Describing anti-branes in F-theory using Lefschetz fibrations

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a thesis submitted to the Departments of Physics & Mathematics
at Utrecht University for the degrees of

Master in Physics & Master in Mathematics

Version: 0.9

Universiteit Utrecht

Acknowledgements

I would like to start by acknowledging Gil and Thomas. We sat down together in a room, there was a two hour conversation of which I could not understand anything and out we came with a ridiculously ambitious and therefore incredibly exciting topic for a thesis. They guided me through it with enthusiasm. I am grateful for their trust and the shot they gave me.

Next I would like to thank my friends, starting with the Theoretical Physics-squad: Koen, Lars, Solange, Kyla, Damian, Thomas, Alex, Guus, Jonas, Sacha and Sandro. They made sure that a day at the university would always be a day well spent; either on studying or on coffee breaks. Then there are Ivo, Aldo, Julius, Jette, Kaj, Rick, Suzan, Mats and Lukas, who made my entire career as a student a joyful experience. I also thank Straight on Stage. They made sure that I always kept my feet on the ground and would not spend too much time on this thesis in my weekends, since they already claimed most of it.

I am extremely grateful for study association A–Eskwadraat. The people there gave me the opportunity to achieve a personal growth I did not think was possible. I also owe a big 'thank you' to my parents for always supporting me. It is nothing short of a blessing to have a safe and warm home to retreat to.

Most of all, I would like to thank Mariken. Generally speaking for the love and support that I could unconditionally rely on. But more specifically for keeping me sane and nourished. I could not have been more lucky to have her, and all these people, in my life.

Utrecht, 2018

Contents

Chapter 1

Introduction

This thesis will concern the part of string theory that is called F-theory. It is a geometric translation of setups in type IIB string theory that uses *elliptic fibrations*. It allows us to access situations that perturbation theory cannot, often in cases that include 7-branes. However, not every object that occurs in type IIB has a translation to F-theory. Anti-branes cannot be incorporated in the elliptic fibrations that are used in F-theory. Elliptic fibrations have too much structure, i.e. there are too much assumptions made in using them, to allow for anti-branes. We will try to find what assumptions in F-theory are made out of convenience rather than absolute necessity.

The type of fibrations that we conjecture to be appropriate to use and able to incorporate anti-branes, are called *achiral Lefschetz fibrations*. They are structures that first arose in algebraic geometry but have got attention from the differential geometry community when a deep link between Lefschetz fibrations and symplectic structures was discovered by Gompf and Donaldson [Don98],[GS99].

Trying to drop assumptions is often a good way to see the underlying structure of the problem. But when in the process of dropping assumptions, we have to maintain a balance. On the one hand, we need to make sure that we drop enough to clear the problem up and on the other hand not drop so much that we erase vital information. This balance is a tricky thing if not handled carefully, so it is wise to formulate clear goals and checks to make sure that we retain our balance. For this project, these are:

1. Achiral Lefschetz fibrations should be able to describe all possbile types of 7-branes (including anti-branes) that occur in type IIB string theory.

2. Achiral Lefschetz fibrations should be able to distinguish between all the different types of 7-branes.

3. Achiral Lefschetz fibrations should be able to allow for the process of pair creation and annihilation that comes with branes and anti-branes.

4. There should be a symmetry between branes and anti-branes: it is an arbitrary choice which of the two is 'anti'. Achiral Lefschetz fibrations should reflect this symmetry.

5. The mutual appearance of a brane and its anti-brane should break supersymmetry completely.

Organization of the thesis

Since F-theory is a branch of string theory, we start by introducing string theory in Chapter 2. Chapter 3 is then concerned with the introduction of F-theory. At the end of the chapter, we already propose a slight abstraction in the mathematical direction in how we think about F-theory. This then is the bridge to the mathematical part of the thesis, being Chapters 4 and 5. In Chapter 4 we introduce our main mathematical object of study, namely (achiral) Lefschetz fibrations. We prove their basic properties, show how their monodromy looks and prove that they have (log-)symplectic structures. In Chapter 5 we develop a lot of mathematical formalism for treating 4-manifolds. This gives us greater flexibility in treating 4-dimensional manifolds, which helps us at certain points and may be very useful in future research. In particular, it gives us a way in purely mathematical terms of creating and annihilating pairs of singularities, so it shows that achiral Lefschetz fibrations can incorporate brane-anti-brane creation and annihilation. In Chapter 6, we summarize the new view of F-theory that we have built. In Chapter 7, we list ideas for further research and put our work in perspective.

We have tried to keep the mathematics out of the chapters that introduce the physics (2 and 3) and have tried to keep the physics out of the chapters that introduce the mathematics (4 and 5). In Chapter 6, we freely use both to give the most complete summary.

A word for the physicists

We propose a new (and more mathematical) view on F-theory. In order to do so, we have to introduce a lot of mathematics. However, if one is not interested in learning the mathematics, already knows F-theory and wants to just understand the new point of view and possible implications, one can read only Chapters 4 (skipping the proofs) and 6.

If one is however interested in going through the mathematics, one will be rewarded not just with some mathematical facts, but with a new toolkit for low-dimensional geometry. In order to go through the mathematics, knowledge of differential geometry (manifolds, differential forms, fiber bundles) is assumed. Mathematical subjects that are critical for understanding the thesis or cannot generally be assumed to be part of the knowledge of a theoretical physicist, are briefly treated in Appendix A.

A word for the mathematicians

String theory is a very involved field of work, hence it is nearly impossible to know all the details. Since in this thesis the physical details are not very important, we have tried to keep their treatment to an absolute minimum. We hope that in this way the overall picture will become ~~more clear~~, which is infinitely more important for understanding this work than knowing the details. Having a bachelor or big interest in physics and a solid mathematical background should be enough to be able to grasp the overall picture.

There is no new mathematics developed in this thesis. As a mathematical document, it is best used as an introduction to Lefschetz fibrations, Kirby diagrams and surgery that is hopefully easier to read than [GS99].

Chapter 2

An overview of string theory

We will introduce string theory in a very brief manner, introducing the necessary vocabulary and concepts, but taking huge leaps and skipping details, ultimately selling it short as the beautiful theory that it is. For a far more elaborate treatment, see for example [GGSW88], [Ton09], [BBS06], [BLT12].

2.1 The step towards string theory

String theory seeks to resolve the gigantic gap between the two theories of nature that we have. On the one hand, we have quantum field theory, describing the electromagnetic, weak and strong force as a sea of interactions between matter and 'force carrier particles', often called gauge bosons. And on the other hand we have Einstein's theory of general relativity, which states that the force of gravity is not merely some particles interacting: it stems from the curved nature of spacetime itself.

If we want to find a unifying theory of nature – and we do – then we have to find an overarching framework explaining both theories in the appropriate limit. One way to go about this, is to say that while Einsteins description is elegant, nothing is keeping us from discarding the interpretation and only keeping the equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{2.1}$$

Maybe it is the case that gravity is also merely some gauge bosons, called *gravitons*, transmitting the force, so we can write down a quantum field theory for it. This conceptual leap is not trivial, yet it is still the easy part. Because when trying to do the math, we run into a technical and very dubious property of quantum field theory: it gives a lot of divergent integrals that can only be regularized perturbatively; that is order by order, where the order is the amount of interactions in the process we are trying to calculate.

But the properties of the Einstein equation, specifically the fact that its relevant field, the spacetime metric $g_{\mu\nu}$, is a 2-tensor (as can be seen by the fact that it carries two indices) makes it problematic. We do not know how we can tame the infinities that this gives rise to. We do not even know whether it can be done.

String theory proposes a new model of matter: instead of looking at point particles, as we do in QFT, we look at strings. Where particles have a *worldline* (the path through spacetime that they sweep out by going forward in time), strings have a *worldsheet*. String theories in general contain both *open* and *closed* strings, which will have fundamentally different properties. We parametrize a string with a timelike coordinate τ and a spatial coordinate σ (with periodic boundary conditions for the closed string). These parametrize the worldsheet W. One then proposes an action for the string, called the *Polyakov action.*

$$S = \frac{-1}{4\pi\alpha'} \int_W d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^{nu} \eta_{\mu\nu}$$

Here α' is the *Regge slope*, related to the tension of the string, g is the metric on the worldsheet and the fields X, that are scalar fields from the worldsheet point of view, can be interpreted as the coordinates of the embedding of the string in spacetime. This action can be quantized to give us a quantum description of the string.

We now skip a lot of work and just point out the consequences of this. The only way to get a consistent theory is if the dimension of spacetime is 26, instead of our usual 4 dimensions. And even then, there is a particle, the tachyon, with negative mass, making the theory unstable. This is where we make the step to the theories that string theorists now work on: superstring theories. To be able to discuss them, we first introduce supersymmetry.

2.2 Supersymmetry

Supersymmetry, often abbreviated to SUSY, is a fundamental concept in the high-energy physics of today. One could write a book about the theory in itself, but we will restrict ourselves to a very short overview. A good reference for such a book and extensive treatment on any claim made in this section is [FVP12].

2.2.1 Global supersymmetry

Supersymmetry is an idea born from theory. Physicists knew that the Poincaré group (Lorentz transformations plus translations) was a global symmetry of nature. There could also be local symmetries, called *gauge symmetries*. The question arose whether there could possibly be more symmetries other than the trivial product of the Poincaré group and the gauge symmetries. The Coleman-Mandula theorem says that it is not possible, but the Haag-Lopuszanski-Sohnius theorem exploits the only possible loophole in it: by also allowing anti-commuting symmetry generators, the Poincaré group could be extended. These generators form supersymmetry and apart from gauge symmetries, it is the only possible other symmetry that nature can have.

Let us start by being brutally honest: supersymmetry has yet to be observed. There is no proof that it exists in nature and hence is nothing more than a convenient assumption for theoretical physics. But if it exists, it would explain a lot of questions that we still have, like the Hierarchy problem (see [Mar10]).

Because the generators of supersymmetry anti-commute, they change the spin of the fields on which they act. Therefore, supersymmetry is a symmetry between bosons and fermions. We can choose how much supersymmetry generators we want in our theory. The most elementary form is $\mathcal{N} = 1$, where the generators are Q^α, where α is a spinor index. For *extended supersymmetry*, we have $\mathcal{N} \geq 2$, often allowing only powers of 2, so $\mathcal{N} = 2, 4, 8$. Here, the generators are Q_i^α where $i = 1, \ldots, \mathcal{N}$. A supersymmetry generator acting on a field changes the spin of the field by $\frac{1}{2}$. Hence in $\mathcal{N} = 1$, spin-0 fields (scalar fields) are related to spin-$\frac{1}{2}$ fields (Dirac fields). For $\mathcal{N} = 2$, they are both also related to spin-1 fields (vector fields), etc. This is also why $\mathcal{N} > 8$ is not considered, because it would necessarily lead to particles of spin larger than 2, for which no interacting field theory is known.

2.2.2 Supergravity

Supergravity (SUGRA for short) is the theory obtained from adding supersymmetry and gravity together. Making this combination necessarily turns supersymmetry into a local symmetry.

Arguably the most important supergravity theory is 11-dimensional $\mathcal{N}=1$ supergravity. For a while, this was seen as a candidate for the theory of everything. After a while, a few problems were discovered, but today it is relevant for *M-theory*. We will come back to this in Section 2.4. The bosonic part of the 11D supergravity action is given by

$$S_{\mathrm{M}} = \frac{1}{2\kappa_{11}^2}\int R\star 1 - \frac{1}{2}G_4\wedge\star G_4 - \frac{1}{6}C_3\wedge G_4\wedge G_4. \tag{2.2}$$

Here R is the scalar curvature, $\star$ is the Hodge star, $G_4 = dC_3$ is the field strength associated to the 3-form gauge field C_3 and κ_{11} is the 11-dimensional gravitational coupling constant. The important thing to note is that the bosonic part only contains gravity and a 3-form field. This action is unique: no other 11-dimensional SUGRA theory exists.

In 10 dimensions, there exist various SUGRA theories. Also, in 10 dimensions, we have *chiral supersymmetry.* Since a spinor in even dimensions can be split up into two Weyl spinors using the chiral projections, and charge conjugation in this number of dimensions leaves chirality invariant, we have to specify how many supercharges of each chirality we want to have. So instead of speaking of the value of $\mathcal{N}$, we speak of $(\mathcal{N}_L,\mathcal{N}_R)$ and we have $\mathcal{N}=\mathcal{N}_L+\mathcal{N}_R$.[1] There are two $\mathcal{N}=2$ SUGRA theories in 10 dimensions: type IIA and type IIB. They are, respectively the $(1,1)$ and $(2,0)$ theories. We give the action for the type IIB theory, since this will be the relevant theory for us. This theory contains the dilaton ϕ, the metric g and the Kalb-Ramond field B_2 in the so called NSNS sector. The RR sector consists of the fields C_0, C_2 and C_4. The field strengths that we will use in the action are defined as follows:

$$H_3 = dB_2, \quad F_1 = dC_0, \quad F_3 = dC_2 - C_0 dB_2, \quad F_5 = dC_4 - \frac{1}{2}C_2\wedge dB_2 + \frac{1}{2}B_2\wedge dC_2.$$

The bosonic part of the type IIB supergravity action is then given by

$$\begin{aligned} S_{\mathrm{IIB}} = \frac{1}{2\kappa_{10}^2}&\int e^{-2\phi}\left(R\star 1 - \frac{1}{2}H_3\wedge\star H_3 + 4d\phi\wedge\star d\phi\right) \\ &- \frac{1}{4\kappa_{10}^2}\int F_1\wedge\star F_1 + F_3\wedge\star F_3 + \frac{1}{2}F_5\wedge\star F_5 + C_4\wedge H_3\wedge F_3. \end{aligned} \tag{2.3}$$

This action has a problem however. We need a self-duality constraint to get the right number of degrees of freedom,

$$F_5 = \star F_5. \tag{2.4}$$

[1] This is also the case in 2 and 6 dimensions.

This constraint can however only be implemented after varying the action, since otherwise the term $F_5 \wedge \star F_5$ would vanish.

But we do not have all the fields that people talk about in type IIB. By dualizing $dC_p = \star dC_{8-p}$, we can also get C_6 and C_8. But since these are related to the fields we already had, they do not present new degrees of freedom in the theory.

2.3 Superstring theories

The string theory from Section 2.1 is not what we are after. First of all, there is a tachyon in the spectrum. Secondly, it does not contain fermions. Certainly, if we wanted a theory of everything, we would want fermions in our theory. The tool we use to get them is to introduce supersymmetry on the world sheet.

This changes the situation in a big way: the dimension of spacetime required for a consistent theory is no longer 26, it is now 10. Also, the spectrum we get when performing this in full generality is inconsistent with itself and therefore we need to drop some of it. This can be done using the *GSO-projection*. The big upshot is that in this procedure the tachyon also vanishes. So this takes away some of the concern. But it comes at a price: the GSO-projection automatically gives us a spacetime supersymmetric spectrum, which means that having supersymmetry in nature is a requirement for superstring theories to work. But supersymmetry has yet to be observed. And that is not all: while the GSO-projection is consistent, we do have to make a choice. This gives rise to five different superstring theories: type I, type IIA, type IIB and two types of heterotic string theories: $E_8 \times E_8$ and $SO(32)$.

It is not a coincidence that there is a type IIA/B superstring theory and a type IIA/B supergravity theory. The supergravity theories are the *low-energy limit* of the superstring theories. This means that when considering an energy scale so small that the massive modes of the string cannot arise, we are only left with the massless modes and these behave according to the supergravity theory. In particular, the massless field content of type IIB string theory and the field content of type IIB SUGRA are the same. This way supergravity can give us a lot of information about the behaviour of the superstring theory.

2.3.1 Dualities

At this point, we have five possible theories of everything. This poses a problem: we cannot have five theories of everything. The one theory that is the right one, should describe the physics we know in the appropriate limit. But we have yet to succeed in this. Luckily for us, we might not have to make a choice. The different theories are not completely unrelated. There is a network of operations, called *dualities*, that link the theories to one another.

A mathematician would say that a duality is an isomorphism between theories. And the mathematician would be right, because a duality means that our description changes, but the physical results of the theory are all exactly the same. A simple, non-stringy example would be this: describe our whole universe in terms of anti-matter, i.e. take the complex conjugate of every field. It would lead to exactly the same predictions for every experiment; it is just another description of the same nature.

Superstring theories are connected via a web of dualities, so it is possible that they are all different limits of the same theory. We will see in Section 2.4 that this may indeed be the case. But first we have to describe the two most important dualities.

T-Duality

Suppose we take one of the nine spatial dimensions in our ten dimensional theory, and make it a circle of radius R (this is a specific form of *compactification*, which we will come back to in Section 2.3.2). Assuming the string is closed, the momentum of the string is only allowed to take discretized values, proportional to n/R for $n \in \mathbb{Z}$. The contribution of this to the energy of the string will therefore be proportional to $(n/R)^2$. The string can also wind around this circle m times, which adds energy proportional to $(mR)^2$.

If we now consider the same situation, but replace R with $1/R$, and interchange the winding number m and momentum number n, we get back the same. So compactifying on a large circle leads to the same observables as compactifying on a small circle. If we would not have been so hand-wavy about this, we would see that we however did change something, so here is the precise formulation: type IIA on a small circle is the same as type IIB on a large circle, and vice versa. T-duality also relates the two heterotic types in the same way.

> **Definition 2.3.1.** Two theories are T-dual to one another when they lead to the same physics if one is compactified on a circle of radius R and the other on a circle of radius $\frac{1}{R}$.

S-Duality

Since strings are to describe particles, and particles interact, strings should also interact. And just like when particles interact, a factor called the *coupling constant* makes an appearance. Quantum field theories are nearly always treated by perturbation theory in their coupling constant: the more interactions are considered, the higher the power of the coupling constant that appears. This only gives reasonable results if the coupling constant is small enough.

The coupling constant g_s in string theory is not a constant: it depends on the value of the massless string mode ϕ via $g_s = e^{\phi}$. So changing the dilaton to minus itself, swaps a small coupling constant for a large one.

> **Definition 2.3.2.** Two theories are S-dual to one another when they lead to the same physics if one has coupling g_s and the other has coupling $\frac{1}{g_s}$.

If two theories are related by S-duality, this gives us a range of new possibilities: something we could not compute perturbatively in the strongly coupled theory, can easily be computed in its weakly coupled equivalent. This duality relates one of the heterotic strings to type I and relates type IIB to itself. The latter will be further explored and generalized in Section 2.7.

2.3.2 Compactification

String theory owes us an explanation: if we believe it to be the theory of everything, then where are the six extra dimensions it proposes? One of the most prominent possibilities is that these dimensions are compact and extremely tiny, often called the *internal space* or *internal manifold*.

For example, if one of the spatial dimensions were a circle with a radius far smaller than our experiments can currently measure, we would only see the effective theory in our daily lives; the extra effects that arise thanks to this hidden dimension only become relevant on a much smaller scale. This type of compactification often goes by the name of *Kaluza-Klein compactification.*

But we do not need to hide one dimension; we need to hide six. This gives us a range of possibilities for the internal manifold. A priori, it seems we can pick any six-dimensional compact manifold we like. This is not the case if we want some properties of the string to survive. String theory in 10-dimensional flat space has *Weyl invariance*: its metric can be rescaled by some positive function without changing the theory. If we want this to survive for a general 10-dimensional metric $G_{\mu\nu}$, the *beta function* for this field, which describes how the strength of the coupling changes with the relevant energy scale, should be zero. This amounts to asking the metric to be Ricci-flat, i.e. the Ricci scalar is identical to zero.

Often one also has a preference for how many supersymmetry should 'survive' the compactification. Most people want $\mathcal{N} = 1$ supersymmetry in four dimensions, which means that the compact manifold should have $SU(3)$ structure. This leads to the famous choice of Calabi-Yau manifolds as compact spaces.[2]

The properties of this internal space hugely influence our four-dimensional world. For example, the size determines the mass of a lot of fields.

The compactification procedure

To say that six dimensions are very tiny means that at every point of our 4-dimensional spacetime M, there sits a this internal space X. The easiest case is when our 10-dimensional spacetime $\mathcal{M}$ takes the form $\mathcal{M} = X \times M$. Now consider a general theory, i.e. the general action

$$S = \int_{\mathcal{M}} \mathcal{L}.$$

Here $\mathcal{L}$ is the lagrangian of the theory. If $\mathcal{M} = X \times M$, the action can be written as

$$S = \int_M \int_X \mathcal{L}.$$

Performing the intergral over X, we get a 4-dimensional theory. Almost always this cannot simply be done. The 10-dimensional fields occurring in the original action have to be expanded into the directions of X in order to be able to do this. This however

[2] For more the definition and basic properties of Calabi-Yau manifolds and why we need them, see chapter 14 in [BLT12].

requires extremely detailed knowledge of the internal space. Often at this point, we make a *compactification ansatz*: a choice of fields on X that we use to expand our 10-dimensional fields in. This choice de facto amounts to a choice of which part of the field content of the compactified theory we want to find. The most popular choice of compactification ansatz is to only look at massless fields. For this we need relatively little knowledge about X and it still gives us a very useful low-energy effective theory. However, other choices are possible and sometimes necesssary depending on what one wants to study.

There is a caveat here: we cannot just pick any compactification ansatz. In order to obtain useful results, it should be made sure that the fields that are dropped cannot influence the fields that are maintained. We want what is called a *consistent truncation*. Several criteria can be formulated for this. See [KPMT13].

2.4 M-theory

In 1995, Edward Witten coined the idea that there may be a theory underlying these five superstring theories. When looking in type IIA, one can interpret the value of the dilaton as representing the size of some eleventh dimension that is curled up into a circle. This 11-dimensional theory was dubbed M-theory, the meaning of the M being left ambiguous on purpose. It can be interpreted as Mother, Matrix, Mystery or Membrane.

M-theory is a theory of which we know very little. It is no longer a theory of strings, as we have derived that string theories should be 10-dimensional, and M-theory is 11-dimensional. We only know its massless part: just as the massless part of a superstring theory gives us 10-dimensional supergravity, the massless part of the M-theory action is 11-dimensional supergravity.

Since we don't know the full quantum description of M-theory, nearly all the information about it has been obtained via the dualities to superstring theories. So to learn more about M-theory, it is important to still do work in the superstring theories.

As mentioned above, M-theory compactified on a circle leads to type IIA, as was shown in [DHIS87]. The dilaton in type IIA is then related to the size of the circle. This means that the stronger the coupling in type IIA, the bigger the circle in M-theory. Therefore, weak type IIA and M-theory are related by S-duality.[3]

[3] It is often said that if $g_{\text{IIA}} \to 0$, the circle on the M-theory side of the duality "decompactifies". That is, it becomes so large that it easily becomes part of the external spacetime rather than the internal

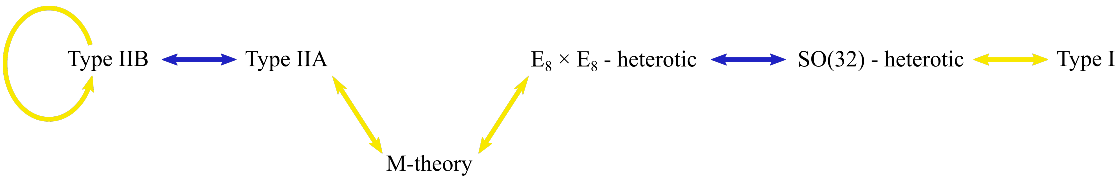


Figure 2.1: The relations between different theories. The yellow arrows denote S-duality, the blue arrows T-duality.

Since type IIA on a circle is the same as type IIB on a dual circle via T-duality, we have a duality between M-theory on a torus and type IIB on a circle. If the radius of the circle in type IIB becomes very large, the area of the torus in M-theory diminishes. Therefore, "uncompactified" type IIB corresponds to M-theory on a torus with vanishing area.

2.5 More details on type IIB superstring theory

Because our main subject will be F-theory and this concerns type IIB superstring theory, we need some more knowledge of this specific theory. We already know the massless spectrum, i.e. contents of the type IIB supergravity. This consists of the fields: $\phi, G, B, C_0, C_2, C_4$. The ϕ and C_0 are real scalars, called the *dilaton* and the *axion*, respectively. These will be combined into the central object of F-theory, the *axio-dilaton* $\tau = C_0 + ie^{-\phi}$, a complex scalar, the value of which defines the values of both scalar fields.

Remember that S-duality linked type IIB to itself. This is not a coincidence, as we will now see that there actually are a lot more transformations that leave type IIB invariant: the whole group $\mathrm{SL}(2,\mathbb{Z})$.[4] It is often called the S-duality group, where the meaning of S-duality is slightly generalized with respect to how it was introduced.

Actually, the supergravity action for type IIB is invariant under $\mathrm{SL}(2,\mathbb{R})$, where an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts as

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} C_2' \\ B' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B \end{pmatrix} \tag{2.5}$$

and leaves the other fields untouched. However, due to quantum effects, we cannot have $\mathrm{SL}(2,\mathbb{R})$ as symmetry group. Notice how it interchanges B and C_2 with arbitrary real

space.

[4] $\mathrm{SL}(n,\mathbb{K})$ is the group of $n \times n$ matrices with coefficients in $\mathbb{K}$ and determinant 1.

coefficients. This is a problem, since it violates the Dirac charge quantization condition (see section 18.3 in [BLT12]). Therefore, only the subgroup $\mathrm{SL}(2,\mathbb{Z})$ can possibly survive.

Now we can see what this symmetry-group does to the axion and dilaton. We note that $\mathrm{SL}(2,\mathbb{Z})$ is generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The first matrix sends $\tau \to \tau + 1$, i.e. $C_0 \to C_0 + 1$. So a discrete shift in C_0 without touching ϕ leaves type IIB invariant. The second matrix sends $\tau \to -\frac{1}{\tau}$. This does not look like S-duality as was introduced in Section 2.3.1, but it is when $C_0 = 0$. Because then $e^{-\phi} \to e^{\phi}$ and hence it interchanges strong and weak coupling. These kind of $\mathrm{SL}(2,\mathbb{Z})$-transformations will play a central role in F-theory.

There is an important remark to be made here: these transformations relate strongly coupled to weakly coupled theories. But since we cannot calculate much in strongly coupled theories, we cannot prove that this duality holds. Instead, it holds for the low-energy part of the theory, the supergravity action, and a few special states that we have control over in strong coupling, and is conjectured to hold for the whole theory. We will go into the physical interpretation of this symmetry in Section 2.7.

> **Conjecture 2.5.1.** Type IIB string theory is invariant under the $\mathrm{SL}(2,\mathbb{Z})$ action given in equation (2.5).

This conjecture is widely accepted in string theory and one of the fundamentals of F-theory.

From now on, we consider type IIB string theory, unless stated otherwise.

2.6 Branes

2.6.1 D-branes

In the first section, we claimed that it is relevant to look at both closed strings and open strings. This is indeed the case, but the boundary conditions for the open string deserve some extra attention, because they will draw our attention to completely new objects: D-branes.

Let us look at the Polyakov action for open strings. If we want to find the equations of motion for this, we should vary the action. Doing so, explicitly from some worldsheet coordinate τ_i to τ_f, we find

$$\delta S = \frac{1}{2\pi\alpha'}\left(\int_\Sigma d^2\sigma(\partial^\alpha\partial_\alpha X)\cdot\delta X + \left[\int_0^\pi d\sigma\partial_\tau X\cdot\delta X\right]_{\tau=\tau_i}^{\tau=\tau_f} - \left[\int_{\tau_i}^{\tau_f} d\tau\partial_\sigma X\cdot\delta X\right]_{\sigma=0}^{\sigma=\pi}\right)$$

The first term gives the equation of motion. The second term is zero, since we always assume variations to be zero at the starting and ending configuration. The third term is something new, and for it to vanish, we need to impose one of two boundary conditions for each spacetime direction μ separately:

- Neumann boundary conditions, given by $\partial_\sigma X^\mu = 0$ at $\sigma = 0, \pi$. This implies that the endpoints of the string move at the speed of light.
- Dirichlet boundary conditions, given by $\delta X^\mu = 0$ at $\sigma = 0, \pi$. This implies that the endpoints are restricted to some subspace of spacetime.

Suppose we have Neumann boundary conditions in the first p directions and Dirichlet boundary conditions in the other directions. So, for a spacetime of dimension D, we have

$$\begin{aligned}\partial_\sigma X^a &= 0 \text{ for } a = 0, \dots, p\\ X^I &= c^I \text{ for } I = p+1, \dots, D-1\end{aligned}$$

This defines some $p+1$-dimensional subspace on which the string has to end. It is called a Dp-brane, or D-brane when we do not specify the dimension.

> **Definition 2.6.1.** A *Dp-brane* is a $p+1$ dimensional object in spacetime on which open strings can end.

The Dirichlet boundary conditions seem weird, and they were not taken seriously for some time. They can however be physically justified. By considering the D-brane not as a solid, immovable object, the strings ending on the D-brane get a natural interpretation.

2.6.2 The D is for dynamical

These D-branes are not just a quirk of the boundary conditions: they are fundamental objects of string theory. One could even say that string theory is a misnomer, as it is just as much a theory of higher-dimensional objects such as the D-branes. But by the time people realized this, string theory had already gotten its name.

The way to see this is as follows: we know that supergravity is the low-energy limit of string theory. The type IIB supergravity action admits not only a vacuum as solution to its equations of motion, but also branes. These are actually the *solitons* of the supergravity theory. When one compares the brane-solutions of supergravity with the D-branes that we just introduced, they turn out to coincide. Where the vacuum without D-branes is invariant under $\mathcal{N} = 2$ supersymmetry, the state including the brane is only invariant under $\mathcal{N} = 1$ supersymmetry. This is why D-branes are often called *BPS-states*[5]. So D-branes are dynamical objects. But how do we see this in string theory?

The picture which we just used to introduce D-branes, as objects on which strings can end, is actually just a weakly coupled perturbation around the non-perturbative state containing only the D-brane. The open strings ending on a Dp-brane have $9-p$ massless, real scalars in their spectrum: the X^i in the Polyakov action where i is an index that does not lie along the D-brane. These can be interpreted as fluctuations of the D-brane, indeed making it into a dynamic object. Actually, these massless scalars are the Goldstone bosons of the translational symmetry that is broken by the appearance of the D-brane.

