ is *analytic* if for every point $x \in X$ there exists a formal power series $\sum_{k=0}^{\infty} f_{k,x}$ from T to V and a neighbourhood $U \subseteq X$ of x such that

$$f(x+h) = \sum_{k=0}^{\infty} f_{k,x}(h)$$

for all $h \in T$ such that $x + h \in U$.

The following proposition states that any convergent power series defines an analytic function. It coincides with Theorem 5.2 in [BS71a].

Proposition B.1.22.
If $\sum_{i=0}^{\infty} f_i$ is a formal power series from a Fréchet space T to a complete locally convex vector space V which converges on an open subset $U \subseteq T$, then the function $f : U \to V$ given by $f(x) = \sum_{k=0}^{\infty} f_k(x)$ is analytic.

The statements up to this point have been valid for both real and complex vector spaces. Although we are mainly interested in real analytic maps, it will be useful to also consider their complex analytic extensions. Every real formal power series $\sum_{n \in \mathbf{N}_0} f_n$ with $f_n \in \mathrm{L}(V, W)$ can be extended to a complex power series $\sum_{n \in \mathbf{N}_0} \tilde{f}_n$ with coefficient $\tilde{f}_n \in \mathrm{L}(V \otimes \mathbf{C}, W \otimes \mathbf{C})$ given by $\tilde{f}_n(v \otimes z) = f_n(v) \otimes z$.

The following proposition is Theorem 7.1 in [BS71a].

Proposition B.1.23.
Let $X \subseteq T$ be a real Fréchet domain and let $f : X \to V$ be an analytic map with values in the complete real locally convex vector space V. There exists a complex analytic map $\tilde{f} : \tilde{X} \to V \otimes \mathbf{C}$ defined on a complex Fréchet domain $\tilde{X} \subseteq T \otimes \mathbf{C}$ with $X \subseteq \tilde{X}$ such that $f = \tilde{f}|_X$.

Like in the finite-dimensional case, complex analytic functions can be characterised in different ways. The following proposition combines Theorem 6.3 and Theorem 3.1 from [BS71a].

Proposition B.1.24.
Let $X \subseteq T$ be a complex Fréchet domain and let $f : X \to V$ be a function to a complete complex locally convex vector space V. The following statements are equivalent:

- **(i)** *f is analytic;*
- **(ii)** *f is continuous and weakly complex differentiable: for every $z \in X$ and every $\dot{z} \in T$, the limit of $\frac{1}{\tau}(f(z + \tau \dot{z}) - f(z))$ for $\tau \to 0 \in \mathbf{C}$ exists.*
- **(iii)** *f is continuous and weakly differentiable as a map between real vector spaces and satisfies the Cauchy–Riemann equation $\mathrm{D}f \circ J = J \circ \mathrm{D}f$, where $J : V \to V$ describes multiplication by i.*

Analytic functions are smooth and locally given by their Taylor series. The following proposition is obtained by combining Corollary 5.1, Theorem 7.1 and Proposition 6.4 in [BS71a].

Proposition B.1.25.
Let $X \subseteq T$ be a Fréchet domain and let $f : X \to V$ be an analytic function to a complete locally convex vector space V. Then f is smooth and for every $x \in X$ there exists a neighbourhood U_x of $x \in X$ such that

$$f(x+h) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{D}^k f(x)(h, \dots, h)$$

for all $h \in T$ such that $x + h \in U_x$. Moreover, the functions

$$X \times \prod^{k} T \longrightarrow V, \qquad (x, h_1, \dots, h_k) \mapsto \mathrm{D}^k f(x)(h_1, \dots, h_k)$$

are analytic for every $k \in \mathbf{N}$.

It follows that analytic maps with connected domains are uniquely determined by their Taylor series at a single point.

Corollary B.1.26.
Let $X \subseteq T$ be a Fréchet domain containing a point $x \in X$ and let V be a locally convex vector space. Two analytic functions $f, g : X \to V$ are equal if and only $\mathrm{j}_x^\infty f = \mathrm{j}_x^\infty g$.

Proposition B.1.27.
For any Fréchet domain $X \subseteq T$ containing 0 and any $x \in X$ there exists a zero neighbourhood $N \subseteq T$ such that for any power series $\sum_n f_n$ that converges on X, the function $f : h \mapsto \sum_n f_n(x+h)$ is given by a power series which converges on $x + N$.

Analytic maps can be composed to obtain new analytic maps.

