Electromagnetic duality, del Pezzo surfaces and Instantons

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Acknowledgements

This thesis was written as part of a double master program in Theoretical Physics and mathematical sciences. As such, it contains parts which are mainly physical or mathematical in nature, but also other parts where there may be a less clear distinction. For the mathematical part, the basis of the project was in a seminar organized by dr. Gil Cavalcanti. In this seminar, my colleagues Joost Broens, Ralph Klaasse and Reinier Storm and I presented parts from the book "The Geometry of Four-Manifolds" by Donaldson and Kronheimer. The mathematical theory was studied in this seminar is closely related to gauge theories in physics. The physics part of the thesis was done under supervision of Prof. dr. Stefan Vandoren. This part focusses on electromagnetic duality on four-manifolds, and is therefore related to the seminar in the sense that electromagnetism is an example of a gauge theory.

Along the way the seminar and the physics project have parted a little. As a result, the thesis is mainly focussed on the physics project and an overview is given of the results we learned during the seminar, and at some point a relation between the projects is made. The physics part also contains a lot of mathematics, as sometimes a mathematical formulation of a physics problem provides an elegant solution (for instance for the Dirac quantization condition, Chapter 2).

Joost Broens, who was also supervised by Vandoren and Cavalcanti, and I worked on the same topic of electromagnetic duality. Therefore, our theses contain parts which are similar to each other. However, along the way we have tried to make a distinction between our work by putting focus different parts.

Presenting in the seminar was often a very non-trivial task, but I learned to tackle this challenge during the year. To be able to explain material on the blackboard that had cost a lot of effort to understand yourself, was a nice experience. Also, I really liked the fact that this project maneuvered on the border between mathematics and physics. Connecting knowledge from both areas and formulating physical knowledge I had in mathematical terms gave me a lot of satisfaction.

I thank Stefan Vandoren and Gil Cavalcanti for supervising me in this project. Stefan, I thank you for our pleasant and instructive meetings, where you forced me to formulate the things I had learned as clear as possible; especially for the inspiring meeting we had the day before I presented my thesis. Gil, I thank you for your approach in the seminar that we were really all learning together and especially your relaxed approach when I was stuck on something in the book or when I was 'freakin out' about the last project.

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Contents

Introduction

Gauge theories are of major importance in physics, as they provide a successful description of the standard model interactions. Quantum electrodynamics is the theory of the electromagnetic interaction between fundamental particles. It is a gauge theory with the gauge group $U(1)$ and the electromagnetic four-potential its gauge field that describes the photon as the gauge boson. Yang-Mills theory is the gauge theory based on the non-abelian gauge group $SU(N)$. The success of this theory is found in the standard model. The gauge-theory based on the group $SU(2)$ provides us with a description of the weak interaction; it has 3 gauge bosons being the $W^{\pm}$ and the Z bosons. Quantum Chromodynamics, a theory of the strong interaction, is a gauge-theory with $SU(3)$ gauge and has 8 gauge bosons: the gluons. The complete standard model is described as a gauge theory with gauge group $U(1) \times SU(2) \times SU(3)$.

General relativity tells us that in the presence of matter and energy, space-time is curved. Instead of a flat Minkowski space, space-time is a manifold: locally it looks like a flat space, but it may have curvature which on the global level results in curves or topologically non-trivial shapes.

Since space-time may be curved, it is important to formulate the physical theories in such a way that they apply for any space-time, flat or curved. For gauge theories, the way to approach this is by using the notion of vector bundles. Gauge fields can be regarded as connections on a vector bundle, while matter fields are sections of this vector bundle. In this frame-work, gauge-transformations arise naturally as bundle automorphisms and connections define a covariant derivative of the sections, or fields. Also, the field strength tensor is defined as the curvature form associated the connection.

Electromagnetism can be formulated on a curved space-time in terms of differential geometry. The field strength tensor, which has the electric and magnetic field as its components, is expressed as a two-form on the space-time manifold. Locally, this two form can be written as the exterior derivative of a one-form: the vector-potential. Maxwell's equations are translated into the condition that the two-form must be closed and coclosed. However, when there are fields that describe charged particles on space-time, we interpret the theory as a $U(1)$ gauge theory. If there are charged bosons that are described by a complex scalar field, the vector potential one-form is interpreted as a connection on a line bundle over the space-time manifold. Similarly, if there are fermions that are described by a spinor field, we can interpret the vector potential as a connection on a line bundle which is associated with a $Spin^c$ bundle.

Using this interpretation, we can deduce a quantization result: in the presence of a boson, the flux through a closed two-dimensional surface can only take integer values. This is known as the *Dirac quantization*, named after Dirac who first proved this from general principles of quantum mechanics [**7**]. From a mathematical viewpoint this quantization is very natural since the flux can be identified with the first Chern class, which is integer valued. In the presence of a fermion we have a similar result: the flux takes integer values, or half integer values depending on the existence of a *Spin* structure on the space-time manifold. This was first noticed by Hawking and Pope [**13**].

Maxwell's equations in vacuum, that is, in the absence of charged particles, show a remarkable symmetry: they are invariant under interchanging of the electric and magnetic field. This is called *electromagnetic duality.* The symmetry can be extended to a full $SL(2,\mathbb{R})$ symmetry: any invertible real linear transformation of the pair of fields leaves the equations invariant. These transformations can be encoded in the Möbius transformations of a single parameter τ, that contains the gauge coupling. When there is charged matter on the space-time manifold, we can investigate how much of this symmetry remains. This is done by studying the partition function. By use of the Dirac quantization it is possible to express the partition function as a sum over the classical solutions: it takes the form of a theta function.

It turns out that the original $SL(2,\mathbb{R})$ invariance is broken down to a discrete subgroup: the partition function transforms as a modular form under $SL(2,\mathbb{Z})$ Möbius transformations of the parameter τ. The weights of this transformation depend on topological invariants of the space-time manifold: the Euler characteristic and the signature.

With the Dirac quantization and the expression of the partition function as a theta function at hand, we can calculate the partition function for a variety of four-dimensional manifolds. Del Pezzo surfaces form an interesting family of four-manifolds.

Del Pezzo surfaces where named after the Italian mathematician Pasquale del Pezzo, who studied them already at the end of the 19th century. Their relevance for physics came much later in the area of superstring theory. According to superstring theory, space-time is 10 dimensional. We only experience a four-dimensional space-time because the other 6 dimensions are compactified: they are rolled up into very small manifolds called Calabi-Yau manifolds. Del Pezzo surfaces are of interest in string theory because they form surfaces inside this Calabi-Yau manifolds: in some cases, Calabi-Yau manifolds can be described as fiber bundles over a del Pezzo surface (for instance in [**12**]) or del Pezzo surfaces form special subvarieties of the Calabi-Yau manifolds (for instance in [**5**]). Another link to string theory is the so called 'mysterious duality'. This is a correspondence between the U-duality group of M-theory compactified on a torus T^k, and the diffeomorphism group of the complex projective plane with k points blown up: a del Pezzo surface [**14**]. Lastly, del Pezzo surfaces admit a mirror symmetry: a duality between del Pezzo surfaces and certain Landau-Ginzburg models [**15**].

From this perspective, it is interesting to describe electromagnetism on the four-dimensional del Pezzo surfaces. In order to construct the partition function, it is important to understand the topology of these del Pezzo surfaces. They can be described in terms of a blow-up construction: we replace a point by a copy of the complex projective line. The partition function also depends on a choice of metric on the manifold. This metric may be difficult to construct explicitly, but it turns out we can solve for the metric of a number of parameters. We obtain the partition function as a theta function, depending on a number of parameters.

The equations of motion of the non-abelian $SU(N)$ gauge theories , Yang-Mills theories, in Euclidean space-time, admit a special type of solutions: *instantons.* These instanton fields are characterized by the fact that they have self-dual or anti-self-dual fields strength. The instantons owe their name to the fact that these fields are localized in space and time.

Given a $SU(N)$ vector bundle over a compact space-time manifold, we can have many connections on this bundle. The gauge group of this bundle has an action on the set of connections: it transforms one connection into another. By identifying gauge-equivalent connections, we obtain the space of all gauge-equivalent classes of connections, generically an infinite dimensional object. However, when we focus only on the subset gauge equivalence classes of instantons, we obtain, under some assumptions, a space which has the structure of a finite dimensional smooth manifold. This space is the moduli space of instantons. By integrating certain cohomology classes of the quotient space over this moduli space, one produces diffeomorphism invariants of the underlying manifold. These invariants are called the *Donaldson invariants*.

This thesis is organized as follows. We set out the basic theory of vector bundles, connections and curvature in chapter 1. This framework forms the mathematical basis of gauge theory on a curved space-time.

In chapter 2 we describe electromagnetism on a Riemannian manifold, by fitting this in the mathematical framework of Yang-Mills theory. We prove the Dirac quantization condition for $U(1)$gauge theory with scalar fields and with spinor fields. The latter requires a short discussion of the notion of a $Spin^c$ structure. This discussion is based on parts of [**2**], [**3**] and [**16**].

In chapter 3 we use the Dirac quantization condition to express the partition function in terms of a theta function and study its modular transformation properties. Finally, the partition function is calculated on two different kind of four-dimensional space-times, inspired by general relativity. The material in this chapter is mainly based on the papers on electromagnetic duality by Witten [**25**], Verlinde [**23**] and Olive and Alvarez [**19**].

The partition function for del Pezzo surfaces is calculated in chapter 4. First we give a detailed description of the complex projective plane. We calculate the partition function for this space and check explicitly the modular transformation properties prescribed by the previous chapter. Then we describe how to construct the different del Pezzo surfaces by blowing up points and how to calculate the topology of these spaces. Finally we explicitly construct the partition function in terms of a number of parameters.

In chapter 5, we give a brief overview of the theory of moduli spaces of Yang-Mills instantons. Most of this material is based on the book by Donaldson and Kronheimer [**9**] and on the seminar organized by Gil Cavalcanti.

Finally, in the appendices one finds some mathematical background on principal G-bundles, Spin structures and lattices. This is included in the appendices to fix notation and to clarify some constructions. In appendix D we give a Dutch summary of this thesis, aimed at the reader without a background in physics or mathematics.

CHAPTER 1

Instantons on four-manifolds

In this first chapter, we discuss the concepts of Yang-Mills theory and instantons. In mathematical language, Yang-Mills theory is a theory of vector bundles over manifolds, connections on these vector bundles and their curvature. In physical language, it is a non-abelian gauge theory, which plays a central role in the standard model by describing various interactions between fundamental particles.

1. Bundles and connections

Let G be a Lie group and $P \to X$ be a principal G-bundle over a smooth manifold X. We refer to Appendix A. for more information about principal G-bundles. Associated to such a principal G-bundle is a vector bundle $E \to X$ with structure group G. We can define a connection as follows:

Definition 1.1. A connection A on E is defined by a covariant derivative of sections of E: it is a linear map

$$\nabla_A : \Omega^0_X(E) \to \Omega^1_X(E) \tag{1.1}$$

which satisfies the Leibnize rule: $\nabla_A(f.s) = f\nabla_A s + df.s$, for a section $s \in \Omega^0_X(E)$ and a smooth function $f : X \to \mathbb{R}/\mathbb{C}$.

Here, $\Omega^p_X(E)$ denotes sections of $\Lambda^p T^*X \otimes E$: p-forms with values in E. Let us denote $\mathfrak{g}$ for the Lie algebra of G and $\mathfrak{g}_E$ for the bundle of Lie algebras associated to the adjoint representation of G, so $\mathfrak{g}_E$ is a subbundle of the endomorphism bundle End $E = E \otimes E^*$. Below, we let E be a complex vector bundle of rank n.

If we have the trivial bundle, $E = X \times \mathbb{C}^n$, there is an easy example of a connection: a covariant derivative on this bundle is given by the ordinary differentiation of vector valued functions. This connection is called the *product connection*.