Another way to see that D-branes have a dynamical effect on their surroundings, is by looking at 'disc diagrams'. Here, a D-brane emits a closed string. Since closed strings carry graviton modes (see [Ton09]), we see that D-branes have tension, i.e. energy and interact with their surroundings via gravity.

How did we forget these objects of fundamental importance at the start of developing string theory? This is just because our initial approach to string theory was perturbative, i.e. $g_s \ll 1$. In this limit, all these objects are so heavy that they never arise without us putting them in by hand. One can calculate the tension (which is energy density and therefore mass) of a Dp-brane to be [Joh06]

$$\tau_p = \frac{\sqrt{\pi}}{16\kappa}(4\pi^2\alpha')^{(11-p)/2} \tag{2.6}$$

where $\kappa = \kappa_0 g_s$ with κ_0 a constant related to Newton's gravitational constant in the theory. The important thing is that the tension goes as $1/g_s$, which means that indeed,

[5] A BPS-state satisfies a precise relationship between its mass and its charge, called the BPS-bound. If this relation is not satisfied, all the supersymmetry generators are broken.

in the limit $g_s \to 0$ where perturbative string theory is valid, these objects will never spontaneously occur. We will come back to what happens in the non-perturbative case in Section 2.7.

It has to be noted that D-branes are not the most general branes. It is possible to also have solutions of the supergravity action that contain $p+1$-dimensional objects that do not allow strings to end on it. Examples of these are NS5-branes and the F1, which is just the usual string.

2.6.3 Charge of D-branes

In electromagnetism, the Maxwell equations can be written as

$$dF = \star J_{\text{m}}, \tag{2.7}$$

$$d \star F = \star J_{\text{e}}. \tag{2.8}$$

where J_{m} and J_{e} are 1-forms describing the current and charge density.[6] If we let A denote the potential of F, i.e. locally $F = dA$, the language used for this is that the object on which $J_{\text{e}}/J_{\text{m}}$ lives is *electrically/magnetically charged* under the field A, is an *electric/magnetic source* for A, or we say that there is a *electric/magnetic coupling* of the currents to the field A. The fact that a particle electrically couples to the field A can be seen from the action of the particle, in which the following term will appear:

$$S_{\text{int}} = e \int A. \tag{2.9}$$

Here, the integral is over the worldline of the particle. This makes sense, since A is a 1-form.

Note that if we want the magnetic and electric fields to change places, we have to send $F \to \star F$. This is the *electromagnetic duality.* Under this transformation, an object that was previously coupled electrically to A, is now coupled magnetically.

Polchinki discovered that D-branes are charged under RR-fields, so C_0, C_2 and C_4 [Pol95]. The field under which it is charged depends on the dimension of the brane. Just as A is a 1-form being integrated over the worldline, we need to integrate the p-form C_{p+1} over a $p+1$-dimensional worldvolume. Indeed, in the action of a Dp-brane (which has a $p+1$

[6] Note that we here also allow magnetic monopoles to exist. But we can set $J_m = 0$ if we do not want them.

dimensional world volume) there appears the term

$$S_{\text{int}} = \mu_p \int C_{p+1} \tag{2.10}$$

We are going to say that a Dp-brane has C_{p+1} charge 1. This is of course a normalization, but the charge of a D-brane is uniquely defined since it has to satisfy the BPS-bound, so there is no assumption hidden here.

By the same reasoning as the electromagnetic duality and by the way we introduced C_6 and C_8 in section 2.2.2, we see that a Dp-brane hence is a magnetic source of C_{7-p}. So for example a D7-brane couples electrically to C_8 and magnetically to C_0. We will make it a convention to talk about electric and magnetic coupling in such a way that we always consider the field C_p with $p \leq 4$. Since type IIB supergravity only has C_p for p even, we see that the theory can only contain stable[7] Dp-branes for p odd. Conversely, type IIA only has C_p for p odd and hence contains only stable Dp-branes for p even.

To summarize:

> **Theorem 2.6.2.** *A Dp-brane is electrically charged under C_{p+1} and magnetically under C_{7-p}. Type IIB superstring theory only contains stable Dp-branes for p odd.*

The string is electrically charged under the NSNS 2-form B. This can also be dualized and we find that there is an object with a 6-dimensional world volume that is magnetically charged under the B-field. This is the NS5-brane that we mentioned earlier.

2.6.4 (p, q)-strings and branes

The $\mathrm{SL}(2,\mathbb{Z})$ symmetry of type IIB leads to more types of objects, as can be seen as follows. Suppose we have a string. This is charged under the B-field, but not under the C_2-field. Since an $\mathrm{SL}(2,\mathbb{Z})$-transformation interchanges these charges, we can get (p, q)-strings for any two relatively prime integers p, q that have charge p under the B-field and charge q under the C_2-field. In this language, the usual string hence is a $(1, 0)$-string.

What is the physical interpretation of these (p, q)-strings? Intuitively speaking, it should again be a one-dimensional object that is made up of p things with charge 1 under the

[7] Branes can be unstable and decay into lower dimensional branes. The BPS-property prohibits this from happening, but a Dp-brane for p even cannot be charged under any of the RR-fields that we have in type IIB, so it cannot satisfy the BPS-property and hence is unstable.

B-field and q things with charge 1 under the C_2 field. The fundamental string and the D1-brane satisfy these properties, respectively. And indeed, (p, q)-strings can be described as 'bound states' of the fundamental string and the D1-brane, as follows: Suppose we put a fundamental string and a D1-brane (sometimes called an F-string and D-string) parallel to each other. To minimize the energy of this system, the string will break with its two new end-points now lying on the D-string. These endpoints are charged under RR-field C_2, so there is a flux between these endpoints on the D-string. When they move off to infinity, we only see the flux left on the D-string. This is precisely a (1,1)-string: a bound state of an F-string with a D-string such that it has a unit charge for both the B and the C_2 field. It is important to note that this state is supersymmetric, just like the F-string and the D-string themselves. For more details and the construction for general p and q, see [Wit96].

Since we can think of D-branes (which in our case will always be D7-branes) as branes on which our usual strings, i.e. $(1, 0)$ strings, can end, we make the definition that a (p, q)-brane is a brane on which (p, q)-strings can end. This is sensible, since this can be transformed by an $\mathrm{SL}(2, \mathbb{Z})$-transformation back into our usual configuration of strings ending on a D-brane. But as we will see in Chapter 3, one has to consider these objects, since it generally is impossible to choose an $\mathrm{SL}(2, \mathbb{Z})$-transformation that transforms them the branes at once into D-branes.

2.6.5 Anti-D-branes

Just like we have antimatter in QFT, there are *anti-branes* in String theory. We consider these objects because of the same reason we consider anti-matter: you cannot create one without the other [MD02]. Also, they are useful objects, since they can be used e.g. to break supersymmetry, as we will see. We denote an anti-Dp-brane by $\overline{\mathrm{D}p}$-brane.

Since branes and anti-branes can annihilate and leave behind a stable vacuum, we see that they should have opposite charges. This is of course a known feature of ordinary antimatter. As we saw in Section 2.6.3, the RR-charge of a Dp-brane is determined by the integral of C_{p+1} over its worldvolume. An opposite charge therefore implies that the orientation on the worldvolume of the $\overline{\mathrm{D}p}$-brane is opposite to that of the Dp-brane. The point-particles one is used to working with are actually no different. The same logic applies and this is why we draw anti-matter in Feynman diagrams with the arrows going back in time: the orientation on their worldline is opposite to the orientation of ordinary matter.

For a one dimensional worldvolume, the only option for orientation reversal is an opposite parametrization of the time direction. But for a higher dimensional worldvolume, we could equally well flip the parametrization of a spatial coordinate. So a $\overline{\mathrm{D}p}$-brane for $p \geq 1$ is just a Dp-brane that is rotated by π.

We can actually get evidence for the system's urge to annihilate the case of $p = 1$. We can calculate the lowest mass state of a string stretching between the two branes. This turns out to be [Has12]

$$m^2 = -\frac{1}{2l_s^2}. \tag{2.11}$$

This gives us a tachyonic mode in the system. Remember that a tachyon signals an instability in the system; the system will tend to go to a true vacuum. We can interpret this as the urge of the D1-brane and the $\overline{\mathrm{D1}}$-brane to annihilate, getting us back to a stable vacuum.

Because $\overline{\mathrm{D}}$-branes are just like ordinary D-branes, they also break half of the supersymmetry and are BPS states. But a more careful analysis (as done in Chapter 13 in [BLT12]) shows that $\overline{\mathrm{D}}$-branes preserve precisely the other half of the supersymmetry. So putting a D-brane and a $\overline{\mathrm{D}}$-brane parallel to each other results in a complete breaking of supersymmetry.

Important to note is that being an anti-brane is a relative property: two branes can be each others anti-brane, but which one is the 'anti' one is a matter of choice. It is in fact just a matter of choosing an orientation.

One might be tempted to think that anti-branes are among the possible (p, q)-branes. After all, $-\mathrm{id} \in \mathrm{SL}(2, \mathbb{Z})$ so this would invert all the charges and hence maybe give an anti-brane. We will see in Remark 3.2.2 that this is not the case. Of course, there are also anti-(p, q)-branes, since an $\mathrm{SL}(2, \mathbb{Z})$ transformation must send a pair consisting of a D-brane and a $\overline{\mathrm{D}}$-brane to a (p, q)-brane and its anti-brane.

2.7 Interpretation of SL$(2, \mathbb{Z})$-invariance

As noted above, 'string theory' is a misnomer, since D-branes are just as fundamental as strings. And since (p, q)-strings and branes are just transformed fundamental strings and D-branes, they also have a fundamental role to play. This gives us an interpretation for the $\mathrm{SL}(2, \mathbb{Z})$ invariance of type IIB.

How can a theory that we build perturbatively look the same for every SL$(2,\mathbb{Z})$ transformation that we apply to it? To see this, we first consider the simplest case $g_s \to 1/g_s$, i.e. S-duality. Remember that this coincides with the element $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in the case $C_0 = 0$. The easiest way to describe this, is to make sure that the metric does not depend on the dilaton. We do this by going to *Einstein frame*, which is nothing more than to introduce $g^{(\mathrm{E})}_{\mu\nu} = e^{-\phi/2} g_{\mu\nu}$. We now look at the D1-brane, an object that has the same dimension as the string. In this case, the tensions of the fundamental string and the D1-brane are $g_s^{\frac{1}{2}}/2\pi\alpha'$ and $g_s^{-\frac{1}{2}}/2\pi\alpha'$ respectively. This is already suggestive and indeed the following is the case: when looking at the limit $g_s \to \infty$, the D1-brane becomes the lightest object in the theory and takes the role of the string. And in turn, the fundamental string becomes heavy and will not naturally occur in the perturbative theory of D1-strings. So we can do everything like we did in the development of perturbative string theory but with the roles reversed. This is the reason that S-duality relates type IIB to itself.

We can also see why the RR 2-form C_2 and the NSNS 2-form B should be interchanged, as is indeed done by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$. The fundamental string is charged under B, while the D1-string is charged under C_2. Since the role of the fundamental string and the D1-string is reversed after the transformation, so should their charges.

Remark 2.7.1. This can also be used to see why type IIA does not relate to itself via S-duality. This theory only contains Dp-branes for p even, and hence there is no object that can take the role of the fundamental string.

More generally, we can look at the aforementioned (p,q)-strings. For every relatively prime combination of p and q, there is a value of g_s in which case the (p,q)-string becomes the lightest mode of the theory. This means that we could have started developing type IIB string theory from every one of these perspectives and they all would have led to the same conclusion. It is somehow beautiful that strings do not have a more privileged position in the theory than the other objects: it is just a matter of perspective.

We can also give this symmetry a geometric interpretation via M-theory. Remember that M-theory on a torus is dual to type IIB on a circle. When one keeps track of the dualities, it can be seen that the symmetry group of the complex structure parameter of this M-theory torus (which is SL$(2,\mathbb{Z})$) is the same as the SL$(2,\mathbb{Z})$ symmetry of type IIB [Sch95]. The shaky thing about this equivalence however, is that the area of the torus has to go to zero to get type IIB without the circle. Of course, the complex structure parameter of a torus that does not have an area sounds ill-defined, but it works since the limit of the area going to zero is applied only after all the equivalences have been gone through.

Chapter 3

F-theory: a more geometric approach

We now turn to F-theory. The F supposedly stands for father, a whimsical reference to M-theory. Although it was not the point of view when it was introduced, we now look at it as a powerful, geometric reformulation of type IIB string theory.

The theory cleverly exploits the $\mathrm{SL}(2,\mathbb{Z})$ symmetry of type IIB to construct a description of type IIB in which we can solve much more general configurations than the perturbative description allows us to do, using geometric tools. For example, it can incorporate 7-branes, which cannot be described perturbatively since their presence has a big influence on the space.

3.1 Describing 7-branes

Let us first see why 7-branes are problematic. Recall from Section 2.6.3 that a D7-brane is charged magnetically under C_0. Suppose we have a D7-brane, then there are two spatial dimensions it does not fill. Those we describe with the complex coordinate u. In this plane, the D7-brane functions as a magnetic source for C_0. Schematically:

$$dF_1 = \delta_{\mathrm{D7}}$$

where F_1 is the field strength associated to C_0. We will now see that the presence of a 7-brane will make sure that C_0 cannot be globally defined: Integrating over a disc, we get

$$\int_{D^2} dF_1 = \int_{D^2} \delta_{\mathrm{D7}} = 1.$$

If we insist on $F_1 = dC_0$ globally, we will have to let go of the notion of C_0 having a single, well-defined value at each point. Then we get

$$1 = \int_{D^2} dF_1 = \oint_{S^1} F_1 = \int_{\partial S^1} C_0 = C_0(x) - C_0(x).$$

So C_0 can now only be defined up to shifts by integer values. Mathematically, this is all just a consequence of the Greens function for the Laplacian in 2 dimensions being a logarithm and hence forcing a branch cut. Since C_0 is contained in the axio-dilaton τ, this will also no longer be single-valued. Instead, we will have

$$\tau \to \tau + 1$$

whenever we walk around the D7-brane. We call this kind of behaviour *monodromy*, a concept that we will introduce in more generality in Section 4.2. We can relate this to the SL$(2,\mathbb{Z})$ transformations encountered in Section 2.5, namely the action of the element $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ on τ. This monodromy suggests that near the location of the brane, call it u_0, we have

$$\tau \sim \frac{1}{2\pi i} \log(u - u_0). \tag{3.1}$$

Notice that when we approach the D-brane, we have $\tau \to i\infty$. Referring back to the physical meaning of τ, we see that this corresponds to $g_s \to 0$. So near the 7-brane, we have weak coupling.

But that is just the friendly part of this setup. We can see that (3.1) can only hold for $|u| \ll 1$ since otherwise we would get a negative value for $g_s = e^{\phi}$. Also, we have seen that, due to the monodromy, τ is not even single-valued, so it is getting hard to write down a global solution for the τ profile. What we however can do, is write down a solution for τ up to SL$(2,\mathbb{Z})$ transformations. This is most easily done by giving a solution for $j(\tau)$ where j is some function with the property

$$j(\tau_1) = j(\tau_2) \quad \Longleftrightarrow \quad \tau_1 = T(\tau_2) \text{ for some } T \in \mathrm{SL}(2,\mathbb{Z}) \tag{3.2}$$

Such a function exists and is called the j-invariant

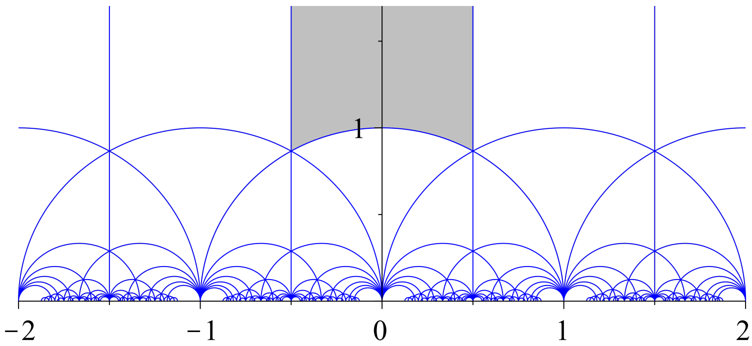


Figure 3.1: The grey area is a possible choice of fundamental region for the j-invariant. Image taken from Wikipedia [Hul18].

Remark 3.1.1 (the j-invariant)**.** The function j is defined on the upper half complex plane and maps to the Riemann sphere (the complex plane including ∞, i.e. $\mathbb{C}\mathrm{P}^1$). Its primary characteristic is that it is invariant under $\mathrm{SL}(2,\mathbb{Z})$. That is, if $T \in \mathrm{SL}(2,\mathbb{Z})$, then $j(T\tau) = j(\tau)$, where T acts on τ as described in equation (2.5). It is surjective, but of course not injective. However, it is injective when we restrict to the *fundamental region*: a connected subset of the upper half plane such that none of the points in it are related via $\mathrm{SL}(2,\mathbb{Z})$ and any other point outside of it is related to one of the points in it. Another way of saying this, is that it is a connected subset of the upper half complex plane containing precisely one point from every orbit. A picture of it can be found in Figure 3.1.

In terms of $q = e^{2\pi i\tau}$, we can write

$$j(q) = \frac{1}{q} + 744 + \mathcal{O}(q) \tag{3.3}$$

The simplest solution that has the form (3.1) near the brane is given by

$$j(\tau(u)) = \frac{1}{u - u_0}. \tag{3.4}$$

The property of the j-invariant function that $j(\frac{1}{2}(1+i\sqrt{3})) = 0$ can be used to conclude that in this solution, we have $g_s = \frac{2}{\sqrt{3}} \approx 1.15$ for large distances from the brane. So far away, there is strong coupling. This is another reason that configurations containing 7-branes are hard to treat perturbatively.

And that is not all the big influence that this D-brane will have. Since, in the u-plane, it will introduce a *deficit angle*: from the point of view of the u-plane, it is a point mass.

And as derived in [GSVY90], this gives a metric that is flat, but only so in terms of the coordinate $\tilde{u} = u^{1-\theta}$ for some θ. This means that in this new coordinate, a circle is only $2\pi(1-\theta)$ radians, hence the name deficit angle. It is clear that this is of global influence on our space. This is what is often called the *backreaction* of 7-branes on the space. Again, we see that we encounter this phenomenon with 7-branes because we have a small number of codimensions.

3.2 The idea of F-theory

F-theory will make use of a geometrical interpretation of $\mathrm{SL}(2,\mathbb{Z})$. This is the following: one can define a complex structure on a torus T^2 by identifying the torus with $\mathbb{C}/L(\omega_1,\omega_2)$. Here $L(\omega_1,\omega_2)$ is the lattice defined by $\omega_1,\omega_2 \in \mathbb{C}$ by

$$L(\omega_1,\omega_2) = \{n\omega_1 + m\omega_2 \mid n,m \in \mathbb{Z}\}.$$

If we choose $\omega_1 = 1$, which we can always do via a holomorphic transformation – which leaves the complex structure invariant – then the structure is fully defined by $\omega_2 \in \mathbb{C}$. The following proposition holds:

> **Proposition 3.2.1.** Let a complex structure on the torus be given by $\omega \in \mathbb{C}$. Then any complex structure on the torus given by $T(\omega)$ for some $T \in \mathrm{SL}(2,\mathbb{Z})$ is equivalent. Here T acts on ω by (2.5).

Now we apply this geometric interpretation to string theory: we look at the axio-dilaton $\tau = C_0 + ie^{-\phi}$ not as a field having a value at every point in spacetime, but as the complex structure of a torus at that point. So instead of considering a 10-dimensional spacetime, we attach a torus to every point in spacetime, i.e. we consider a 12-dimensional space which fibers[1] over the spacetime in such a way that at every point the fiber is a torus. The complex structure of this torus over point x is $\tau(x)$. Note that by the previous proposition, this τ is only defined up to an $\mathrm{SL}(2,\mathbb{Z})$ transformation. But this is not at all a problem, and maybe even good, since type IIB string theory is invariant under these transformations. The torus is often called the auxiliary torus, since it is a bookkeeping device and not a physical part of the spacetime.

[1] A *fibration* of a space X over a space M can be thought of as a projection $\pi\colon X \to M$ that can twist and turn. A *fiber* over a point p is $\pi^{-1}(p)$, i.e. the set of all points in X that get mapped to p by π. More treatment follows in Section 3.5 and Chapter 4.

Determining the profile of τ given some setup is now a problem of determining the geometry of the 12-dimensional space. This opens the door to a whole new set of tools to tackle problems with. It should be noted that the existence of a complex structure on the torus is a big assumption from a mathematical perspective.

One could ask what constraints τ has to satisfy. From supersymmetry we can derive that τ has to be a holomorphic function.

3.2.1 Monodromy around branes

We can use the results from 3.1 to see how the appearance of 7-branes influences the geometry of our 12-dimensional space. To do this, we temporarily forget about the 8 dimensions that the 7-branes lie in, and consider only the 2 other spacetime dimensions and the torus over every point, so a 4-dimensional space.

Around a D7-brane, we have seen that τ transforms according to the element $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$. For a general (p,q)-brane, we can transform it to a D7-brane via the transformation $\left(\begin{smallmatrix} p & s \\ q & r \end{smallmatrix}\right)^{-1}$, where $r, s \in \mathbb{Z}$ are such that $pr - qs = 1$. We then know how the monodromy around it looks and have to transform it back afterwards. So the monodromy around a (p,q)-brane is given by

$$M_{(p,q)} = \begin{pmatrix} p & s \\ q & r \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & s \\ q & r \end{pmatrix}^{-1} = \begin{pmatrix} 1 - pq & p^2 \\ -q^2 & 1 + pq \end{pmatrix}. \tag{3.5}$$

If we walk around in the reverse direction, we get the inverse of this matrix. If we walk around a (p,q)-brane and then a (p',q')-brane, we get the monodromy $M_{(p,q)}M_{(p',q')}$.

Remark 3.2.2. Because branes and anti-branes can annihilate, the monodromy around them should be zero. This can be used to see that an anti-brane is not a (p,q)-brane for any $p, q \in \mathbb{Z}$. Note that

$$M_{(p,q)}^{-1} = \begin{pmatrix} 1 + pq & -p^2 \\ q^2 & 1 - pq \end{pmatrix}.$$

Suppose that this is equal to $M_{(p',q')}$ for some $p', q' \in \mathbb{Z}$, then that would imply that $p'^2 = -p^2$ and $q'^2 = -q^2$, which is impossible for $(p,q) \neq (0,0)$. So anti-branes are not (p,q)-branes.

At the location of the brane, τ does not have a value. This is also reflected geometrically, as at the location of the branes the torus over them 'pinches', so the fiber is not even a torus.[2]

Turning the logic upside down: if we know the profile of τ, we can pinpoint where precisely what kind of branes are. This is the subject of the next example.

3.3 The K3 example

F-theory uses elliptic fibrations. We will explicitly define this notion in Section 3.5. It is a special kind of torus fibration. This amount of structure makes working with it and determining the τ profile much easier. We will now work out the example of F-theory on a K3-surface. This is an especially friendly elliptically fibered space of 4 dimensions. Since it is good to have seen, but not extremely relevant for our purposes, we will make some claims. For more details, see [Den08].

The K3 can be described as lying in the space

$$\{(u,v,x,y,z) \in \mathbb{C}^5 \mid (u,v) \neq 0, (x,y,z) \neq (0,0,0)\}/\sim$$

where the equivalence relation is given by

$$\begin{aligned}(u,v,x,y,z) &\sim (\lambda u, \lambda v, \lambda^4 x, \lambda^6 y, z)\\ &\sim (u,v,\mu^2 x, \mu^3 y, \mu z)\end{aligned}$$

for $\lambda, \mu \in \mathbb{C} - \{0\}$. The equation defining the K3 is then

$$y^2 = x^3 + f(u,v)xz^4 + g(u,v)z^6.$$

Where f, g are polynomials. Note that this is a well-defined equation only when f and g are polynomials in u, v of degrees 8 and 12, respectively. The object defined by such a formula is Calabi-Yau if the weighed degree of the defining polynomial (the power of e.g. λ that appears when we insert the point $(\lambda u, \lambda v, \lambda^4 x, \lambda^6 y, z)$) equals the sum of the weights of all the coordinates, for each equivalence separately. In this case

[2] When we introduced F-theory, we attached a fiber to every point. But this was incorrect, since we indeed add pinched fibers over the locations of the branes. The simply reason is that it is impossible to have a regular torus over the location of the fiber and also have a monodromy around it.

$1+1+4+6=12$ and $0+0+2+3+1=6$, so this works out. A complex 2-dimensional Calabi-Yau is always a K3.

To see that this is indeed a fibration with tori as fibers, we define $\pi\colon \mathrm{K3} \to \mathbb{C}\mathrm{P}^1$ by

$$\pi\colon (u,v,x,y,z) \mapsto (u:v).$$

This is well-defined and the fiber is a torus, since it is a complex 1-dimensional Calabi-Yau – of which the torus is the only one – as can be checked by the same rule.

We need to deduce an explicit form of τ from this abstract description in terms of polynomials. For this we use the holomorphic coordinate on T^2, namely $z = x + \tau y$ and define the form $\Omega_1 = dz$. We choose a basis of 1-cycles, (A, B) on the T^2 and then τ is given by

$$\tau = \frac{\oint_B \Omega_1}{\oint_A \Omega_1}$$

What happens if we choose another basis? Suppose we take $(A+B, B)$ as basis. Then we would get

$$\tau = \frac{\oint_B \Omega_1}{\oint_A \Omega_1} + 1.$$

This is fine: every other choice of basis will give us another answer, but all of them will be related to each other by $\mathrm{SL}(2,\mathbb{Z})$-transformations. This is actually to be expected, since they all describe the same complex structure.

One can do the computation for τ in this case, which results in

$$j(\tau) = \frac{4(24f)^3}{\Delta}, \quad \Delta = 27g^2 + 4f^3.$$

We call Δ the discriminant of the elliptic curve. When the discriminant is zero, the elliptic curve becomes singular, generically by a 1-cycle collapsing to zero size (i.e. pinching). In this case, we can determine how many 7-branes there are in our setup: Δ is a homogeneous polynomial of degree 24, and generically will have 24 distinct zeroes. Notice that near a generic zero u_0 that is not $(u:v) = (1:0)$, we can set $v = 1$ and write $j(\tau) \sim \frac{1}{u-u_0}$. This is solved by

$$\tau(u) \sim \frac{1}{2\pi i} \log(u - u_0), \tag{3.6}$$

up to an $\mathrm{SL}(2,\mathbb{Z})$ transformation, which is the same as we found in equation (3.1). Therefore, we have a monodromy $\tau \to \tau + 1$ around u_0 and this signals the presence of a D7-brane at u_0.

Does this mean there are 24 D7-branes in our setup? No. However, it does mean that there are 24 7-branes in our setup. Let us see why they are not all D-branes. The reason is actually pretty simple, and we can see it both from a mathematical and physical perspective. Physically, a D7-brane is a $(0, 1)$-brane and hence carries charge. If all the branes are D-branes and they all carry the same charge, the net charge is not zero and we have field lines that cannot end anywhere. This is problematic on a compact manifold such as $\mathbb{CP}^1$, because they also cannot go out to infinity. This will lead to tadpoles and is not a stable state of the system.

Mathematically, a path around all the branes, i.e. singular fibers, on $\mathbb{CP}^1 \cong S^2$ is also a path around none of the branes. Therefore the total monodromy has to be zero. But we know the monodromy corresponding to going around N D7-branes, $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$, which certainly is not the identity. Hence only having D-branes leads to a contradiction.

The other branes are the more general (p, q)-branes, introduced in Section 2.5. They come from when we naively claimed the form of τ around a brane, but only did so up to an $\mathrm{SL}(2, \mathbb{Z})$ transformation. Locally, we can always forget about the other branes in our setup, apply an $\mathrm{SL}(2, \mathbb{Z})$ transformation and then consider a D-brane. But it is impossible to choose an $\mathrm{SL}(2, \mathbb{Z})$ transformation such that all the branes are D-branes. Making the one a D-brane will ruin the "D-ness" of another.

3.4 The M-theory/F-theory duality

In Section 2.4, we found that M-theory on a torus is dual to type IIB on a circle. To make things more explicit, we consider M-theory on a 9-dimensional base space B_9 times the torus T^2. We consider type IIB on $B_9 \times S^1$. If we follow the dualities carefully, we can see that the complex structure τ of the torus on the M-theory side is determined by $\tau = C_0 + ie^{-\phi}$, i.e. the value of the axio-dilaton on the type IIB side.

F-theory is concerned with a varying axio-dilaton profile. It is not immediately clear that in this case the duality still holds, but following an adiabatic argument we can assume that it does [VW96].

Suppose that we have a varying axio-dilaton profile. Then the complex structure of the torus on the M-theory side also varies. To treat this case in general, we no longer talk about the space $B_9 \times T^2$, but turn to the more general notion of an elliptic fibration

$X_{11} \to B_9$ (c.f. Defintion 3.5.1). We now reason the following:

> **Theorem 3.4.1.** *M-theory on an elliptic fibration $X_{11} \to B_9$ is dual to F-theory on $X_{11} \times S^1 \to B_9 \times S^1$, where the fiber lives in the auxiliary dimensions of F-theory and the radius of S^1 is inversely related to the area of the fiber on the M-theory side.*

Proof. The complex structure $\tau(z)$ of the torus fiber is now a function of the base space. Since it coincides with the axio-dilaton, we see that the complex structure of the torus fiber in the auxiliary dimensions of F-theory should also be τ. But in F-theory, the torus lies over the full 10-dimensional space of type IIB, which is $B_9 \times S^1$. So the relevant elliptic fibration for the F-theory side is $X_{11} \times S^1 \to B_9 \times S^1$.