Proposition B.1.28.
Sums, products and compositions of analytic maps are analytic:

- *Given a Fréchet domain X and a locally convex vector space V, the sum $f + g : X \to V$ of two analytic functions from X to V is analytic.*
- *Given a Fréchet domain X and locally convex vector spaces V and W, the product $f \otimes g : X \to V \otimes W$ of two analytic functions from X to V and W respectively is analytic.*

- *Given Fréchet domains X and Y and a locally convex vector space V, the composition $g \circ f : X \to V$ of analytic functions $f : X \to Y$ and $g : Y \to Z$ is analytic.*

Showing that the sum of two analytic functions is analytic is straightforward and analyticity of the composition $g \circ f$ is proven in Theorem 6.4 (complex case) and Theorem 7.3 (real case) of [BS71a]. To prove the latter fact, Proposition B.1.24 is used in the complex case and the real case is deduced from this through Proposition B.1.23. Analyticity of the product $f \otimes g$ in Proposition B.1.28 can be demonstrated in a similar fashion by also employing Proposition B.1.7.

Analytic manifolds modelled on Fréchet spaces can be defined similarly to how smooth manifolds were defined in Definition B.1.10. An atlas $(\varphi_\lambda)_{\lambda \in \Lambda}$ is called analytic whenever the transition functions $\varphi_{\kappa,\lambda} = \varphi_\lambda \circ \varphi_\kappa^{-1}$ are analytic functions and two such atlases are considered equivalent whenever their union is an analytic atlas as well. An analytic structure can now be defined as an equivalence class of analytic atlases.

Definition B.1.29.
An *analytic Fréchet manifold* is a Hausdorff space endowed with an analytic structure.

Analytic maps between analytic manifolds are now defined as functions $f : M \to N$ for which the composition $\varphi \circ f \circ \psi^{-1}$ is analytic for any two analytic charts φ and ψ on M and N respectively.

B.2. Mapping spaces

In this section we introduce topologies on the spaces of maps of class C^k from a Fréchet domain $X \subseteq T$ to an arbitrary locally convex vector spaces with $k \in \mathbf{N}$. A number of properties of these spaces are discussed.

B.2.1. Spaces of continuous maps

Given two topological spaces X and Y, we endow the space $C^0(X, Y)$ of continuous maps from X to Y with the compact-open topology. We note that $C^0(X, Y)$

is a locally convex vector space whenever Y is a locally convex vector space and that its topology coincides with the topology of uniform convergence on compact subsets of X. A basis of zero neighbourhoods for this topology is formed by the subsets

$$\mathcal{N}(K_X, U_Y) = \left\{ f \in \mathrm{C}^0(X,Y) \,\middle|\, f(K_X) \subseteq U_Y \right\} \subseteq \mathrm{C}^0(X,Y),$$

for $K_X \subseteq X$ compact and $U_Y \subseteq Y$ an (absolutely convex) zero neighbourhood.

Given two locally convex vector spaces V and W, we endow the space $\mathrm{L}(V,W)$ of continuous linear maps from V to W with the topology of uniform convergence on precompact subsets. This topology is generated by the zero neighbourhoods

$$\mathcal{N}(K_V, U_W) = \left\{ f \in \mathrm{C}^0(V,W) \,\middle|\, f(K_V) \subseteq U_W \right\}$$

for $K_V \subseteq V$ precompact and $U_W \subseteq W$ an (absolutely convex) zero neighbourhood.

The canonical inclusion map $\mathrm{L}(V,W) \to \mathrm{C}^0(V,W)$ is always continuous, but that the aforementioned topology on $\mathrm{L}(V,W)$ is generally finer than the associated subspace topology. In light of Proposition A.2.3, these two topologies do however coincide whenever V is complete.

Proposition B.2.1 (The exponential law).
Let X and Y be metrisable spaces and let V be a locally convex vector space. Given any map f from $X \times Y$ to V (not necessarily continuous), the following statements are equivalent:

- **(i)** *the map $f: X \times Y \to V$ is continuous;*
- **(ii)** *there exists a (unique) continuous map $f_X: X \to \mathrm{C}^0(Y,V)$ which is given by $f_X(x)(y) = f(x,y)$ for all $(x,y) \in X \times Y$.*

The correspondence $f \longleftrightarrow f_X$ describes an isomorphism of topological vector spaces between $\mathrm{C}^0(X \times Y, V)$ and $\mathrm{C}^0(X, \mathrm{C}^0(Y,V))$.

Proof: Once existence is established, uniqueness of f_X is immediate because it is explicitly defined in terms of f.