Given a connection A on E and a Lie-algebra bundle-valued one-form $a \in \Omega^1_X(\mathfrak{g}_E)$, the operator $\nabla_A + a$ is again a covariant derivative

$$\nabla_A + a : \Omega^0_X(E) \to \Omega^1_X(E) \tag{1.2}$$

where a acts on sections $s \in \Omega^0_X(E)$ via the contraction

$$\Omega^0_X(E) \times \Omega^1_X(\mathrm{End}E) \to \Omega^1_X(E), \quad (s, a) \mapsto a(s). \tag{1.3}$$

Conversely, the difference of two connections on E is defined as an element of $\Omega^1_X(E)$. This means that the space of all connections on E, $\mathscr{A}$, is an infinite dimensional affine space modeled on $\Omega^1_X(\mathfrak{g}_E)$.

Things become clearer when we study the connections in local trivializations of the bundle. Suppose that we have an open neighbourdhood U over which E is trivial, $E|_U \cong U \times \mathbb{C}^n$ via a trivialization $\tau : E|_U \to \mathbb{C}^n$. Using this trivialization τ, we can compare a connection A on E with the product connection on $U \times \mathbb{C}^n$ over U. In terms of covariant derivatives, we write

$$\nabla_A = d + A^\tau, \tag{1.4}$$

where the *connection form* A^τ is a $\mathfrak{g}$-valued 1-form. Given local coordinates x_i on U, we can write this covariant derivative in coordinates as

$$\nabla_A = \sum_i \nabla_i dx_i, \tag{1.5}$$

where

$$\nabla_i = \frac{\partial}{\partial x_i} + A_i^\tau \tag{1.6}$$

for matrix-valued functions A_i^τ on U.

It is important to note that these connection matrices A^τ depend on the choice of trivialization τ.

Definition 1.2. The *gauge group of* E, denoted $\mathscr{G}$, is the group of all automorphisms $u : E \to E$ that respect the structure on the fibers and cover the identity, i.e. act fiberwise.

There is a pointwise exponential map $\exp : \Omega^0_X(\mathfrak{g}_E) \to \mathscr{G}$. The gauge group acts on the set of connections by

$$\nabla_{u(A)} s = u \nabla_A (u^{-1} s). \tag{1.7}$$

We can expand this as $u\nabla_A u^{-1} = \nabla_A - (\nabla_A u)u^{-1}$, where we can take the covariant derivative of u by regarding it as a section of the bundle $\mathrm{End}(E)$. Therefore,

$$u(A) = A - (\nabla_A)u^{-1}. \tag{1.8}$$

If u is an automorphism of the trivial bundle $X \times \mathbb{C}^n$, and τ is a trivialization of E, then $(u\tau)$ is a new trivialization and we have

$$\begin{aligned} A^{u\tau} &= A^\tau - \{(d + [A^\tau, \cdot])u\}u^{-1} \\ &= A^\tau - \{du + A^\tau u - uA^\tau\}u^{-1} \\ &= uA^\tau u^{-1} - (du)u^{-1}. \end{aligned} \tag{1.9}$$

Hence, A^τ and $A^{u\tau}$ are different matrices that represent the same connection.

Concluding, we can say that a connection on a bundle over X is given by the following data. First, the bundle can be defined in terms of an open cover $\mathscr{U} = \{U_\alpha\}$ of X and transition maps

$$u_{\alpha\beta} : U_\alpha \cap U_\beta \to U(n) \tag{1.10}$$

such that $u_{\alpha\beta} = u_{\beta\alpha}^{-1}$ and $u_{ab}\, u_{bg}\, u_{ga} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$. (See appendix A.) The connection is given by matrix valued one-forms A_α on the open neighbourdhoods U_α such that

$$A_\alpha = u_{\alpha\beta} A_\beta u_{\alpha\beta}^{-1} - (du_{\alpha\beta})u_{\alpha\beta}^{-1}. \tag{1.11}$$

2. Curvature

We can extend the de Rham complex

$$\Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \dots \xrightarrow{d} \Omega^p_X \xrightarrow{d} \Omega^{p+1}_X \xrightarrow{d} \dots \tag{1.12}$$

with exterior derivatives d_A, defined by:

(1) $d_A = \nabla_A$ on $\Omega^0_X(E)$
(2) Leibniz rule: $d_A(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^p \omega \wedge d_A \theta$, for $\omega \in \Omega^p_X, \theta \in \Omega^q_X(E)$

The ordinary exterior derivative satisfies $d^2 = 0$, but this need not be the case for d_A. The Leibniz rule gives us that

$$d_A^2(\omega \wedge \theta) = d_A^2 \omega \wedge \theta + \omega \wedge d_A^2 \theta, \tag{1.13}$$

$$d_A^2(f\omega) = ddf \wedge \omega + f d_A^2 \omega = f d_A^2 \omega, \tag{1.14}$$

for $\omega \in \Omega^p_X, \theta \in \Omega^q_X(E)$ and f a smooth function on X. Hence, d_A^2 is an algebraic operator on $\Omega^p_X(E)$ that commutes with multiplication by smooth functions.

DEFINITION 1.3. The *curvature* $F_A \in \Omega^2_X(\mathfrak{g}_E)$ of a connection A is defined by

$$d_A d_A s = F_A s. \tag{1.15}$$

If we vary the connection A with an element $a \in \Omega^1_X(\mathfrak{g}_E)$, the curvature changes as

$$F_{A+a} = F_A + d_A a + a \wedge a, \tag{1.16}$$

where $a \wedge a$ denotes the combination of the wedge product with multiplication in $\mathfrak{g}_E \subset \operatorname{End} E$.

In a local trivialization, the curvature is given by a matrix of 2-forms

$$F_A^\tau = dA^\tau + A^\tau \wedge A^\tau. \tag{1.17}$$

In local coordinates, we can write the curvature matrix as $F_A^\tau = \sum_{i,j} F_{ij} dx_i \wedge dx_j$, where

$$\begin{aligned} F_{ij} &= [\nabla_i, \nabla_j] = [\frac{\partial}{\partial x_i} + A_i^\tau, \frac{\partial}{\partial x_j} + A_j^\tau] \\ &= \frac{\partial A_j^\tau}{\partial x_i} - \frac{\partial A_i^\tau}{\partial x_j} + [A_i^\tau, A_j^\tau]. \end{aligned} \tag{1.18}$$

Under bundle automorphisms, the curvature transforms as

$$F_{u(A)} = u F_A u^{-1}. \tag{1.19}$$

Note that the set of connections with zero curvature, called *flat* connections, is preserved by the gauge group $\mathscr{G}$.

The curvature satisfies the *Bianchi identity*:

$$d_A F_A = 0, \tag{1.20}$$

which follows easily from the definition of the curvature:

$$(d_A F_A)s = d_A(F_A s) - F_A(d_A s) = d_A(d_A^2 s) - d_A^2(d_A s) = d_A^3 s - d_A^3 s = 0, \tag{1.21}$$

for any section s.

EXAMPLE 1.4. Consider a complex vector bundle of rank 1, with Hermitian metric, i.e. a line bundle L. This has structure group $G = U(1)$ and we can identify the Lie algebra $\mathfrak{g}$ with $i\mathbb{R}$. Then, in a local trivialization, a connection on L is represented by a purely imaginary one-form A and the curvature is given by $F = dA$, as the commutator term vanishes since $U(1)$ is abelian.

We can write a gauge transformation as $u = \exp(i\chi)$, where χ is a real-valued function on X, and the connection transforms as

$$A \to A - id\chi \tag{1.22}$$

under this gauge transformation. Now the curvature is a purely imaginary two-form, which we can write as $F = -2\pi i\omega$. Then ω is a real two-form which is closed by the Bianchi identity. Therefore it defines a de Rham cohomology class $[\omega] \in H^2(X;\mathbb{R})$. Now if we have another connection $A' = A + a$, the curvature is $F' = F + da$, hence $[\omega'] = [\omega]$. We see that the cohomology class $[\omega]$ does not depend on the connection, so it only depends on the bundle L. We can identify $[\omega]$ with the first Chern class $c_1(L)$, since line bundles are classified by their first Chern class (Appendix A.).

3. (Anti-)Self-dual connections

If X is an oriented Riemannian manifold, we can consider the Hodge $*$-operator that transforms p-forms into $n-p$-forms ($n = \dim X$). It is defined by

$$\alpha \wedge *\beta = (\alpha, \beta) d\mu, \tag{1.23}$$

where $(\cdot,\cdot)$ is the natural metric on the $p-$forms and $d\mu$ is the Riemannian volume element. Now if X is a four-manifold, this operation takes two-forms to two-forms and $*^2 = 1_{\Lambda^2}$. The *self-dual* and *anti-self-dual* forms, Ω^+_X, Ω^-_X respectively, are the ± 1 eigenspaces of $*$: they are sections of the rank-3 bundles $\Lambda^\pm$, where

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-, \quad \alpha \wedge \alpha = \pm|\alpha|^2 d\mu, \text{ for } \alpha \in \Lambda\pm. \tag{1.24}$$

We can introduce a formal adjoint operator for the exterior derivative,

$$d^* : \Omega^p \to \Omega^{p-1}, \tag{1.25}$$

such that

$$\int_X (d\alpha, \beta) d\mu = \int_X (\alpha, d^*\beta) d\mu. \tag{1.26}$$

For oriented X, the adjoint operator can be expressed as $d^* = \pm * d *$. If X is compact, the Hodge theorem says that each real cohomology class has a unique harmonic representative α:

$$d\alpha = d^*\alpha = 0. \tag{1.27}$$

It is clear that the $*$ operator preserves these harmonic forms, so we get a decomposition

$$H^2(X;\mathbb{R}) = \mathscr{H}^+ \oplus \mathscr{H}^-, \tag{1.28}$$

where $\mathscr{H}^\pm$ are the self-dual and anti-self-dual harmonic 2-forms. These are the maximal positive and negative subspaces for the intersection form

$$Q : H^2(X;\mathbb{R}) \times H^2(X;\mathbb{R}) \to \mathbb{R}, \quad (\alpha, \beta) \mapsto \int_X \alpha \wedge \beta \tag{1.29}$$

on de Rham cohomology and $\dim \mathscr{H}^+ = b^+$, $\dim \mathscr{H}^- = b^-$. Note that the self-dual and anti-self-dual forms are orthogonal with respect to the intersection form:

$$\alpha \wedge \beta = \alpha \wedge *\beta = (\alpha, \beta) d\mu = (\beta, \alpha) d\mu = \beta \wedge *\alpha = -\beta \wedge \alpha = -\alpha \wedge \beta = 0. \tag{1.30}$$

The above splitting can be extended to bundle-valued two-forms:

$$\Omega^2_X(\mathfrak{g}_E) = \Omega^+_X(\mathfrak{g}_E) \oplus \Omega^-_X(\mathfrak{g}_E), \quad \Omega^\pm_X = \Gamma(\Lambda^\pm \otimes \mathfrak{g}_E). \tag{1.31}$$

Then, the curvature two-form F_A splits as

$$F_A = F_A^+ \oplus F_A^-. \tag{1.32}$$

We say a connection is *self-dual* is $F_A^- = 0$, and *anti-self-dual* if $F_A^+ = 0$.

DEFINITION 1.5. An *instanton* is a self-dual or anti-self-dual connection on a vector bundle over an oriented Riemannian manifold.

EXAMPLE 1.6. If $X = \mathbb{R}^4$, the anti-self-dual condition $F_A^+ = 0$ translates into

$$\begin{aligned} F_{12} + F_{34} &= 0 \\ F_{14} + F_{23} &= 0 \\ F_{13} + F_{42} &= 0. \end{aligned} \tag{1.33}$$

4. Yang Mills theory and instantons

Suppose we are given vector bundle E over a closed, oriented four-manifold X, with structure group $SU(n)$. The Lie algebra $\mathfrak{su}(n)$ consists of traceless skew-hermitian matrices $A^\dagger = -A$, such that $\mathrm{Tr}(A^2) = -|A|^2$:

$$\mathrm{Tr}(a^2) = a_{ij}a_{ji} = -a_{ij}^* a_{ij} = -|a_{ij}|^2. \tag{1.34}$$

Then, for the curvature we have

$$\begin{aligned} \mathrm{Tr}(F_A \wedge F_A) &= \mathrm{Tr}(F_A^+ \wedge F_A^+) + \mathrm{Tr}(F_A^- \wedge F_A^-) = (F_A^+, F_A^+)d\mu - (F_A^-, F_A^-)d\mu \\ &= -(|F_A^+|^2 - |F_A^-|^2)d\mu \end{aligned} \tag{1.35}$$

$$\begin{aligned} \mathrm{Tr}(F_A \wedge *F_A) &= \mathrm{Tr}(F_A^+ \wedge F_A^+) - \mathrm{Tr}(F_A^- \wedge F_A^-) = (F_A^+, F_A^+)d\mu + (F_A^-, F_A^-)d\mu \\ &= -(|F_A^+|^2 + |F_A^-|^2)d\mu = -|F_A|^2 d\mu. \end{aligned} \tag{1.36}$$

It can be shown that the quantity $\int_X \mathrm{Tr}(F_A)^2 = \int_X |F_A^-|^2 - \int_X |F_A^+|^2$ does not depend on the connection. It can be recognized as the second Chern class of the bundle E:

$$8\pi^2 c_2(E) = \int_X \mathrm{Tr}(F_A^2). \tag{1.37}$$

This is also called the *instanton number*. This quantity is positive (negative) when A is anti-self-dual (self-dual).