As described in Section 2.4, the S^1 on the type IIB side comes from one of the circles of the torus on the M-theory side. Between these two theories, a T-duality has to be applied to the circle. This makes it so that the larger the S^1 on the type IIB side, the smaller one of the components of the M-theory's torus. So the area of the torus fiber is inversely proportional to the radius of the S^1. □

When people consider so called *F-theory compactifications*, it may at first not be clear what is meant, since F-theory is not a theory in that it describes a new type of physics. M-theory however, can be compactified as described in Section 2.3.2. So when F-theory is compactified, this physically makes sense as a compactification of M-theory that this duality identifies it with.

3.5 The mathematical side of F-theory: elliptic fibrations

F-theory makes use of the mathematics of elliptic fibrations. We will explicitly define this notion, since it shows what is generally assumed as mathematical structure when working with F-theory.

> **Definition 3.5.1.** Let Y and B be complex manifolds. An *elliptic fibration* is a surjective, holomorphic map $\pi\colon Y \to B$ such that the fiber $\pi^{-1}(b)$ for generic $b \in B$ is topologically a torus.

In this definition, "generic" means that this property holds except for a finite or countable number of points.

There are a few remarks to be made about this definition. First, note that this definition needs a complex structure on both Y and B. Second, it is often assumed that there is a (meromorphic) section. F-theory without sections has already been studied to some extend, see [AGEGK14].

In the K3-example, we used that π was holomorphic quite extensively, as it allowed us to write a polynomial equation for the fibers, immediately detecting that the singular fibers were located at the point where the discriminant was zero. This indicates the usefulness of such a structure when one wants to get concrete results. But it has a severe drawback.

The mathematician Kodaira classified all the possible singular fibers that can occur on an elliptic surface (a complex surface admitting an elliptic fibration). None of these possibilities describe something else than stacks of general (p, q)-branes. So it is impossible to find a singular fiber describing an anti-brane in the framework of elliptic surfaces. This is not to say that it is useless to look at elliptic surfaces; the K3-example just showed how powerful it can be. But it does say that all the nice structure comes with restrictions.

3.6 Dropping the complex structure

Since the aim of this thesis is to describe anti-branes, we now already see that we must drop some of the structure from the elliptic fibrations. We show that we can drop the assumption of a complex structure on the torus, but still have the group $\mathrm{SL}(2, \mathbb{Z})$ arise in a natural way from the it.

We can look at the space of all diffeomorphisms from the torus to itself. This is huge and infinite dimensional, but we declare a lot of them equivalent, namely if they can be smoothly deformed into one another. More formally:

> **Definition 3.6.1.** An *isotopy* between two embeddings $\varphi_0, \varphi_1 \colon Y \to X$ is a smooth map $\Phi\colon I \times Y \to X$ such that each $\Phi(t, -) = \varphi_t \colon Y \to X$ is an embedding. If an isotopy between two maps exists, we call them isotopic.

So a map from the torus to itself that just wobbles and stretches it a little bit, but does not really introduce a twist, is just isotopic to the identity. We call a diffeomorphism a *large diffeomorphism* if it is not isotopic to the identity.

Definition 3.6.2. The *mapping class group* of a space X, denoted by $\mathcal{M}(X)$, is the group of isotopy classes of diffeomorphisms from X to itself. The group structure is given by composition of maps. If X is a surface of genus g, we denote $\mathcal{M}_g = \mathcal{M}(X)$.

The mapping class group of the torus can be rigorously proved to be $\mathcal{M}_1 \cong \mathrm{SL}(2,\mathbb{Z})$ [FM11]. But here we will just give the idea: suppose we have a diffeomorphism of the torus. Since the torus is just $S^1 \times S^1$, this diffeomorphism will be an embedding of $\{x\} \times S^1$ and $S^1 \times \{y\}$ for each $x, y \in S^1$. So these circles have to get mapped to circles. Since we are working up to isotopy, we do not care about where they land precisely. We do care how often these circles wind around the torus, since different winding numbers cannot be deformed into one another. So say that our diffeomorphism would send the circle $\{x\} \times S^1$ to a circle that winds around the first component of the torus p times, and around the other q times. We claim that this map is then isotopic to the diffeomorphism $\psi\colon T^2 \to T^2$ given by $\psi(\theta,\varphi) = (\theta + p\varphi, q\varphi)$. If p and q are not relatively prime, this is not injective and hence not a diffeomorphism. So the elements of $\mathcal{M}_1$ can be represented by pairs of relatively prime integers (p, q). For every such pair, there exist unique integers $r, s \in \mathbb{Z}$ such that $pr - sq = 1$. So the matrix $\left(\begin{smallmatrix} p & s \\ q & r \end{smallmatrix}\right)$ has determinant 1 and hence lies in $\mathrm{SL}(2,\mathbb{Z})$. Every element of $\mathrm{SL}(2,\mathbb{Z})$ arises in this way.

If we also look at what composing two of these diffeomorphisms does to the winding numbers, we see that this respects the same group structure as the matrix multiplication of $\mathrm{SL}(2,\mathbb{Z})$. So we conclude that the mapping class group $\mathcal{M}_1 \cong \mathrm{SL}(2,\mathbb{Z})$.

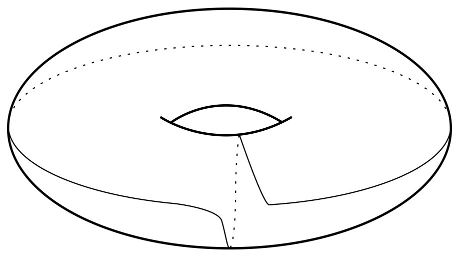


Figure 3.2: An example of a path winding around both components of the torus once. This would be the image of a map that is isotopic to $\psi(\theta_0, \varphi) = (\theta_0 + \varphi, \varphi)$ for some given θ_0.

Not having a complex structure robs us of the opportunity to talk about the value of τ at a certain point. But as argued above, it was necessary to drop something and we will see later that it looks like this is enough.

Chapter 4

Introduction to Lefschetz fibrations

Now we arrive at the more mathematical side of this thesis. We just noted that elliptic fibrations are powerful but restrictive. Here we develop the theory of Lefschetz fibrations. These are a related but much less restrictive type of fibrations. They originated in algebraic geometry, but Gompf and Donaldson have proven their usefulness in differential geometry as well [Don98], [GS99].

We largely follow the approach of [GS99] in this and the following section.

4.1 Definition and basic properties

We define Lefschetz fibrations for all manifolds of even dimension, but will soon restrict to manifolds of dimension 4, which are relevant for our purposes.

> **Definition 4.1.1.** Let X^{2n} and Σ^2 be compact, connected, oriented, smooth manifolds. A *Lefschetz fibration* on X is a map $\pi\colon X \to \Sigma$, such that for any critical point and its value, there are orientation preserving, complex coordinate charts centered at them, such that π takes the form $\pi(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$.

Remark 4.1.2. Some terminology: we call the space $\pi^{-1}(p) \subseteq X$ the *fiber* over $p \in \Sigma$. For an overview of the words regular/critical point/value, see Appendix A.

Remark 4.1.3. Some authors include in their definition of a Lefschetz fibration the condition that π is *injective on its critical points*, i.e. each critical point has its own corresponding critical value. This is just another way of saying that there is at most one critical point per fiber. This can always be achieved by slightly perturbing the map π.

We will now treat some basic properties of Lefschetz fibrations. First of all, there are finitely many critical points of π. To see this, we denote the set of critical points of π by K. By the demanded local form of π around critical points, we see that in particular, we have an open neighbourhood U_x around each $x \in K$, such that $K \cap U_x = \{x\}$. We can add opens to this collection of $\{U_x\}$ that do not contain critical points so that they have no intersection with K. Then, by compactness of X, we can choose a finite subcover, which in particular must contain all U_x for all $x \in K$. Therefore, there are only finitely many critical points of π.

Secondly, we can prove that apart from this finite number of fibers over the critical values, π is a fiber bundle.

Lemma 4.1.4. Let $\pi\colon X \to \Sigma$ be a Lefschetz fibration. Let Σ^* be the set of regular values of π and let $\Gamma = \pi^{-1}(\Sigma - \Sigma^*)$ be the set of singular fibers. Then $\pi\big|_{X-\Gamma}\colon X - \Gamma \to \Sigma^*$ is a fiber bundle with connected base space.

Proof. By Ehresmann's fibration theorem, $\pi\big|_{X-\Gamma}$ defines a fiber bundle if it is a proper[1] surjective submersion. The properness follows from π being continuous and X being compact. It is a submersion because we deleted the points on which π is not, namely the critical points. Surjectiveness can be reasoned as follows: submersions send open sets to open sets, so $\pi(X - \Gamma) \subseteq \Sigma^*$ is open. Since X is compact, $\pi(X) \subseteq \Sigma$ is also compact and hence closed. That means that $\pi(X - \Gamma) = \pi(X) - \pi(\Gamma) \subseteq \Sigma^*$ is also closed. So $\pi(X - \Gamma) \subseteq \Sigma^*$ is a connected component. Since there are finitely many critical values, Σ^* is connected. Therefore, $\pi\big|_{X-\Gamma}$ is surjective. □

This lemma in particular implies that all the fibers over regular values are the same and are of dimension $2n - 2$. As a result, we say that a Lefschetz fibration has F as its fiber when its regular fiber is diffeomorphic to F. The fibers over the critical values are of a different form and hence are called *singular fibers*.

[1] A map is proper if the inverse image of any compact subspace is compact.

We see that each fiber F is compact and canonically oriented. The compactness follows from F being a closed subspace of a compact space. The orientation follows by virtue of π being a submersion at every point except its critical points (where the fibers are also no longer manifolds, so we cannot talk about orientations there) as follows: at a point $x \in F$, pick $2n-2$ vectors $v_1, \dots, v_{2n-2} \in T_xX$ that get mapped by $d\pi$ to a basis of $T_{\pi(x)}\Sigma$. A basis $\{w_1, w_2\}$ of $T_xF = \ker d\pi$ will give us a basis $\{v_1, \dots, v_{2n-2}, w_1, w_2\}$ of T_xX. Since X is oriented, we can say whether this is a positive or negative basis, calling the basis $\{w_1, w_2\}$ positive (negative) whenever the basis for T_xX is positive (negative).

The following proposition is a direct analogue for Lefschetz fibrations of a famous result for fibrations. See Appendix A for a definition of π_n.

Proposition 4.1.5. Let $\pi\colon X \to \Sigma$ be a Lefschetz fibration with fiber F. The maps $F \hookrightarrow X \to \Sigma$ induce an exact sequence

$$\pi_1(F) \to \pi_1(X) \to \pi_1(\Sigma) \to \pi_0(F) \to 0.$$

Here exactness of the map $\pi_1(\Sigma) \to \pi_0(F)$ is defined as exactness between pointed sets.

Proof. This sequence is derived in the same way as the exact sequence for fiber bundles: the first two maps are the maps induced by inclusion and π. The map $\pi_1(\Sigma) \to \pi_0(F)$, which we will call δ, is defined using the homotopy lifting property (HLP). Every fiber bundle has this property, so we first restrict to the case of the fiber bundle $\pi\big|_{X-\Gamma}\colon X - \Gamma \to \Sigma^*$. Given any path $\gamma\colon I \to \Sigma^*$, the HLP implies that we can *lift* this to a path $\tilde{\gamma}\colon I \to X - \Gamma$ such that $\pi \circ \tilde{\gamma} = \gamma$. We can choose the starting point of $\tilde{\gamma}$, as long as it sits in $\pi^{-1}(\gamma(0))$. We choose a preferred connected component of F, such that $\pi_0(F)$ becomes a pointed set, and let $\tilde{\gamma}(0)$ lie in this component. We then define $\delta([\gamma]) = [\tilde{\gamma}(1)]$. In words: it sends a loop γ to the connected component in which the lift of γ ends. The fact that this is well-defined can also be proved using the HLP. Exactness is reasoned as follows:

- At $\pi_1(X)$: A loop that lies in the fiber will always be projected to the constant path. Conversely, a loop that is send to the constant path by the projection must have been contained in a single fiber.
- At $\pi_1(\Sigma)$: Given a loop in X that we call γ, its projection $\pi \circ \gamma$ has naturally γ as its lift. Since γ is a closed path, its begin and end point are the same and thus lie in the same connected component of the fiber. Conversely, if the lift of a path

in Σ has its starting and ending point in the same component of the fiber, we can ~~assume it~~ to be a loop. Hence the path in Σ was given by the projection of its lift.

- At $\pi_0(F)$: Since X is connected, there is a path γ in X connecting any given connected components of F. Projecting this path to Σ, it becomes a loop with γ as its lift. So $\delta([\gamma])$ is the required connected component.

This proof works for the case where γ does not go through critical values. If γ does go through a critical value, we can homotope it a little bit such that it no longer does so. Since it is only γ up to homotopy that is relevant, this is not a problem. However, we have to check that the different ways of homotoping γ away from the critical value, leave $\delta([\gamma])$ invariant. However, in Section 4.1.1 we will see how the singular fibers look and from that we can conclude that singular fibers have the same number of connected components and hence going through a singular fiber will not change the connected component that our path lies in. So this is indeed well defined. Nothing changed in the reasoning about exactness, so this proves the proposition. □

From this proposition, we deduce that we can always assume the fibers of Lefschetz fibrations to be connected. Because F is compact, $\pi_0(F)$ is always a finite set and we call its number of elements n. Hence if it $n \neq 1$, then $\pi_1(X)$ gets mapped to a subgroup of $\pi_1(\Sigma)$ of index n. By replacing Σ with its n-sheeted covering space Σ', we obtain a new Lefschetz fibration with connected fibers. Geometrically, we replace Σ by a space that contains n copies of Σ and place each connected component of F over a different copy of Σ.

From now on, we will work with Lefschetz fibrations of 4-dimensional manifolds. Also we will assume the fiber to be connected.

Since in this case a regular fiber F is an oriented surface, we know it will always be a surface of some genus g. This genus of the regular fibers cannot vary, since by Lemma 4.1.4, π is a fiber bundle except for finitely many points. Therefore, we call a Lefschetz fibration with regular fibers of genus g a genus-g Lefschetz fibration.

4.1.1 Vanishing cycles

A Lefschetz fibration distinguishes itself from a usual fiber bundle by the appearance of critical points of π. Because of our allowance for these critical points, there can be finitely many fibers that do not look like the others. We will find a description for these singular fibers in this section.

Our demanded local form around a critical point can be transformed to $\pi(z_1, z_2) = z_1 z_2$, as it gives back the demanded form after a linear transformation $(z_1, z_2) \to (z_1 - iz_2, z_1 + iz_2)$. Given a value $t \in \mathbb{C}$, $t \neq 0$ that is small enough, we can write $\pi^{-1}(t) = \{(z_1, \frac{t}{z_1})\} \subseteq \mathbb{C}^2$. In particular, this submanifold has a non-contractible circle embedded in it, whose radius gets smaller when t gets smaller. In this parametrization, it is given by $(z, \frac{t}{|t|}\overline{z})$. When we finally reach the point $t = 0$, we no longer have a smooth submanifold, but a transverse intersection of two planes in the origin, where the radius of our circle has become zero[2].

Because of the four dimensional nature, it is hard to imagine what is happening. But one can get an intuitive picture by taking every parameter to be real and reading 'circle' and 'plane' as S^0 and 'line' respectively. See Figure 4.1 for this point of view.

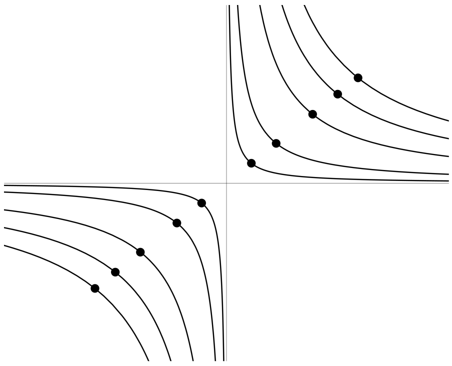


Figure 4.1: A drawing of the "side view" of $\pi^{-1}(t)$ for five different values of t. The thick points are S^0: they begin to lie closer to each other as $t \to 0$, so the cycle starts to vanish.

This circle is called the *vanishing cycle*. If the generic fiber is a torus, the singular fiber can be for example a pinched torus or it can be a torus with a sphere attached to one point. See Figure 4.2 for a drawing of these two cases.

The case of a torus with a sphere attached is a special case of what happens when the vanishing cycle is ~~a *separating cycle*~~, ~~i.e. when deleting it from the fiber would leave two connected components~~. In this case, the resulting singular fiber will be the image of an immersion of a disconnected surface: ~~two surfaces intersecting each other transversally in a point~~. If none of the vanishing cycles are ~~separating cycles~~, we call ~~a~~ Lefschetz fibration *relatively minimal*. For our purposes, only relatively minimal Lefschetz fibrations will

[2]This gives us slightly more, namely that the singular fiber has a positive transverse self-intersection.

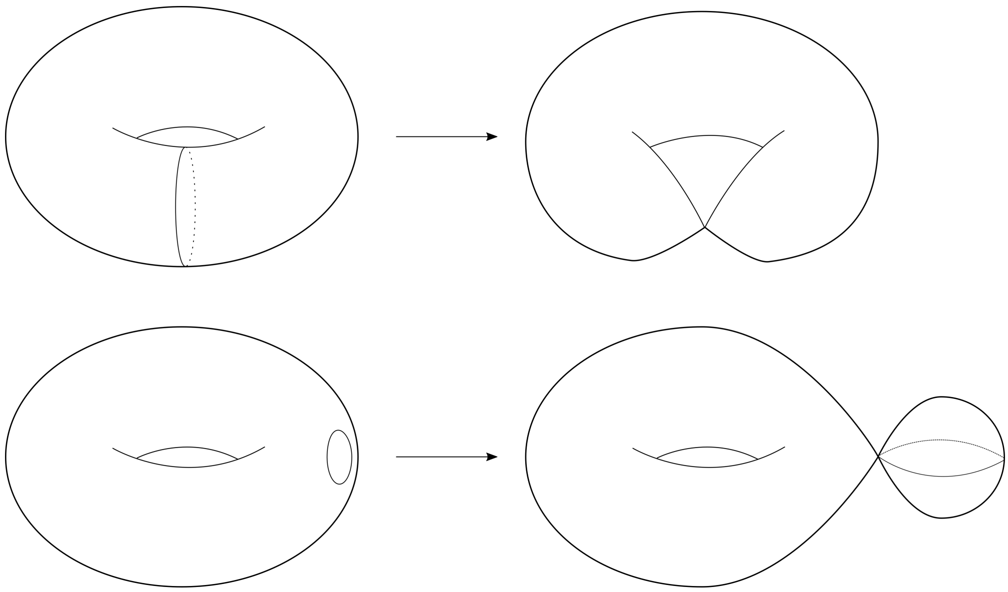


Figure 4.2: Two examples of vanishing cycles and their resulting singular fiber. The second case is an example of a separating cycle.

be relevant.

4.2 Monodromy

Before properly introducing the concept of monodromy, we first turn to the example of a helicoid, see Figure 4.3. This helicoid is a fiber bundle over $\mathbb{C} - \{0\}$ where the projection π is simply projecting down to the plane. The fiber F over some point x is isomorphic to $\mathbb{Z}$. Now suppose we have a path γ in $\mathbb{C} - \{0\}$ that starts and ends at x. Although the path in $\mathbb{C} - \{0\}$ is a loop, a lift of it to the helicoid need not be. In this case, it is a loop if γ does not encircle 0, but is no longer a loop if γ does. A way to describe this, is to associate a map to γ that goes from the fiber to itself. We call this $\Psi(\gamma)$ and it is given by sending $y \in \pi^{-1}(x)$ to $\eta(1) \in \pi^{-1}(x)$ where η is a lift of γ such that $\eta(0) = y$. Hence if $\Psi(\gamma)$ is not the identity, we know that lifts of γ cannot be loops. This map $\Psi(\gamma)$ is the monodromy associated to the path γ.

We can do the same for any fiber bundle, so the general notion of monodromy is as follows. Let $\pi \colon E \to B$ be a given fiber bundle with fiber F. Recall from Section 3.6 that we denote the mapping class group of F by $\mathcal{M}(F)$. Choose a base point x_0 in B and let φ be the identification of F with the fiber over x_0. Then we define the monodromy

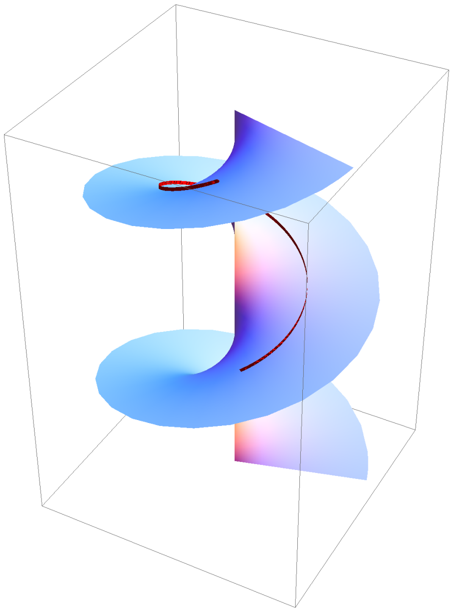


Figure 4.3: A helicoid with a lift of the path that goes around 0 once.

representation $\Psi\colon \pi_1(B) \to \mathcal{M}(F)$ as follows: For each loop $\gamma\colon I \to B$, the pull-back bundle $\pi_\gamma\colon \gamma^*(E) \to I$ is trivial (since its base space is contractible) and hence induces a diffeomorphism $\pi_\gamma^{-1}(0) \to \pi_\gamma^{-1}(1)$ that is well-defined up to isotopy.[3] We use φ to identify both fibers with F and hence obtain an element $\Psi(\gamma) \in \mathcal{M}(F)$. This is a well-defined anti-homomorphism from $\pi_1(B)$ to $\mathcal{M}(F)$, reversing the order of products instead of preserving them. But the only reason for this is that we denote concatenation of paths from left to right and concatenation of functions from right to left. This therefore has its origin in poorly compatible definitions and does not say anything significant about the underlying mathematical structure.

> **Definition 4.2.1.** Let $\pi\colon E \to B$ be a fiber bundle with fiber F. The monodromy representation of the fiber bundle is the homomorphism from $\pi_1(B) \to \mathcal{M}(F)$ as described above. We call the image of a class of paths $[\gamma] \in \pi_1(B)$ under this homomorphism the monodromy associated to γ.

For a Lefschetz fibration $\pi\colon X \to \Sigma$, we define the monodromy representation as the monodromy representation of the fiber bundle over Σ^*, the regular values of π. Remember that this is indeed a fiber bundle by Lemma 4.1.4.

[3] By varying the path, we might get a different diffeomorphism. But the varying of the path will lead to an isotopy between the two diffeomorphisms.

It is a fact [Mat96] that for π a relatively minimal Lefschetz fibration that is injective on its critical points, the monodromy representation $\Psi\colon \pi_1(\Sigma^*) \to \mathcal{M}_g$ determines π up to isomorphism, except in the cases of sphere and torus fiber bundles over closed surfaces. Examples that show that there is more than one possibility in these cases are given by $S^2 \times S^2 \to S^2$ and $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2 \to S^2$ for the sphere bundles and $S^2 \times T^2 \to S^2$ and $S^3 \times S^1 \to S^2$ for the torus bundles. We will encounter this last example again in Section 6.3.1.

We can characterize which monodromy representations can be realized by Lefschetz fibrations. Given a relatively minimal Lefschetz fibration $\pi\colon X \to \Sigma$ and a disc $D \subseteq \Sigma$, we can consider the monodromy of the bundle $\pi\big|_{\partial D}$, provided that this oriented circle avoids the critical values of π. If D does not contain any critical values, then the path ∂D is nullhomotopic in Σ^*, hence the monodromy over it is zero. If D however contains a single critical value p, the monodromy is non-trivial.[4][5] In this case, the monodromy representation of $\pi\big|_{\partial D}$ is fully defined by the monodromy of the path going around ∂D once in the counter clockwise direction. We will call this path σ_p. To describe what we get in these situations, we define Dehn twists.

> **Definition 4.2.2.** A right-handed *Dehn twist* $\psi\colon F \to F$ on a circle γ in an oriented surface F is a diffeomorphism obtained by identifying a tubular neighbourhood $\nu\gamma$ of γ with $S^1 \times I$, settting $\psi(\theta, t) = (\theta + 2\pi t, t)$ on $\nu\gamma$ and gluing smoothly into $\mathrm{id}_{F-\nu\gamma}$.

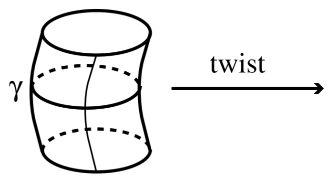


Figure 4.4: An illustration of a Dehn twist over the cycle γ.

Intuitively, this can be seen as cutting the surface F along γ, twisting 360° to the right and regluing. See Figure 4.4. The monodromy in a Lefschetz fibration around a vanishing cycle C, is the Dehn twist along C.

[4] Since the critical values form a discrete subset of Σ, it is always possible to pick D in such a way that it only contains one critical value.

[5] In a Lefschetz fibration that is not relatively minimal, the monodromy can still be trivial. Since it is the monodromy that is relevant for F-theory and hence our purposes, Lefschetz fibrations that are not relatively minimal are not relevant for us.

> **Theorem 4.2.3.** *Let $\pi\colon X \to \Sigma$ be a Lefschetz fibration with regular fiber F and let $p \in \Sigma$ be a critical value of π with vanishing cycle $C \subseteq F$. Then the monodromy over the path σ_p is given by a right-handed Dehn twist along C.*

The proof of this theorem will be based on the theory that is developed in the next chapter. One can skip it at first reading and always come back if interested.

Proof. We look at the disk D enclosed by σ_p in Σ. Over this disk, the Lefschetz fibration $\tilde{\pi}\colon \pi^{-1}(D) \to D$ has only a single singularity. The boundary of $\pi^{-1}(D)$ is what is important: this is the torus bundle over S^1 that can be described as $\dfrac{T^2 \times [0,1]}{(x,0) \sim (\psi(x),1)}$, where ψ is the monodromy. We can assume without loss of generality that the Kirby diagram of $\pi^{-1}(D)$ looks like Figure 5.23. The influence on the boundary of adding a handle is described by surgery on the attaching region of that handle. So we have to prove that surgery on $S^1 \times T^2$ on a -1-framed circle C that lies in a single copy of T^2, called F, gives us a Dehn twist as monodromy. Surgery in this case means the removal of a tubular neighbourhood of C, i.e. removing $\nu C \cong S^1 \times D^2$, and gluing a solid torus back in via a diffeomorphism φ of the boundary torus. The -1-framing specifies this diffeomorphism.

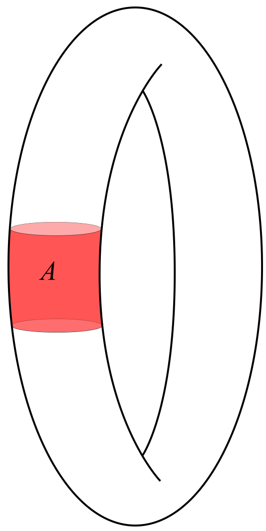


Figure 4.5: Illustration of the fiber F with A in it. The solid torus can be imagined on the inside of A.

Only the isotopy class of this diffeomorphism matters. We know that $\mathcal{M}_1 = \mathrm{SL}(2,\mathbb{Z})$. Now we have to find the element that corresponds to this -1-framing. Figuring out the map, we find that it is the element $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. Since we can isotope the solid torus without changing the diffeomorphism type of the result, we can make it so that it intersects F in a neighbourhood $A \cong C \times (-\epsilon, \epsilon)$. The diffeomorphism φ can be chosen to be the identity on the boundary torus except at A. On A it is a Dehn twist (as can be seen by looking at its effect on homology). Therefore, we have introduced a Dehn twist on the fiber that contains C. $\square$

The reason we only have right-handed Dehn twists is the condition of orientation preservation in the definition of a Lefschetz fibration.

4.2.1 Possible monodromies

We can calculate the possible monodromies around a singularity in a genus-1 Lefschetz fibration. The result will be the first strong indication of Lefschetz fibrations being useful in F-theory: it will achieve goal 2 from chapter 1 for us.

The torus T^2 has two basis elements for its first homology class, which we will call a and b. Since, as noted in Section 3.6, the result of the monodromy associated to some path γ on a and b fully describes the element of $\mathcal{M}_1 = \mathrm{SL}(2,\mathbb{Z})$ that is associated γ, it is enough to see what Dehn twists do to these basis elements.

To get a picture: let α, C be given 1-cycles in the torus. We consider effect on α of the Dehn twist along C. If α intersects C once, the Dehn twist will add the winding of C to α. If α did not intersect C, it would be left invariant. However, if α intersected C twice, the winding of C would also be added to α twice.

We can turn this reasoning into a formula and obtain the *Lefschetz-Picard formula.* It states that the Dehn twist ψ over the circle C sends an element $[\alpha] \in H_1(T^2)$ to

$$\psi_*([\alpha]) = [\alpha] - (\alpha \cdot C)[C]. \tag{4.1}$$

Here $[C]$ means the homology class of C and $\alpha \cdot C$ is the *intersection number* of the cycles C and α. The only thing that is important to know about the intersection number for this computation is that in this case $a \cdot b = 1$ and $b \cdot a = -1$.[6] We now use this formula on the most general vanishing cycle possible: $[C] = pa + qb$ for some $p, q \in \mathbb{Z}$ relatively prime. We show the effect of the Dehn twist on a. The computation for b is completely analogous.

$$\begin{aligned} \psi_*(a) &= a - (a \cdot (pa + qb))[pa + qb] \\ &= a - qpa - q^2 b \\ &= (1 - pq)a - q^2 b. \end{aligned}$$

The result is that the monodromy around a singularity with vanishing cycle $C = pa + qb$ is given by

$$\begin{pmatrix} 1 - pq & p^2 \\ -q^2 & 1 + pq \end{pmatrix} = M_{(p,q)} \tag{4.2}$$

[6]In general the intersection number is graded commutative, so it is anti-commutative for 1-cycles, commutative for 2-cycles, etc.