Assume that f is continuous and let $f_X: X \to \mathrm{C}^0(Y,V)$ be given by $f_X(x) = f(x,\cdot)$. To demonstrate that f_X is continuous, pick a point $x \in X$ and let $\mathcal{N}(K_Y, U_V) \subseteq \mathrm{C}^0(Y,V)$ be a zero neighbourhood that is determined by some compact subset $K_Y \subseteq Y$ and an absolutely convex open zero neighbourhood $U_V \subseteq V$.

Continuity of f tells us that we can find a neighbourhood $U_X^y \times U_Y^y \subseteq X \times Y$ of (x, y) for every $y \in K_Y$ such that $f(U_X^y \times U_Y^y) \subseteq f(x, y) + \frac{1}{2}U_V$. Since K_Y is compact, we can pick elements $y_1, y_2, \ldots, y_N \in K_Y$ for some $N \in \mathbf{N}_0$ such that $K_Y \subseteq \bigcup_{i=1}^N U_Y^{y_i}$. Let $U_X := \bigcap_{i=1}^N U_X^{y_i}$ denote the intersection of the corresponding neighbourhoods in X.

Given $x' \in U_X$ and $y \in K_Y$, let $i \in \{1, 2, \ldots, N\}$ be such that $y \in U_Y^{y_i}$. Both (x, y) and (x', y) are then elements $U_X^{y_i} \times U_Y^{y_i}$, from which we infer that

$$\big(f_X(x') - f_X(x)\big)(y) = \big(f(x', y) - f(x, y_i)\big) + \big(f(x, y_i) - f(x, y)\big)$$

is an element of $\frac{1}{2}U_V + \frac{1}{2}U_V \subseteq U_V$. Consequently, $f_X(x') - f_X(x) \in \mathscr{N}(K_Y, U_V)$ for all $x' \in U_X$. Since $x \in X$ was arbitrary and any neighbourhood of $f_X(x)$ contains one of the form $f_X(x) + \mathscr{N}(K_Y, U_V)$, continuity of f_X follows.

To prove the converse, assume that f_X exists and is continuous, and let $(x_n, y_n)_{n\in\mathbf{N}}$ be a convergent sequence in $X \times Y$ with limit (x, y). Let $U_V \subseteq V$ be a zero neighbourhood in V and let $K_Y = \{y_n \,|\, n \in \mathbf{N}\} \cup \{y\}$. Because $K_Y \subseteq Y$ is compact, we can find an index $n_0 \in \mathbf{N}$ such that $f_X(x_n) - f_X(x) \in \mathscr{N}(K_Y, \frac{1}{2}U_V)$ for all $n \geq n_0$, which in particular implies that $f(x_n, y_n) - f(x, y_n) \in \frac{1}{2}U_V$ for such n. Continuity of $f_X(x)$ moreover tells us that we can choose an $n_1 \geq n_0$ such that $f_X(x)(y_n) - f_X(x)(y) \in \frac{1}{2}U_V$ whenever $n \geq n_1$. Now, given $n \in \mathbf{N}$,

$$f(x_n, y_n) - f(x, y) = \big(f(x_n, y_n) - f(x, y_n)\big) + \big(f(x, y_n) - f(x, y)\big),$$

which is an element of $\frac{1}{2}U_V + \frac{1}{2}U_V \subseteq U_V$ if $n \geq n_1$. The neighbourhood U_V was arbitrary, so $f(x_n, y_n)$ in fact converges to $f(x, y)$ as $n \to \infty$. Because $X \times Y$ is metrisable and f sends convergent sequences in $X \times Y$ to convergent sequences in V, we deduce that f is indeed continuous.

We thus obtain a canonical bijection between $\mathrm{C}^0(X, \mathrm{C}^0(Y, V))$ and $\mathrm{C}^0(X \times Y, V)$. It is a topological isomorphism because the topologies on these spaces are generated by the open subsets $\mathscr{N}(K_X, \mathscr{N}(K_Y, U_V)) \subseteq \mathrm{C}^0(X, \mathrm{C}^0(Y, V))$ and $\mathscr{N}(K_X \times K_Y, U_V)$ for $K_X \subseteq X$ and $K_Y \subseteq Y$ compact and $U_V \subseteq V$ open, and these are identified by this correspondence. ☐

Proposition B.2.1 will be used frequently in appendix B.2.2 or section 3.1.