We can define the *Yang-Mills functional* on the space $\mathscr{A}$ of connections A on E as the square of the L^2 norm of the curvature

$$S(A) = \|F_A\|^2 = \int_X |F_A|^2 d\mu = -\int_X \mathrm{Tr}(F_A \wedge *F_A). \tag{1.38}$$

A connection A extremizes this functional if

$$\begin{aligned} \delta S = S(A+a) - S(A) &= 2\int_X (F_a, d_A a)d\mu + O(\|a\|^2) \\ &= 2\int_X (d_A^* F_A, a)d\mu + O(\|a\|^2) = 0, \end{aligned} \tag{1.39}$$

for all infinitesimal variations a, so the Euler-Lagrange equations are given by

$$d_A^* F_A = 0. \tag{1.40}$$

We see that the self-dual and anti-self-dual connections are solutions to these Euler-Lagrange equations. Conversely,

$$\int_X |F_A|^2 d\mu = \int_X |F_A^{\pm}|^2 d\mu \mp \int_X \mathrm{Tr}(F_A)^2 \geq \mp \int_X \mathrm{Tr}(F_A^2), \tag{1.41}$$

and we have an equality precisely when A is (anti-)self-dual.

The theory described here is the mathematical framework that underlies non-abelian gauge theories in physics: *Yang-Mills theory.* This becomes much clearer when we express things in local coordinates.

Let T^a be a basis of traceless skew-hermitian matrices for $\mathfrak{g}$ such that $\mathrm{Tr}(T_aT_b) = -\frac{1}{2}\delta_{ab}$ and $[T_a, T_b] = f_{ab}^c T_c$. Then, a connection A is a one-form $A = A_\mu^a T_a dx^\mu$. This defines the covariant derivative $d_A = D_\mu dx^\mu$, with $D_\mu = \partial_\mu + [A_\mu, \cdot]$. The curvature is then $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ where $F_{\mu\nu} = F_{\mu\nu}^a T_a = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = [D_\mu, D_\nu]$. Also, we write $*F = \frac{1}{2}\tilde{F}_{\mu\nu}dx^\mu \wedge dx^\nu$ where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$.

Now,

$$\begin{aligned} \mathrm{Tr}(F \wedge *F) &= \frac{1}{8}F_{\mu\nu}^a F^{\rho\sigma\ b}\mathrm{Tr}(T_aT_b)\epsilon_{\alpha\beta\rho\sigma}\ dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta \\ &= -\frac{1}{16}F_{\mu\nu}^a F^{\rho\sigma\ b}\delta_{ab}\epsilon_{\alpha\beta\rho\sigma}\epsilon^{\alpha\beta\mu\nu}\ dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \\ &= -\frac{1}{8}F_{\mu\nu}^a F^{\rho\sigma\ a}(\delta_\rho^\mu\delta_\sigma^\nu - \delta_\sigma^\mu\delta_\rho^\nu)\ d^4x \\ &= -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu\ a}\ d^4x. \end{aligned} \tag{1.42}$$

We see that in local coordinates,

$$S = -\frac{1}{4}\int_X d^4x F_{\mu\nu}^a F^{\mu\nu\ a}, \tag{1.43}$$

which can be recognized as the action of Yang-Mills theory.

The Euler-Lagrange equation $d_A^* F_A = 0$ becomes

$$\begin{aligned} 0 &= *d_A * F_A = \frac{1}{2} * D_\alpha F^{\mu\nu}\epsilon_{\mu\nu\rho\sigma}dx^\alpha \wedge dx^\rho \wedge dx^\sigma \\ &= \frac{1}{2}D_\alpha F^{\mu\nu}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\rho\sigma\beta}dx^\beta \\ &= \frac{1}{2}D_\alpha F^{\mu\nu}(\delta_\mu^\alpha\delta_\nu^\beta - \delta^\alpha\nu\delta_\mu^\beta)dx^\beta \\ &= D_\mu F^{\mu\nu}dx^\nu, \end{aligned} \tag{1.44}$$

so

$$D_\mu F^{\mu\nu} = 0. \tag{1.45}$$

Similarly, the Bianchi identity reads

$$D_\mu \tilde{F}_{\mu\nu} = 0. \tag{1.46}$$

Definition 1.7. A *Yang-Mills instanton* is a solution to the classical equations of motion (1.45) given by the Yang-Mills action (1.43) on a Euclidean space-time, such that the action is finite.

REMARK 1.8. Yang-Mills theory is the quantum field theory we obtain by quantizing the theory described by the action (1.43). Yang-Mills plays an essential role in the standard model, where it describes the various fundamental forces (except gravity). First of all, if we take $G = U(1)$, we have the theory of electromagnetism as described above, which leads to Quantum Electrodynamics. This describes the electromagnetic force. Second, the so called *electroweak unification* can be described by $G = SU(2) \times U(1)$, which is a theory that incorporates the electromagnetic and the weak interaction. Third, $G = SU(3)$ leads to the theory of Quantum Chromodynamics, which is a theory that describes the strong interaction.

CHAPTER 2

Maxwell's theory and the Dirac quantization condition on four-manifolds

In this section, we consider the theory of electromagnetism formulated by Maxwell, on a four-manifold. It is described in terms of a two-tensor $F_{\mu\nu}$, or equivalently, a real-valued two-form F, called the field strength tensor. Its components can be recognized as the electric and magnetic field. In the presence of scalar or spinor fields, the two-form F can be recognized as the curvature of a connection on a line bundle. Therefore, electromagnetism can be viewed as an abelian version of Yang-Mills theory. The cocycle condition of the line bundle imposes that the flux of F through a two-dimensional surfaces can only take integer values, a result which is known as *the Dirac quantization condition*. The discussion below is based on parts of [**2**], [**3**] and [**16**].

1. (Co)Homology

Let M be a compact, connected, oriented Riemannian four-manifold, representing space-time. Let $H_k(M;\mathbb{Z})$ denote the k-th homology group of M with integer coefficients, that is, the k-cycles modulo the k-boundaries. Likewise let $H^k(M;\mathbb{Z})$ denote the k-th cohomolgy group. We will assume that these groups are finitely generated.

The elements of finite order in these groups form the torsion subgroups, $T_k(M;\mathbb{Z})$ and $T^k(M,\mathbb{Z})$: a k-cycle α has finite order N if N is the smallest integer such that

$$N\alpha = \partial\beta, \tag{2.1}$$

for some $k-1$-chain β. The quotients

$$F_k(M;\mathbb{Z}) = H_k(M;\mathbb{Z})/T_k(M;\mathbb{Z}), \quad F^k(M;\mathbb{Z}) = H^k(M;\mathbb{Z})/T^k(M;\mathbb{Z}) \tag{2.2}$$

are then finitely generated, free abelian groups, hence are of the form

$$F_k(M;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}\oplus ... \oplus\mathbb{Z} = \oplus_{b_k}\mathbb{Z}. \tag{2.3}$$

Here b_k and b^k are the k-th Betti numbers of M. Now the *universal coefficient theorem* for cohomology implies that

$$F^k(M;\mathbb{Z}) \cong F_k(M;\mathbb{Z}), \quad T^k(M;\mathbb{Z}) \cong T_{k-1}(M;\mathbb{Z}). \tag{2.4}$$

Since we assume our space-time manifold to be connected, compact and oriented, it will satisfy Poincaré duality:

$$H^k(M;\mathbb{Z}) \cong H_{m-k}(M;\mathbb{Z}), \tag{2.5}$$

where m is the dimension of M. The above statements imply that $b_k = b^k = b_{m-k} = b^{m-k}$.

If we have cycles Σ, Σ' whose dimensions add up to m, generically they intersect in a finite number of points. Since M is oriented, we can count these points with signs (by matching the orientations) and the resulting integer is called the *intersection number* $I(\Sigma,\Sigma')$ of Σ and Σ'.

This intersection number depends only on the homology classes of the cycles: if $\Sigma = \partial V$, another cycle Σ' will intersect it an even number of times, with opposite orientations such that the intersection number vanishes. Moreover, only on the free parts of the cycles since the intersection number vanishes on torsion elements:

$$I(\alpha,\gamma) = 1/N\, I(N\alpha,\gamma) = 1/N\, I(0,\gamma) = 0. \tag{2.6}$$

If we consider 4-manifolds, the 2-cycles will satisfy the above properties. Let $\Sigma_1, \Sigma_2, ..., \Sigma_{b_2}$ be a basis of the integer lattice $F_2(M;\mathbb{Z})$ and let

$$I(\Sigma_i,\Sigma_j) = (Q^{-1})_{ij}. \tag{2.7}$$

Then Q^{-1} is obviously a symmetric $b_2 \times b_2$ matrix with integer entries. Also, Poincaré duality implies that it has determinant ± 1. Therefore, Q has the same properties:

$$Q = Q^T, \quad \det Q = \pm 1. \tag{2.8}$$

From the above we see that the free homology $F_2(M;\mathbb{Z})$ forms an integer lattice with quadratic form defined by Q^{-1}. In general, this quadratic form is indefinite of type (b^+, b^-), where $b_2 = b^+ + b^-$ and we define

$$\sigma(M) = b^+ - b^-, \tag{2.9}$$

which is called the signature of M.

2. Flux and gauge transformations

Let F be a real valued, closed 2-form on M which defines the electromagnetic field strength. The flux through a 2-cycle Σ is defined as

$$\int_\Sigma F. \tag{2.10}$$

The flux depends only on the homology class of Σ and the cohomology class of F, since

$$\int_{\partial V} F = \int_V dF = 0, \quad \int_\Sigma dB = \int_{\partial\Sigma} B = 0, \tag{2.11}$$

for boundaries ∂V and coboundaries dB and where we used Stokes' theorem. Moreover the flux through torsion cycles vanishes:

$$\int_\alpha F = \frac{1}{N}\int_{N\alpha} F = \frac{1}{N}\int_{\partial\beta} F = 0, \tag{2.12}$$

for a torsion cycle α of order N, hence the flux depends only on the free part of the homology class of Σ. Now F defines an element of the de Rham cohomology, where real coefficients are used, so there is no reason to suspect that the fluxes will be integral. However, we will see that if there are other fields defined on M, the flux will only take integer values.

Let us take a cover $\mathscr{U} = \{U_\alpha\}$ of M consisting of contractible open neighbourhoods U_α. Since M is compact, we can take this cover to be finite. The Poincaré lemma tell us that F is locally an exact form:

$$F|_{U_\alpha} = dA_\alpha, \tag{2.13}$$

for some one-form A_α[1]. This one-form is called the *vector potential*.

Assuming that the overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta$ are also contractible, in each overlap the 1-forms can only differ by an exact form:

$$A_\alpha - A_\beta = d\chi_{\alpha\beta}, \tag{2.14}$$

for a real valued function $\chi_{\alpha\beta}$ on $U_{\alpha\beta}$. Note that $\chi_{\alpha\beta} = -\chi_{\beta\alpha}$.