This is precisely the monodromy around a (p,q)-brane as obtained in (3.5). So in physical terms: for any (p,q)-brane, there is precisely one vanishing cycle (up to homotopy) such that a singularity with that vanishing cycle has the monodromy of a (p,q)-brane. So Lefschetz fibrations can distinguish between (p,q)-branes for different values of p and q.

4.3 Achiral Lefschetz fibrations

In the definition of a Lefschetz fibration, we assumed X and Σ to be orientable. In achiral Lefschetz fibrations, we drop this assumption.

Definition 4.3.1. Let X^{2n}, Σ^2 be compact manifolds. An *achiral Lefschetz fibration* on X is a smooth map $\pi\colon X \to \Sigma$ such that for any critical point x and its value y, there are complex coordinate charts on X and Σ centered at x and y in which π takes the form

$$\pi(z_1, \dots, z_n) = z_1^2 + \cdots + z_n^2.$$

Note that even when X and Σ are oriented, this definition is broader than the definition of a Lefschetz fibration, since the singularities do not have to respect the orientation. We will see the consequences of this for the topology of X in Section 5.3.3.

Note that while a singularity is fully determined by its vanishing cycle in the ordinary Lefschetz case, we can now have two different singularities corresponding to the same vanishing cycle, depending on the orientation of the complex coordinate charts chosen around it. We say that these singularities are of *opposite chirality.*

We will now prove the following:

Proposition 4.3.2. Let an achiral Lefschetz fibration be given. The monodromy around two singularities with the same vanishing cycle but of opposite chirality is the identity.

Proof. In the case of ordinary Lefschetz fibrations, the monodromy around a Lefschetz singularity is a right-handed Dehn twist on the vanishing cycle. However, looking at the proof of that statement (c.f. Theorem 4.2.3), we see that the right-handed Dehn twist arose because of the -1-framing on the added 2-handle. For a singularity of opposite chirality, the framing is $+1$. Therefore, an achiral Lefschetz fibration can also have a left-handed Dehn twist on the vanishing cycle. Saying that two singularities are of opposite chirality is equivalent to saying that the monodromy of one of the two singularities is a right-handed Dehn twist and the other a left-handed Dehn twist. Since these are performed on the same vanishing cycle, we conclude that the total monodromy is the identity map. □

Just for general interest, we include the following theorem:

Theorem 4.3.3 (Theorem 8.4.10 [GS99]). *Let $\pi: X \to S^2$ be a genus-1 achiral Lefschetz fibration. Suppose that X is simply connected and π has critical points of both orientations. Then X is diffeomorphic (possibly reversing orientation) to a connected sum of copies of either $\mathbb{C}P^2$ and $\overline{\mathbb{C}P}^2$ or $E(2)$ and $S^2 \times S^2$.*

Remark 4.3.4. The surface $E(2)$ is a K3 surface.

Remark 4.3.5. The requirement that X is simply connected sounds very strict, but in practice it will not be. Suppose a Lefschetz fibration contains two singularities; both with a basis element of $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ as their vanishing cycle, but not the same. Then the space X will be simply connected. There are a few ways of seeing this: one way would be to identify this configuration as a "cusp fiber". The other can be seen by using the techniques developed in the next chapter: the two 2-handles corresponding to the singularities can both cancel a 1-handle, hence leaving no 1-handles. Since 1-handles are the generators of $\pi_1(X)$, this means that X is simply connected.

4.4 Log-symplectic structures and b-manifolds

Log-symplectic structures are symplectic structures with a certain degeneracy; they are symplectic, except on a codimension 1 submanifold. The orientation induced by this log-symplectic form in some sense 'switches' when passing through its degenerate locus. We first give the definition and the local form. After that, we will set up a beautiful framework, which we will refer to as the 'b-language', to describe these structures in a more natural way.

We introduce these structures because there exists a link between achiral Lefschetz fibrations and log-symplectic structures that is similar to the link between Lefschetz fibrations and symplectic structures. We will treat this link in Section 4.5.

4.4.1 Log-symplectic structures

Log-symplectic structures are defined using Poisson structures, so we define them first.

Definition 4.4.1. A *Poisson structure* on an m-dimensional manifold M, is a bivector $\pi \in \mathfrak{X}^2(M) = \Gamma(\wedge^2 TM)$ such that

$$[\pi, \pi] = 0, \tag{4.3}$$

where $[\cdot, \cdot]$ is the *Schouten-Nijenhuis bracket.*

The Schouten-Nijenhuis bracket is an extension of the traditional Lie bracket to multivectors: $[\cdot, \cdot]\colon \mathfrak{X}^p(M) \times \mathfrak{X}^q(M) \to \mathfrak{X}^{p+q-1}(M)$.

When $m = 2n$, a generic bivector at a point p gives an isomorphism $\pi\colon T_p^*M \to T_pM$. This is the case if and only if $\pi^n = \pi \wedge \ldots \wedge \pi \neq 0$. If a Poisson structure is everywhere invertible (that is, it is an isomorphism between $T_p^*M \to T_pM$), its inverse gives us a symplectic structure. The closedness condition for symplectic structures is equivalent to equation (4.3). For more details, see [FM15].

The log-symplectic case is a slight generalization of the symplectic case, allowing for a degeneracy in the structure.

Definition 4.4.2. A *log-symplectic structure* on M^{2n} is a Poisson structure π for which the zeroes of π^n are non-degenerate.

The definition implies that the zeroes of π^n define a codimension 1 submanifold, which we denote by Z_π. This is called its *zero locus.* On the complement of the zero locus, the Poisson structure is invertible and therefore $M - Z_\pi$ admits a symplectic structure. The complement of the zero locus is therefore sometimes called the *symplectic locus.* We say that a *pair* (M, Z), i.e. a manifold M with a codimension 1 submanifold $Z \subseteq M$, admits a log-symplectic structure when there is such a structure on M such that $Z_\pi = Z$.

We will give a simple, but important example.

Example 4.4.3 (Standard log-symplectic structure)**.** As our manifold, we pick $M = \mathbb{R}^{2n}$ and we define the Poisson structure

$$\pi = x_1 \partial_{x_1} \wedge \partial_{x_2} + \partial_{x_3} \wedge \partial_{x_4} + \cdots + \partial_{x_{2n-1}} \wedge \partial_{x_{2n}}. \tag{4.4}$$

The zeroes of π^n are given by $Z_\pi = \{x_1 = 0\}$. Indeed they are non-degenerate and we can therefore speak of a log-symplectic structure. We can invert this away from the zero locus, to get a symplectic form there, given by

$$\omega = d\log|x_1| \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}. \tag{4.5}$$

Here, we used that $d\log|x_1| = \frac{dx_1}{x_1}$. We now see why it is called a log-symplectic structure. Note that ω is not defined on the singular locus, as should be the case since π is not invertible there. We can also see that the zero locus is not a symplectic submanifold, but it has 'symplectic leaves': for every value of λ, the set $\{(0, \lambda, x_3, \ldots, x_{2n})\} \cong \mathbb{R}^{2n-2}$ is a symplectic submanifold.

In everything that follows, one can always pick this example to see what is going on in practice. In some cases we will work it out ourselves in order to make a construction more explicit.

There is another important reason for considering this example. Just as one has a standard local form on symplectic manifolds by the Darboux theorem, the Weinstein splitting theorem [Wei83] implies that every log-symplectic manifold looks like this around a point in its zero locus. On the symplectic locus we can still use the Darboux theorem to obtain a local image of our structure. Since we now know the local structure of a log-symplectic manifold, we see that it is always the case that the zero locus has codimension 1 subspaces that are its symplectic leaves.

It is important to note that a log-symplectic structure does not induce an orientation, contrary to a symplectic structure.

4.4.2 b-manifolds

In this section, we will introduce more language to talk about log-symplectic structures and achiral Lefschetz fibrations. In this way, we can make more precise what kind of orientation a log-symplectic structure gives, since it cannot be an ordinary one. This framework also helps to develop an intuition for these structures. Our treatment follows

The idea will be that we 'formalize' the part of our Poisson structure that degenerates. So in Example 4.4.3, instead of seeing the vector $x_1\partial_{x_1}$ as zero on the locus $\{x_1 = 0\}$, we will consider it as being a vector like any other in something called the b-tangent bundle. We can then regard log-symplectic structures as ordinary symplectic structures on this modified tangent bundle.

The modified tangent bundle will be an object known as a Lie algebroid, so we will start by defining that.

Definition 4.4.4. Let M be a manifold. A *Lie algebroid* is a vector bundle $L \to M$ together with a bracket $[\cdot,\cdot]\colon \Gamma(L)\times\Gamma(L)\to\Gamma(L)$ on the space of sections such that $\Gamma(L)$ is a Lie algebra, and an *anchor map* $\rho_M\colon L \to TM$ that satisfies the following Leibniz rule for $v, w \in L$ and $f \in C^\infty(M)$,

$$[v, fw] = f[v,w] + (\rho_M(v)f)w.$$

Two examples Lie algebroids are Lie algebra's, where M is the one-point space, and the tangent bundle, where $L = TM$ and $\rho_M = \mathrm{id}$.

Given a pair (M^{2n}, Z) with M connected, we define $\mathcal{V}_b(M) \subseteq \Gamma(TM)$ as the vector fields on M that are tangent to Z. We note that $\mathcal{V}_b(M)$ is a Lie subalgebra, since the Lie bracket of two vectors that are tangent to Z is again tangent to Z[7]. If $Z = \{x_1 = 0\}$, these would be the vector fields spanned by $\langle x_1\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_{2n}}\rangle$. The Serre-Swan theorem now states that $\mathcal{V}_b(M)$ can be described as the sections of a vector bundle over M called ${}^bTM$.

Definition 4.4.5. Let (M, Z) be a pair. The *b-tangent bundle* ${}^bTM \to M$ is the vector bundle over M such that $\mathcal{V}_b(M) = \Gamma({}^bTM)$. A *b-manifold* is a pair (M, Z) with the bundle ${}^bTM$.

It may not be immediately clear, but when we spoke of 'formalizing' a certain vector at the start of this subsection, this was the step we referred to. Away from Z, ${}^bTM$ is isomorphic to TM, so it is a vector bundle of rank $2n$. Over Z, it is spanned by all vectors that are tangent to Z, of which there are $2n-1$, plus some extra vector. If $Z = \{x_1 = 0\}$, we will denote this formal vector by $x_1\partial_{x_1}$[8]. It should however be kept in mind that in ${}^bTM$, this is just a name for a basis vector and hence never zero, not even over Z.

This b-tangent bundle ${}^bTM$ is a Lie algebroid over M, with the anchor map $\rho_M\colon {}^bTM \to TM$ being the natural inclusion. This is an isomorphism away from Z, but on Z, it

[7] This can be shown by applying the vector to the function of which Z is a level set: if the vector applied to it is 0, the vector is tangent.

[8] More generally, we could write $\sigma(x)\partial_{x_1}$ for some smooth function σ that vanishes only on Z. But in order for this to be an element of the basis of ${}^bTM$, we should be able to build the vector field $x_1\partial_{x_1}$ from it. This forces σ to be transversal to Z or, equivalently, have a zero of first order at Z. Therefore, we may as well just write $\sigma(x) = x_1$.

'realizes' that this formal vector cannot possibly be tangent to Z and sends it to zero.

Note that we did not use log-symplectic structures to introduce this concept: a submanifold Z was all that we needed. But the connection with log-symplectic structures can already be seen: in this new tangent bundle, the log-symplectic structure $\pi = x_1\partial_{x_1} \wedge \partial_{x_2} + \cdots$ (c.f. equation (4.4)) now is invertible over Z_π, since its first term is not zero in $\bigwedge^2 {}^bTM$. Therefore it will give us a symplectic form on the b-tangent bundle, allowing us to use symplectic techniques to study what actually is only a log-symplectic structure. We make this more precise in Proposition 4.4.11, but first need to give some extra definitions.

> **Definition 4.4.6.** A *b-map* between two b-manifolds (M, Z_M) and (N, Z_N) is a map $f\colon M \to N$ such that $f^{-1}(Z_N) = Z_M$ and such that f is transverse to Z_N. That is, for $y \in Z_N$ and $x \in f^{-1}(y)$, we have $df_x(T_xM) \oplus T_yZ_N = T_yN$.

> **Definition 4.4.7.** A *b-orientation* on (M, Z) is an orientation for the bundle ${}^bTM$, i.e. a smooth choice of ordered basis.

Note that, away from Z, we get an orientation from such a b-orientation via the anchor map ρ_M. But on Z, one of the vectors in the basis of ${}^bTM$ gets projected to zero by ρ_M. Also, the formal vector gets a minus sign when crossing Z. So a choice of ordering for the basis $(x_1\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_{2n}})$ is after applying ρ_M never the same as an ordinary orientation. See Figure 4.6.

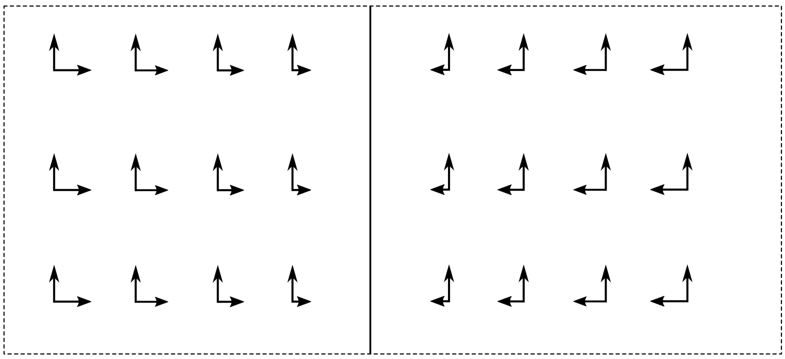


Figure 4.6: An image of a b-orientation after ρ_M has been applied to it. The line in the middle is the zero locus.

This leads to the fact that some manifolds that do not admit orientations, do admit b-orientations (see Example 4.4.8). But also the other way around: not every orientable manifold admits a b-orientation.[9]

Example 4.4.8 (b-orientation on the Moebius strip)**.** We construct a b-orientation on perhaps the most famous of the unorientable surfaces: the Moebius strip. We parametrize the circle lying in the strip by θ. As the zero locus Z, we take a line perpendicular to the circle that lies within the strip, so over the full width of the strip. Around this zero locus, we trivialize the Moebius strip and obtain a coordinate t that parametrizes the width on the strip and θ parametrizing a piece of the circle, such that $Z = \{\theta = 0\}$. Picking an orientation on one side of Z, the problem with orienting the Moebius strip is that when we go around, it comes back with the t-vector upside down. But when we pick the b-orientation that is given in local coordinates by $(\theta\partial_\theta, \partial_t)$, we see that also the θ-vector gets turned around when passing through Z (as in Figure 4.6), effectively giving us the same b-orientation. The clash at Z between orientations that we chose and orientations that went around the strip once, is no longer there. Therefore, the Moebius strip does admit a b-orientation.

Now that we have some intuition for the workings of b-manifolds, we can introduce more objects on them that are the b-counterparts of things we know from ordinary manifolds, starting with b-forms.

Definition 4.4.9. A b-k-form on a b-manifold (M, Z) is a section of the k-th exterior power of the b-cotangent bundle. That is, ${}^b\Omega^k(M) = \Gamma(\wedge^k({}^bTM)^*)$.

It is possible to define an exterior derivative ${}^bd$ on these forms, since everything we need for that is a Lie bracket and a notion of applying vectors to functions, which is given by the anchor ρ_M. And now that we have that, we can speak of closed b-forms and introduce one of the key objects of this part.

Definition 4.4.10. A *b-symplectic form* is a closed, non-degenerate b-two-form.

A b-symplectic form is the b-counterpart of a log-symplectic structure. We will make this more precise, but first go back to our elementary case, Example 4.4.3. In the b-tangent bundle, we have the formal element $x_1\partial_{x_1}$. In the dual of this bundle, the b-cotangent bundle, we have a dual element to this, which we for the moment call ξ. Then $\xi(x_1\partial_{x_1}) = 1$. Over the zero locus ξ will be a formal element, but away from the

[9] It should be noted that we have defined the b-tangent bundle only for when Z is a codimension 1 submanifold and hence explicitly not empty. One could also choose to leave the option of $Z = \emptyset$ open and therefore allow for the possibility of a b-orientation being an actual orientation. If this choice is made, the case $Z \neq 0$ would be called *bona-fide* b-structures.

zero locus, it gets identified with the actual dual vector to $x_1\partial_{x_1}$, which is $\frac{dx_1}{x_1}$. Therefore, we will write ξ as $\frac{dx_1}{x_1} = d\log|x_1|$. Again, we keep in the back of our heads that over Z, this should be not be considered as the actual expression for the covector.

Having the basis $(d\log|x_1|, dx_2, \dots, dx_{2n})$ for $({}^bTM)^*$, it is easy to write down a b-symplectic form: $\omega = d\log|x_1| \wedge dx_2 + dx_3 \wedge dx_4 + \dots dx_{2n-1} \wedge dx_{2n}$. This is precisely the inverse of the Poisson structure π on the b-tangent bundle.

This is no coincidence, as the following construction shows. Given a log-symplectic structure π on M, it gives a Poisson structure of maximal rank on ${}^bTM$. This means that it is invertible, and inverting it gives us a b-symplectic form.

Conversely, suppose we are given a b-symplectic form ω on (M, Z). Then we invert it to get a Poisson structure ${}^b\pi = \omega^{-1}$ on ${}^bTM$. Note that then ${}^b\pi \in \Gamma(\wedge^2({}^bTM))$. Therefore, $\pi_\omega := \rho_M({}^b\pi) \in \Gamma(\wedge^2 TM)$. This is a Poisson structure that vanishes linearly over Z, hence a log-symplectic structure on M. We obtain the following proposition. For more details, see [GMP14].

Proposition 4.4.11. A b-two-form on a b-manifold (M, Z) is b-symplectic if and only if its dual vector π_ω (constructed above) is log-symplectic with $Z_{\pi_\omega} = Z$.

Corollary 4.4.12. A manifold M has a log-symplectic structure with zero locus Z if and only if (M, Z) has a b-symplectic structure.

b-Lefschetz fibrations

We can also introduce the "b-analogue" of Lefschetz fibrations. We will temporarily use p instead of π to avoid confusion with the Poisson structure.

Definition 4.4.13. A b-Lefschetz fibration is a b-map $p\colon (X^{2n}, Z_X) \to (\Sigma^2, Z_\Sigma)$ between compact connected b-oriented manifolds so that for each critical point x, there exist complex coordinate charts compatible with orientations induced by the b-orientations centered at x and $p(x)$ in which p takes the form $p(z_1, \dots, z_n) = z_1^2 + \cdots z_n^2$.

The following proposition shows that a b-Lefschetz fibration gives rise to an achiral Lefschetz fibration (c.f. Definition 4.3.1).

Proposition 4.4.14. Let $p\colon X^{2n} \to \Sigma^2$ be an achiral Lefschetz fibration between compact connected manifolds which is injective on its critical points. Assume that the generic fiber F is orientable and $[F] \neq 0 \in H_{2n-2}(X;\mathbb{R})$. Then there exists a hypersurface $Z_\Sigma \subseteq \Sigma$ so that $p\colon (X^{2n}, Z_X) \to (\Sigma^2, Z_\Sigma)$ is a b-Lefschetz fibration. Here $Z_X = p^{-1}(Z_\Sigma)$.

Remark 4.4.15. Suppose we have an achiral Lefschetz fibration with a pair of singularities of opposite chirality x_1, x_2. If we apply this proposition to it and obtain a b-Lefschetz fibration, then x_1, x_2 must be in different connected components of $\Sigma - Z_\Sigma$. Because if they were not, the b-orientation on both these points would induce the same orientation and hence they are not of opposite chirality.

4.5 Symplectic structures

From the perspective of differential geometry, Lefschetz fibration suddenly became interesting when it turned out that they were likely to give a topological classification of symplectic manifolds.[10] Donaldson proved that every symplectic manifold admits a Lefschetz pencil, a structure which after a finite number of blow-ups is a Lefschetz fibration. Gompf then proved the following a year later.

Theorem 4.5.1 (Theorem 10.2.8 [GS99])**.** *Let X be a closed 4-manifold that admits a Lefschetz fibration $\pi\colon X \to \Sigma$, and let $[F]$ denote the homology class of the fiber. Then X admits a symplectic structure with symplectic fibers iff $[F] \neq 0 \in H_2(X;\mathbb{R})$. If $e_1 \dots e_n$ is a finite set of sections of the Lefschetz fibration, the symplectic form ω can be chosen in such a way that all these sections are symplectic.*

This sparked research into relations between variations on both structures. The following is the case for achiral Lefschetz fibrations:

Theorem 4.5.2. *Let X^4 and Σ^2 be compact connected manifolds and $\pi\colon X \to \Sigma$ be an achiral Lefschetz fibration with generic fiber F. If F is orientable and $[F] \neq 0 \in H_2(M;\mathbb{R})$, then X has a log-symplectic structure whose singular locus has one connected component and for which the fibers are symplectic submanifolds of the symplectic leaves of the Poisson structure.*

[10]For the definition of a symplectic manifold, see Appendix **??**.

Remark 4.5.3. The only case when the condition $[F] \neq 0$ can *possibly* fail to hold, is in the case of a genus-1 Lefschetz fibration. This has the following origin: $\ker(d\pi)$ defines a vector bundle over X without its critical points, which extends to X and can be made into a complex line bundle using the Hodge star as complex structure. The first Chern class of this bundle, coincides with the Euler class of the fiber when we restrict c_1 to this fiber. But the Euler class, when evaluated on the fundamental homology class of the fiber $[F]$, should give us the Euler characteristic of the fiber. Since the Euler characteristic of a genus g surface is $2-2g$, the only case in which this is zero, is $g=1$.

We will prove Theorem 4.5.1. After that, there are a few comments about the proof of Theorem 4.5.2.

Before we start, we will give an idea of the proof: To construct a symplectic form on X, we want to construct a form η that is symplectic along all the fibers, and has some controlled form around the critical points. We can then also pick a symplectic form ω_Σ on Σ, and pull that back and add them to make sure it is non-degenerate in every direction: $\omega := \pi^*\omega_\Sigma + t\eta$. For some small $t > 0$, this will be symplectic.

To construct η, we create a form ζ that is already symplectic along the fibers. We modify it with some exact form (so we don't change its cohomology class) in a way so we can control it around the critical points, giving us η. The modification is obtained by forming a symplectic structure ω_y on each fiber F_y separately, then extending them to a neighbourhood around the fiber, taking a finite subcover of these opens and gluing everything together with a partition of unity subordinate to the finite subcover.

We also note that $[F] \neq 0$ is necessary, since if we have some symplectic structure ω that restricts to a symplectic structure on F, we have $\langle [\omega], [F] \rangle > 0$, which cannot be the case for $[F] = 0$[11].

To see that the requirement $[F] \neq 0$ is also sufficient is the hard part. We assume that the fibers are connected and we perturb π slightly if necessary to make sure that it becomes injective on its set of critical points, i.e. contains no two critical points in the same fiber.

Lemma 4.5.4. There exists a closed 2-form ζ on X such that if F_0 is a closed surface contained in a fiber, then $\int_{F_0} \zeta > 0$.

[11] here $\langle \cdot, \cdot \rangle$ is the pairing between homology and cohomology, in this case given by integrating: $\int_F \omega$.

Proof. Let F be an arbitrary fiber. If it is regular, we see that it is an oriented surface that we have assumed to be connected. Therefore, the only closed surface contained in it, is F itself. Since $[F] \neq 0$, there exists an $a \in H^2_{\mathrm{dR}}(X)$ such that $\langle a, [F] \rangle > 0$[12].

Turning to the case that F is singular, we still have that it only contains itself as a closed surface, except for when we have a separating vanishing cycle. In this case, F is the wedge sum of two closed, oriented surfaces F_0 and F_1 (i.e. they kiss each other at one point). Again, we have $s = \langle a, [F] \rangle > 0$. If $\langle a, [F_0] \rangle =: r \leq 0$, we can redefine a to $a' = a + (-r + \frac{1}{2}s)\mathrm{PD}[F_1]$, where PD stands for the Poincaré dual.

Claim: We have $\langle a', [F_i] \rangle > 0$ for $i = 0, 1$ and $\langle a', [G] \rangle = \langle a, [G] \rangle$ for G any other closed surface contained in a fiber.

Proof. We denote $\alpha \in \mathrm{PD}[F_1]$, $\beta \in \mathrm{PD}[F_0]$ and $\iota_0 \colon F_0 \hookrightarrow X$. Then

$$\begin{aligned}
\langle a', [F_0] \rangle &= \langle a, [F_0] \rangle + (-r + \frac{1}{2}s)\langle \mathrm{PD}[F_1], [F_0] \rangle \\
&= r + (-r + \frac{1}{2}s)\langle \mathrm{PD}[F_1], [F_0] \rangle \\
&= r + (-r + \frac{1}{2}s)[F_0] \cdot [F_1] \\
&= \frac{1}{2}s > 0
\end{aligned}$$

That $[F_0] \cdot [F_1] = 1$ follows from the analysis of singular fibers above. To analyse $\langle a', [F_1] \rangle$, which will contain the self-intersection of $[F_1]$, we start by noting the following: any fiber (singular or not) can be moved around to not intersect itself, so $[G]^2 = 0$ (the square denotes the intersection product with itself) for all fibers G. So in this case

$$0 = [F]^2 = ([F_0] + [F_1])^2 = [F_0]^2 + 2[F_0] \cdot [F_1] + [F_1]^2 = [F_0]^2 + [F_1]^2 + 2$$

We have studied the vanishing cycle and seen that the local form of π around these critical points is symmetric and it does not care about which part is $[F_0]$ and which $[F_1]$. Therefore, the self-intersections of these two have to be the same, and we conclude $[F_1]^2 = -1$.

By the transversal intersection of $[F_0]$ and $[F_1]$, we have $\langle a, [F] \rangle = \langle a, [F_0] + [F_1] \rangle$ and

[12] note that the assumption $[F] \neq 0$ is in $H^2_{\mathrm{dR}}(X)$ and therefore forms in a are not only defined on the fiber, but on the whole of X

the same for a'. Therefore $\langle a, [F_1]\rangle = s - r$ and we can write

$$\langle a', [F_1]\rangle = \langle a, [F_1]\rangle + (-r + \frac{1}{2})[F_1]^2 = s - r - \frac{1}{2}s + r = \frac{1}{2}s$$

By adding the two, we see that $\langle a', [F]\rangle = s$, so it is unaffected. We also see why (closed surfaces in) other fibers are not effected, since they do not intersect with $[F_1]$. This proves the claim. □

We see that we only have to modify a for singular fibers, of which there are only finitely many. Therefore, after a finite number of modifications, we have a class $a \in H^2_{\mathrm{dR}}(X)$ such that $\langle a, [F_0]\rangle$ for each closed surface F_0 contained in a fiber. Picking some $\zeta \in a$ proves the lemma. □

Proof of Theorem 4.5.1. We denote $F_y := \pi^{-1}(y)$ the fiber over $y \in \Sigma$. We construct a 2-form ω_y on F_y. We do this by choosing disjoint open balls U_j in X around the critical points, such that on each U_j the projection π is given as $\pi(z_1, z_2) = z_1^2 + z_2^2$, which can be done by definition of a Lefschetz fibration. Now define $\omega_{U_j} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ on U_j, where $z_i = x_i + iy_i$ for $i = 1, 2$. Since $F_y \cap U_j$ is given by $\pi\big|_{U_j}^{-1}(z)$ for some $z \in \mathbb{C}$, it is a holomorphic curve and therefore $\omega_{U_j}\big|_{U_j \cap F_y}$ is a symplectic form. Since F_y is a 2-dimensional manifold and has an orientation, we can extend this form to a volume form (i.e. symplectic form). So we now have symplectic forms ω_y on F_y for all $y \in \pi(\bigcup_j U_j)$. We define ω_y on the remaining $y \in \Sigma$ such that ω_y is symplectic along F_y. By rescaling ω_y away from $\bigcup_j U_j$, we can assume that $[\omega_y] = [\zeta\big|_{F_y}]$ holds on $H^2_{\mathrm{dR}}(F_y)$ for each $y \in \Sigma$. This can be done since $[\zeta\big|_{F_y}]$ is defined by its action on homology classes, but the U_j do not contain whole 2-cycles.

We will now extend all the ω_y to closed 2-forms η_y on neighbourhoods of F_y. Take a neighbourhood $W_y \subseteq \Sigma$ of y containing no critical values except (possibly) y. Lifting this gives the neighbourhoods of the fibers $F_y \supseteq \tilde{W}_y = \pi^{-1}(W_y)$. When we make sure that the W_y are deformation retractable to y, we can deformation retract $\tilde{W}_y$ to F_y when y is not critical, and to $F_y \cup \mathrm{cl}(U_j)$ when y is critical. We call this deformation retract r. We can define the η_y as the pull-back $r^*(\omega_y)$ when y not critical and as $r^*(\omega_y \text{ or } \omega_{U_j})$ when y is critical. This weird notation in the last expression must be interpreted as follows: the form ω_y was chosen to extend ω_{U_j} to F_y, so it is the pull-back of a form that we defined in parts.

Now we choose a finite subcover of $\{W_y\}$ of Σ, which we denote by W_i for $i = 1, \ldots, n$. We fix a partition of unity $\{\rho_i\}$ subordinate to this finite cover $\{W_i\}$. As we have remarked before, by rescaling we can obtain that η_i and $\zeta\big|_{\tilde{W}_i}$ represent the same cohomology class on F_i. And since $H^2_{\mathrm{dR}}(\tilde{W}_i) = H^2_{\mathrm{dR}}(F_i)$ because they are related by a deformation retraction, we have that $[\eta_i - \zeta\big|_{\tilde{W}_i}] = 0 \in H^2_{\mathrm{dR}}(\tilde{W}_i)$. Therefore, $\eta_i - \zeta\big|_{\tilde{W}_i} = d\theta_i$ for some $\theta_i \in \Omega^1(\tilde{W}_i)$.