B.2.2. Spaces of continuously differentiable maps

Let $X \subseteq T$ be a Fréchet domain and let $f : X \to V$ be function of class C^k from X to a locally convex vector space V. The k-jet of f is a continuous map from

Definition B.2.2.
The compact-open C^k-topology on the space $\mathrm{C}^k(X,V)$ of C^k maps is the initial topology for the canonical inclusion map

$$\mathrm{j}^k\colon \mathrm{C}^k(X,V) \longhookrightarrow \mathrm{C}^0(X,\mathrm{J}^k(T,V)), \qquad f \mapsto \mathrm{j}^k f$$

where $\mathrm{C}^0(X,\mathrm{J}^k(T,V))$ is endowed with the compact-open topology.

Note that since $\mathrm{J}^0(T,V) \simeq V$, the C^0-topology described by Definition B.2.2 coincides with the compact-open topology on $\mathrm{C}^0(T,V)$, which was described in appendix B.2.1. Since $\mathrm{J}^k(T,V) \simeq \prod_{p=0}^k \mathrm{L}(\bigodot^p T, V)$ and $\mathrm{L}(\bigodot^p T, V)$ is an embedded subspace of $\mathrm{L}^p(T,V) \simeq \mathrm{L}(\bigotimes^p T, V)$, the C^k-topology is also characterised by the fact that the canonical inclusion map

$$\mathrm{C}^k(X,V) \longhookrightarrow \prod_{i=1}^{p} \mathrm{C}^0(X,\mathrm{L}^p(T,V)), \qquad f \mapsto (\mathrm{D}^p f)_{p=0}^k$$

is an embedding of topological vector spaces.

It is not difficult to show that for any collection $(V_i)_{i\in I}$ of locally convex vector spaces, the spaces

$$\mathrm{C}^k\big(X, \textstyle\prod_{i\in I} V_i\big) \simeq \displaystyle\prod_{i\in I} \mathrm{C}^k(X,V_i)$$

are canonically isomorphic. We can moreover use results from Appendix A to demonstrate continuity of several canonical maps between such mapping spaces when this topology is used.

The sum rule follows simply from the fact that $\mathrm{C}^k(X,V)$ is a topological vector space.

Proposition B.2.3.
For any $k \in \mathbf{N}_0 \cup \{\infty\}$, addition describes a morphism

$$\mathrm{C}^k(X,V) \times \mathrm{C}^k(X,V) \longrightarrow \mathrm{C}^k(X,V) \qquad (f_1, f_2,) \mapsto f_1 + f_2.$$

of topological vector spaces

One can moreover show that postcomposition by a morphism of locally convex vector spaces is a continuous operation. The following lemma is generalised by Proposition 3.1.8.

Lemma B.2.4.
Postcomposition by a continuous multilinear map $\alpha \in \mathrm{L}(V_1, \dots, V_q; W)$ of locally convex vector spaces describes a morphism

$$\alpha_*\colon \prod_{i=1}^{q} \mathrm{C}^k(X, V_i) \longrightarrow \mathrm{C}^k(X, W), \qquad (f_1, \dots, f_q) \mapsto \alpha \circ (f_1, \dots, f_q)$$

of topological vector spaces for every $k \in \mathbf{N}_0 \cup \{\infty\}$.

Proof: The case $k = 0$ is easy to verify. For the general case we note that α_* is part of a commuting diagram

$$\begin{array}{ccc} \mathrm{C}^k(X, V) & \xrightarrow{\quad\alpha_*\quad} & \mathrm{C}^k(X, W) \\ \big\downarrow \mathrm{j}^k & & \big\downarrow \mathrm{j}^k \\ \prod_{p=0}^{k} \mathrm{C}^0\big(X, \mathrm{L}(\bigotimes^p T, V)\big) & \xrightarrow{\quad a\quad} & \prod_{p=0}^{k} \mathrm{C}^0\big(X, \mathrm{L}(\bigotimes^p T, W)\big), \end{array}$$

where a is such that $a(\tilde{f})(x)(\dot{x}_{\{1,\dots,p\}}) = \alpha(\tilde{f}(x)(\dot{x}_{\{1,\dots,p\}}))$. This map is continuous because it can be described using the $k = 0$ case of this lemma applied to the continuous linear maps from $\mathrm{C}^0\big(X, \mathrm{L}(\bigotimes^p T, V)\big)$ to $\mathrm{C}^0\big(X, \mathrm{L}(\bigotimes^p T, W)\big)$ induced by α through Proposition A.2.23.

Continuity of α_* now follows from continuity of a and the fact that both of the maps labelled j^k are embeddings. □

Lemma B.2.4 can be used to show that differentiation defines a continuous operatorc.