Remark 2.1. The family of one-forms $\{A_\alpha\}$ defines a Čech 0-chain with values in the 1-forms, defined on the open cover $\mathscr{U} = \{U_\alpha\}$. The difference $A_\alpha - A_\beta$ is then precisely the coboundary $(\delta A)_{\alpha\beta}$, a Čech 1-chain with values in 1-forms, such that (2.14) is an equality of Čech 1-chains:

$$(\delta A)_{\alpha\beta} = (d\chi)_{\alpha\beta} = d\chi_{\alpha\beta}. \tag{2.15}$$

Consider the problem of well-defining the line integral

$$\int_\Gamma A, \tag{2.16}$$

where Γ is a path in M. Suppose that the path Γ goes through an overlap $U_{\alpha\beta}$, let I, F be the begin and endpoint of γ and let P be a point on γ in the overlap as in figure 1.

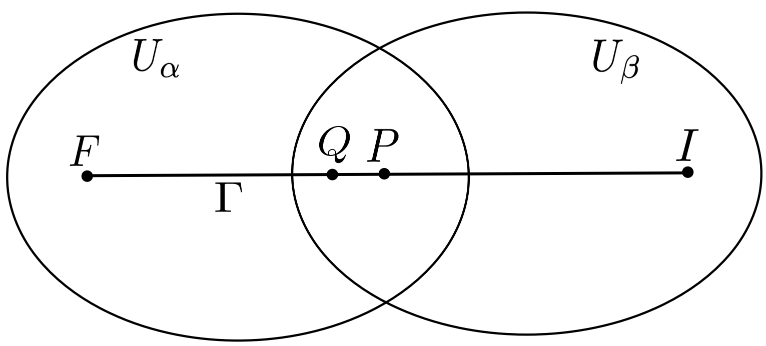

Figure 1. A path Γ from a point $I \in U_\beta$ to a point $F \in U_\alpha$, passing through a point $P \in U_{\alpha\beta}$.

A first guess for the line integral would be

$$I_P := \int_P^F A_\alpha + \int_I^P A_\beta. \tag{2.17}$$

However, this expression depends on the choice of the point P: if we choose another point Q on Γ in the overlap, we have

$$I_Q - I_P = -\int_P^Q (A_\alpha - A_\beta) = -\int_P^Q d\chi_{\alpha\beta} = \chi_{\alpha\beta}(P) - \chi_{\alpha\beta}(Q). \tag{2.18}$$

We see that

$$\int_\Gamma A = I_Q + \chi_{\alpha\beta}(Q) = \int_Q^F A_\alpha + \chi_{\alpha\beta}(Q) + \int_I^Q A_\beta \tag{2.19}$$

[1] A word of caution: the index α in A_α refers to the member of $\mathscr{U}$ on which the 1-form is defined and is not a Lorentz-index!

does not depend on the choice of Q, hence correctly defines the line integral. The next step is to consider triple overlaps, by considering a third open set U_γ and point $R \in U_{\alpha\gamma}, P \in U_{\beta\gamma}$ as in Figure 2.

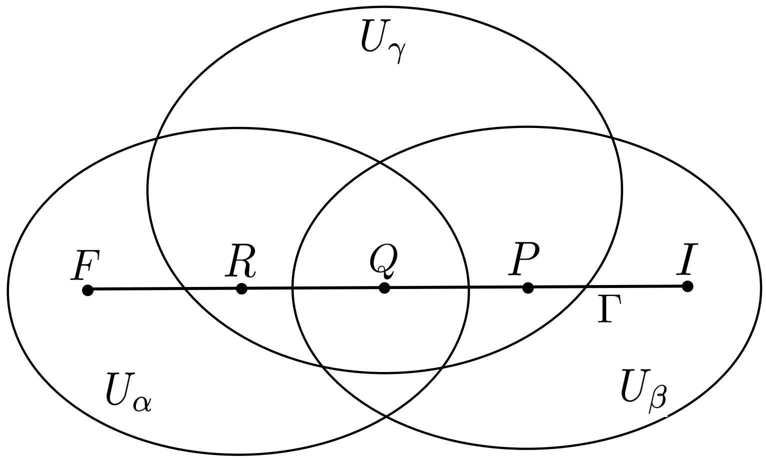

Figure 2. The same path Γ connecting I and F, passing through the points $P \in U_{\beta\gamma}, Q \in U_{\alpha\beta\gamma}$ and $R \in U_{\gamma\alpha}$.

If we ignore U_γ at first, the line integral is given by equation (2.19). Now we can use the gauge transformation (2.14) to rewrite the line integral between P and R to obtain:

$$\int_\Gamma A = \int_R^F A_\alpha + \chi_{\alpha\gamma}(R) + \int_P^R A_\gamma + \chi_{\gamma\beta}(P) + \int_I^P A_\beta + (\chi_{\alpha\beta}(Q) + \chi_{\beta\gamma}(Q) + \chi_{\gamma\alpha}(Q)). \tag{2.20}$$

Note that the only Q dependence is in the last term. Now consider the gauge transformations in each double overlap:

$$\begin{aligned} A_\alpha - A_\beta &= d\chi_{\alpha\beta} \\ A_\beta - A_\gamma &= d\chi_{\beta\gamma} \\ A_\gamma - A_\alpha &= d\chi_{\gamma\alpha}, \end{aligned} \tag{2.21}$$

which add up to

$$d(\chi_{\alpha\beta} + \chi_{\beta\gamma} + \chi_{\gamma\alpha}) = 0. \tag{2.22}$$

Now if we also assume that each triple overlap $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$ is contractible, we can again invoke the Poincaré lemma to find that, on $U_{\alpha\beta\gamma}$,

$$\chi_{\alpha\beta} + \chi_{\beta\gamma} + \chi_{\gamma\alpha} = c_{\alpha\beta\gamma}, \tag{2.23}$$

where $c_{\alpha\beta\gamma}$ is a constant.

Remark 2.2. Adding the equations (2.21) to zero is equivalent with stating $\delta(\delta A) = 0$. On the right hand side, we therefore obtain the equation $\delta d\chi = d\delta\chi = 0$, where we switch the order of the Čech and de Rham differentials acting on the Čech 1-chain $\{\chi_{\alpha\beta}\}$. Using the Poincaré lemma in each $U_{\alpha\beta\gamma}$ as above, we obtain the Čech 2-chain $\{c_{\alpha\beta\gamma}\}$. Then equation (2.23) reads $\delta\chi = c$ and we immediately find $\delta c = 0$, such that c defines a Čech cocycle.

Using this terminology, we see that the method used above illustrates a constructive way to prove an isomorphism between de Rham cohomology and Čech cohomology. A cocycle representative F of a de Rham cohomology class defines a Čech cocycle $\{c_{\alpha\beta\gamma}\}$ and vice versa. One still has to show that this construction is independent of the choices made. The discussion above is based on cocycles of degree 2, but the construction can be done for cohomology in any degree. This is for instance explained in [**3**].

Next, we consider the flux through a 2-dimenional surface Σ. We would like to break up the integral as surface integrals over each patch $U_\alpha \cap \Sigma$ covering Σ:

$$\int_\Sigma F = \sum_\alpha \int_{U_\alpha \cap \Sigma} F. \tag{2.24}$$

This is not correct, since we are counting contributions from the overlaps double in this way. However, we can subdivide our surface into non-overlapping regions $V_\alpha \subset U_\alpha \cap \Sigma$ as shown in figure 3.

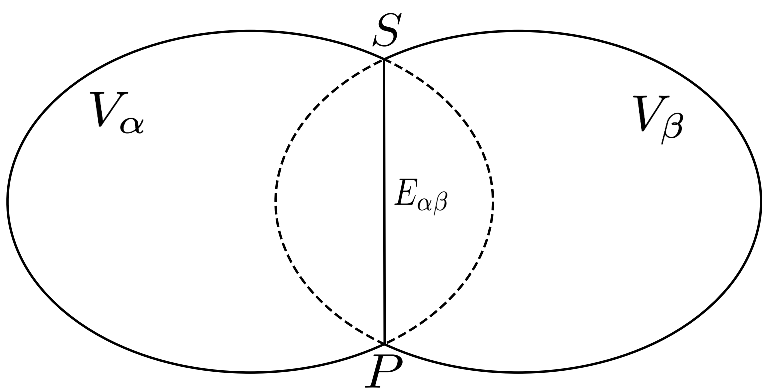

Figure 3. The overlapping neighborhoods U_α and U_β are replaced by two non-overlapping neighborhoods V_α and V_β, sharing a common border $E_{\alpha\beta}$.

By use of Stokes' theorem, we write

$$\int_\Sigma F = \sum_\alpha \int_{V_\alpha} F = \sum_\alpha \int_{V_\alpha} dA_\alpha = \sum_\alpha \int_{\partial V_\alpha} A_\alpha. \tag{2.25}$$

Since the contourintegrals over ∂V_α and ∂V_β result in line integrals over the edge $E_{\alpha\beta}$ in opposite directions, its contribution to the contour integral may be expressed as

$$\int_{E_{\alpha\beta}} (A_\alpha - A_\beta) = \int_{\partial E_{\alpha\beta}} \chi_{\alpha\beta} = \chi_{\alpha\beta}(S) - \chi_{\alpha\beta}(P), \tag{2.26}$$

where we again used Stokes' theorem.

The next step is again to consider triple overlaps. Assume for a moment that our surface Σ can be covered by three open sets U_α, U_β and U_γ (it is equal to the union of these three), which have a triple overlap $U_{\alpha\beta\gamma}$. We replace these open neighborhoods with three non-overlapping neighborhoods, as shown in Figure 4.

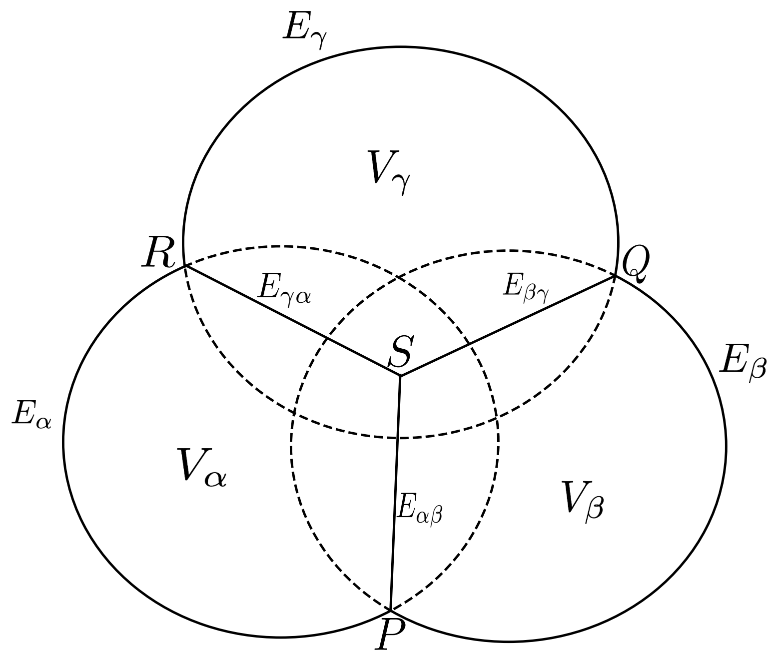

Figure 4. Three neighborhoods $U_\alpha, U_\beta, U_\gamma$ with non-empty triple intersection, replaced by non-overlapping neighborhoods $V_\alpha, V_\beta, V_\gamma$.