We are now ready to define the global 2-form η by patching everything together: $\eta = \zeta + d(\sum_i (\rho_i \circ \pi)\theta_i)$. Then $d\eta = d\zeta = 0$, so it is closed. Also

$$\eta\big|_{F_y} = \zeta\big|_{F_y} + \sum_i \rho_i(y) d\theta_i\big|_{F_y} = \zeta\big|_{F_y} + \sum_i \rho_i(y)(\eta_i\big|_{F_y} - \zeta\big|_{F_y}) = \sum_i \rho_i(y)\eta_i\big|_{F_y}$$

for every $y \in \Sigma$. Since all the η_i are compatibly[13] oriented area forms on F_y, they are symplectic along the fiber. Summarizing: η is a closed 2-form on X which is symplectic along the fibers.

We now introduce the symplectic form ω_Σ on Σ. Since Σ is orientable, we can pick such a symplectic form. It can even be made compatible with the complex structures at $\pi(U_j)$ for all j.[14] That is, $\omega_\Sigma(iv, iw) = \omega_\Sigma(v, w)$ and $\omega_\Sigma(v, iv) > 0$.

We can now define $\omega = t\eta + \pi^*\omega_\Sigma$, which we can prove is symplectic on the whole of X for t small enough. To start things off, it is obviously closed. So we only have to worry about non-degeneracy. We start checking the case $x \in X - \bigcup_j U_j$. Here x is certainly regular. Denote the fiber that it lies in by F_y. We split

$$T_x X = T_x F_y \oplus (T_x F_y)^{\perp_\eta}$$

where $(T_x F_y)^{\perp_\eta}$ denotes the symplectic complement[15] This splitting is allowed since η is symplectic along the fibers and hence this indeed spans the whole space. Because $T_x F_y \cap (T_x F_y)^{\perp_\eta} = \{0\}$, we have that $\pi^*\omega_\Sigma$ is non-degenerate on $(T_x F_y)^{\perp_\eta}$, since ω_Σ is non-degenerate and $ker d\pi = T_x F_y$. We now let $v + w, v' + w' \in T_x F_y \oplus (T_x F_y)^{\perp_\eta}$ be given. Then

$$\omega_t(v + w, v' + w') = t\eta(v, v') + t\eta(w, w') + \pi^*\omega_\Sigma(w, w'),$$

[13] all fibers got a canonical orientation from the Lefschetz fibration, and every η_i was defined with respect to this orientation

[14] It is a general fact that for any almost-complex structure, there is a compatible symplectic form. The fact that we can make it compatible with the structure on $\pi(U_j)$ for each j can be proven by picking a partition of unity.

[15] The symplectic complement $V^{\perp_\omega}$ of $V \subseteq W$ is defined as $V^{\perp_\omega} = \{w \in W \mid \omega(v, w) = 0 \text{ for all } v \in V\}$.

since $\eta(v, w') = \eta(w, v') = 0$ and $d\pi(v) = d\pi(v') = 0$. Because of the non-degeneracy of η on T_xF_y and that of ω_Σ on $(T_xF_y)^{\perp_\eta}$, we can pick v', w' such that $\eta(v, v') > 0$ and $\pi^*\omega_\Sigma(w, w') > 0$. Therefore, for $t > 0$ small enough, we have

$$\omega_t(v + w, v' + w') = t\eta(v, v') + t\eta(w, w') + \pi^*\omega_\Sigma(w, w') > 0.$$

And hence ω_t is non-degenerate on T_xF_y. Since $X - \bigcup_j U_j$ is compact and ω_t varies smoothly over X, we can choose a t such that this holds for every $x \in X - \bigcup_j U_j$.

Now we treat the case that $x \in U_j$ for some j. On U_j, we constructed η in such a way that $\eta\big|_{U_j} = \omega_{U_j} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Also π has its standard form $\pi(z_1, z_2) = z_1^2 + z_2^2$. Then for any vector $v = (v_{x_1}, v_{y_1}, v_{x_2}, v_{y_2}) \in T_xX$, we have

$$\begin{aligned}\omega_t(v, iv) &= t\eta(v, iv) + \pi^*\omega_\Sigma(v, iv) \\ &= t\left(v_{x_1}v_{x_1} - (-v_{y_1}v_{y_1}) + v_{x_2}v_{x_2} - (-v_{y_2}v_{y_2})\right) + \omega_\Sigma(d\pi(v), id\pi(v)) \\ &= t\|v\|^2 + \omega_\Sigma(d\pi(v), id\pi(v)) > 0\end{aligned}$$

Here we used that $d\pi(iv) = id\pi(v)$ because π is a holomorphic map. The fact that $\omega_\Sigma(d\pi(v), id\pi(v)) > 0$ comes from the compatibility of ω_Σ with the complex structure. This proves that ω_t is non-degenerate on $\bigcup_j U_j$.

To prove the statement about symplectic sections, we note that $\pi^*\omega_\Sigma$ is symplectic along the sections – sections are transverse to the fibers – and we can choose t even smaller to make sure that this term dominates in ω_t. This proves the theorem. □

The proof of Theorem 4.5.2 can be done analogous to the proof of Theorem 4.5.1, as done in [Cav13]. There is also another way, pursued in [CK16]. Here it is shown that b-Lefschetz fibrations give rise to b-symplectic structures, which are equivalent to log-symplectic structures by Proposition 4.4.11. By Proposition 4.4.14, an achiral Lefschetz fibration gives us a b-Lefschetz fibration. Ergo an achiral Lefschetz fibration gives a log-symplectic structure.

There is of course some logic behind the correspondence found in Theorem 4.5.2: log-symplectic structures and achiral Lefschetz fibrations both do not induce orientations. And, when taking a step to the b-language, they both induce a codimension 1 space $Z \subseteq X$ that is a zero locus.

Chapter 5

Introduction to 4-manifold theory

In the previous chapter we introduced Lefschetz fibrations: a certain type of maps $\pi\colon X^{2n} \to \Sigma^2$. As we will argue in Chapter 6, these are viable candidates to replace elliptic fibrations in F-theory, but we will need to work with $n = 2$. This means we will have to deal with 4-dimensional manifolds.

Our first goal will be the development of Kirby diagrams and their application to Lefschetz fibrations. Kirby diagrams are a way of drawing 4-dimensional manifolds and using them, we will be able to make drawings of Lefschetz fibrations. This gives us greater insight in what influence the existence of singularities in the fibration has on the topology of X.

The second goal of this chapter is the introduction of surgery. This ~~is an~~ operation creates a new manifold out of a given manifold. This will allow us to create new singularities in a given Lefschetz fibration. Kirby diagrams can then be used to figure out what the newly obtained manifold is.

A much more complete guide to this subject is [GS99]. We will treat parts of chapter 4, 5 and 8 here, introducing the preliminaries only if they are strictly needed. We will skip most of the proofs and focus on developing the machinery and intuition. However, because it is hard to judge what might become relevant in further research, we do aim to build an understanding and not just a list of results.

The theory we will develop in this chapter is not needed if the reader is willing to accept the results that follow from it that are used in Chapter 6. It will not severely impair one's understanding of what is written there.

5.1 Handles and handlebodies

5.1.1 Basic definitions and classifying framings

The idea of handles is to build a smooth manifold in a step by step manner, starting from a (possible empty) manifold. The readers that are familiar to CW-complexes will quickly see the analogy.[1] We start by introducing handles. Note that D^n is the n-dimensional disk with boundary.

> **Definition 5.1.1.** An n-dimensional k-handle h is a copy of $D^k \times D^{n-k}$, attached to the boundary of a manifold X along $\partial D^k \times D^{n-k}$ by an embedding $\varphi\colon \partial D^k \times D^{n-k} \to \partial X$.

Figure 5.1: A kettlebell: where other people think about lifting weights when seeing this, a geometer would think of handles.

Technically, a manifold with a handle added to it is not smooth, since it has a corner. However, there is a canonical way to smooth these and we will therefore not care about that fact.

There is some terminology regarding handles that we should introduce: The number k is called the *index* of the handle. The set $D^k \times \{0\}$ is called the *core* of the handle, $\{0\} \times D^{n-k}$ the *cocore*, φ the *attaching map*, the set $\partial D^k \times D^{n-k}$ (or often its image under φ) the *attaching region*. This is the thickened version of the *attaching sphere* $\partial D^k \times \{0\}$. Lastly, the sphere $\{0\} \times \partial D^{n-k}$ is called the *belt sphere*.

[1]The biggest differences between CW-cells and handles come from the fact that all the cells are "thickened" to all be of the same dimension and the fact that attaching new handles requires a little more information because of this and because we are in the smooth category.

Example 5.1.2. A simple example of 3-dimensional handles is the kettlebell, see Figure 5.1 (note: we think of it as filled, hence 3-dimensional). A 0-handle has attaching region $\partial D^0 \times D^n = \emptyset$, and hence adding a 0-handle is just taking a disjoint union with D^n. So for building the kettlebell, we start with the empty set and add a 0-handle. A 1-handle has attaching sphere $S^0 = \{\pm 1\}$, so it can be attached at two disks on the sphere. The core of this 1-handle is the handle without its thickness like if we would have attached a piece of string instead of metal/plastic. The cocore is the slice in the exact middle of the handle. The belt sphere is the boundary of this slice, say where you would pick it up using only your thumb and index finger.

We will generally be interested in the diffeomorphism type of $X \cup_\varphi h$, hence it suffices to specify φ up to isotopy (c.f. Definition 3.6.1): by the isotopy extension theorem, an isotopy between φ and φ' would lead to a diffeomorphism between $X \cup_\varphi h$ and $X \cup_{\varphi'} h$. Also, by the tubular neighbourhood theorem, $\varphi\colon \partial D^k \times D^{n-k} \to \partial X$ can be determined up to isotopy from an embedding $\varphi_0\colon \partial D^k \times \{0\} \to \partial X$ together with an identification f of the normal bundle of $\mathrm{Im}\varphi_0$ with $\partial D^k \times \mathbb{R}^{n-k}$. So $X \cup_\varphi h$ is determined by:

1. An embedding $\varphi_0\colon S^{k-1} \to \partial X$ with trivial normal bundle.
2. A *framing* f of $\varphi_0(S^{k-1})$ that identifies the normal bundle of $\varphi_0(S^{k-1})$ with $S^{k-1} \times \mathbb{R}^{n-k}$.

and the diffeomorphism type of $X \cup_\varphi h$ is determined by the isotopy class of (φ_0, f).[2]

The part of this that is hardest to imagine is the framing. We can classify these framings via homotopy groups: pick a certain framing f_0 and let another framing f be given. When comparing f to f_0, we get an element of $\mathrm{GL}(n-k)$ at each point in S^{k-1}. By composing with a self-diffeomorphism of D^{n-k}, we can make sure that this element of $\mathrm{GL}(n-k)$ is the identity at a certain point $p \in S^{k-1}$. Note that this is the same as a map $S^{k-1} \to \mathrm{GL}(n-k)$ that is the identity at p. Therefore it represents an element in $\pi_{k-1}(\mathrm{GL}(n-k)) = \pi_{k-1}(O(n-k))$.[3] If two of these maps are homotopic, we can get an isotopy of the framing. Conversely, if two framings are isotopic, they will induce homotopic maps. Hence we have a well-defined injective map from isotopy classes of framings of S^{k-1} to $\pi_{k-1}(O(n-k))$. To see that it is also surjective, note that we can

[2]This pair is isotopic to another pair (φ_0', f') if there exist isotopies between φ_0 and φ_0' and between f and f' such that at every point along the way they are a combination of an embedding and a normal framing of that embedding.

[3]$\pi_{k-1}(\mathrm{GL}(n-k))$ denotes the $(k-1)$ homotopy group of the space $\mathrm{GL}(n-k)$. Since this space is homotopy equivalent to $O(n-k)$ via the Gram-Schmidt procedure, the groups are isomorphic. For a small treatment of these groups, see A.4, although one can easily continue by accepting the claims we make as facts.

pick a representing element of a homotopy class and obtain a framing f by applying this representing element to f_0. It is important to note that although we have a bijection, it depends on our choice of reference frame f_0. But we at least know what possible framings we can have and we will use this to treat the cases that we will use later on:

- A 1-handle is attached via two disjoint disks. So for ∂X connected, compact and non-empty, there is a unique isotopy class of embeddings $\varphi\colon S^0 \times \{0\} \to \partial X$. We have $\pi_0(O(n-1)) = \mathbb{Z}_2$ for $n \geq 2$ and so there are two possible framings. If X was orientable, then the possible framings are distinguished by whether $X \cup_\varphi h$ is orientable or not. In particular, a 0-handle is orientable (since it is just D^n) and in $n = 2$, we would have that adding a 1-handle to a 0-handle is either an annulus or a Moebius strip.
- In the case of 2-handles, we have $\pi_1(O(n-2)) = \mathbb{Z}_2$ for $n > 4$ and $\pi_1(O(n-2)) = \mathbb{Z}$ for $n = 4$. We will encounter this 'framing number' in the 4-dimensional case often and one can visualize it as how many times the 2-handle twists around its attaching circle S^1. We will see this in Section 5.2.2.
- For $(n-1)$-handles with $n \neq 2$, there is a unique framing since $\pi_{n-2}(O(1)) = 0$ except for $\pi_0(O(1)) = \mathbb{Z}_2$.
- For n-handles, the same reasoning holds and hence there is also a unique framing. There is another important thing to mention about this case: an n-handle can only be attached to a manifold X with a boundary component diffeomorphic to S^{n-1}. For $n \leq 4$, it is known that any self-diffeomorphism of S^{n-1} is isotopic to either the identity or a reflection. Therefore, there is a unique way to attach an n-handle to an S^{n-1} boundary component.

Note that we now indeed have treated all the relevant cases for 4-manifolds: 0, 1, 2, 3 and 4-handles.

5.1.2 Handle decompositions

We can not only build new manifolds using these techniques, but also analyse given ones by looking whether they can be build using handles.

> **Definition 5.1.3** (Definition 4.2.1 [GS99]). Let X be a compact n-manifold with boundary ∂X decomposed as a disjoint union $\partial_+ X \sqcup \partial_- X$ of two compact sub-

manifolds (either of which may be empty). If X is oriented, orient $\partial_\pm X$ such that $\partial X = \partial_+ X \sqcup \overline{\partial_- X}$ in the boundary orientation. A *handle decomposition* of X (relative to $\partial_- X$) is an identification of X with a manifold obtained from $I \times \partial_- X$ by attaching handles, such that $\partial_- X$ corresponds to $\{0\} \times \partial_- X$ in the obvious way. X is called a *relative handlebody* built on $\partial_- X$, or a *handlebody* if $\partial_- X = \emptyset$. If X is a handlebody, we denote by X_m the handlebody where only the k-handles for $k \leq m$ have been attached.

Example 5.1.4. The sphere S^n is a handlebody. Starting with a 0-handle D^n, we add the n-handle via the obvious attaching circle $S^{n-1} \subseteq D^n$. This effectively builds the sphere by gluing the northern hemisphere on the southern (or the other way around).

Example 5.1.5. The torus is a handlebody obtained from starting with a single 0-handle, attaching two 1-handles and then a 2-handle (in the same way as the CW-structure of the torus). We can draw it in the square with sides identified. See Figure 5.2.

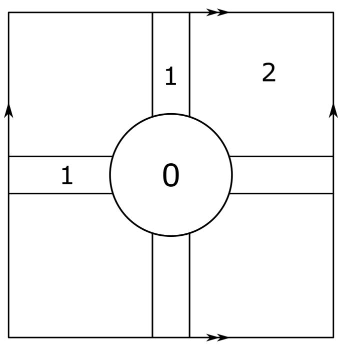


Figure 5.2: A handle decomposition of the torus.

The following proposition shows that we can always add handles without paying much attention to the order in which we do it.

Proposition 5.1.6 (Proposition 4.2.7 [GS99])**.** Any handle decomposition of a compact pair $(X, \partial_- X)$ can be modified (by isotoping attaching maps) so that the handles are attached in order of increasing index. Handles of the same index can be attached in any order or simultaneously.

We can also *dualize* the handle decomposition of $(X, \partial_- X)$ as follows: instead of starting with $I \times \partial_- X$, we start with $I \times \overline{\partial_+ X}$. We glue 1-handles on this by reversing the roles of the core and the cocore (so using $D^k \times D^{n-k} = D^{n-k} \times D^k$) of what were previously $n-1$-handles. Continuing this process gives us the same manifold X, but with a dual description of its handlebody decomposition.

5.1.3 Handle moves

Handle moves are operations we can perform on a our handlebody decomposition that do not change the diffeomorphism type of the result. There are two moves: cancellation and sliding.

We have seen in Example 5.1.4 that the sphere can be build using two handles. However, we could also build S^2 for example by starting with a 0-handle and attaching a 1-handle. This gives us $S^1 \times I$. We can glue two 2-handles on this: one on top, one on the bottom. This gives us the same sphere. So somewhere there are unnecessary handles. How can we see which ones we can ignore?

A *cancelling pair of handles* is a pair consisting of a $k-1$ and a k-handle. Intuitively, it happens when the k-handle is attached to the manifold and the $k-1$-handle in such a way that they together form a blob on the manifold such that the result is just diffeomorphic to the original manifold, see Figure 5.3. Easy to visualize examples are $k=1$ and $k=2$. When $k=1$, we have a 0-handle (a disjoint ball) and a 1-handle that connects the manifold to the disjoint ball. This gives us a weird spike on the manifold, but nothing that a diffeomorphism cannot bring back to the manifold we started with.

For $k=2$ we have a 1-handle like on the kettlebell in Example 5.1.2 and add to it a 2-handle with the attaching circle running once over the 1-handle. This means we fill up the hole that we would usually put our fingers through. It is again just a ball, which is the same as the 0-handle without 1 or 2-handles attached.

The following proposition makes precise when we can cancel handles. The reader should try to see why this holds for the $k = 1, 2$ cases.

> **Proposition 5.1.7** (Proposition 4.2.9 [GS99]). A $(k-1)$-handle h_{k-1} and a k-handle h_k can be cancelled, provided that the attaching sphere of h_k intersects the belt sphere of h_{k-1} transversely in a single point.

Another important move is the following:

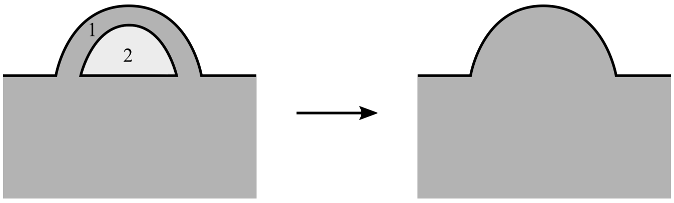


Figure 5.3: An illustration of cancelling 1- and 2-handles in the 2-dimensional case. Notice how the attachment circle of the 2-handle intersects the belt sphere of the 1-handle in exactly one point.

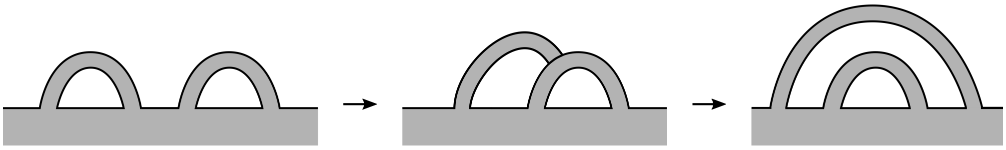


Figure 5.4: A picture of a 1-handle sliding over another 1-handle.

> **Definition 5.1.8** (Definition 4.2.10 [GS99])**.** Given two k-handles h_1 and h_2 ($0 < k < n$) attached to ∂X, a *handle slide* of h_1 over h_2 is performed by isotoping the attaching sphere A of h_1 in $\partial X \cup h_2$), pushing it through the belt sphere B of h_2.

Note that by Proposition 5.1.6 these two manifolds should not be different. After all, we could have attached the handles simultaneously, robbing us of the possibility of sliding one handle over the other. This may seem nearly trivial, but using only one handle slide is enough to prove that the Klein bottle is diffeomorphic to $\mathbb{RP}^2 \# \mathbb{RP}^2$.

One can prove that handle cancellation and handle sliding form a complete set of moves:

> **Theorem 5.1.9** (Theorem 4.2.12 [GS99])**.** *Given any two handle decompositions for a compact manifold X, it is possible to get from one to the other by a sequence of handle slides, creating/annihilating handle pairs and isotopies within the same level.*

> **Proposition 5.1.10** (Proposition 4.2.13 [GS99])**.** If X^n is a compact and connected manifold, then $(X, \partial_- X)$ admits a handle decomposition with exactly one 0-handle (if $\partial_- X = \emptyset$) or no 0-handles (if $\partial_- X \neq \emptyset$). We can also assume that there is exactly one n-handle (if $\partial_+ X = \emptyset$) or no n-handles (if $\partial_+ X \neq \emptyset$).

5.2 Diagrams

5.2.1 Heegaard diagrams

We will develop the machinery of Heegaard diagrams. Not that we will ever use it, but it is a simple introduction into the logic of Kirby diagrams, that we will develop in the next section and use later on.

Heegaard diagrams are a way of drawing 3-dimensional handlebodies. We start by assuming compactness and $\partial_- X = \emptyset$, so we have a unique 0-handle. This handle is a D^3 and we can attach other handles to its boundary S^2. When we consider S^2 as $\mathbb{R}^2 \cup \{\infty\}$, we can make a drawing of the attaching regions of the other handles.

We start with the 1-handles: these are attached via pairs of disks. We have to choose whether or not we attach them in a way such that the resulting manifold is orientable. We choose to make it orientable (we will not go into the non-orientable case, but one can easily develop the same kind of drawing for that case). The two discs hence have to be identified in an orientable way, which we can pick to be the reflection $(x, y) \mapsto (-x, y)$ without loss of generality. So the attachment of a 1-handle to D^3 (which is again the kettlebell of Example 5.1.2) is drawn in Figure 5.5

Figure 5.5: Heegaard diagram of the kettlebell: a 0-handle with a 1-handle attached to it.

The 2-handles have attaching circle S^1. We have seen that in this case (they are $(n-1)$-handles) there is a unique framing, hence we know precisely how to attach the 2-handle when we know the attaching sphere. Therefore we can draw the attachment circles in the diagram to signal the attachment of a 2-handle. But 2-handles can also be attached over a 1-handle, so the attachment circles will not necessarily be circles in the diagram: they are arcs with endpoints on the boundaries of the attaching regions of the 1-handles. When we identify the two boundaries via the reflection, the arcs will become circles. Hence the attachment of two 2-handles to a kettlebell could look like Figure 5.6.

If $\partial_+ X = \emptyset$, we should also add a 3-handle. However, as noted before, there is a unique way to do this, so often we do not bother with the 3-handle. If we want to add a 3-handle, we write "+ 3-handle" next to the diagram.

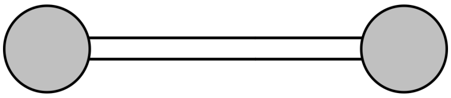


Figure 5.6: Heegaard diagram of the kettlebell with two 2-handles attached. Both run over the 1-handle.

When there is more than one 1-handle, it should be clear from the diagram which disks together form a 1-handle. They are not necessarily all mirrored in the same axis. We could for example also have the diagram in Figure 5.7.

These diagrams are useful to indicate the diffeomorphism type of our manifold. Since for the diffeomorphism type it is irrelevant where the 1 and 2-handles are precisely located, we can always freely move the disks and lines. As long as we do so in a continuous way, the result will be a diffeomorphic manifold.

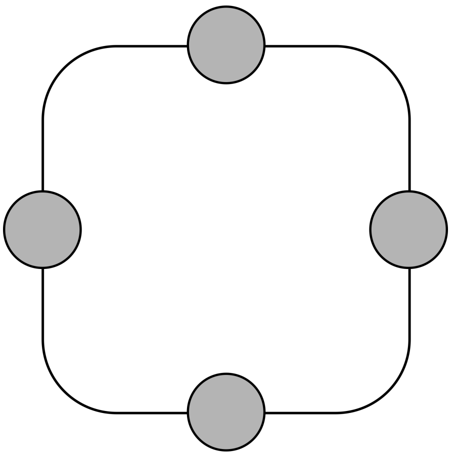


Figure 5.7: Heegaard diagram of $T^2 \times D^1$. The top and bottom disk form a 1-handle and the left and right disk form a 1-handle. The attachment circle of the 2-handle runs as follows: starting from the top right piece and going clockwise, we stop on the disk on the right, go over the 1-handle to the left, come out of the top of the left disk. Then we go to the left of the top disk, over the other 1-handle and come out of the left of the bottom disk. We then go to the bottom of the left disk, again over the first 1-handle, come out of the bottom of the right disk, walk to the right of the bottom disk and again go over the other 1-handle. Then we are back where we started. Compare this to Figure 5.2.

Handle moves

It is also good to see what cancelling handles and handle slides look like in Heegaard diagrams. In a Heegaard diagram we already assume that there is only one 0-handle, hence cancelling 0 and 1-handles will not occur. We can however have cancelling 1 and 2-handles. We already noted in Section 5.1.3 how a 2-handle should be added to a kettlebell to again make it into a D^3, and otherwise we could also use Proposition 5.1.7. The Heegaard diagram can be found in Figure 5.8

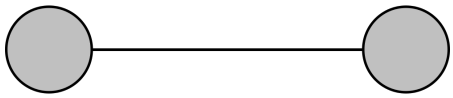


Figure 5.8: Heegaard diagram of a pair of cancelling 1 and 2-handles: note that since the attachment circle of the 2-handle only goes over the 1-handle once, it intersects the belt sphere of the 1-handle in precisely one point.

If there are other ~~1-handles attached~~ to this 1-handle, we simply connect the lines going in and out. This makes sense, since these lines are actually running over the 1-handle, but together with the 2-handle, this is merely a pimple on our manifold. So the lines actually just run over S^2. For an example, see Figure 5.9

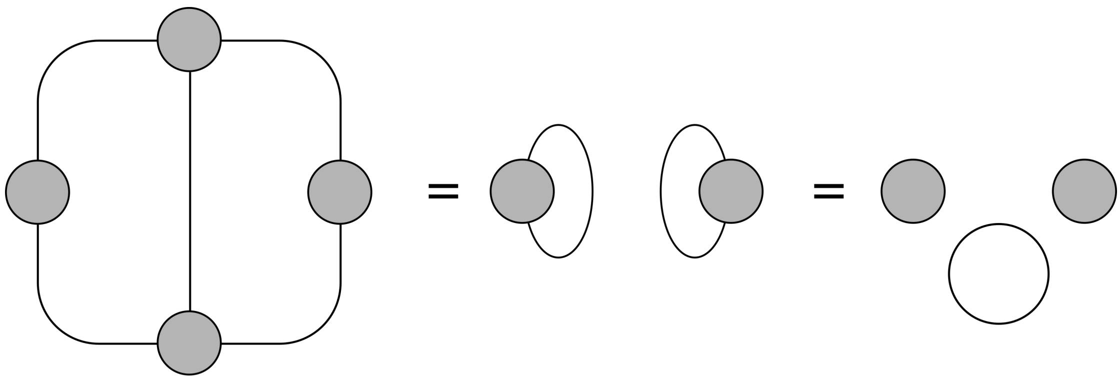


Figure 5.9: Heegaard diagram with a cancelling 1 and 2-handle: the 1-handle and arc from top to bottom. The first equality follows from the cancelling of the handles. The second equality is gained by just moving the attaching circle of the 2-handle off the 1-handle.

We can also have cancelling 2 and 3-handles. This happens when we have an isolated circle in our diagram and have actually added a 3-handle to the diagram. One can prove that, using the other operations, the situation of cancelling 2 and 3-handles can always be brought to this case. See Figure 5.10.

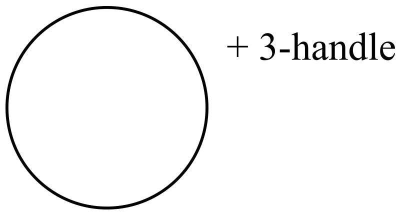


Figure 5.10: Heegaard diagram of a pair of cancelling 2 and 3-handles.

To see how 1-handle slides work, we note that it merely changes the attachment circles of the 2-handles. If they only run over 1-handle h_1 and then we slide h_1 over another 1-handle h_2, the attachment circle will now run over h_1 and h_2. In diagram form, we use the arrow to indicate how we slide which handle over which. See Figure 5.11.

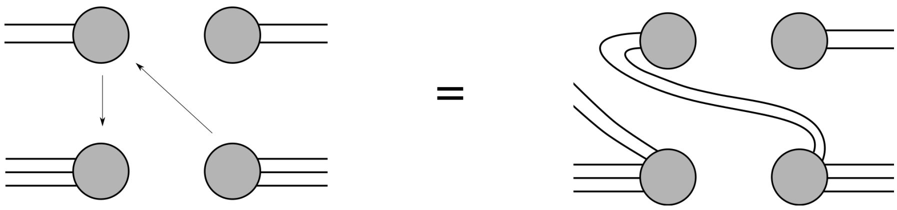


Figure 5.11: Heegaard diagram of before and after performing a 1-handle slide. Just as the arrows indicate, the upper 1-handle is slid from left to right over the lower 1-handle.

We can also slide 2-handles. Before we do this, we make the following definition.

Definition 5.2.1. Let h be a 2-handle. Then its attaching circle is given by the image of $\partial D^2 \times \{0\}$. We define a *parallel curve* or *parallel knot* to it as the image of $\partial D^2 \times \{x\}$ for some $x \in D^{n-2} - \{0\}$.

The parallel curve is a visual representation of the framing on the 2-handle when $n = 3, 4$. Since in 3 dimensions the framing of a 2-handle is unique, it is rather useless. But we use it to slide 2-handles in our diagrams and already using it will make the extension to the 4-dimensional case easy.