Proposition B.2.5.
For any $k \in \mathbf{N} \cup \{\infty\}$ and any $l \in \mathbf{N}_0$ with $l \leq k$, the map

$$\mathrm{D}^l\colon \mathrm{C}^k(X, V) \longrightarrow \mathrm{C}^{k-l}\big(X, \mathrm{L}^l(T, V)\big), \qquad f \mapsto \mathrm{D}^l f$$

describing l-fold differentiation is a morphism of topological vector spaces.

Proof: The derivative D^l is part of a commuting diagram

$$\begin{array}{ccc} \mathrm{C}^k(X,V) & \xrightarrow{\mathrm{D}^l} & \mathrm{C}^{k-l}(X,\mathrm{L}^l(T,V)) \\ \big\downarrow{\scriptstyle \mathrm{j}^k} & & \big\downarrow{\scriptstyle \mathrm{j}^{k-l}} \\ \prod_{p=0}^k \mathrm{C}^0(X,\mathrm{L}^p(T,V)) & \xrightarrow{d_*} & \prod_{p=l}^k \mathrm{C}^0\big(X,\mathrm{L}^{p-l}(T,\mathrm{L}^l(T,V))\big), \end{array}$$

where d_* is given by $d_*(\tilde{f})(x)(\dot{x}_{\{1,\ldots,l\}})(\dot{x}_{\{l+1,\ldots,l+p\}}) = \tilde{f}(x)(\dot{x}_{\{1,\ldots,l+p\}})$. It is induced by the canonical isomorphisms $d\colon \mathrm{L}^p(T,V) \to \mathrm{L}^l(T,\mathrm{L}^{p-l}(T,V))$ for $p \in \mathbf{N}_0$ with $l \le p \le k$ and the zero morphism on $\mathrm{L}^p(T,V)$ for $p < l$. Now Lemma B.2.4 allows us to conclude that also d_* is continuous and continuity of D^l then follows from the fact that j^k and j^{k-l} are topological embeddings. □

The product rule Proposition B.1.7 between mapping spaces that we can now show is continuous for the chosen topology.

Proposition B.2.6 (Product rule).
For any $k \in \mathbf{N}_0 \cup \{\infty\}$ and any sequence $V_1, V_2, \ldots, V_q$ of locally convex vector spaces, the product map

$$\otimes\colon \prod_{i=1}^q \mathrm{C}^k(X,V_i) \longrightarrow \mathrm{C}^k\big(X, \textstyle\bigotimes_{i=1}^q V_i\big) \qquad (f_1,f_2,\ldots,f_q) \mapsto \textstyle\bigotimes_{i=1}^q f_i$$

is continuous.

Proof: This follows simply from Lemma B.2.4 and continuity of the tensor product map $\otimes\colon \prod_{i=1}^q V_i \to \bigotimes_{i=1}^q V_i$. □

The following corollary is a direct consequence of Proposition B.2.3 and Proposition B.2.6.

Corollary B.2.7.
For any $k \in \mathbf{N}_0 \cup \{\infty\}$, pointwise multiplication describes a topological algebra structure on $\mathrm{C}^k(X,\mathbf{R})$ and a topological $\mathrm{C}^k(X,\mathbf{R})$-module structure on $\mathrm{C}^k(X,V)$.

Rather importantly, we can show that precomposition by a smooth map is a continuous operation.

Proposition B.2.8 (Precomposition)**.**
Let $X \subseteq S$ and $Y \subseteq T$ be Fréchet domains and let V be an arbitrary locally convex vector space. Precomposition by a map $f : X \to Y$ of class C^k describe a morphisms

$$f^*\colon \mathrm{C}^k(Y,V) \longrightarrow \mathrm{C}^k(X,V), \qquad g \mapsto g \circ f$$

of topological vector spaces.

Proof: For $k = 0$, continuity of f^* follows from the fact $f^*(\mathscr{N}(f(K_X), U_V))$ is contained in $\mathscr{N}(K_X, U_V)$ for any compact subset $K_X \subseteq X$ and any open subset $U_V \subseteq V$, and the fact that $f(K_X) \subseteq Y$ is also compact.