We write

$$\begin{aligned}\int_\Sigma F &= \sum_\alpha \int_{\partial V_\alpha} = \int_{E_\alpha} A_\alpha - \chi_{\alpha\beta}(P) + \int_{E_\beta} A_\beta - \chi_{\beta\gamma}(Q) + \int_{E_\gamma} A_\gamma - \chi_{\gamma\alpha}(R) \\ &\quad +(\chi_{\alpha\beta}(S) + \chi_{\beta\gamma}(S) + \chi_{\gamma\alpha}(S)) \\ &= \text{"}\oint_{\partial\Sigma} A\text{"} + c_{\alpha\beta\gamma},\end{aligned} \tag{2.27}$$

where "$\oint A$" is the corrected contour integral which is independent of our choice of point P, Q, R on the boundary $\partial\Sigma$. We can now easily generalize this construction to an arbitrary open cover $\mathscr{U}$:

$$\int_\Sigma F = \text{"}\oint_{\partial\Sigma} A\text{"} + \sum_{\Sigma\cap U_{\alpha\beta\gamma}} c_{\alpha\beta\gamma}, \tag{2.28}$$

where we define the contour integral "$\oint_{\partial\Sigma} A$" as

$$\text{"}\oint_{\partial\Sigma} A\text{"} = \sum_k \int_{P_{k-1,k}}^{P_{k,k+1}} A_k - \chi_{k,k+1}(P_{k,k+1}), \tag{2.29}$$

which is a generalization of the above expression for the contour integral, where we have labeled open sets $U_0, ..., U_N$ covering $\partial\Sigma$ and have chosen a point $P_{k,k+1}$ in each overlap $\partial\Sigma \cap U_k \cap U_{k+1}$. In particular, when Σ is a cycle, i.e. $\partial\Sigma = 0$, we see that

$$\int_\Sigma F = \sum_{\Sigma\cap U_{\alpha\beta\gamma}} c_{\alpha\beta\gamma}. \tag{2.30}$$

This expression for the flux will play an important role in the derivation of the Dirac quantization condition. We will see that the presence of scalar or spinor fields on M impose restrictions on the possible values of the $c_{\alpha\beta\gamma}$, which lead to different quantization results.

3. The Dirac quantization condition for scalar fields

Suppose that next to the electromagnetic fields, we have a complex scalar field ϕ living on our manifold. This means the theory we are describing is scalar electrodynamics, which is governed by a Lagrangian of the form

$$\mathscr{L} = \frac{1}{2}|D_\mu \phi|^2 + U(|\phi|^2) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{2.31}$$

where $D_\mu = \partial_\mu - i\frac{q_b}{\hbar}A_\mu$ is the covariant derivative. We write q_b for the coupling constant, i.e. the electric charge carried by the bosonic field ϕ. We will however treat this scalar field as a fixed background field: it will not show up in the path integral.

Now in each open neighbourhood U_α we have a well-defined vector potential 1-form A_α and a complex valued function ϕ_α. On the overlaps $U_{\alpha\beta}$, the fields are related by $U(1)$ gauge transformations:

$$A_\alpha - A_\beta = d\chi_{\alpha\beta}, \quad \phi_\alpha = e^{i\frac{q_b}{\hbar}\chi_{\alpha\beta}}\phi_\beta. \tag{2.32}$$

In the triple overlap regions $U_{\alpha\beta\gamma}$, the three fields ϕ_α, ϕ_β and ϕ_γ are defined and are related by the gauge transformations (2.32). To be self consistent, the gauge transformations should satisfy

$$\begin{aligned} \phi_\alpha &= e^{i\frac{q_b}{\hbar}\chi_{\alpha\beta}}\, e^{i\frac{q_b}{\hbar}\chi_{\beta\gamma}}\, e^{i\frac{q_b}{\hbar}\chi_{\gamma\alpha}}\, \phi_\alpha \\ &= e^{i\frac{q_b}{\hbar}(\chi_{\alpha\beta}+\chi_{\beta\gamma}+\chi_{\gamma\alpha})}\, \phi_\alpha, \end{aligned} \tag{2.33}$$

evaluated at any point in $U_{\alpha\beta\gamma}$. This means that

$$c_{\alpha\beta\gamma} = \chi_{\alpha\beta}(P) + \chi_{\beta\gamma}(P) + \chi_{\gamma\alpha}(P) \in \frac{2\pi\hbar}{q_b}\mathbb{Z}, \tag{2.34}$$

for any $P \in U_{\alpha\beta\gamma}$.

The mathematical formulation of the above statements is as follows. We consider a complex line bundle $L \to M$ over M, which has gauge group $G = U(1)$ and transition maps $g_{\alpha\beta} : U_{\alpha\beta} \to U(1)$. See appendix A. These transition functions satisfy the cocycle condition:

$$g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1. \tag{2.35}$$

The vector potential A above is then defined as a connection on L, which in a local trivialization is represented by a 1-form $\tilde{A}_\alpha = i\frac{q}{\hbar}A_\alpha$. Here, we identify the Lie-algebra $\mathfrak{g}$ of $U(1)$ with $i\frac{q}{\hbar}\mathbb{R}$ to explicitly bring in the physical constants. The covariant derivative defined by this connection is then precisely the one appearing in the Lagrangian (2.31).

Under a change of local trivialization, a connection transforms as (1.11), which now reads

$$\tilde{A}_\alpha = \tilde{A}_\beta - (dg_{\alpha\beta})\, g_{\alpha\beta}^{-1}, \tag{2.36}$$

since the gauge transformations are complex numbers, hence commute with A.

The scalar field ϕ above is precisely a section of the line bundle and is locally given by complex functions $\phi_\alpha : U_\alpha \to \mathbb{C}$. Under a change of trivialization this section transforms as

$$\phi_\alpha = g_{\alpha\beta}\,\phi_\beta. \tag{2.37}$$

Now if we write our transition functions as

$$g_{\alpha\beta} = \exp(i\frac{q_b}{\hbar}\chi_{\alpha\beta}), \quad \chi_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{R}, \tag{2.38}$$

the change under local trivializations (2.36) and (2.37) precisely produces the gauge transformations (2.32). Then, cocycle conditon implies that

$$c_{\alpha\beta\gamma} \in \frac{2\pi\hbar}{q_b}\mathbb{Z}, \tag{2.39}$$

where $c_{\alpha\beta\gamma}$ is defined as above.

Since the flux through a 2-dimensional surface could be expressed in terms of the $c_{\alpha\beta\gamma}$ (equation 2.28), we conclude that the flux through a 2-cycle Σ is quantized:

$$\frac{q_b}{2\pi\hbar}\int_\Sigma F = \frac{q_b}{2\pi\hbar}\sum_{\Sigma\cap U_{\alpha\beta\gamma}} c_{\alpha\beta\gamma} \in \mathbb{Z}. \tag{2.40}$$

This is known as the *Dirac quantization condition* and was first proven by Dirac in [**7**].

REMARK 2.3. We see above that F is the curvature of a connection A on a line bundle L. Following Example 1.4, the curvature F defines a de Rham cohomology class $\frac{q}{2\pi\hbar}F$, which we can identify with the first Chern class of L. Since $c_1(L) \in H^2(M;\mathbb{Z})$, we have that $\langle c_1(L), \Sigma\rangle = \int_\Sigma \frac{q}{2\pi\hbar}F \in \mathbb{Z}$ for a integer homology class $\Sigma \in H_2(M;\mathbb{Z})$.

This can also be seen from the discussion in this section: the Čech 2-cocycle $\{c_{\alpha\beta\gamma}\}$ is precisely the image of the Čech 1-cocycle $\{g_{\alpha\beta}\}$ under the coboundary map in (A.10). Therefore, $\{c_{\alpha\beta\gamma}\} \in H^2(M;\mathbb{Z})$ is the first Chern class of L, and the sum over the $U_{\alpha\beta\gamma}\cap\Sigma$ is the evaluation of this cocycle on the integer cycle Σ.

4. The Dirac quantization condition for spinor fields

In the previous section we saw that the presence of a complex scalar field ϕ on the manifold M resulted in a quantization of the flux through cycles. Now we investigate what happens in the presence of a complex spinor field ψ, so we are describing quantum electrodynamics on M:

$$\mathscr{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2.41}$$

We write q_f for the charge carried by the fermonic field ψ, the coupling constant for the coupling to the gauge field. Just as in the scalar case, we treat the spinor field as a background field. Again, we cover M with finitely many open sets U_α and on each open neighbourdhood we have a complex spinor wave function ψ_α and a 1-form A_α. Now next to the $U(1)$ gauge freedom,

$$\begin{aligned} \psi_\alpha &\to e^{i\frac{q_f}{\hbar}\chi}\psi_\alpha, \\ A_\alpha &\to A_\alpha + d\chi, \end{aligned} \tag{2.42}$$

the spinors also transform under local $SO(4)$ transformations:

$$\begin{aligned} F_\alpha &\to LF_\alpha, \\ \psi_\alpha &\to S(L)\psi_\alpha, \end{aligned} \tag{2.43}$$

where F_α is an oriented frame (vierbein) and $L \mapsto S(L)$ is a lift $SO(4) \to Spin(4)$. This lift has an ambiguity in sign, because $Spin(4)/\mathbb{Z}_2 \cong SO(4)$.
In each overlap $U_{\alpha\beta}$, the fields are related by gauge transformations:

$$\begin{aligned} A_\alpha &= A_\beta + d\chi_{\alpha\beta} \\ F_\alpha &= L_{\alpha\beta} F_\beta \\ \psi_\alpha &= S(L_{\alpha\beta}) e^{i\frac{q_f}{\hbar}\chi_{\alpha\beta}} \psi_\beta. \end{aligned} \tag{2.44}$$

Just as in the scalar case, we have self-consistency conditions in triple overlaps $U_{\alpha\beta\gamma}$:

$$\begin{aligned} L_{\alpha\beta}\, L_{\beta\gamma}\, L_{\gamma\alpha} &= I \\ \psi_\alpha &= e^{i\frac{q_f}{\hbar}(\chi_{\alpha\beta}+\chi_{\beta\gamma}+\chi_{\gamma\alpha})} S(L_{\alpha\beta}) S(L_{\beta\gamma}) S(L_{\gamma\alpha}) \psi_\alpha. \end{aligned} \tag{2.45}$$

This means that

$$S(L_{\alpha\beta}) S(L_{\beta\gamma}) S(L_{\gamma\alpha}) = \pm I =: \epsilon_{\alpha\beta\gamma} I \tag{2.46}$$

where we cannot determine the sign since it depends on the choice of lift $SO(4) \to Spin(4)$ and hence

$$e^{i\frac{q_f}{\hbar}(\chi_{\alpha\beta}+\chi_{\beta\gamma}+\chi_{\gamma\alpha})} = e^{i\frac{q_f}{\hbar}c_{\alpha\beta\gamma}} = \epsilon_{\alpha\beta\gamma}. \tag{2.47}$$

We saw before that the flux through Σ was given in terms of the $c_{\alpha\beta\gamma}$'s, hence we find that

$$e^{i\frac{q_f}{\hbar}\int_\Sigma F} = \prod_{U_{\alpha\beta\gamma}\cap\Sigma\neq\emptyset} \epsilon_{\alpha\beta\gamma} =: (-1)^{w(\Sigma)}. \tag{2.48}$$

To find out how this sign depends on the cycle Σ we should formulate the above in more mathematical language in terms of a *Spin^c-structure*.

4.1. Spin^c-structures. The notion of a *Spin^c* structure is essentially the 'complex analogue' of a *Spin* structure, and we will see that the requirements for existence of a *Spin^c* structure are less restrictive then for the existence of a *Spin* Structure.

Definition 2.4. The *Spin^c* group is defined as

$$Spin^c(n) := Spin(n) \times_{\mathbb{Z}_2} U(1) = Spin(n) \times U(1)/(-1,-1). \tag{2.49}$$

Note that the *Spin^c* group fits in the short exact sequence

$$0 \to \mathbb{Z}_2 \to Spin^c(n) \xrightarrow{\xi} SO(n) \times U(1) \to 1, \tag{2.50}$$

where the subgroup $\mathbb{Z}_2 \subset Spin^c(n)$ is generated by the element $[(-1,1)] = [(1,-1)]$. Just as in the case of a *Spin*-bundle, we look for a principal $Spin^c(n)$-bundle over M which admits a bundle mapping

$$P_{Spin^c} \xrightarrow{\xi} P_{SO(n)} \times P_{U(1)}, \tag{2.51}$$

which is *Spin^c*-equivariant in the sense that $\xi(pg) = \xi(p)\xi(g)$ for all $p \in P_{Spin^c(n)}$ and $g \in Spin^c(n)$. As in Appendix A, we consider the exact sequence

$$H^1(M; Spin^c(N)) \xrightarrow{\xi} H^1(M; SO(n)) \oplus H^1(M; U(1)) \xrightarrow{w_2+\tilde{c_1}} H^2(M; \mathbb{Z}_2), \tag{2.52}$$

which is determined by the coefficient sequence (2.50). The coboundary map associates to a pair $(P_{SO(N)}, P_{U(1)})$ an element $w_2(P_{SO(N)}) + \tilde{c_1}(P_{U(1)})$ where $\tilde{c_1}$ is the mod 2 reduction of the first Chern class of $P_{U(1)}$. Under the isomorphism appendix this map can be written as

$$H^1(M; SO(n)) \oplus H^2(M; \mathbb{Z}) \xrightarrow{w_2+\rho} H^2(M; \mathbb{Z}_2), \tag{2.53}$$

where ρ is the mod 2 reduction. Remember (appendix A.) that a compatible set of choices of transition functions exists if and only if this boundary map vanishes. We conclude that given a bundle $P_{SO(n)}$, we can find the bundle in (2.51) if w_2 is de mod 2 reduction of an integral class.