Suppose we have two 2-handles h_1 and h_2. To slide h_1 over h_2 in the diagram, we draw the parallel curve K' to the attaching circle of h_2. Now we take the connected sum of the attaching circle of h_1 and K'.[4] Here is why this works: the attaching sphere of a 2-handle is $\partial D^2 \times \{0\}$, i.e. it is the boundary of the core of the handle. Inside h_2, we can pick a disk that is parallel to its core, so $D^2 \times \{x\}$ for some $x \in \partial D^1$. The parallel

[4]If the reader is not familiar with the connected sum: in this case it is just removing a small piece from both curves and connecting the leftovers in the simplest way.

curve K' is precisely $\partial D^2 \times \{x\}$. By definition, a handle slide is when we isotope the attaching sphere of h_1 through the belt sphere of h_2. So we isotope the attaching sphere of h_1 over the disk bound by the parallel curve and let it land on the 0-handle again. The result is precisely the described connected sum.

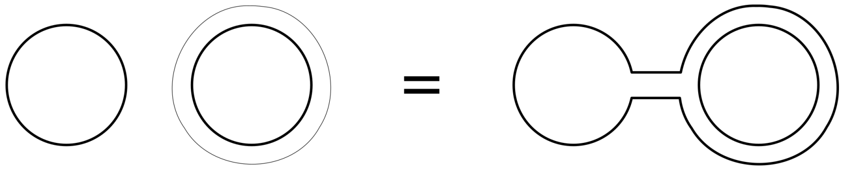


Figure 5.12: Heegaard diagram of the 2-handle slide. On the left hand side, we have drawn the parallel curve K' with the thin line. On the right hand side, we have actually performed the 2-handle slide.

Example 5.2.2. To make a Heegaard diagram of D^3 is of course easy: it is the empty diagram. But we can see in Figure 5.13 also a diagram for the ball. There are two pairs of cancelling handles in the diagram. Note that both the attaching circles of both 2-handles intersect the belt-sphere of the 1-handle in one point, allowing one of them to cancel against the 1-handle. This leaves us with one 2-handle, which together with the 0-handle makes a thickened S^2, and a 3-handle that can only be added by filling up this sphere. This again satisfies the conditions of Proposition 5.1.7 and hence the 2- and 3-handle cancel.

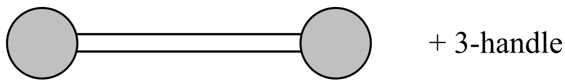


Figure 5.13: Heegaard diagram of D^3, the 1-handle cancels agains one of the 2-handles, which leaves a 2-handle and a 3-handle that also cancel.

5.2.2 Kirby diagrams

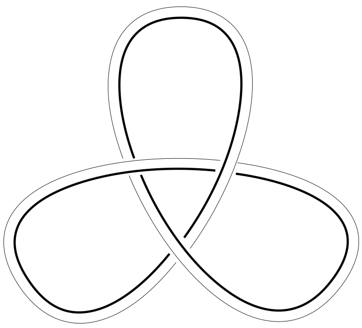
Now we are ready to repeat the same trick, only for 4-dimensional handlebodies. Again we assume compactness and $\partial_- X = \emptyset$ such that there is a unique 0-handle. This time, the 0-handle is D^4 and hence its boundary is $S^3 = \mathbb{R}^3 \cup \{\infty\}$. So we draw the attaching regions of the remaining handles in $\mathbb{R}^3$. The attaching region of a 1-handle is $D^3 \sqcup D^3$, so this time we draw it as two balls and mirror it in a plane. See Figure 5.14.

As before, the 2-handle is attached via a circle. Again, these are arcs that can end on the balls of 1-handles. But this time, the arcs are in $\mathbb{R}^3$ and hence can be knotted and

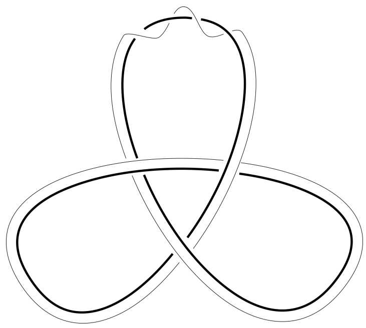
Figure 5.14: Kirby diagram of a 1-handle added to the 0-handle D^4.

linked. Also, we have to deal with framings. As we have said before, the framings for 2-handles in the 4-dimensional case are classified by $\pi_1(O(2)) = \mathbb{Z}$.

To get a sense of framings over a knot K, we will visualize them using the parallel knot from Definition 5.2.1. We can then see that it indeed makes sense that the framings are categorized by $\mathbb{Z}$: what framing represents the 0-element is not clear and we have to choose. But after that, we can add twists around the attaching circle as many times as we want. We adopt the convention that a right-handed twist corresponds to a positive element and a left-handed twist to a negative element.



(a) The trefoil knot with the blackboard framing.



(b) The trefoil knot with the blackboard framing plus 2.

Figure 5.15: Two different framings for the trefoil knot, indicated by a parallel knot.

We can make more canonical choices regarding the framing for 2-handles. But for that we need the linking number.

> **Definition 5.2.3** (Proposition 4.5.2 [GS99]). For K_1 and K_2 two knots in S^3, we define their *linking number* $\ell k(K_1, K_2)$ as the signed number of times that K_2 crosses underneath K_1.

We have to explain what "signed" means in this definition. Also, we have to take care of the different ways to draw an equivalent knot. To do so, we first introduce the concept of a *link*. A link is a collection of m disjoint knots in some 3-manifold M for some m. A *link diagram* of a link $L \subseteq \mathbb{R}^3$ is a generic projection of L into $\mathbb{R}^2$. This gives us an immersion of $\coprod_{i=1}^{m} S^1$ which is bijective except at transverse double points (knots crossing under or over one another) together with information which specifies which knot crosses underneath. Such a diagram determines L up to isotopy. Some diagrams represent the same link however, but if and only if they are related by the Reidemeister moves. These are presented in Figure 5.16.

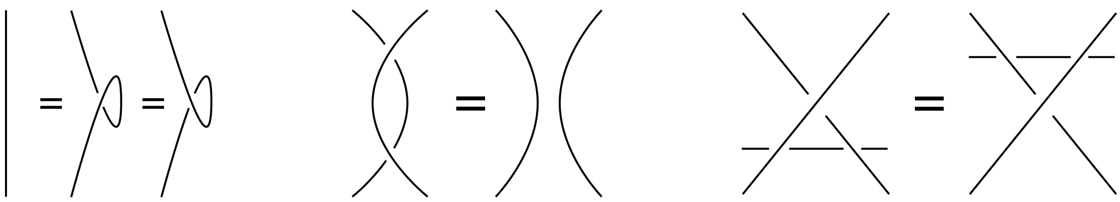


Figure 5.16: The Reidemeister moves.

By picking an orientation on the link, we can assign a sign to each crossing. We choose the conventions depicted in Figure 5.17. Now we have the tools to determine the linking number of two knots. One could check as an exercise that $\ell k(K_1, K_2) = \ell k(K_2, K_1)$, i.e. the linking number is symmetric. It is easy to see that the Reidemeister moves leave the linking number invariant.

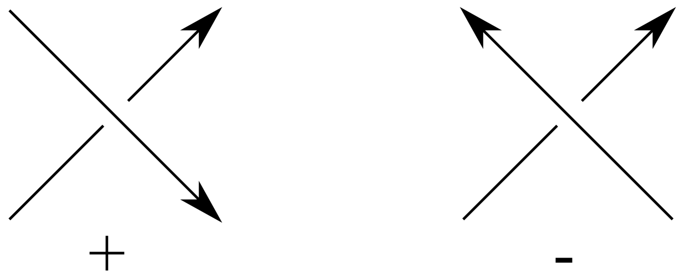


Figure 5.17: The signs of crossings in a link diagram.

Definition 5.2.4 (Definition 4.5.7 [GS99])**.** For a framed knot K in the boundary of a 0-handle, we define the *framing coefficient* to be the integer $\ell k(K, K')$ where K' is a parallel copy of K determined by the framing and the orientations of K and K' are chosen to be parallel.

This gives us the possibility to speak of an n-framed knot and, in particular, a 0-framed knot. The 0-framing may not always be the most intuitive framing in a drawing. For example: the trefoil knot in Figure 5.15a has framing coefficient 3.

We denote the framing coefficient next to an attaching circle in a Kirby diagram, as in Figure 5.18. Note that this definition does not hold for attaching circles that run over

1-handles. The general case is more involved and we will develop a way to deal with it in Section 5.4.

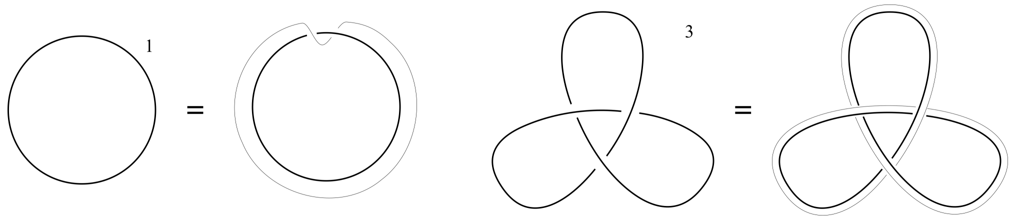


Figure 5.18: Examples of 2-handles with framing coefficients and the parallel knots corresponding to these framings.

One more important framing to introduce is the *blackboard framing*: this is the framing where the parallel knot K' lies in the plane of the paper or blackboard on which we are drawing. An example of this framing is Figure 5.15a. In this way the framing cannot twist around the knot, but beware that it depends on the way we are drawing the knot. See for example Figure 5.19.

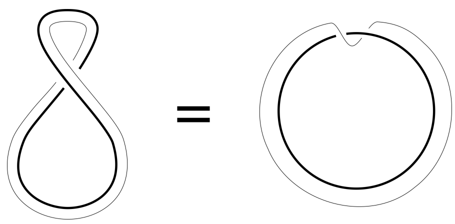


Figure 5.19: The blackboard framing depends on how we draw the knot.

Luckily for us, when we have dealt with the 2-handles, we are pretty much done. To see this, first consider the case $\partial_+ X = \emptyset$. Then there is a unique 4-handle by Proposition 5.1.10. Considering the dual handle decomposition, we see that this 4-handle together with the 3-handles, must have boundary $\# m S^1 \times S^2$ for some m. So $\partial X_2 = \# m S^1 \times S^2$. It is a fact that any self-diffeomorphism of $\# m S^1 \times S^2$ can be extended to a self-diffeomorphism of $\natural m S^1 \times D^3$.[5] So in whatever way we decide to attach the 3-handles, they will all be equivalent. Therefore we do not need to ~~worry~~ about the 3-handles and 4-handles and just have to mention how many of them we attach. If $\partial_+ X \neq \emptyset$, we know that there is no 4-handle. If $\partial_+ X$ is connected and X is simply connected, then an argument by Trace [Tra82] proves that in this case X is completely defined by X_2 and the number of 3-handles. So also in this case it is often enough to solely know the number of 3-handles. We will again denote it next to the diagram whether and how

[5] This is the *boundary connected sum*. For a definition, see A

many 3-handles and 4-handles we attach.

Handle moves

Handle moves for Kirby diagrams are very much like the ones in Heegaard diagrams. Slides of 1-handles work exactly the same. See Figure 5.20.

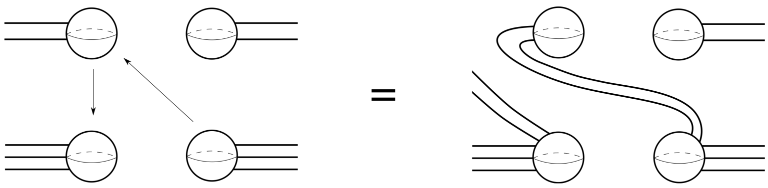


Figure 5.20: An example of a 1-handle slide in a Kirby diagram.

Sliding 2-handles is slightly more involved, but again the same principle as in the Heegaard case applies. Let us consider the 2-handles h_1 and h_2, with attaching circles K_1 and K_2 respectively, and the framing indicated by parallel knots K_1' and K_2'. By the same reasoning as in the Heegaard case, we have to take the connected sum of K_2' and K_1. This time, if the knots are oriented, the parallel knot K_2' can be oriented either parallel to K_2 or opposite to it. This corresponds to *handle addition* or *subtraction*.[6] We also have to determine the new framing of h_1. The most elementary and general way to do this is by drawing the parallel knot of h_1 too, to indicate the framing. This will also be slid over a disk bounded by a parallel knot K_2'': the parallel knot of K_2'. One could take $K_2'' = K_2$ for example. When carefully drawn, one can see the new framing.

This method always works, but we can do it faster if the knots lie in S^3, i.e. they don't go over 1-handles. The derivation requires more prior knowledge, but the formula can be used by everyone: In this case, we can apply homology. Readers familiar with CW-complexes (which are homotopy equivalent to our handlebodies) know that the 2-handles will form a basis of $H_2(X)$, say $\alpha_1, \ldots, \alpha_m$. A handle slide of h_1 over h_2 is then a change of basis: we get $\alpha_1', \alpha_2, \ldots, \alpha_m$ where $\alpha_1' = \alpha_1 \pm \alpha_2$, where $+$ is for addition and $-$ for subtraction. The intersection form on $H_2(X)$ is equal to the matrix $(\ell k(K_i, K_j))_{i,j}$ and hence the self-intersection (which is the framing coefficient) of α_1' is given by

$$\alpha_1'^2 = (\alpha_1 \pm \alpha_2)^2 = \alpha_1^2 + \alpha_2^2 \pm 2\alpha_1 \cdot \alpha_2 = n_1 + n_2 \pm 2\ell k(K_1, K_2). \tag{5.1}$$

[6]Note that we could not do the subtraction in the Heegaard diagrams, since we would not have been able to connect K_2' to K_1 while preserving the orientations given to both.

Here n_1 and n_2 are the framing coefficients of h_1 and h_2. This will tackle most cases.

A cancelling 1-handle and 2-handle still look the same. But when cancelling a 1-handle that has other 2-handles on it, we need to be careful to take into account the framing. This can be seen by first sliding these extra 2-handles over the 2-handle we want to cancel; they pick up the framing of the about-to-be-cancelled 2-handle (but we have not properly defined this framing yet). Then we have an isolated instance of a 1-handle with a 2-handle running over it once: they can be cancelled and forgotten about.

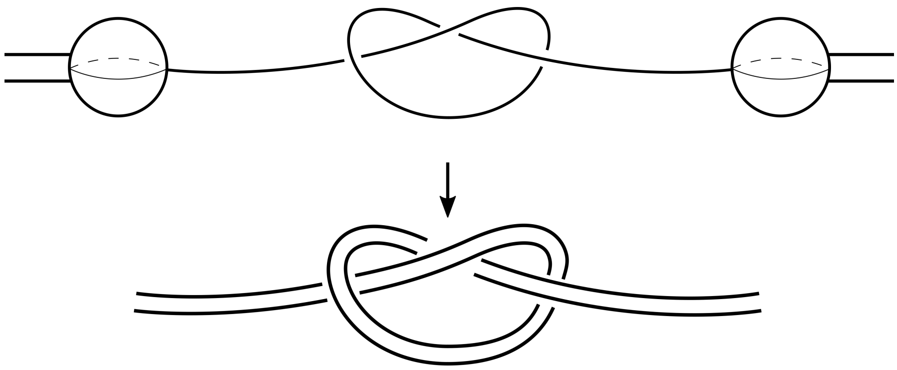


Figure 5.21: The cancellation of a 1-handle and a 2-handle. We use the blackboard framing on the cancelling 1-handle. If we would have added twists to the framing, the remaining 2-handles would be twisted in the same way because of the handle slides.

And for cancelling 2 and 3-handles the same reasoning holds: it can only happen if the 2-handle is attached via a isolated circle. Only this time we also need the isolated circle to have the 0-framing. Cancelling 3 and 4-handles do not occur, since we know whether we should have none or one 4-handle.

5.3 Application to Lefschetz fibrations

Now we come to our main application of all the theory we have built in this chapter: analysing Lefschetz fibrations. As shown in Chapter 4, we lose little by assuming that our fibers are connected and that every fiber contains at most one singularity.

5.3.1 Critical points and 2-handles

Every critical point corresponds to a 2-handle, as can be seen as follows: Suppose we have a Lefschetz fibration $\pi\colon X^4 \to \Sigma^2$. Near a critical point, we have complex coordinate charts such that it looks as $\pi(z_1, z_2) = z_1^2 + z_2^2$. For a certain t that we ~~choose~~ to be real (we can always do this by multiplying π with a complex number of length 1) we have the fiber $F_t = \pi^{-1}(t)$. The critical value lies at zero, so the singular fiber is denoted by $F_0 = \pi^{-1}(0)$. The vanishing cycle, as defined in Section 4.1.1, can be obtained by intersecting F_t with $\mathbb{R}^2$, that is

$$F_t \cap \mathbb{R}^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = t\}.$$

This bounds a disk $D_t \subseteq X$. By definition of the vanishing cycle, F_0 can be obtained from F_t by shrinking D_t to zero size. A regular neighbourhood νF_0 of F_0 (think of it as F_0 but slightly thickened at each point) is then obtained from a regular neighbourhood νF_t of F_t by adding a regular neighbourhood of D_t. But this is the same as attaching a 2-handle to νF_t. To visualize the attachment, the attaching circle for the 2-handle is ∂D_s for some s if we choose νF_t in such a way that $F_s \subseteq \partial \nu F_t$, which we can always do. The core of the handle is then D_s.

So a singular fiber indeed corresponds to a 2-handle in X. But we still have to figure out its framing. This time, instead of a parallel knot that indicates the framing, we will use a vector. The principle is the same: the vector is the difference between the parallel knot and the knot itself. So at each point on the knot, it points to where the parallel knot sits. The other way around, given a such a vector at every point of our knot, we can obtain a parallel knot by transporting our knot along the vector at each point. Therefore, the two notions are equivalent.

The easiest way to figure out the framing in this case, is by comparing it to another framing that we already have: the framing on ∂D_s determined by F_s. That is, the framing given by a nowhere zero normal vector field on ∂D_s in F_s. At a point $(\sqrt{s}\cos\theta, \sqrt{s}\sin\theta) \in \partial D_s \subseteq \mathbb{R}^2$, the vector $(-\sin\theta, \cos\theta)$ is tangent to ∂D_s. Since F_s is holomorphic in the local coordinates that we have, the vector field $v(\theta) = (-i\sin\theta, i\cos\theta)$ is still tangent to F_s, but no longer to ∂D_s. This is our reference framing, but it is not the framing by which the 2-handle is attached. The framing used for the attachment is the product framing: the framing on ∂D_s that one gets by simply embedding D_s into $\mathbb{R}^2$ and that into $\mathbb{R}^2 \times i\mathbb{R}^2$. When we look at how the vanishing cycle shrinks when approaching the singular fiber, we see that this is the framing that is used to attach the 2-handle.

We compare the product framing to our reference framing. Since the reference framing map $v\colon \partial D_s \to i\mathbb{R}^2 - \{0\}$ has degree 1 (that is, goes around 0 once when we do a full round over its domain), this has one right twist relative to the product framing. That is the same as saying that the product framing has framing -1 relative to the reference framing.

There is one slight caveat that we still need to address: the orientation we gave to the handle in the product framing using $\mathbb{R}^2 \times i\mathbb{R}^2$ is reversed from the orientation on $\mathbb{C}^2$. However, to get from the boundary of the handle to $\partial \nu F_t$, the orientation again has to be reversed. So everything works out.

5.3.2 Drawing the Kirby diagrams

Let $\pi\colon X^4 \to D^2$ be a Lefschetz fibration. As proclaimed in the previous chapter, we focus on the case where the fiber has genus 1.

We start with the case where there is no singular fiber. In this case, the Lefschetz fibration is simply the trivial fibration $D^2 \times T^2 \to D^2$. So we have to draw the Kirby diagram for this. This we can do in a simple way: while $T^2 \times D^2$ may be 4-dimensional, but it is merely the thickened version of T^2. Therefore, we can use the handle structure of the torus to see the Kirby diagram. It is the Kirby version of Figure 5.7. Strictly speaking we do not know how to treat the framing when the attaching circle runs over 1-handles. But we let 0 indicate the blackboard framing, which in this case is the correct choice.[7] So we have Figure 5.22 as result.

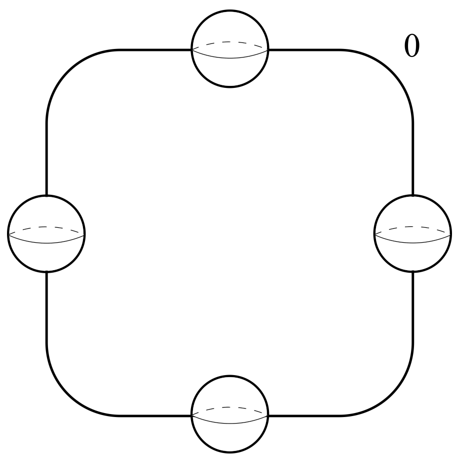


Figure 5.22: The Kirby diagram of the fibration $D^2 \times T^2 \to D^2$.

[7] This requires too much introduction to argue properly. So in short: the framing coefficient relative to the blackboard framing equals, in this case, the euler number $e(X)$. For a trivial fibration, this is 0.

When we have a single singular fiber, we can use the previous section to see what we have to do: attach a 2-handle along the vanishing cycle. Hence we have to locate the vanishing cycle in the diagram. This will be something that does not cross itself (since the cycle has to be able to shrink to zero size) but in general running over the 1-handles. The framing for this is -1 relative to the framing induced by the fiber T^2. The framing induced by the fiber is given by a parallel knot that lies in the same fiber, next to the vanishing cycle. In the Kirby diagram, this is represented by the blackboard framing. So the framing we need is -1 relative to the blackboard framing. The result can be for example Figure 5.23.

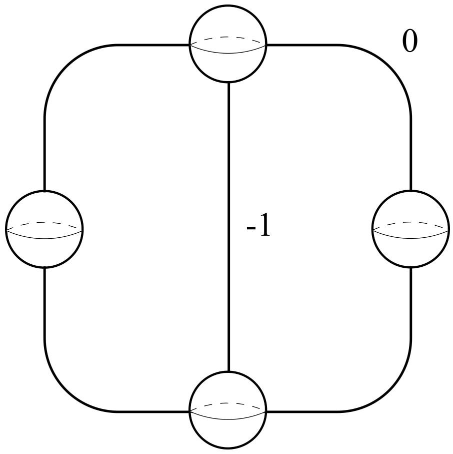


Figure 5.23: The Kirby diagram of a possible Lefschetz fibration with one singularity over D^2.

When there are multiple critical points, say n, we assume that they all lie in different fibers. We can already guess that we will probably have to add n 2-handles. The question is: along what vanishing cycles?

We call the critical values $t_1, \ldots, t_n$ and the singular fibers $F_1 = \pi^{-1}(t_1), \ldots, F_n = \pi^{-1}(t_n)$. We choose a regular fiber $F_0 = \pi^{-1}(t_0)$ so that we have something to compare the singular ones with. We define V_i to be a neighbourhood of t_i. We let a_i be an embedded path connecting t_0 to t_i for each i. A small neighbourhood of these paths is denoted by A_i. Now that we have n paths leaving $t_0 \in D^2$, we can assume without loss of generality that they are ordered in such a way that the numbering increases when going anti-clockwise.

Just as with the definition of monodromy in Section 4.2, $\pi^{-1}(a_i)$ gives us a trivial fiber bundle over the interval. This way, we can carry the identification of F_0 with the torus to a regular fiber in V_i, giving us the possibility of talking about 'the' vanishing cycle γ_i of that singular fiber. Together with the used identification, this completely characterizes the singular fiber. If we now define

$$V = V_0 \cup \bigcup_{i=1}^{n} (A_i \cup V_i) \subseteq D^2,$$

we see that $\pi^{-1}(V)$ is diffeomorphic to X, since V is just a very deformed disk and $X - \pi^{-1}(V)$ only contains regular fibers. Hence we can describe X as

$$X = D^2 \times T^2 \cup \bigcup_{i=1}^{n} H_i.$$

Here H_i is a 2-handle attached along the vanishing cycle γ_i. This has to respect the same framing condition as derived in the case of only one singular fiber. Since the singular fibers all lie in different neighbourhoods and are not intertwined, we can add them to the Kirby diagram in separate 'layers' with index increasing towards the reader. See Figure 5.24.

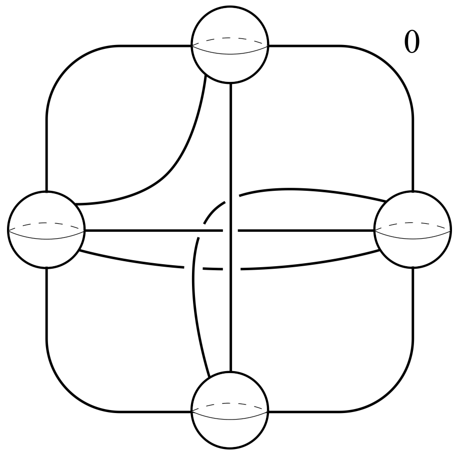


Figure 5.24: The Kirby diagram of a possible Lefschetz fibration with four singularities over D^2. Notice that all the attaching circles lie in their own layer. We have left out the framing coefficients.

The precise diagram that we get by this method depends on the choice of paths a_i. A choice of paths is always related to the other possible choices by (possibly multiple) *elementary transformations.* An elementary transformation is shown in Figure 5.25. By elementary transformations, we can achieve any permutation of the numbering of the fibers. But because the paths a_i change, so does the identification with the reference fiber F_0 and hence so does the vanishing cycle.

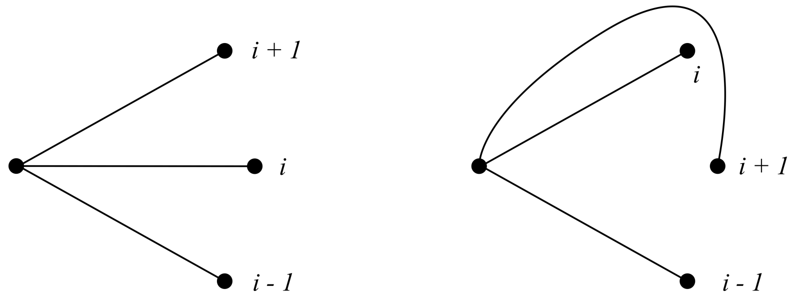


Figure 5.25: An example of an elementary transformation.

5.3.3 Achiral Lefschetz fibrations

Since achiral Lefschetz fibrations are our main object of interest, we want to know how those Kirby diagrams look. We can apply the same reasoning as above to see what the diagrams of achiral Lefschetz fibrations (as introduced in Definition 4.3.1) look like. Since the reasoning is so very similar, we skip it and state the result: everything works the same, but whereas singular fibers corresponded to a 2-handle added with a -1-framing in the previous case, they can now have either a $+1$ or -1-framing.

We saw in Proposition 4.3.2 that the monodromy around two critical points with the same vanishing cycle and opposite chirality is trivial. So we can try to insert such a pair of singularities in the achiral Lefschetz fibration. We know that a singularity corresponds to a 2-handle and that one should have framing $+1$ and the other -1 if they are to have opposite chirality. Therefore, let C be a circle embedded in a regular fiber. By adding two 2-handles along C, one with framing $+1$ and another with framing -1, we obtain a new space X'. Although the space has changed, it still has an achiral Lefschetz fibration over the same base space but now with an extra pair of singularities.

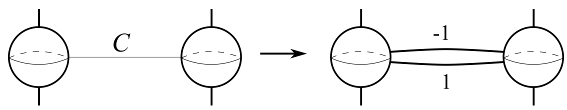


Figure 5.26: An example of replacing a cycle C by two 2-handles of opposite orientation.

This construction can be made more precise and also formulated without the use of handles. This we will do in Section 5.4.1. It will allow us to figure out more about the diffeomorphism type of X' in relation to that of X and construct a natural inverse operation. See for example the following proposition.

Proposition 5.3.1. Let $\pi\colon X \to \Sigma$ be an achiral Lefschetz fibration. The insertion of two singularities as above along a nullhomotopic cycle C changes X to either $X\#(S^2 \times S^2)$ or $X\#\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$.

Proof. As we will see in Section 5.4.1, the procedure of inserting these pairs of singularities coincides with surgery on C. This proposition is then a consequence of Proposition 5.2.3 in [GS99]. □

In the next chapter, this insertion of pairs of singularities will allow us to talk about pair creation and annihilation of branes.

5.4 Surgery

Surgery is an operation that creates a new manifold out of an old one. It can be seen as what happens to the n-dimensional boundary of an $n+1$-dimensional manifold when a $k+1$-handle is added to the manifold. We will use it as a way to add or subtract singularities from achiral Lefschetz fibrations.

Definition 5.4.1 (Definition 5.2.1 [GS99]). Let $\varphi\colon S^k \to M^n$ $(-1 \le k < n)$ be an embedding of a k-sphere in an n-manifold, with a normal framing f on $\varphi(S^k)$. Then the pair (φ, f) determines an embedding $\hat{\varphi}\colon S^k \times D^{n-k} \to M$ (uniquely up to isotopy). *Surgery* on (φ, f) is the procedure of removing $\hat{\varphi}(S^k \times \mathrm{int}D^{n-k})$ and replacing it by $D^{k+1} \times S^{n-k-1}$, with gluing map $\hat{\varphi}\big|_{S^k \times S^{n-k-1}}$.

Remark 5.4.2. In the above definition, the convention $S^{-1} = \emptyset$ should be used.

As can be seen, the construction relies on the fact that

$$\partial(S^k \times D^{n-k}) = S^k \times S^{n-k-1} = \partial(D^{k+1} \times S^{n-k-1})$$

The diffeomorphism type of the result of the surgery is uniquely determined by the isotopy class of (φ, f).

By definition, there is a canonical embedding of $S^{n-k-1} \times D^{k+1}$ in the manifold after the surgery. If we would again perform surgery, but now on this sphere together with this framing, we would obtain the original manifold. This is called *reversing the surgery.*

The embedding $\hat{\varphi}$ from the definition defines a framing on S^k. In the case that $n = 4$, framings are classified by the groups listed in Table 5.1.

k	Group of framings $\pi_k(O(4-k))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	0
3	0
4	0

Table 5.1: The possible framings on embeddings of $S^k \hookrightarrow M^4$.