The general case now follows from the fact that the composition $\mathrm{j}^k \circ f^*$ can be written as a composition $F \circ f^* \circ \mathrm{j}^k$:

$$\begin{array}{ccccc}
\mathrm{C}^k(Y,V) & & \xrightarrow{\quad f^* \quad} & & \mathrm{C}^{k-l}(X,V) \\
\big\downarrow \mathrm{j}^k & & & & \big\downarrow \mathrm{j}^k \\
\mathrm{C}^0(Y,\mathrm{J}^k(T,V)) & \xrightarrow{f^*} & \mathrm{C}^0(X,\mathrm{J}^k(T,V)) & \xrightarrow{F} & \mathrm{C}^0(X,\mathrm{J}^k(S,V)).
\end{array}$$

Here f^* again denotes precomposition by f and F is the function given by $F(\tilde{h})(x) = \tilde{h}(x) \diamond \mathrm{j}^p_x f$, so that

$$F(\tilde{h})(x)(\dot{x}_1,\ldots,\dot{x}_p) = \sum_{I_1,\ldots,I_m} \tfrac{1}{m!}\,\tilde{h}(x)\big(\textstyle\bigodot_{j=1}^m \mathrm{D}^{\#I_j} f(x)(\dot{x}_{I_j})\big)$$

for $\tilde{h} \in \mathrm{C}^0(X,\mathrm{J}^k(T,V))$, $x \in X$ and $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_p \in S$. That precomposition by f is continuous follows from the case $k = 0$, and continuity of F can be derived from Proposition 3.1.3. ☐

The analogue of Proposition B.2.8 for postcomposition by a map $g\colon Y \to V$ of class C^k also holds if $k = 0$, but it is generally false for $k > 0$.

One reason to care about Proposition B.2.8 is the following corollary.

Corollary B.2.9.
Let $X \subseteq S$ and $Y \subseteq T$ be Fréchet domains, let V be an arbitrary locally convex vector space and let $f : X \to Y$ be a diffeomorphism. The maps

$$f^*\colon \mathrm{C}^\infty(Y,\mathbf{R}) \longrightarrow \mathrm{C}^\infty(X,\mathbf{R}) \qquad \textit{and} \qquad f^*\colon \mathrm{C}^\infty(Y,V) \longrightarrow \mathrm{C}^\infty(X,V)$$

describe an isomorphism between the topological $C^\infty(X,\mathbf{R})$*-module* $C^\infty(X,V)$ *and the topological* $C^\infty(Y,\mathbf{R})$*-module* $C^\infty(Y,V)$.

B.3. Regular Fréchet Lie groups

Another topic that we should briefly mention is regular Fréchet Lie groups. The definition of a Fréchet Lie group is, unsurprisingly, the same as its finite-dimensional analogue, but some complication arise if one attempts to set up Lie theory for Fréchet Lie groups. This is mainly due to the fact that paths in the Lie algebra of a Fréchet Lie group are not guaranteed to integrate to a path in the group itself due to the failure of the existence and uniqueness theorem for ordinary differential equations on Fréchet manifolds. Regular Fréchet Lie group are those Fréchet Lie groups for which this is possible.

Definition B.3.1.
A *Fréchet Lie group* is a smooth Fréchet manifold G endowed with a group structure for which multiplication $m\colon G\times G\to G$ and inversion $i\colon G\to G$ are smooth maps. A group morphism between two Fréchet Lie groups is a called a Lie group morphism whenever it is smooth.

Suppose we are given a Fréchet Lie group G and an interval I of positive length. Any smooth path $\gamma\colon I\to G$ in the Lie group G can be differentiated to obtain a path in its Lie algebra,

$$\delta\gamma\colon I\longrightarrow \mathrm{Lie}(G),\qquad t\mapsto L^*_{\gamma(t)}\gamma'(t)=\tfrac{\mathrm{d}}{\mathrm{d}s}\gamma(t)^{-1}\gamma(t+s)\Big|_{s=0},$$

which is called the (left) *logarithmic derivative* of γ. Right logarithmic derivatives can be defined similarly, but will not be discussed.

One can ask whether a given a path $\xi\colon I\to\mathrm{Lie}(G)$ in the Lie algebra of G can be integrated to a path $\gamma\colon I\to G$ in the Lie group, i.e. if there exists a path γ that has ξ as its logarithmic derivative. If such a path exists, one can show that it is unique up to multiplication by a (fixed) group element (see e.g. Lemma 7.4 in [Mil84], or the second half of the proof of Lemma B.3.3 below). Any two paths $\gamma_0,\gamma_1\colon[0,1]\to G$ integrating $\xi\colon[0,1]\to\mathrm{Lie}(G)$ that both start at the identity element e_G thus necessarily coincide.