DEFINITION 2.5. Let $P_{SO(n)}$ be a principal $SO(n)$-bundle over M. A ***Spin^c*-structure** on $P_{SO(n)}$ consist of a principal $U(1)$-bundle $P_{U(1)}$ and a principal $Spin^c(n)$-bundle with a $Spin^c(n)$-equivariant map

$$P_{Spin^c(n)} \longrightarrow P_{SO(n)} \times P_{U(1)}. \tag{2.54}$$

From the above argument we can conclude:

THEOREM 2.6. *$P_{SO(n)}$ carries a $Spin^c(n)$-structure if and only $w_2(P)$ is the mod 2 reduction of an integral class.*

Analogous to the definition of a spin-manifold we can define the notion of a $Spin^c$-manifold:

DEFINITION 2.7. An oriented Riemannian manifold with a $Spin^c(n)$-structure on its tangent bundle is called a ***Spin^c*-manifold**.

Then Theorem 2.6 has an immediate consequence:

COROLLARY 2.8. *An orientable manifold M admits $Spin^c$-structure if and only if $w_2(M)$ is the mod 2 reduction of an integral class.*

REMARK 2.9. A lift of the second Stiefel-Whitney class has components in $H^2(M;\mathbb{Z})$ and $T^3(M;\mathbb{Z})$, which can be seen as follows. Since $w_2 \in H^2(M;\mathbb{Z}_2)$, it satisfies $\delta w_2 = 0$ mod 2. We can then either have $\delta w_2 = 0$ in which case w_2 lifts to $H^2(M;\mathbb{Z})$, or $\delta w_2 = 2\lambda$ for some 3-cochain λ. Since $\delta^2 = 0$ we see that $\delta\lambda = 0$ such that $\lambda \in T^3(M;\mathbb{Z})$. For the space-time four-manifolds that we are considering (compact, oriented), we have the isomorphisms

$$T^3(M;\mathbb{Z}) = T_2(M;\mathbb{Z}) = T^2(M;\mathbb{Z}) = T_1(M;\mathbb{Z}), \tag{2.55}$$

where the middle equality is due to Poincaré duality and the other two equalities are due to the universal coefficient theorem. Because of these isomorphisms, we can always say that w_2 is the mod 2 reduction of an integral class, since T^3 and T^2 are isomorphic. Therefore, the space-time M admits a $Spin^c$-structure.

Let us consider the problem of constructing a spinor bundle (a bundle of irreducible complex modules for the Clifford algebra over a manifold M. We can do this locally: let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M such that $U_{\alpha_1} \cap ... \cap U_{\alpha_k}$ is contractible for all $\alpha_1, ..., \alpha_k$. On each U_α, the bundles can be trivialized and we can find a complex spinor bundle of the form $U_\alpha \times V$ where V is an irreducible complex Clifford module.

Now to compare the fibers over different neighbourhoods, we try to find transition functions $\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to Spin(n)$, such that $\xi \circ \tilde{g}_{\alpha\beta} = g_{\alpha\beta} : U_{\alpha\beta} \to SO(n)$ are the corresponding transition functions for $P_{SO}(M) = P_{SO}(TM)$. As described in appendix B, the existence of such a spin bundle is equivalent to the vanishing of the Čech cocycle

$$w_{\alpha\beta\gamma} := \tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} : U_{\alpha\beta\gamma} \longrightarrow \mathbb{Z}_2 = \ker(\xi) \tag{2.56}$$

in $H^2(M;\mathbb{Z}_2)$.

Now suppose that the class $[w] \in H^2(M;\mathbb{Z}_2)$ is the mod 2 reduction of an integral class $W \in H^2(M;\mathbb{Z})$ and let λ be the complex line bundle corresponding to W, that is, $c_1(\lambda) = W$. We try to find a square root of λ: a line bundle $\lambda^{1/2}$ with $(\lambda^{1/2})^2 = \lambda$. Let $\gamma_{\alpha\beta} : U_{\alpha\beta} \to U(1)$ be the transition functions for λ. Since $U_{\alpha\beta}$ is contractible, we can take a square root $\tilde{\gamma}_{\alpha\beta} := \gamma_{\alpha\beta}^{1/2} : U_{\alpha\beta} \to U(1)$. These transition functions are compatible if the Čech cocycle

$$w'_{\alpha\beta\gamma} := \tilde{\gamma}_{\alpha\beta}\, \tilde{\gamma}_{\beta\gamma}\, \tilde{\gamma}_{\gamma\alpha} : U_{\alpha\beta\gamma} \to \mathbb{Z}_2 = \ker(\sigma) \tag{2.57}$$

vanishes. Here $\sigma(z) = z^2$ in the exact sequence

$$0 \to \mathbb{Z}_2 \to S^1 \xrightarrow{\sigma} S^1 \to 0. \tag{2.58}$$

As in Appendix A, we obain a long exact sequence where the class $[w'] \in H^2(M;\mathbb{Z})$ is just the coboundary of $\lambda \in H^1(M;S^1)$. We get the following commutative diagram:

$$\begin{tikzcd}
H^1(M;S^1) \arrow[r, "\sigma"] \arrow[d, "\cong"] & H^1(M;S^1) \arrow[r, "w'"] \arrow[d, "\cong"] & H^2(M;\mathbb{Z}_2) \arrow[d, "="] \\
H^2(M;\mathbb{Z}) \arrow[r, "2"] & H^2(M;\mathbb{Z}) \arrow[r, "\rho"] & H^2(M;\mathbb{Z}_2)
\end{tikzcd}$$

Since the diagram commutes, we see that $[w'] = \rho(c_1(\lambda)) = \rho(W) = [w]$ such that $[w] + [w'] = 0$ mod 2. By adjusting by coboundaries, we can choose $\tilde{g}_{\alpha\beta}$ and $\tilde{\gamma}_{\alpha\beta}$ such that $w_{\alpha\beta\gamma} \equiv w'_{\alpha\beta\gamma}$. Then the transition functions

$$G_{\alpha\beta} = \tilde{g}_{\alpha\beta} \times \tilde{\gamma}_{\alpha\beta} : U_{\alpha\beta} \longrightarrow Spin(n) \times_{\mathbb{Z}_2} S^1 = Spin^c(n) \tag{2.59}$$

satisfy $G_{\alpha\beta}\, G_{\beta\gamma}\, G_{\gamma\alpha} \equiv 0$, hence they determine a global bundle. We see that while we cannot construct the spinor bundle and $\lambda^{1/2}$ separately, we can construct their product.

4.2. Quantized flux. We return to the physics picture. The vierbein F_α defined above represents a section of the principal bundle $P_{SO}(TM)$. The $SO(4)$ gauge transformations on this vierbein represent the action of the structure group $SO(4)$ and we see that the vierbeins on different trivializations are related by transition functions on the overlaps.

The spinor field ψ represents a section of the complex spinor bundle associated to $P_{Spin^c}(TM)$. The gauge transformations that relate the spinor fields over different overlaps,

$$\psi_\alpha = S(L_{\alpha\beta})\, e^{i\frac{q_f}{\hbar}\chi_{\alpha\beta}}\, \psi_\beta, \tag{2.60}$$

now play the role of the transition functions $G_{\alpha\beta}$ defined above. The relation $G_{\alpha\beta}\, G_{\beta\gamma}\, G_{\gamma\alpha} \equiv 0$, or $w_{\alpha\beta\gamma} \equiv w'_{\alpha\beta\gamma}$ is then translated into the relation

$$e^{i\frac{q_f}{\hbar}c_{\alpha\beta\gamma}} \equiv \epsilon_{\alpha\beta\gamma} \tag{2.61}$$

on each triple overlap $U_{\alpha\beta\gamma}$. We can now distinguish 2 cases:

(1) If the second Stiefel-Whitney class of M vanishes, M admits a *Spin* structure and the $Spin^c$-structure above actually comes from a *Spin* structure. In this case we can choose

the transition functions such that $w_{\alpha\beta\gamma} \equiv 0$, which translates into $\epsilon_{\alpha\beta\gamma} \equiv 1$ on $U_{\alpha\beta\gamma}$. This implies that $\frac{q}{\hbar} c_{\alpha\beta\gamma} \in 2\pi\mathbb{Z}$ so that

$$\frac{q_f}{2\pi\hbar} \int_\Sigma F = \frac{q_f}{2\pi\hbar} \sum_{U_{\alpha\beta\gamma} \cap \Sigma \neq \emptyset} c_{\alpha\beta\gamma} \in \mathbb{Z} \tag{2.62}$$

and $w(\Sigma) = 0 \bmod 2$ where $w(\Sigma)$ is defined as in (2.48).

(2) If $w_2(M)$ does not vanish, we have that $w_{\alpha\beta\gamma} \neq 0$, hence $\epsilon_{\alpha\beta\gamma} = -1$ on some (possibly all) $U_{\alpha\beta\gamma}$. On these triple overlaps we have then $\frac{q_f}{\hbar} c_{\alpha\beta\gamma} \in \pi + 2\pi\mathbb{Z}$. Now we have that

$$\begin{aligned} e^{\pi i w(\Sigma)} &= (-1)^{w(\Sigma)} = \prod_{U_{\alpha\beta\gamma} \cap \Sigma \neq \emptyset} \epsilon_{\alpha\beta\gamma} = \prod_{U_{\alpha\beta\gamma} \cap \Sigma \neq \emptyset} e^{i \frac{q_f}{\hbar} c_{\alpha\beta\gamma}} \\ &= \exp\{i \frac{q_f}{\hbar} \sum_{U_{\alpha\beta\gamma} \cap \Sigma \neq \emptyset} c_{\alpha\beta\gamma}\} = \exp\{i \frac{q_f}{\hbar} \int_\Sigma F\} \end{aligned} \tag{2.63}$$

where we use the definition (2.48) in the second, relation (2.61) in the third and relation (2.30) in the last equality. This implies that

$$1 = \exp\{i \frac{q_f}{\hbar} \int_\Sigma F - \pi i w(\Sigma)\} \tag{2.64}$$

and we see that

$$\frac{q_f}{2\pi\hbar} \int_\Sigma F - \frac{1}{2} w(\Sigma) \in \mathbb{Z}. \tag{2.65}$$

REMARK 2.10. From the relation (2.65) we obtain that, if w_2 does not vanish, it is impossible for the charge q_f to vanish. This reflects the fact that w_2 forms the obstruction for the existence of a *Spin*-structure on M, which would describe a *neutral* spinor. Therefore, if w_2 does not vanish, there only exists a *charged* spinor on M, described by a *Spin*^{c}-structure.