We will primarily be concerned with the cases $k = 1, 2$. We see that we only have to worry about the framing on a circle that we will perform surgery on. But even then, it only matters whether the framing number is odd or even.

Effect on Kirby diagrams

We can represent certain surgeries on 4-dimensional manifolds by Kirby diagrams. The interesting cases are $k = 1, 2$. We start with the simplest case: consider D^4 with a 1-handle attached to it. This is diffeomorphic to $S^1 \times D^3$. For the same reason D^4 with a 2-handle attached to it is diffeomorphic to $S^2 \times D^2$. These two manifolds have the same boundary and surgery on $S^1 \subseteq S^1 \times D^3$ gives $S^2 \times D^2$ and vice versa. So surgery on the proper sphere turns a 1-handle into a 2-handle or the other way around.

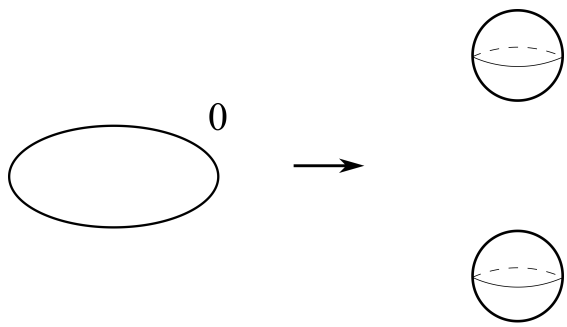


Figure 5.27: Surgery on the sphere consisting of the core of the 2-handle together with the disk in S^3 bounded by the attaching circle gives a 1-handle.

For the more general case of a 4-manifold M and another point of view, we note that 1-handles and 2-handles can cancel. Therefore, adding a 1-handle to M is the same as removing the corresponding 2-handle. This 2-handle has a cocore D^2 whose boundary

S^1 lies in ∂M, whose interior lies in int M, has 0 framing and is unknotted. Hence we can represent it by a circle in the Kirby diagram of M. We denote such a circle by a circle without framing and with a dot, as in Figure 5.28. Such a circle thus means that we push the disk that it bounds into the interior of M and then remove a tubular neighbourhood of that disk. Intuitively speaking, we can add a 1-handle by either gluing it on our manifold, or by digging a tunnel such that the 1-handle becomes the overpass.

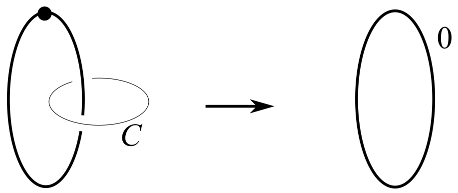


Figure 5.28: Surgery on the cycle C with 0 framing gives a 0-framed 2-handle.

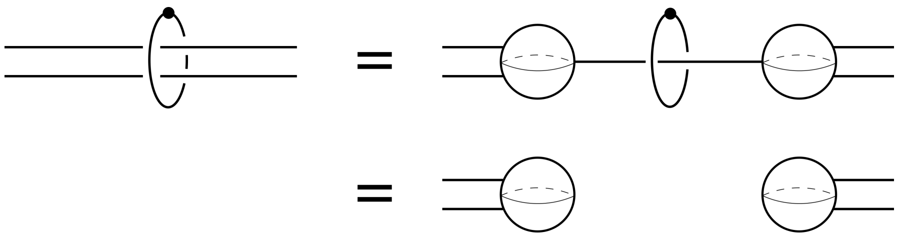


Figure 5.29: Inserting a cancelling pair of 1- and 2-handles, we see that the dotted circle is the boundary of the cocore of the cancelling 2-handle. So deleting the disk inside it is deleting the 2-handle.

We can now conclude that surgery on a 0-framed 2-handle is the same as replacing the 2-handle with a dotted circle, since they both result in a 1-handle. This gives us a new way to denote 1-handles: instead of drawing pairs of balls where lines enter the one and exit the other, we draw a dotted circle around these lines, as on the left hand side of Figure 5.29. This notation has the advantage that all 2-handles are now knots in S^3 and hence framings can be uniquely defined.

5.4.1 Surgery on fibrations

When we have a 4-dimensional manifold admitting an achiral Lefschetz fibration (remember that we always take the generic fiber to be a torus), we can ask whether the result of performing surgery on this manifold still admits such a fibration and what changed about it. We show that it does, when performed on the proper embeddings of S^k, and that it gives us a way to add and cancel singularities. More precisely, we get:

Theorem 5.4.3. *Given an achiral Lefschetz fibration, surgery on a circle C that lies in a regular fiber gives us an achiral Lefschetz fibration with two extra singularities of same vanishing cycle and opposite chirality. Conversely, given an achiral Lefschetz fibration with two singularities of same vanishing cycle and opposite chirality, surgery on the sphere between these singularities gives us an achiral Lefschetz fibration without these two singularities. In these processes, the base space is never changed.*

Remark 5.4.4. We should explain "the sphere between these singularities". Suppose we have an achiral Lefschetz fibration $\pi\colon X \to \Sigma$ that has two singularities over y_1 and y_2 of opposite chirality and with the same vanishing cycle. We pick a path γ in Σ that goes from y_1 to y_2 and track the vanishing cycle C along this path. This gives us space $L \cong C \times [0,1]/C \times (\{0\} \cup \{1\}) \cong S^2 \subseteq X$.

There are two approaches to this. We first show it by analysing Kirby diagrams, giving us a short but also slightly magical proof: it is not clear how the surgery precisely influences the fibers, just that it works. The other approach gives a far more detailed picture but is also more explicit. It relies on constructing fibration structures on $S^2 \times D^2$ and $S^1 \times D^3$.

Using Kirby diagrams

Let C be a given cycle in a regular fiber with framing $+1$ or -1.[8] Using our new-found knowledge on how surgery on C influences the Kirby diagram, we can show that the result of the surgery again has the Kirby diagram of an achiral Lefschetz fibration.

In the first step of the proof, shown in Figure 5.30, we *twist the 1-handle.* By turning the sphere on the right a full 360°, we can make the framing on C the 0-framing. This move can be derived from handle cancellation and handle slides, see Figure 5.43 in [GS99].

In this proof we assumed (for the sake of being able to make drawings) that C runs over only one 1-handle. By an $\mathrm{SL}(2,\mathbb{Z})$ transformation, we can always make sure that this happens. Note that C cannot be knotted, since it has to lie in a single fiber. Also note that the result obtained in Figure 5.30 is indeed the Kirby diagram of a Lefschetz fibration, since the one we started with was and the extra 2-handles have the correct framing (either $+1$ or -1) and the attaching circles lie in a single fiber, since C did as

[8] Since $\pi_1(O(3)) = \mathbb{Z}/2$, it does not matter.

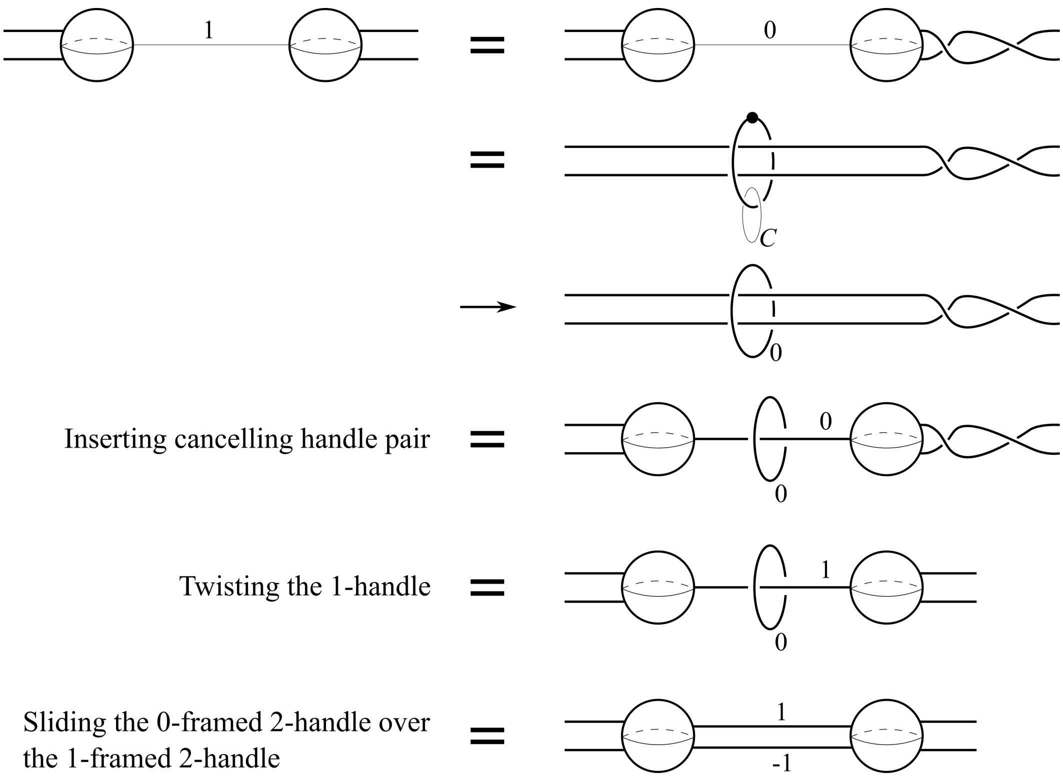


Figure 5.30: The thin line represents the given cycle C. The arrow denotes surgery on C. The two external lines of the diagram are indicating that we are only working locally within the diagram of the given Lefschetz fibration and are there to indicate what happens to the other handles.

well.

The surgery the other way around is given by the same proof, but reading it backwards. The surgery is now performed on the 0-framed 2-handle in the third line. This proves Theorem 5.4.3

Explicit construction

We describe fibration structures on $S^2 \times D^2$ and $S^1 \times D^3$ and show that, by gluing these in with a fiber preserving map, the result of surgery again admits a Lefschetz fibration. This construction is taken from, and described in more detail in, [Mat85].

The construction will be very explicit. We start by constructing the space $N(S_0)$, which will be equivalent to $S^2 \times D^2$. Let $0 < 2\epsilon < \delta < 1$. We define the spaces

$$
\begin{aligned}
U &= \{(u_1, u_2) \in \mathbb{C}^2 \mid |u_1 u_2| \leq \epsilon, |u_1| < 2, |u_2| \leq \delta\}, \\
V &= \{(v_1, v_2) \in \mathbb{C}^2 \mid |v_1 v_2| \leq \epsilon, |v_1| < 2, |v_2| \leq \delta\}, \\
U_0 &= \{(u_1, u_2) \in U \mid |u_1| > \frac{1}{2}\} \subseteq U, \\
V_0 &= \{(v_1, v_2) \in V \mid |v_1| > \frac{1}{2}\} \subseteq V.
\end{aligned}
$$

The map $\phi\colon U_0 \to V_0$ given by

$$\phi(u_1, u_2) = \left(\frac{1}{u_1}, u_2|u_1|^2\right)$$

is an orientation preserving diffeomorphism, which we can use to glue U and V together by identifying U_0 with V_0. We call the resulting manifold $N(S_0)$. The S_0 stands for the 2-sphere that lies in this space, defined by

$$S_0 = \{(u_1, u_2) \in U \mid u_2 = 0\} \cup \{(v_1, v_2) \in V \mid v_2 = 0\} \simeq S^2.$$

Lemma 5.4.5. The manifold $N(S_0)$ is homeomorphic to $S^2 \times D^2$.

Proof. We built $N(S_0)$ out of U and V. Note that U and V are both topologically trivial spaces, so $U \cong V \cong D^4 \cong D^2 \times D^2$. We glued them together along U_0 and V_0. Be restricting the norm of u_1 from below, the topology of U_0 differs from that of U in that $u_1 \neq 0$. So $U_0 \cong S^1 \times D^3$. The same holds for V_0. We then obtain $N(S_0)$ by identifying U_0 and V_0. We essentially identify two disks D^2 along their boundary, so we get S^2, but everything is slightly thickened to 4 dimensions by a product with D^2. This leads to a D^2 bundle over S^2, the zero section of which is given by S_0. ~~This bundle is not necessarily trivial.~~ However, we can find two linearly independent, non-vanishing sections of this bundle by picking $s_1(u_1) = (u_1, r)$ over U for some $0 < r < \delta$ and over V we write $s_1(v_1) = (v_1, \rho(v_1)r)$ such that ~~g~~$(v_1) > 0$ for any v_1. The second section is constructed in the same way, by starting with $s_2(u_1) = (u_1, ri)$. So the bundle is trivial and hence $N(S_0) \cong S^2 \times D^2$. $\square$

We will now construct a fibration on it. Let D_ϵ denote the 2-dimensional disk of radius

ϵ. By defining the maps $f_U : U \to D_\epsilon$, $f_V : V \to D_\epsilon$ as

$$f_U(u_1, u_2) = u_1 u_2, \qquad f_V(v_1, v_2) = \overline{v_1} v_2$$

and noting that $f_V \circ \phi = f_U$, we get a well-defined smooth map $f : N(S_0) \to D_\epsilon$.

We claim that $f : N(S_0) \to D_\epsilon$ is a fibration, so we study its fibers. Let $0 \neq z \in D_\epsilon$ be given. Then $f^{-1}(z) \cong S^1 \times [0,1]$ is an annulus. This can best be seen by writing the local expressions of f in polar coordinates. The fiber $f^{-1}(0)$ (see Figure 5.31) is given by the union of S_0 and the two disks

$$D_+ = \{(u_1, u_2) \in U \mid u_1 = 0\},$$
$$D_- = \{(v_1, v_2) \in V \mid v_1 = 0\}.$$

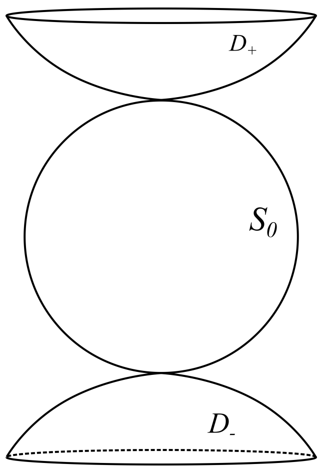


Figure 5.31: A drawing of $S_0 \cup D_+ \cup D_-$. One can imagine it as the shape of a christmas cracker.

We see that the singular fiber has two singularities and that by definition there exist complex coordinate charts around these singularities such that $f(u_1, u_2) = u_1 u_2$ for one and $f(v_1, v_2) = \overline{v_1} v_2$ for the other. This shows that f has Lefschetz singularities of opposite chirality.

We now turn to constructing $N(C_0)$, where C_0 is a circle. This we simply define as

$$N(C_0) = D_\epsilon \times C_0 \times [0,1].$$

The trivial projection $g : N(C_0) \to D_\epsilon$ turns this into a fibration of which every fiber is the annulus $C_0 \times [0,1]$.

Now that we have constructed a fibration on the spaces $S^1 \times D^3$ and $S^2 \times D^2$ that are relevant for surgery, we need to construct a map that identifies their boundaries, so that we know how to glue in $S^2 \times D^2$ after deleting $S^1 \times D^3$ or vice versa. We call this map $h\colon \partial N(S_0) \to \partial N(C_0)$. The important part of this map is that it exists and that it is fiber preserving, i.e. $g\big|_{\partial N(C_0)} \circ h = f\big|_{\partial N(S_0)}$. The definition of h can be found in Appendix B.

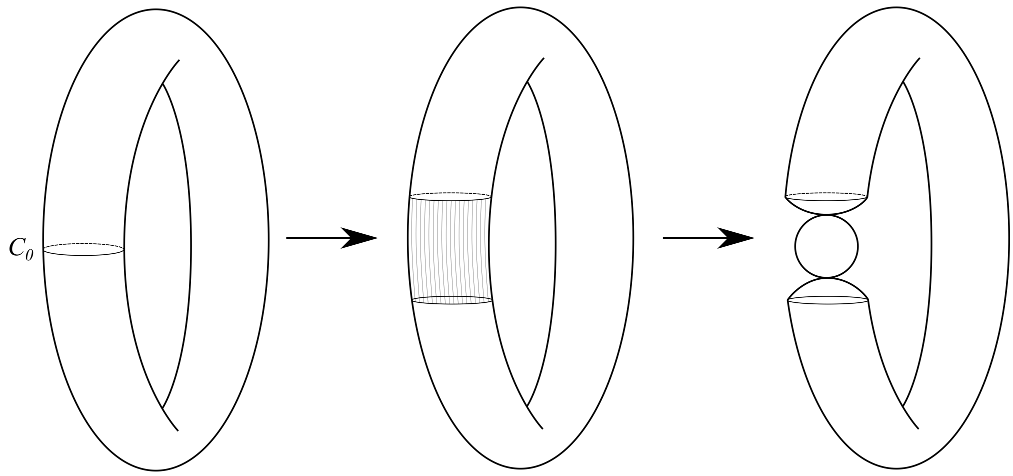


Figure 5.32: The result of surgery on a circle C_0 that lies in a regular fiber. The resulting fiber has two singularities. Going from right to left would illustrate surgery on the sphere.

We are now ready to apply this construction to a Lefschetz fibration $\pi\colon X \to \Sigma$. For a proof of the identifications made in the following, see Appendix B. Suppose that we have a small disk $D \subseteq \Sigma$ such that there are no critical values contained in D. Let 0 denote the origin of D in Σ. Then $F_0 = f^{-1}(0)$ is a regular fiber and hence a torus. Let C be a circle in F_0. Then surgery on C with framing induced by the fiber cuts a small tube out of every fiber (see Figure 5.32). Using our construction, we can describe this by identifying D with D_ϵ and $C \times D^3$ with $N(C)$. The small tube in every torus is then the annulus that is a fiber of $N(C)$. By surgery on C, we replace $N(C)$ by $N(S_0)$ in X. This is a fibration over the same disk, and also has an annulus as general fiber. But the fiber over 0 is now $S_0 \cup D_+ \cup D_-$. This means that in the resulting Lefschetz fibration, the regular fiber F_0 is replaced by singular fiber F_0' containing a pair of singularities of opposite chirality. We can always slightly permute the fibration or the construction to stop these singularities from being in the same fiber, if we desire so.

Conversely, when we are given a Lefschetz fibration $\pi\colon X \to \Sigma$ with a small disk $D \subseteq \Sigma$

such that the fiber over 0 is the only singular fiber and contains two singularities of opposite chirality. Performing surgery on the sphere between these two singularities, which we identify with S_0, cuts $S_0 \cup D_+ \cup D_-$ out of the singular fiber and an annulus out of the regular fibers. We then have to glue in $S^1 \times D^3 \cong N(C_0)$, which fibers over the same disk D, which replaces $S_0 \cup D_+ \cup D_-$ with an annulus, turning the singular fiber into a regular torus. This again proves Theorem 5.4.3.

Chapter 6

Lefschetz fibrations in F-theory

In this chapter we combine our previous work to properly introduce our proposed new point of view. We then prove some properties of it, using the goals and checks we formulated in Chapter 1. We will refer back to them when they are treated.

F-theory in the literature almost exclusively uses elliptic fibrations. This works nice, since a lot of algebraic geometry can be used to immediately derive information about the τ-profile. Notice, referring back to Section 3.3, how quickly we knew the amount of branes in our system when we knew the polynomial description of the K3.

But our mission is a different one. It is not finding what is nice to work with, but what is necessary to work with. By focussing on this, we create as much space in our theory as we can to allow for an F-theoretic description of anti-branes. Thinking about what the F-theory picture needs to detect branes, we get two requirements

1. A fibration $p\colon X^{12} \to M^{10}$ where X is the total space and M the spacetime.
2. A notion of monodromy on the fibers of this fibration, whose elements lie in $\mathrm{SL}(2,\mathbb{Z})$.

We will show that these things can also be achieved by looking at (achiral) Lefschetz fibrations. Because Lefschetz fibrations are maps from a $2n$-dimensional space to a 2-dimensional one, while we want 2-dimensional fibers according to requirement 1, we consider the case of a 4-dimensional space on which we apply these fibrations, forgetting about the 8-dimensional transverse space (as was also done in the example in Section 3.3). This already satisfies the first requirement. Remember that we always assume (achiral) Lefschetz fibrations to go from 4 to 2 dimensions and have genus 1.

6.1 Monodromies

Let us start by seeing how achiral Lefschetz fibrations can give us the correct monodromies around singularities. That is, if we consider monodromy on the right object.

There is a subtle but important distinction between the monodromy of an elliptic fibration and the monodromy of Lefschetz fibrations. The elliptic fibration has a τ-profile of which we can speak. That is, there is a complex number indicating the values of the fields C_0 and ϕ that is well-defined up to $\mathrm{SL}(2,\mathbb{Z})$ transformations. We also used this in Section 3.1 where we determined the coupling constant far away from the brane.

We already noted in Section 3.6 that there is another way to think about monodromy. Lefschetz fibrations do not have a complex structure, hence we cannot speak of the complex structure parameter of the torus fiber, or a τ-profile. Here, the monodromies are elements of $\mathcal{M}_1 \cong \mathrm{SL}(2,\mathbb{Z})$, the mapping class group of the torus, as was shown in Section 4.2. So knowing the geometry of a Lefschetz fibration does not give us as much information as it does in the case of an elliptic fibration: there is no explicit value for the fields contained in it. But it is also much less restrictive. As proved in Section 4.2.1, both types of fibrations their monodromy allow for all the different (p,q)-branes. So Lefschetz fibrations allow for all the relevant monodromies and can distinguish between them. This was one of the main checks (number 2) we had given ourselves in Chapter 1.

2 Achiral Lefschetz fibrations should be able to distinguish between all possible types of 7-branes. ✓

6.2 Incorporating anti-branes

Of course, the 'Lefschetz-fibration point of view' has to give us something extra, otherwise it is of no use. We claim that it indeed gives us extra freedom and that we can use that to describe anti-branes.

> **Conjecture 6.2.1.** Achiral Lefschetz fibrations are able to describe both D-branes and $\overline{\mathrm{D}}$-branes.

We start by noting an essential difference between achiral Lefschetz fibrations and elliptic fibrations. Since elliptic fibrations carry a complex structure, they have an orientation. In contrast, the whole point of achiral Lefschetz fibrations is that we drop the assumption of orientability on the associated spaces. Let us argue how this gives us the freedom to talk about anti-branes.

~~Remember~~ from Proposition 4.4.14 that an achiral Lefschetz fibration $\pi\colon X \to \Sigma$ induces a b-orientation on (X, Z_X) and (Σ, Z_Σ) for some subspaces $Z_X \subseteq X$ and $Z_\Sigma \subseteq \Sigma$ of codimension 1. Suppose that Σ is orientable (which is to ask that spacetime is orientable), then a choice of orientation on Σ gives us a way to assign a sign to each connected component of $\Sigma - Z_\Sigma$.

Definition 6.2.2. Let (Σ, Z_Σ) be a b-manifold with a b-orientation (c.f. Definitions 4.4.5 and 4.4.7) and assume that we have an orientation on Σ. We call a connected component of $\Sigma - Z_\Sigma$ *positive* if the b-orientation agrees with the given orientation on this connected component. We call it *negative* if this is not the case.

In this way, we have a Lefschetz fibration over each connected component of $\Sigma - Z_\Sigma$, hence we can detect what types of branes live there. But globally, whether this component is positive or negative is just a choice of orientation on Σ. We say that singularities in positive components correspond to branes, while the singularities in negative components correspond to anti-branes. Notice the parallel with what was said in section 2.6.5: two branes can be each others anti-brane, but which one is the 'anti' one is a matter of choice.

Remark 6.2.3. We could have chosen a more direct approach in defining positive and negative singularities. When we assume that X and Σ are orientable, we can call a singularity positive or negative depending on whether the complex coordinate charts in which the achiral Lefschetz fibration gets its local form $\pi(z_1, z_2) = z_1^2 + z_2^2$ are compatible with the orientations. This however requires us to have an orientation on both X and Σ. In our definition, we do not require an orientation on X and if we do have it, our definition of positive and negative singularities can also be expressed using compatibility with the coordinate charts.

Although it is a conjecture and it is not at all proven that this definition allows for all the properties that we want anti-branes to have, we do see that there is enough freedom in achiral Lefschetz fibrations to incorporate these extra objects. So we go ahead and give ourselves these two checkmarks:

1 Achiral Lefschetz fibrations should be able to describe all possbile types of 7-branes (including anti-branes) that occur in type IIB string theory. ✓

4 There should be a symmetry between branes and anti-branes: it is an arbitrary choice which of the two is 'anti'. Achiral Lefschetz fibrations should reflect this symmetry. ✓

We still need to gather a lot of evidence for it and describe how known phenomena can be incorporated. Also, the monodromy around a brane and its anti-brane should of course be zero.

6.2.1 Brane creation and annihilation

One of the key properties of particles and anti-particles, but also branes and anti-branes, is that pairs of them can be created or annihilated. If we really are to conjecture that achiral Lefschetz fibrations can describe anti-branes, we should be able to create and annihilate pairs of singularities of opposite chirality in achiral Lefschetz fibrations, without influencing the other singularities.

Mathematically, we have already described in Theorem 5.4.3 how pair creation and annihilation of singularities works: by surgery on a cycle C in a regular fiber, we obtain a new space. This has again an achiral Lefschetz fibration, but now there are two extra singular fibers of opposite chirality and the same vanishing cycle C: a brane and an anti-brane. Conversely, surgery can also cancel two of such singularities.

By choosing C we can vary the monodromy that we get around these branes. We have already seen in Section 4.2.1 that it is possible to create any kind of (p, q)-brane and anti-(p, q)-brane this way.

3 Achiral Lefschetz fibrations should be able to allow for the process of pair creation and annihilation that comes with branes and anti-branes. ✓

6.3 The search for a potential for brane-anti-brane pairs

It is now time for physics to reap the fruits of our labour. Using our conjectured correspondence between achiral Lefschetz fibrations and (anti-)branes, we can construct spaces X such that F-theory compactified on these spaces corresponds to the compactification of type IIB with branes and anti-branes. If we perform this compactification with the right compactification ansatz, we will get a potential for brane-anti-brane pairs in our action.

Alas, we did not succeed in doing so, but have some thoughts on it that we will share here.

6.3.1 An example: S^4

We can construct an achiral Lefschetz fibration $S^4 \to S^2$ containing two singularities with the same vanishing cycle of opposite chirality.

We start with the Hopf fibration $S^3 \to S^2$ with fiber S^1. Note that there are no singular fibers in this. From this we can get a fibration $S^3 \times S^1 \to S^2$ by first projecting to the first component and then applying the Hopf fibration, giving us a torus fiber over every point. We define $C = \{p\} \times S^1$ for some $p \in S^3$ and perform surgery on C. This gives us two singularities with vanishing cycle C and opposite chirality, as described in Section 5.4.1. Note that it was required that C lies in a fiber, which it does since the S^1 component is always projected to a point. The manifold obtained by the surgery is S^4, as we will now show.

First we find the Kirby diagrams for $S^3 \times S^1$ and S^4. Then we identify C, perform the surgery on it and conclude that the result is S^4.

The Kirby diagram of S^4 was already obtained in Example 5.1.4, so it is an empty diagram with a 4-handle added to it. The diagram for $S^3 \times S^1$ can be obtained as follows[1]: we again decompose S^3 and S^1 as handlebodies into $D^3_- \cup D^3_+$ and $D^1_- \cup D^1_+$, where we interpret D^1_- and D^3_- as 0-handles, D^1_+ as a 1-handle and D^3_+ as a 3-handle. So we can write

$$S^3 \times S^1 = (D^3_- \times D^1_-) \cup (D^3_- \times D^1_+) \cup (D^3_+ \times D^1_-) \cup (D^3_+ \times D^1_+).$$

In this expression, we can interpret the first term as a 4-dimensional 0-handle, the second as a 1-handle since it is the thickened version of the 1-handle in the handlebody decomposition of S^1 and its attaching region is hence $D^3 \times S^0$, the third as a 3-handle for the same reason and also the fourth as a 4-handle. This gives us the diagram in Figure 6.1.

[1]Again, for the people that know CW-complexes, this is the same procedure as obtaining a CW-structure for the product of two CW-complexes.

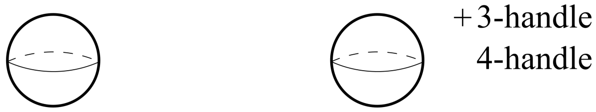


Figure 6.1: Kirby diagram of $S^3 \times S^1$.

We now identify $C = \{p\} \times S^1$. This can be rewritten as $C = \{p\} \times (D^1_- \cup D^1_+)$, which is to say that C is simply given by a line between the two balls in the Kirby diagram, running back over the 1-handle. Surgery on it, as in Figure 5.30 then gives two 2-handles of framing $+1$ and -1. We then slide the 2-handle with framing -1 over the other 2-handle, cancel the 1-handle and the 1-framed 2-handle and cancel the 0-framed 2-handle and the 3-handle. All that is left over is a 0-handle and a 4-handle, so that is S^4. See Figure 6.2.

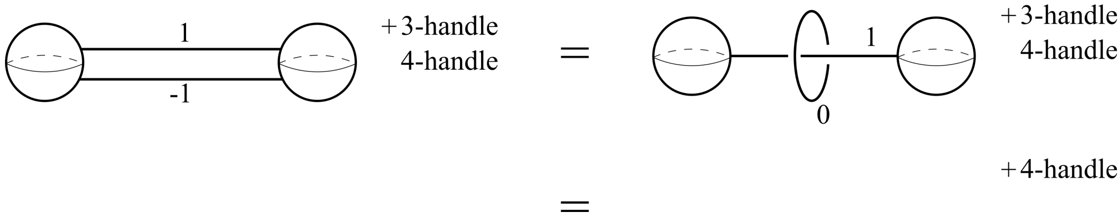


Figure 6.2: The result of surgery on C is S^4.

This gives us an achiral Lefschetz fibration $S^4 \to S^2$ with two singular fibers of opposite chirality.

Let us argue why this example may be important. It is always nice to have a not too complicated example. In this case, we only have two singularities of opposite chirality, the minimal amount needed to study such a setup. Also, the manifold S^4 does not admit a complex structure, which would have immediately ruled it out as a candidate for an elliptic fibration. So it really is a new example.