Definition B.3.2.
A Fréchet Lie group G is *regular* if for every path $\xi\colon [0,1] \to \mathrm{Lie}(G)$, the initial value problem

$$\xi(t) = L^*_{\gamma(t)}\gamma'(t), \qquad \gamma(0) = \mathrm{e}_G$$

has a (unique) solution $\gamma_\xi\colon [0,1] \to G$, and the *evolution map*

$$\mathrm{evol}_G\colon \mathrm{C}^\infty([0,1], \mathrm{Lie}(G)) \longrightarrow G \qquad \xi \mapsto \gamma_\xi(1),$$

is smooth.

Note that if a solution γ_ξ exists, it is automatically unique as a consequence of Lemma B.3.3 below, so requiring this as part of the definition is optional.

The concept of regularity for Lie groups was first introduced by Omori, Maeda, Yoshioka and Kobayashi in [OMYK82], although their definition is somewhat stronger than the one provided here. Definition B.3.2 is due to Milnor [Mil84] and can also be found in e.g. [KM97a] or [Nee06]. While Regular Lie groups have some of the nice properties of finite-dimensional Lie groups, many other elementary results for finite-dimensional Lie groups fail. For example: while there is a one-to-one correspondence between morphisms of simply connected regular Fréchet Lie groups and morphisms of their Lie algebras (cf. Theorem 8.1 in [Mil84]), it is not true that every closed Lie subalgebra of the Lie algebra of a regular Fréchet Lie group is the Lie algebra of an immersed Lie subgroup (or of any Lie group for that matter).

We think of $\mathrm{evol}_G(\xi)$ as a (left) product integral of the curve ξ and note that it coincides with the usual integral when G is a Fréchet space (endowed with the associated Abelian Lie group structure). Although regularity was defined using left logarithmic derivatives, right logarithmic derivatives could have been used instead and the resulting definition would be equivalent.

Regularity of a Lie group G also implies the existence of an indefinite version of the evolution map,

$$\mathrm{Evol}_G\colon \mathrm{C}^\infty([0,1], \mathrm{Lie}(G)) \longrightarrow \mathrm{C}^\infty([0,1], G), \qquad \xi \mapsto \gamma_\xi.$$

Since the curve $s \mapsto \gamma(t\,s)$ solves the differential equation $\delta\gamma(t\,s) = t\,\xi(t\,s)$, the indefinite evolution map can also be expressed through the identity

$$\mathrm{Evol}_G(\xi)(t) = \mathrm{evol}_G(s \mapsto t\,\xi(t\,s)),$$

which one can use to prove that Evol_G is smooth whenever evol_G is. It is in fact a diffeomorphism from the space of smooth paths in $\mathrm{Lie}(G)$ to the space of smooth paths in G that start at the identity element.

We observe that regularity of G in particular implies the existence of a exponential map $\exp_G\colon \mathrm{Lie}(G) \to G$, which assigns to any Lie algebra element $X \in \mathrm{Lie}(G)$ the integral $\mathrm{evol}_G(X)$ of the constant curve $t \mapsto X$. Although regularity of G does imply smoothness of $\exp_G$ and one can easily verify that $\mathrm{T}_0 \exp_G = \mathrm{id}_{\mathrm{Lie}(G)}$, this exponential map may not be local diffeomorphism near $0 \in \mathrm{Lie}(G)$ since the inverse function theorem cannot be applied in this setting. A simple counter-example is the diffeomorphism group of the circle, for which one can in fact show that the image of the exponential map does not contain a neighbourhood of the identity element (see e.g. Counterexample 5.5.2 in [Ham82] for more details).

It is not known whether any non-regular Fréchet Lie groups exist and this question has been an open problem since the concept was first introduced. While there is a priori little reason to expect the initial value problem (B.3.2) to always admit a solution, finding a counter-example and then proving the non-existence of such solutions is not an easy task. Most (infinite-dimensional) Fréchet Lie groups that arise in (finite-dimensional) differential geometry are known to be regular. Such groups are often related to diffeomorphism groups or Lie group(oid)-valued mapping spaces, which allows their evolution maps to be constructed using the flow of appropriate vector fields on the underlying finite-dimensional spaces.

We shall be interested in smooth right actions

$$\alpha\colon M \times G \longrightarrow G$$

and the corresponding infinitesimal action, which is the Lie algebra morphism

$$\alpha_*\colon \mathrm{Lie}(G) \longrightarrow \mathfrak{X}(M), \qquad \alpha_*(X)(x) = \mathrm{T}_{(x,\mathrm{e}_G)}\alpha(X).$$

obtained through differentiation.