4.3. Identifying $w(\Sigma)$ with the intersection number. Since we have spinor fields on M, we can evaluate the index of the Dirac operator D_A acting on them [**16**]:

$$\mathrm{Ind}(\not{D}_A) = \left\{ e^{\frac{1}{2}c} \hat{A}(M) \right\} [M] = -\frac{1}{8}\sigma(M) + \frac{q_f}{8\pi^2\hbar^2} \int_M F \wedge F, \tag{2.66}$$

where c is the first Chern class of the bundle λ above, so $\frac{1}{2}c = \frac{q_f}{2\pi\hbar}[F]$ and we use that the $\hat{A}$ *class* is given by $\hat{A}(M) = 1 - \frac{1}{24}p_1(X) = 1 - \frac{1}{8}\sigma(M)$ [**9**]. We can evaluate the integral by making use of the *Riemann bilinear identity*:

$$\int_M F \wedge F = \sum_{i,j=1}^{b_2} \int_{\Sigma_i} F \; Q_{ij} \; \int_{\Sigma_j} F, \tag{2.67}$$

where $\Sigma_1, ..., \Sigma_{b_2}$ is a basis of 2-cycles for $F_2(M;\mathbb{Z})$ as defined above. The flux through each Σ_i is quantized as in (2.65), so we can write

$$\frac{q_f}{2\pi\hbar} \int_{\Sigma_i} F = m_i + \frac{1}{2} w_i, \tag{2.68}$$

where $m_i \in \mathbb{Z}$ and $w_i := w(\Sigma_i)$ as defined above. We can now evaluate the integral such that the index becomes

$$\mathrm{Ind}(\not{D}_A) = -\frac{1}{8}\sigma + \frac{1}{2}(m + \frac{w}{2})^T Q (m + \frac{w}{2}) = \frac{1}{8}(w^T Q w - \sigma) + \frac{1}{2}(m^T Q m + m^T Q w), \tag{2.69}$$

where we have arranged the quantities m_i and w_i in column vectors w and m. In this treatment, the integers m_i are fixed integers determined by the particular field strength F. Therefore, the second term in (2.69) depends on F, whereas the first term only depends on the topology of M and the homology classes of the Σ_i. Since the index has to be an integer for every choice of F, we conclude that the individual terms in (2.69) must be integer, such that

$$w^T Q w = \sigma + 8\mathbb{Z}, \tag{2.70}$$
$$m^T Q m = m^T Q w + 2\mathbb{Z}. \tag{2.71}$$

The second equation shows that w is a characteristic vector for the integer valued unimodular matrix Q (see Appendix C). Since this has to hold for every choice of integers m_i, we can insert the choice $m_i = (Q^{-1})_{ik}$ for some $k \in \{1, ..., b_2\}$ into (2.71) to obtain:

$$m^T Q m = (Q^{-1})_{ki}\, Q_{ij}\, (Q^{-1})_{jk} = \delta_{kj}\, (Q^{-1})_{jk} = (Q^{-1})_{kk} \tag{2.72}$$
$$m^T Q w = (Q^{-1})_{ki}\, Q_{ij}\, w_j = \delta_{kj} w_j = w_k, \tag{2.73}$$

hence we see that

$$w_k = -(Q^{-1})_{kk} + 2\mathbb{Z} = -I(\Sigma_k, \Sigma_k) + 2\mathbb{Z}. \tag{2.74}$$

We conclude that $w(\Sigma) = -I(\Sigma, \Sigma)$ mod 2 for a general 2-cycle Σ.

5. The field strength F

According to the discussion above, in the flux through a cocycle Σ becomes quantized when there are background scalar or spinor fields on M. In the case of a scalar field, carrying a charge q_b, we have seen that the flux becomes quantized:

$$\frac{q_b}{2\pi\hbar} \int_\Sigma F = m(\Sigma) \in \mathbb{Z}, \tag{2.75}$$

for all $\Sigma \in H^2(X; \mathbb{Z})$. We can choose closed 2-forms $F^1, ..., F^{b_2}$, representing a basis of $F^2(X; Z)$ which is dual to the basis Σ_i:

$$\frac{q_b}{2\pi\hbar} \int_{\Sigma_i} F^j = \delta_i^j. \tag{2.76}$$

Due to the quantization condition we can now expand F in term of the basis:

$$F = \sum_{i=1}^{b_2} m_i F^i, \quad m_i = m(\Sigma_i). \tag{2.77}$$

In the case of a spinor field, which couples to the gauge field with a charge q_f, we can distuinguish 2 cases:

- The intersection form is even, $I(\Sigma, \Sigma) \in 2\mathbb{Z}$ for all $\Sigma \in H^2(M; \mathbb{Z})$. This implies that $w(\Sigma) = 0$ mod 2 for all Σ and we get

$$\frac{q_f}{2\pi\hbar} \int_\Sigma F = m(\Sigma) \in \mathbb{Z} \tag{2.78}$$

  for all $\Sigma \in H^2(M; \mathbb{Z})$. Note that this implies that the second Stiefel-Whitney class vanishes, hence that M actually admits a *Spin* structure.

As in the scalar case, we can expand F as

$$F = \sum_{i=1}^{b_2} m_i F^i, \tag{2.79}$$

where we choose the basis F^j such that

$$\frac{q_f}{2\pi\hbar} \int_{\Sigma_i} F^j = \delta_i^j. \tag{2.80}$$

- The intersection form is not even: $I(\Sigma, \Sigma)$ is odd for some 2-cycle Σ. Then the flux is quantized as

$$\frac{q_f}{2\pi\hbar} \int_{\Sigma} F + \frac{1}{2} I(\Sigma, \Sigma) = m(\Sigma) \in \mathbb{Z}. \tag{2.81}$$

In this case,

$$F = \sum_{i=1}^{b_2} (m_i + \frac{w_i}{2}) F^i, \tag{2.82}$$

where $w_i = w(\Sigma_i)$.

Remark 2.11. In the presence of scalar *and* spinor fields, there must be a compatibility condition between the bosonic and fermonic charges, depending on the parity of the intersection form. If $I(\Sigma, \Sigma)$ is even for all Σ, we obtain from (2.75) and (2.78) that

$$\frac{q_b}{q_f} = \frac{m_b}{m_f} \in \mathbb{Q}, \tag{2.83}$$

where $m_b, m_f \in \mathbb{Z}$, so the ratio between the charges can be any rational number.

If $I(\Sigma, \Sigma)$ is odd for some 2-cycle Σ, we obtain from (2.75) and (2.81) that

$$\frac{q_b}{q_f} = \frac{2m_b}{2m_f + 1}, \tag{2.84}$$

where $m_b, m_f \in \mathbb{Z}$. We see that the ration between the charges is restricted to the fraction of an even and an odd integer.

CHAPTER 3

The Maxwell partition function and electromagnetic duality on four-manifolds

We consider electromagnetism on a four-manifold Minkowskian manifold $\mathcal{M}$, which is described by a gauge theory with the Abelian gauge group $G = U(1)$. In vacuum, the classical Maxwell's equations admit a symmetry of interchanging the electric and the magnetic field, which is called *electromagnetic duality*. This symmetry can be extended to an action of $SL(2,\mathbb{R})$. We investigate how much of this symmetry remains in the quantum theory when there are background matter fields. We do a Wick rotation to describe the theory on a Riemannian manifold M and we study the partition function, which can be expressed as a sum over the classical solution to the equations of motion, with the help of the Dirac quantization condition. We will see that the symmetry is reduced to an action of (a subgroup of) $SL(2,\mathbb{Z})$. Most of the discussion is based on parts of [**23**], [**25**] and [**19**].

1. Electromagnetic duality

The idea of electromagnetic duality arose already when Maxwell wrote down his equations. Maxwell's equations in vacuum,

$$\nabla \cdot \vec{E} = 0, \quad \nabla \times \vec{E} = -\partial_t \vec{B}, \tag{3.1}$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{B} = \partial_t \vec{E}, \tag{3.2}$$

are invariant under the transformation

$$(\vec{E}, \vec{B}) \to (\vec{B}, -\vec{E}). \tag{3.3}$$

When we group $\vec{E}$ and $\vec{B}$ together in single complex vector field $\vec{E} + i\vec{B}$, Maxwell's equations can be grouped together to just two equations:

$$\nabla \cdot (\vec{E} + i\vec{B}) = 0, \tag{3.4}$$

$$\nabla \times (\vec{E} + i\vec{B}) = i\partial_t(\vec{E} + i\vec{B}). \tag{3.5}$$

These equations now exhibit the symmetry

$$\vec{E} + i\vec{B} \to e^{i\phi}(\vec{E} + i\vec{B}). \tag{3.6}$$

Note that (3.3) is a special case of this symmetry. Note that the energy and momentum density of the electromagnetic field,

$$\frac{1}{2}|\vec{E} + i\vec{B}|^2 = \frac{1}{2}(E^2 + B^2), \tag{3.7}$$

$$\frac{1}{2i}\overline{(\vec{E} + i\vec{B})} \times (\vec{E} + i\vec{B}) = \vec{E} \times \vec{B}, \tag{3.8}$$

are invariant under (3.6).

However,

$$\frac{1}{2}(\vec{E} + i\vec{B})^2 = \frac{1}{2}(E^2 - B^2) + i\vec{E} \cdot \vec{B}, \tag{3.9}$$

is not invariant under the rotation (3.6). In a Lorentz covariant formulation of Maxwell's theory, the electric and magnetic fields combine together in the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_{nu} A_\mu$ and the real and imaginary parts of (3.9) can be recognized as the terms in the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} F_{\mu\nu} {}^*F^{\mu\nu}, \tag{3.10}$$

where ${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. We conclude that the rotation (3.6) is a symmetry of the equations of motion, but not of the action. We can extend this idea as follows.

Let us consider the action

$$S(\tau) = \frac{-1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g} \left[\frac{4\pi}{g^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{2\pi} \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \right]. \tag{3.11}$$

We can write in terms of differential forms $F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$:

$$S(\tau) = \frac{1}{4\pi} \int_{\mathcal{M}} \left[\frac{4\pi}{g^2} F \wedge *F + \frac{\theta}{2\pi} F \wedge F \right] = \frac{1}{4\pi} \int_M F \wedge \hat{\tau} F, \tag{3.12}$$

where the operator $\hat{\tau} = \tau_1 + *\tau_2 = \frac{\theta}{2\pi} + *\frac{4\pi}{g^2}$ and $*$ is the Hodge star operator. Here g is the gauge coupling constant and θ is often called the *theta angle*. The space-time manifold $\mathcal{M}$ is supposed to have Minkowskian signature. The real parameters τ_i can be combined into a single complex parameter

$$\tau = \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}. \tag{3.13}$$

The Lagrangian density appearing in this action is generalization of (3.10) by adding coupling constants and written in terms of differential forms instead of in coordinates.

REMARK 3.1. The action (3.12) is written in terms of a dimensionless field strength tensor F, which is related to the field strength tensor of the previous chapter by a factor $\frac{q}{\hbar}$. As a result, the fluxes will be dimensionless, such that the Dirac quantization condition becomes

$$\frac{1}{2\pi} \int_\Sigma F \in \mathbb{Z}, \tag{3.14}$$

in the presence of scalar fields, and a similar expression in the presence of spinor fields. Note also that the charge q, which played the role of the gauge coupling constant, is now replaced by the more conventional gauge coupling constant g.

The equations of motions of the Maxwell action (3.12) are

$$d * F = 0. \tag{3.15}$$

Also, due to the Bianchi identity, F is a closed form:

$$dF = 0. \tag{3.16}$$

We can rewrite these two equations as

$$\begin{cases} d\hat{\tau} F = 0 \\ dF = 0. \end{cases} \tag{3.17}$$

These equations of motion are invariant under the linear transformations

$$\begin{aligned}\hat{\tau}F \to \hat{\tau}'F' &= a\hat{\tau}F + bF \\ F \to F' &= c\hat{\tau}F + dF\end{aligned} \tag{3.18}$$

where a, b, c, d are real constants. Under these transformations, τ transforms as

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}. \tag{3.19}$$

Analogous to the invariance of (3.7) under the rotation (3.6), we can investigate the effect of the transformation (3.18) on the symmetric energy momentum tensor $T_{\mu\nu}$, which can be written as

$$T_{\mu\nu} = \frac{1}{g^2}[\frac{1}{2}F_{\mu\rho}\ F_{\sigma\nu}\ g^{\rho\sigma} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}] = \frac{\tau_2}{4\pi}(F_{\mu\rho}g^{\rho\sigma}F_{\sigma\nu} + *F_{\mu\rho}\ g^{\rho\sigma}\ * F_{\sigma\nu}), \tag{3.20}$$

where we have written $F = F_{\mu\nu}dx^\mu \wedge dx^\nu/2, *F = *F_{\mu\nu}dx^\mu \wedge dx^\nu/2$. The densities (3.7) form components of this tensor, so it is natural to ask under which conditions the energy momentum tensor remains invariant under (3.18). The energy momentum tensor changes as

$$T_{\mu\nu} \to (ad - bc)T_{\mu\nu}, \tag{3.21}$$

and we conclude that $T_{\mu\nu}$ is invariant if and only if $ad - bc = 1$. These transformations form the group $SL(2, \mathbb{R})$. The natural question to ask next is: what happens when we are not in vacuum but there are charged particles on $\mathcal{M}$. In the previous chapter we have seen that the presence of charged background fields leads to the quantization of fluxes trough two-cycles, this will play a role in the discussion below. We will investigate how the transformations (3.18) translate into a transformation of the partition function of this Maxwell action.