Furthermore, the manifold S^4 is of course a manifold that is studied from all different angles. For example, we know the eigenvalues and multiplicity of the Laplacian acting on p-forms for any value of p, see [IK79]. We also know its topology, characteristic classes, etcetera. This information is all relevant for compactifying on it. For example, a lot is known about M-theory on $AdS_7 \times S^4$ [Mal99].

But maybe the most important reason is that this example can be made much more concrete. There is an explicit map for the Hopf fibration and using the explicit construction of surgery on fibrations by Matsumoto from Section 5.4.1, we can write down a concrete map for the fibration $S^4 \to S^2$. This construction would lead to the singularities both being in the same fiber, but by permuting the map, we can get both singularities in different fibers. We did not have time in this project to pursue this line of attack, but encourage any reader that is feeling brave to do so.

6.3.2 The Lagrangian sphere

In Remark 5.4.4, we constructed a sphere between two singularities with the same vanishing cycle (and possibly of opposite chirality, but this is not necessary). One direction along this sphere lies in the direction of the fiber, the other in the direction of the space Σ. Looking back at how we obtained our log-symplectic structure ω on X from the achiral Lefschetz fibration, we see that $\omega\big|_L = 0$. Therefore, we call this sphere the *Lagragian sphere*. This terminology is taken from symplectic geometry.

Of course, the size[2] of this sphere is related to the distance between the branes. We hence feel like the form(s) needed for our compactification ansatz should be related to this sphere.

6.3.3 Forms on an achiral Lefschetz fibration

On an achiral Lefschetz fibration $\pi\colon X \to \Sigma$, we get a log-symplectic form. But there are more forms that arise from the achiral Lefschetz fibration. These live on Z.

A cosymplectic structure on Z

According to Theorem 4.5.2, we get a log-symplectic structure ω with zero locus Z from the achiral Lefschetz fibration. On this codimension 1 zero locus Z, there exists a cosymplectic structure.

> **Definition 6.3.1.** A cosymplectic structure on a manifold Z^{2n-1} is a pair of closed forms $\theta \in \Omega^1(Z)$ and $\sigma \in \Omega^2(Z)$ such that $\theta \wedge \sigma^{n-1} \neq 0$.

The following theorem is taken from [Cav13]. We do not define the notion of a modular vector field, but such a vector field always exists and hence this theorem tells us that Z admits a cosymplectic structure.

[2] This is intuitively speaking. The natural way of measuring this volume would be using the symplectic structure. But since it is a lagrangian subspace, it has zero volume.

Theorem 6.3.2. *Let (M, π) be a log-symplectic manifold, let Z be a connected component of the zero locus and ξ a modular vector field of π. Then the pair (π, ξ) determines the following structure on Z:*

1. *The normal bundle of Z as a vector bundle*
2. *A closed 1-form $\theta \in \Omega^1(Z)$ such that $\theta(\xi) = -1$*
3. *A closed 2-form $\sigma \in \Omega^2(Z)$ such that $\iota_\xi \sigma = 0$ and (θ, σ) is a cosymplectic structure on Z.*

Further, any log-symplectic structure inducing the data above on Z is equivalent to a neighbourhood of the zero section of the normal bundle of Z endowed with the structure

$$d \log |x| \wedge \theta + \sigma$$

where $|\cdot|$ is the distance to the zero section measured with respect to a fixed fiberwise linear metric on $\mathcal{N}_Z$.

This form of the log-symplectic structure around Z is a more global version of the local form of a log-symplectic structure on Z given in 4.4.3. For the standard log-symplectic structure

$$\omega = d \log |x_1| \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}$$

we have $\theta = dx_2$ and $\sigma = dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}$.

Cosymplectic structures are very strong and sometimes described as the odd-dimensional equivalent of Kähler structures. Although it is not clear what is the role of Z in the physics and hence it is not clear for what situations this structure is relevant, it is a strong thing to have and might be useful e.g. in the specific cases where our total space X is given as $X = Z^{2n-1} \times M^1$.

Chapter 7

Discussion and further questions

Our mission has been an ambitious one and a lot of questions have been left unanswered. In this chapter we will try to formulate some questions that further research could focus on and also take a critical look at our own results

Evidence for Conjecture 6.2.1

In the previous chapter, we made the conjecture that achiral Lefschetz fibrations are able to describe anti-branes in F-theory, contrary to the often used elliptic fibrations in F-theory. This is a rather big claim and it is hard to say what is necessary to prove it. In a way, the role of the fibration in F-theory is bookkeeping. But we want the rules of this bookkeeping to be compatible with the laws that physics gives us. An example of that is that branes and anti-branes should be able to annihilate. We have seen that this can be done in achiral Lefschetz fibrations, so this is reassuring. But it is in no way a proof. We should be critical of ourselves and test this conjecture against all the properties of anti-branes that we know of. The breaking of supersymmetry would certainly be one of those.

Compactification on F-theory including anti-branes

To get information from this new formulation that is relevant for physics, we should be able to perform F-theory compactifications on configurations that contain anti-branes.

This could lead to a potential for a brane-anti-brane pair, as we tried to find in Section 6.3. This is hard, because it is not clear what are relevant compactification ansatzes for this case.

Breaking supersymmetry

One of the goals we formulated in Chapter 1 was that the mutual appearance of a brane and an anti-brane should break supersymmetry completely. We did not have time to see how this would manifest itself in our formulation of F-theory. Specifically: what kind of properties would the achiral Lefschetz fibration $\pi\colon X \to \Sigma$ have such that supersymmetry would break? Is it a property of the space X or of the map π? It cannot be Σ since that does not change whether we have a brane-anti-brane pair or not. This is strongly correlated to the problem of finding a compactification ansatz, since breaking of supersymmetry should also be visible in the action.

The symplectic structure

As we showed in Section 4.5, Lefschetz fibrations and symplectic structures are intertwined. It would only seem logical that, if achiral Lefschetz fibrations are the natural language for F-theory, the (log-)symplectic structure arising from it should also play a role. It is not clear whether this structure corresponds to something physical.

In b-Lefschetz fibrations and log-symplectic structures, there is also the zero locus Z. This separates the branes from the anti-branes. Does Z have a physical meaning? Could it maybe be interpreted as an O-plane in some cases? Z is in our work pretty much free to choose without changing anything about the singularities. Hence if Z would have a physical meaning, it would be after making some restrictions on it (for example requiring some geometric structures).

It is noteworthy that elliptic fibrations do not necessarily have a symplectic structure. This will actually allow us to speak of the volume of a fiber in F-theory. The fibers are normally just bookkeeping devices using only their complex structure, but a physical meaning of the symplectic structure could also give a physical meaning to the volume of the fibers.

Hyperfibrations

The setup we have worked with is not general: we have structurally ignored the 8 dimensions that the 7-branes lie in. This means that the 7-branes must all be parallel. In general this of course does not have to be the case and we will have to find a type of fibration that is related to achiral Lefschetz fibrations that can go from a 12-dimensional to a 10-dimensional space. A logical candidate for this would be *hyperfibrations*.

Specifically, b-hyperfibrations would make the best candidates, as these are the higher dimensional analogues of b-Lefschetz fibrations. They also give rise to a log-symplectic structure.

The definition of these structures is rather involved and we will not treat it here. It has to be said that things do not get more intuitive when considering these structures. More details on b-hyperfibrations can be found in [CK16].

Appendix A

Mathematical prerequisites

In this appendix we treat a few of the mathematical concepts used in this thesis that may not be familiar to physicists. Still, the introductions given here are very short and non-precise, but hopefully enough to understand how we use them in this thesis.

A.1 Regular/Critical points/values

When talking about Lefschetz fibrations, it is inevitable to talk about regular and critical values and points, so it is good to clear up the language.

Suppose we have two smooth manifolds M, N and a map $f\colon M \to N$. We can define the derivative df of this map by picking local coordinates $x_1, \ldots, x_m$ on M and $y_1, \ldots, y_n$ on N. Then df at the point $x \in M$ is given by the $n \times m$ matrix

$$df_x = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_m}(x) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_m}(x) \end{pmatrix}.$$

We say that x is a *regular point* of f if df_x is surjective and that it is a *critical point* if this is not the case. We call $y \in N$ a *regular value* if all $x \in f^{-1}(y)$ are regular points and we call y a *critical value* otherwise. Equivalently, we call y a critical value if is the image of a critical point and call y a regular value if it is not.

Note that in general a “value” lies in the target space and a point lies in the “domain”.

A.2 Symplectic structures

A symplectic structure is a geometric structure on a manifold. It can be thought of like a metric: it lives on the manifold and is not intrinsic to it. To recap, a manifold is a set of points. That set of points is then given a topology so that it is a topological manifold. That topological manifold is then given a smooth structure, giving us a smooth manifold. The metric, symplectic structure or complex structure are only then given to the manifold, e.g. some smooth manifolds admit multiple non-equivalent complex structures.

Definition A.2.1. A symplectic structure on a manifold M is a 2-form ω such that

1. ω is closed, i.e. $d\omega = 0$
2. ω is non-degenerate, i.e. for all $x \in M$, if $v \in T_xM$ is such that $\omega(v, w) = 0$ for all $w \in T_xM$, then $v = 0$.

Note that, as 2-form, ω is in particular bilinear and anti-symmetric.

Another way of stating non-degeneracy is that the map $T_xM \to T_x^*M$ given by $v \mapsto \omega(v, -)$ is a bijection.

A symplectic manifold has to be even dimensional and orientable, since $\omega \wedge \omega \wedge \ldots \wedge \omega$ is a volume form on it. Also e.g. S^4 cannot have a symplectic structure, since it has no 2-forms that are closed but not exact, and exactness of a 2-form leads to a failure in non-degeneracy on closed manifolds.

A.3 Connected sums

There are two types of connected sums that are used throughout this thesis. We start with the ordinary connected sum.

Definition A.3.1. Let M_1, M_2 be manifolds of dimension n. Choose two embeddings $\iota_i \colon D^n \to M_i$ for $i = 1, 2$. Then we define

$$M_1 \# M_2 = (M_1 - \iota_1(0)) \sqcup (M_2 - \iota_2(0)) / \sim$$

where $\iota_1(tu) \sim \iota_2((1-t)u)$ for all $u \in S^{n-1}$.

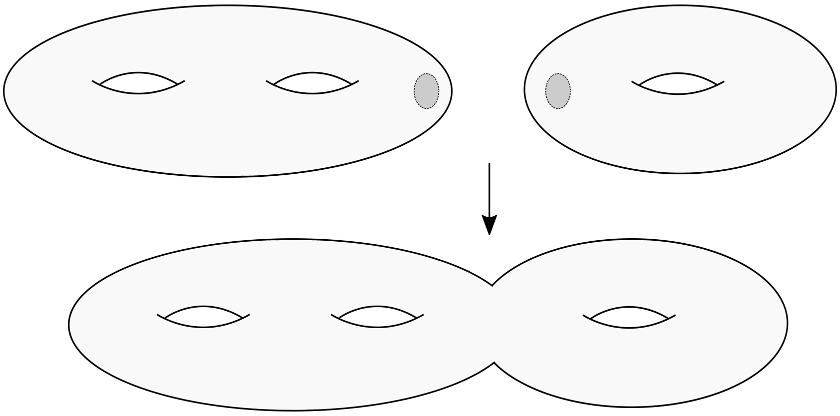


Figure A.1: An illustration of the connected sum of two manifolds. If we imagine the manifolds as surfaces, this illustrates the ordinary connected sum. If we image the manifolds as filled, 3-dimensional objects with the surfaces as their boundary, this illustrates the boundary connected sum.

Topologically, one cuts a disc from each manifold and glues the boundaries together, only in this construction the resulting manifold is again smooth. The fact that the result does not depend on the chosen embeddings is a consequence of the disk theorem.

Then there is the boundary connected sum. This is defined as follows:

> **Definition A.3.2.** Let M_1, M_2 be manifolds with boundary and let $D_i \subseteq \partial M_i$ be disks embedded in the boundary for $i = 1, 2$. Let $\varphi\colon D_1 \to D_2$ be an orientation reversing diffeomorphism. Then we define
>
> $$M_1 \natural M_2 = M_1 \sqcup M_2/(D_1 \sim_\varphi D_2)$$
>
> where the equivalence means we identify the two disks via the map φ.

See Figure A.1 for an illustration of these definitions.

A.4 Homotopy groups

For a complete treatment, see [Hat02].

Homotopy groups are one of the main objects of study in algebraic geometry. For every $n = 0, 1, 2, \ldots$, there exists the homotopy group $\pi_n(X, x_0)$, where X is some topological space and $x_0 \in X$.

The general idea is that $\pi_n(X, x_0)$ is the group of classes of maps $(S^n, s_0) \to (X, x_0)$. This notation means that it is a map from S^n to X and maps the point s_0 to x_0. Two maps belong to the same class if they can be deformed into each other while keeping this point fixed.

All maps in this section are assumed to be continuous.

Homotopy

This notion of "deforming maps" can be made more precise as follows.

> **Definition A.4.1.** Two maps $\varphi_0, \varphi_1 \colon X \to Y$ are *homotopic* if there exists a map $\Phi \colon X \times [0,1] \to Y$ such that $\Phi(0, -) = \varphi_0$ and $\Phi(1, -) = \varphi_1$.

> **Definition A.4.2.** Two maps $\varphi_0, \varphi_1 \colon (X, x_0) \to (Y, y_0)$ are *homotopic relative to a basepoint* if they are homotopic through some map $\Phi \colon X \times [0,1] \to Y$ and $\Phi(x_0, t) = y_0$ for all $t \in [0,1]$.

Homotopy groups

We are now ready to define homotopy groups

> **Definition A.4.3.** For $n \geq 0$, we define the homotopy group $\pi_n(X, x_0)$ as the group of homotopy classes of maps $(S^n, s_0) \to (X, x_0)$.

The constant map is always there and corresponds to the identity group element. Homotopy groups are thus used to see what "holes" there are in our space: images of S^n in X that cannot be deformed into the constant map.

The group structure may not be clear. We will not go into the group structure for $n \geq 2$ here, since it is not used in this thesis. Also, for $n = 0$ there is no group structure, so we will only treat the group structure on $n = 1$.

One can ask whether it makes a difference to the group if we choose x_0 or some other point x_1 as our base point. The answer is that it only makes a difference if there is no path between the two. Since we mainly concern ourselves with connected spaces and these are always path connected in the world of manifolds, we can drop the base point from our notation.

The case $n = 0$

The set $\pi_0(X, x_0)$ is the only one of the homotopy groups that does not admit a group structure. It is easy to see what it represents. The sphere $S^0 \cong \{-1, 1\}$ and we choose without loss of generality $s_0 = 1$. A homotopy between two maps $\varphi_0, \varphi_1 \colon (S^0, s_0) \to (X, x_0)$ is the same as a path between $\varphi_0(-1)$ and $\varphi_1(-1)$. So there are as many homotopy classes of maps $(S^0, s_0) \to (X, x_0)$ as there are connected components of the space X. So if $\pi_0(X) = \{0\}$, then the manifold X is connected.

The case $n = 1$: the fundamental group

The case $n = 1$ is the most famous homotopy group. It is often referred to as the fundamental group. This group consists of homotopy classes of closed paths that can be deformed into each other while keeping the endpoints fixed.

Let us give a simple example of a path that is not homotopic to the constant path. For this we look at $\mathbb{R}^2 - \{0\}$ and pick the path $\gamma(t) = (\sin(t), \cos(t)) \in S^1$, where we parametrize S^1 as $[0, 2\pi]$ with endpoints identified. This path runs around the origin once in the counter-clockwise direction. We cannot hope to contract it to the constant path, since doing so would require us to go through 0 and hence outside our space. So we already see that $\pi_1(\mathbb{R}^2 - \{0\}) \neq 0$.

Now let us first go into the group structure and then continue this example. The multiplication in the group is given by concatenation of paths. So the path $\gamma_1 \cdot \gamma_2$ is the path we get by first walking through γ_1 and then through γ_2. The identity element of this group is the class of the constant path, since this does not change the path it is concatenated with at all.

Back to our example $\pi_1(\mathbb{R}^2 - \{0\})$. We note that we can also run twice around 0 in counter-clockwise direction. This is not homotopic to running around it once. It is however the concatenation of the path that runs around around once with itself. So concatenation of paths that go around in the same direction work as if they are added: concatenating the paths that run around 4 times and 7 times in counter-clockwise direction, gives us a path that runs around 11 times in counter-clockwise direction.

What happens if we concatenate the path going around counter-clockwise once with the path going around once in the clockwise direction? This can be contracted (i.e. there is a homotopy to the constant path), so it is the identity element again. Concatenating

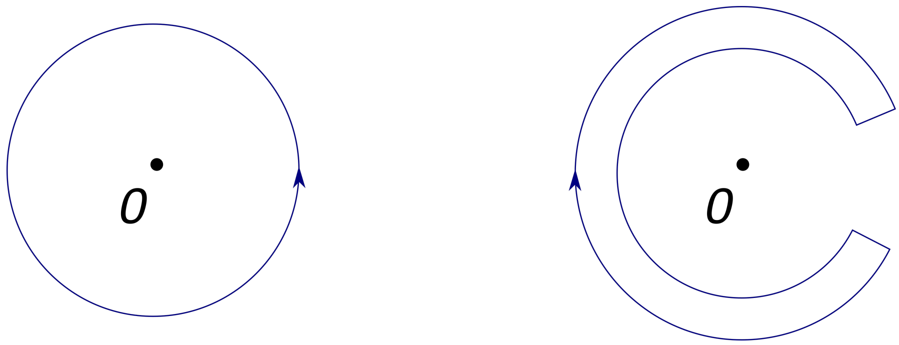


Figure A.2: Illustration of two different paths around the origin. On the left there is the path that goes around once in the counter-clockwise direction. On the right there is this path concatenated with the path that goes around once in the clockwise direction. One can see that this is indeed contractible to the constant path.

paths that run in opposite directions hence works as subtraction.

At this point, it may not come as a surprise that we can indeed make the identification $\pi_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{Z}$. This is the simplest case of a non-trivial fundamental group.

In our chapter on Lefschetz fibrations, we have a 2-dimensional base space Σ of our Lefschetz fibration. The monodromy representation is then defined as a map $\pi_1(\Sigma^*) \to \mathcal{M}(F)$ where Σ^* is obtained from Σ by deleting the critical values. We can now see that locally around a critical value, Σ^* looks like a disk with a point removed. We know that the homotopy classes of paths around this critical point are in correspondence with $\mathbb{Z}$. It now seems only logical how the monodromy representation preserves the group structure: if we run around once and have a monodromy given by $\psi \colon F \to F$, then running around in the opposite direction gives ψ^{-1}. Hence running around 7 times in one direction, then 2 times in the opposite direction and 4 times in the first direction, we get $\psi^4 \circ \psi^{-2} \circ \psi^7 = \psi^9$.

Definition A.4.4. A topological space X is called *simply connected* if $\pi_1(X) = 0$.

If one is busy reading Chapter 5, it is a good exercise to try and see why 1-handles determine the fundamental group in a handlebody. In particular, if a handlebody has no 1-handles, it is simply connected.

Homotopy invariance

There is a notion of two topological spaces being *homotopy equivalent*. Very intuitively speaking, this is slightly weaker than spaces that can be deformed into each other as is often presented in popular science (the donut and coffee mug being the same), which is called a homeomorphism. For homotopy equivalence, we also allow "contractions": the interval is homotopy equivalent to a point, since we can contract it. Hence the cylinder $S^1 \times [0,1]$ is homotopy equivalent to the circle S^1. The precise definition is as follows:

> **Definition A.4.5.** Two topological spaces X, Y are homotopy equivalent if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that fg is homotopic to the identity on Y and gf is homotopic to the identity on X.

This next theorem is our reason for introducing the concept.

> **Theorem A.4.6.** *Suppose that two topological spaces X, Y are homotopy equivalent to each other. Then $\pi_n(X) = \pi_n(Y)$ for all n.*

The space $\mathbb{R}^2 - \{0\}$ that we just considered is homotopy equivalent to the space S^1. Hence we now also know that $\pi_1(S^1) = \mathbb{Z}$.

Appendix B

Details of surgery on fibers

In this appendix, we fill in the gaps left in the proof of Theorem 5.4.3.

First of all, we define the map $h\colon \partial N(S_0) \to \partial N(C_0)$ and prove that it satisfies $g\big|_{\partial N(C_0)} \circ h = f\big|_{\partial N(S_0)}$.

We split $\partial N(S_0)$ into four regions:

$$\begin{aligned}
T_U &= \{(u_1, u_2) \in U \mid |u_1 u_2| \le \epsilon, |u_2| = \delta\} \\
T_V &= \{(v_1, v_2) \in V \mid |v_1 v_2| \le \epsilon, |v_2| = \delta\} \\
U' &= \{(u_1, u_2) \in U \mid |u_1 u_2| = \epsilon\} \\
V' &= \{(v_1, v_2) \in V \mid |v_1 v_2| = \epsilon\}.
\end{aligned}$$

We then define h separately on each of these components

$$\begin{aligned}
h(u_1, \delta e^{i\theta}) &= (u_1 \delta e^{i\theta}, e^{i\theta}, 1) \in D_\epsilon \times C_0 \times \{1\} && \text{for} \quad (u_1, \delta e^{i\theta}) \in T_U, \\
h(v_1, \delta e^{i\theta}) &= (\overline{v_1} \delta e^{i\theta}, e^{i\theta}, 0) \in D_\epsilon \times C_0 \times \{0\} && \text{for} \quad (v_1, \delta e^{i\theta}) \in T_V, \\
h(u_1, u_2) &= \left(u_1 u_2, \epsilon^{-1} u_2 |u_1|, \frac{\epsilon^{-1}\delta - |u_1|}{\epsilon^{-1}\delta - \epsilon\delta^{-1}} \right) \in \partial D_\epsilon \times C_0 \times [0,1] && \text{for} \quad (u_1, u_2) \in U', \\
h(v_1, v_2) &= \left(\overline{v_1} v_2, \epsilon^{-1} v_2 |v_1|, \frac{\epsilon^{-1}\delta - |v_1|^{-1}}{\epsilon^{-1}\delta - \epsilon\delta^{-1}} \right) \in \partial D_\epsilon \times C_0 \times [0,1] && \text{for} \quad (v_1, v_2) \in V'.
\end{aligned}$$

To remind ourselves, we repeat the definition of f and g.

$$f\big|_U(u_1,u_2) = u_1u_2,$$
$$f\big|_V(v_1,v_2) = \overline{v_1}v_2.$$

The map $g\colon N(C_0) = D_\epsilon \times C_0 \times [0,1] \to D_\epsilon$ is given by projection onto the first factor.

It is now easy to see that we have $g\big|_{\partial N(C_0)} \circ h = f\big|_{\partial N(S_0)}$. Hence h is fiber preserving.

We also have to prove that, given an achiral Lefschetz fibration $\pi\colon X \to \Sigma$ with a regular fiber F_0 over the point $x_0 \in \Sigma$ and a circle $C \subseteq F_0$, we have an orientation preserving, smooth embedding $\varphi\colon N(C_0) \to X$ such that

(i) $\varphi(C_0) = C$,

(ii) φ is fiber preserving, i.e. if we identify D_ϵ with a disk in Σ centered at x_0, we have $\pi \circ \varphi = g\colon N(C_0) \to D_\epsilon$.

Because F_0 is a regular fiber, we can choose a neighbourhood $L \cong D^2$ of x_0 such that $\pi^{-1}(L) \cong F_0 \times L \cong F_0 \times D^2$. We can embed $C_0 \times [0,1]$ into F_0 and hence we can embed $N(C_0) = C_0 \times [0,1] \times D^2$ into $\pi^{-1}(L)$ in a fiber preserving way.

Conversely, if F_0 is a singular fiber with two singularities of opposite chirality, then it is made up of two spheres, which we call S and R that intersect each other transversely at two points. We have to find an embedding $\psi\colon N(S_0) \to X$ such that

(i) $\psi(S_0) = S$,

(ii) $\psi(D_+ \cup D_-) = R \cap \psi(N(S_0))$,

(iii) ψ is fiber preserving, i.e. if we identify D_ϵ with a disk in Σ centered at x_0, we have $\pi \circ \psi = f\colon N(S_0) \to D_\epsilon$.

Note that the first two requirements can easily be fulfilled because of the form of the singular fiber. Because $N(S_0) \cong S^2 \times D^2$, the third property is asking for a specific framing on this embedding of S_0. Since $\pi_2(O(2)) = 0$, there is a unique framing for an embedding of a 2-sphere. By the tubular neighbourhood theorem, we can always get an embedding together with a framing. Since the framing is unique, the required embedding exists.

Bibliography

[AGEGK14] Lara B. Anderson, Iñaki García-Etxebarria, Thomas W. Grimm, and Jan Keitel, *Physics of F-theory compactifications without section*, JHEP **12** (2014), 156.

[BBS06] K. Becker, M. Becker, and J. H. Schwarz, *String theory and M-theory: A modern introduction*, Cambridge University Press, 2006.

[BLT12] Ralph Blumenhagen, Dieter Lüst, and Stefan Theisen, *Basic concepts of string theory*, Springer Science & Business Media, 2012.

[Cav13] Gil R. Cavalcanti, *Examples and counter-examples of log-symplectic manifolds*, ArXiv e-prints (2013).

[CK16] Gil R Cavalcanti and Ralph L Klaasse, *Fibrations and log-symplectic structures*, arXiv preprint arXiv:1606.00156 (2016).

[Den08] Frederik Denef, *Les houches lectures on constructing string vacua*, arXiv preprint arXiv:0803.1194 (2008).

[DHIS87] M. J. Duff, P. S. Howe, T. Inami, and K. S. Stelle, *Superstrings in D=10 from supermembranes in D=11*, Physics Letters B **191** (1987), 70–74.

[Don98] S. Donaldson, *Lefschetz fibrations in symplectic geometry*, Proceedings of the International Congress of Mathematicians **2** (1998), 309–314.

[FM11] B. Farb and D. Margalit, *A primer on mapping class groups (pms-49)*, Princeton Mathematical Series, Princeton University Press, 2011.

[FM15] R. L. Fernandez and I. Marcut, *Lectures on Poisson geometry*, Springer, 2015.

[FVP12] Daniel Z. Freedman and Antoine Van Proeyen, *Supergravity*, Cambridge Univ. Press, Cambridge, UK, 2012.

[GGSW88] M.B. Green, M.B. Green, J.H. Schwarz, and E. Witten, *Superstring theory: Volume 1, introduction*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1988.

[GMP14] Victor Guillemin, Eva Miranda, and Ana Rita Pires, *Symplectic and poisson geometry on b-manifolds*, Advances in Mathematics **264** (2014), 864 – 896.

[GS99] R.E. Gompf and A. Stipsicz, *4-manifolds and kirby calculus*, Graduate studies in mathematics, American Mathematical Society, 1999.

[GSVY90] Brian R. Greene, Alfred D. Shapere, Cumrun Vafa, and Shing-Tung Yau, *Stringy Cosmic Strings and Noncompact Calabi-Yau Manifolds*, Nucl. Phys. **B337** (1990), 1–36.

[Has12] Koji Hashimoto, *D-brane: Superstrings and new perspective of our world*, Springer, Heidelberg, Germany, 2012.

[Hat02] A. Hatcher, *Algebraic topology*, Algebraic Topology, Cambridge University Press, 2002.

[Hul18] Hulpke, Alexander, *The j-invariant — Wikipedia, the free encyclopedia*, 2018, [Online; accessed September 17 2018].

[IK79] Ichiro Iwasaki and Kiyoshi Katase, *On the spectra of laplace operator on* $\lambda^* (S^n)$, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), no. 4, 141–145.

[Joh06] C.V. Johnson, *D-branes*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2006.

[KPMT13] Amir-Kian Kashani-Poor, Ruben Minasian, and Hagen Triendl, *Enhanced supersymmetry from vanishing euler number*, Journal of High Energy Physics **2013** (2013), no. 4, 58.

[Mal99] Juan Maldacena, *The large-n limit of superconformal field theories and supergravity*, International journal of theoretical physics **38** (1999), no. 4, 1113–1133.

[Mar10] Stephen P Martin, *A supersymmetry primer*, Perspectives on supersymmetry II, World Scientific, 2010, pp. 1–153.

[Mat85] Yukio Matsumoto, *Torus fibrations over the* 2 *-sphere with the simplest singular fibers*, J. Math. Soc. Japan **37** (1985), no. 4, 605–636.

[Mat96] ———, *Lefschetz fibrations of genus two-a topological approach*, Topology and Teichmüller spaces, World Scientific, 1996, pp. 123–148.

[MD02] Mahbub Majumdar and Anne-Christine Davis, *Cosmological creation of d-branes and anti-d-branes*, Journal of High Energy Physics **2002** (2002), no. 03, 056.

[Pol95] Joseph Polchinski, *Dirichlet branes and ramond-ramond charges*, Physical Review Letters **75** (1995), no. 26, 4724.

[Sch95] John H. Schwarz, *An SL(2,Z) multiplet of type IIB superstrings*, Phys. Lett. **B360** (1995), 13–18, [Erratum: Phys. Lett.B364,252(1995)].

[Ton09] David Tong, *Lectures on string theory*, arXiv preprint arXiv:0908.0333 (2009).

[Tra82] Bruce Trace, *On attaching 3-handles to a 1-connected 4-manifold*, Pacific Journal of Mathematics **99** (1982), no. 1, 175–181.

[VW96] Cumrun Vafa and Edward Witten, *Dual string pairs with N=1 and N=2 supersymmetry in four-dimensions*, Nucl. Phys. Proc. Suppl. **46** (1996), 225–247, [,225(1995)].

[Wei83] Alan Weinstein, *The local structure of poisson manifolds*, J. Differential Geom. **18** (1983), no. 3, 523–557.

[Wit96] Edward Witten, *Bound states of strings and p-branes*, Nucl. Phys. **B460** (1996), 335–350.