Lemma B.3.3.
Let G be a regular Fréchet Lie group and let $\alpha\colon M \times G \to M$ be a smooth right action. For any smooth path $\xi\colon I \to \mathrm{Lie}(G)$ such that $0 \in I$ and any $x_0 \in M$, the initial value problem

$$\tfrac{\mathrm{d}}{\mathrm{d}t} x(t) = \alpha_*(\xi(t))(x(t)), \qquad x(0) = x_0$$

for $x\colon I \to M$ *is uniquely solved by* $x(t) = x_0 \cdot \gamma_\xi(t)$.

Proof: That $t \mapsto x_0 \cdot \gamma_\xi(t)$ solves the initial value problem follows from a direct computation:

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}(x_0 \cdot \gamma_\xi(t)) &= \tfrac{\mathrm{d}}{\mathrm{d}s}(x_0 \cdot \gamma_\xi(t)) \cdot \big(\gamma_\xi(t)^{-1} \cdot \gamma_\xi(t+s)\big)\big|_{s=0} \\ &= \mathrm{T}_{(x_0\cdot\gamma_\xi(t),\mathrm{e}_G)}\alpha\big(\tfrac{\mathrm{d}}{\mathrm{d}s}\gamma_\xi(t)^{-1}\gamma_\xi(t+s)\big|_{s=0}\big) \\ &= \alpha_*(\xi(t))(x_0 \cdot \gamma_\xi(t)).\end{aligned}$$

To show that this solution is unique, assume that we are given another solution $x\colon I \to M$. If we act on this solution by the inverse of $\gamma_\xi(t)$ for every $t \in I$, we obtain a path in M whose derivative is given by

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}(x(t) \cdot \gamma_\xi(t)^{-1}) &= \tfrac{\mathrm{d}}{\mathrm{d}s}x(t+s) \cdot \gamma_\xi(t)^{-1}\big|_{s=0} + \tfrac{\mathrm{d}}{\mathrm{d}s}x(t) \cdot \gamma_\xi(t+s)^{-1}\big|_{s=0} \\ &= \tfrac{\mathrm{d}}{\mathrm{d}s}x(t+s) \cdot \gamma_\xi(t)^{-1}\big|_{s=0} \\ &\quad + \tfrac{\mathrm{d}}{\mathrm{d}s}\big(x(t) \cdot \big(\gamma_\xi(t)^{-1}\gamma_\xi(t+s)\big)^{-1}\big) \cdot \gamma_\xi(t)^{-1}\big|_{s=0} \\ &= \mathrm{T}_{(x(t),\gamma_\xi(t)^{-1})}\alpha\big(\alpha_*(\xi(t))(x(t))\big) \\ &\quad + \mathrm{T}_{(x(t),\gamma_\xi(t)^{-1})}\alpha \circ \mathrm{T}_{(x(t),\mathrm{e}_G)}\alpha \circ \mathrm{T}_{\mathrm{e}_G} i(\xi(t))\end{aligned}$$

for all $t \in I$. These two terms cancel because $\mathrm{T}_{(x,\mathrm{e}_G)}\alpha(X)$ is equal to $\alpha_*(X)(x)$ by definition and one can readily verify that $\mathrm{T}_{\mathrm{e}_G} i = -\mathrm{id}_{\mathrm{Lie}(G)}$. We thus conclude that $x(t) \cdot \gamma_\xi(t)^{-1}$ is constant and, since $\gamma_\xi(0) = \mathrm{e}_G$ and $x(0) = x_0$, hence that $x(t)$ is equal to $x_0 \cdot \gamma_\xi(t)$, for all $t \in I$. □

Corollary B.3.4.
Let G *be a connected regular Fréchet Lie group and let* $\alpha\colon M \times G \to M$ *be a smooth right action. Two points* $x_0, x_1 \in M$ *are in the same orbit if and only if there exists a pair* (ξ, x) *of smooth paths* $\xi\colon [0,1] \to \mathrm{Lie}(G)$ *and* $x\colon [0,1] \to M$ *such that* $x(0) = x_0$, $x(1) = x_1$ *and*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha_*(\xi(t))(x(t)), \tag{B.3.1}$$

for all $t \in [0,1]$.

Proof: If x_0 and x_1 are in the same orbit, then we can find a smooth path $\gamma\colon [0,1] \to G$ such that $\gamma(0) = \mathrm{e}_G$ and $x_0 \cdot \gamma(1) = x_1$ since G is (path) connected. The path $x\colon t \mapsto x_0 \cdot \gamma(t)$ in M is now such that $x(0) = x_0$ and $x(1) = x_1$, and it solves the differential equation (B.3.1) for $\xi = \delta\gamma$. Because G is regular, Lemma B.3.3 tells us that every solution to this differential equation is of this form. □

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