2. The partition function

In vacuum, we have seen that the energy density is invariant under the duality transformation. Therefore, we consider the partition function

$$Z(\tau) = Tr\left(e^{-H(\tau)}\right), \tag{3.22}$$

which can be expressed in the path integral

$$Z(\tau) = \int \mathcal{D}A\ e^{iS_E[A]}. \tag{3.23}$$

The Euclidean Maxwell action is a Wick rotated version of the previous action:

$$S_E(\tau) = \frac{1}{4\pi}\int_M F \wedge \hat{\tau}F \tag{3.24}$$

where the space-time manifold M is now a closed, oriented, smooth Euclidean manifold M and $\hat{\tau} = \tau_1 + i * \tau_2$ with τ as before. The Hodge dual operator $*$ a quires a factor i due to the Wick rotation.

To evaluate the path integral, we expand it around the stationary points of the exponent, that is, field configurations for which the classical action is minimized. These field configurations are precisely the solutions to the equations of motion for the Maxwell action:

$$d * F = 0. \tag{3.25}$$

Since F is also closed, $dF = 0$ by the Bianchi identity, this means that the classical solutions F are harmonic 2-forms. On our manifolds, the Hodge theorem is applicable and states that there is a unique harmonic representative in each cohomology class. This means that we can choose a basis of harmonic representatives $F^1, ..., F^{b_2}$ for $H^2(X;\mathbb{Z})$. Using the Dirac quantization, we see that the classical solutions are precisely given by

$$F = \begin{cases} m_i F^i, & \text{in the presence of scalar fields,} \\ (m_i + \frac{w_i}{2})F^i & \text{in the presence of spinor fields,} \end{cases} \tag{3.26}$$

where $w_i = I(\Sigma_i, \Sigma_i) \mod 2$. We see that each field configuration that is a stationary point for the classical action, is specified by the b_2 integers $m_1, ..., m_{b_2}$. So the expansion around the stationary point results in a sum over the lattice $F^2(X;\mathbb{Z})$. After the expansion around a stationary point, we can perform the Gaussian integral over the quadratic fluctuations. The resulting determinant is equal for all stationary points and is of the form

$$\Delta(\tau) = C\ \tau_2^{\frac{b_1-1}{2}}, \tag{3.27}$$

where C is a constant and b_1 denotes the first Betti number of M [**25**]. We conclude that the partition function is given by this factor Δ multiplying a summation over a lattice, of the exponentiated action which is given in term of the coordinates of the lattice point. Note that since the action is quadratic in the fields, this expansion method is exact.

3. Scalar fields

We start by considering the case of coupling to scalar fields or spinor fields with Q even. We can write the contribution of the first term in the action as

$$\frac{i\tau_1}{4\pi}\int_M F \wedge F = i\pi\tau_1 m^T Q m. \tag{3.28}$$

For the second term, note that if F^i is harmonic, its Hodge dual $*F^i$ is also harmonic. Therefore, it can be expressed in terms of the basis F^j so there should be a matrix G such that

$$* F^i = G^{ij}(Q^{-1})_{jk}F^k. \tag{3.29}$$

Now since the metric on M is Euclidean, the Hodge star satisfies $*^2 = 1$, so the matrix G should satisfy

$$(GQ^{-1})^2 = 1. \tag{3.30}$$

Therefore we can write the second term as

$$\frac{-\tau_2}{4\pi}\int_M F \wedge *F = -\pi\tau_2 m^T G m. \tag{3.31}$$

For general p-forms α, β, the Hodge operator is defined by $\alpha \wedge *\beta = (\alpha, \beta)d\mu$, where $(\cdot, \cdot)$ is the natural metric on the forms and $d\mu$ is the volume form on M. From this we can conclude that the matrix G must be symmetric and positive definite. Putting the terms together, we see that the action, evaluated at a stationary point, takes the form

$$iS(m) = i\pi m^T \Omega(\tau) m, \tag{3.32}$$

where Ω is the $b_2 \times b_2$ matrix $\Omega(\tau) = \tau_1 Q + i\tau_2 G$. The partition function becomes

$$Z(\tau) = \Delta(\tau) \sum_{m_i \in \mathbb{Z}} e^{i\pi m^T \Omega(\tau) m}. \tag{3.33}$$

Let us define the theta function

$$\Theta(\Omega) = \sum_{m_i \in \mathbb{Z}} e^{i\pi m^T \Omega m}, \tag{3.34}$$

such that $Z(\tau) = \Delta(\tau)\Theta(\tau)$. Note that, due to the positive definiteness of G, convergence of this theta function is guaranteed.

REMARK 3.2. Instead of choosing a harmonic representative F^i in each integer cohomology class and express the partition function as a sum of the integer lattice $H^2(M,\mathbb{Z})$, we can choose a basis of harmonic forms by choosing bases of self-dual and anti-self-dual harmonic forms for $\mathscr{H}^{\pm}$. These form another basis for the vector space $H^2(M,\mathbb{R})$ and in this basis the matrices Q and G are expressed as

$$\begin{aligned} Q &= b^+(1) \oplus b^-(-1) \\ G &= b^+(1) \oplus b^-(1). \end{aligned} \tag{3.35}$$

The partition function becomes a sum over the integral lattice Λ_{b^+,b^-} spanned by the (anti)-self-dual harmonic forms:

$$Z(\tau) = \Delta(\tau) \sum_{(m_+,m_-)\in\Lambda_{b^+,b^-}} e^{\pi i \tau (m_+)^2 - \pi i \bar{\tau}(m_-)^2}. \tag{3.36}$$

REMARK 3.3. While the intersection form Q is determined by the topology of M, the G determines on metric on M. In general, a manifold can admit multiple metrics, therefore the partition function will depend on such a choice of metric. According to Verlinde [**23**], the moduli space of possible partition functions is given by the double quotient

$$\mathscr{M}_{b^+,b^-} = SO(b^+) \times SO(b^-) \backslash SO(b^+,b^-)/SO(b^+,b^-,\mathbb{Z}). \tag{3.37}$$

This can be seen as follows. In a basis of (anti-)self-dual harmonic forms, the intersection matrix is written $Q = b^+(1) \oplus b^-(-1)$ and the relation $(GQ^{-1})^2$ translates into $G^TQG = Q$, the defining relation of the group $SO(b^+,b^-)$. Therefore, the possible metrics G are elements of $SO(b^+,b^-)$. Now given such G, the partition function is written as (3.36). Now when we act with an element $A = A_+ \oplus A_- \in SO(b^+) \times SO(b^-)$ on the lattice Λ_{b^+,b^-}, this corresponds to a change of basis for $\mathscr{H}^+$ and $\mathscr{H}^+$ separately and will leave the quadratic forms (3.35) invariant. Therefore, the action will not change when we conjugate G with A: A^TGA and G determine the same partition function.

The quotient by the discrete group in (3.37) corresponds to transformations that preserve the lattice Λ_{b^+,b^-}, therefore conjugating G with these transformations will determine the same partition function.

We conclude that the moduli space of possible partition functions is of dimension

$$\dim SO(b^+,b^-) - \dim(SO(b^+)\times SO(b^-)) = \frac{1}{2}(b^+ + b^-)(b^+ + b^+ - 1) - \frac{1}{2}b^+(b^+ - 1) - \frac{1}{2}b^-(b^- - 1) = b^+ \times b^-, \tag{3.38}$$

or equivalently, that the partition function depends on $b^+ \times b^-$ parameters, or moduli.

Let us investigate some properties of this partition function. First, for $A \in GL(b_2,\mathbb{Z})$ (invertible $b_2 \times b_2$ matrices with integer coefficients), we have that

$$\Theta(A^T\Omega A) = \sum_{m\in\mathbb{Z}^{b_2}} e^{i\pi (Am)^T \Omega (Am)} = \sum_{m' \in A\mathbb{Z}^{b_2}} e^{i\pi m'^T \Omega m} = \Theta(\Omega), \tag{3.39}$$

since for $A \in GL(b_2, \mathbb{Z})$, $Ax = y \in \mathbb{Z}^{b_2}$ always has an integer solution $x \in \mathbb{Z}^{b_2}$, so $A\mathbb{Z}^{b_2} = \mathbb{Z}^{b_2}$. Also, if B is a symmetric matrix with integer entries, which are even on the diagonal, we have

$$\Theta(\Omega + B) = \Theta(\Omega), \tag{3.40}$$

which can be seen by splitting $B = D + T + T^T$, where D is a diagonal matrix and T a strictly upper triangular matrix. Then $m^T B m = m^T D m + 2m^T T m$ and we see that Θ is invariant if D has even entries.

Consider the Poisson resummation formula:

$$\sum_{n_i \in \mathbb{Z}} f(x_i + n_i) = \sum_{m_j \in \mathbb{Z}} e^{2\pi m_j x_j} \hat{f}(m_j), \tag{3.41}$$

where

$$\hat{f}(k_j) = \int d^n x e^{-2\pi i k_j x_j} f(x). \tag{3.42}$$

In our case, $f(x) = e^{i\pi x_j \Omega_{jm} x_m}$, hence

$$\begin{aligned} \hat{f}(k) &= \int d^n x e^{-2\pi i k_j x_j} e^{i\pi x_j \Omega_{jm} x_m} \\ &= \int d^n x e^{-\frac{1}{2}(-2\pi i \Omega_{jm}) x_j x_m - 2\pi i k_j x_j} \\ &= \sqrt{\frac{(2\pi)^n}{\det(-2\pi i \Omega)}} e^{\frac{1}{2}(-2\pi i) k_j k_m (\Omega^{-1})_{jm}} \\ &= \frac{1}{\sqrt{\det(-i\Omega)}} e^{-i\pi k_j (\Omega^{-1})_{jm} k_m}, \end{aligned} \tag{3.43}$$

where $n = b_2$ and we 'completed the square' and used a standard expression for Gaussian integrals. Plugging this back into the resummation formula results in

$$\sum_{n_i \in \mathbb{Z}} e^{i\pi(x_j + n_j)\Omega_{jk}(x_k + n_k)} = \frac{1}{\sqrt{\det(-i\Omega)}} \sum_{m_i \in \mathbb{Z}} e^{2\pi i m_j x_j} e^{-\pi i m_j (\Omega^{-1})_{jk} m_k}. \tag{3.44}$$

Now if we set $x = 0$ in the above expression, we obtain

$$\Theta(-\Omega^{-1}) = \sqrt{\det(-i\Omega)}\Theta(\Omega). \tag{3.45}$$

We note that the matrix $\Omega(\tau)$ has some special properties:

$$\Omega(\tau + 1) = \Omega(\tau) + Q, \quad \Omega(-1/\tau) = -Q\Omega(\tau)^{-1}Q. \tag{3.46}$$

The first property is obvious, the second property follows from (3.30). Also, we have that

$$\begin{aligned} \sqrt{\det(-i\Omega(\tau)} &= \sqrt{\det(-i\Omega(\tau)Q^{-1}Q)} \\ &= \sqrt{\det\left(\tau_1 I + i\tau_2 GQ^{-1}\right)}\sqrt{\det(-iQ)} \\ &= \tau^{b^+/2}\bar{\tau}^{b^-/2}(-i)^{(b^+ - b^-)/2} = \tau^{b^+/2}\bar{\tau}^{b^-/2}e^{-\frac{2\pi i \sigma}{8}}, \end{aligned} \tag{3.47}$$

where we used that Q and GQ^{-1} have $b^\pm$ eigenvalues ± 1 and $\sigma = b^+ - b^-$ is the signature of M.

Now using the transformation properties (3.39), (3.40) and (3.45) we obtain

$$\begin{aligned} \Theta(-1/\tau) &= e^{-\frac{2\pi i\sigma}{8}}\tau^{b^+/2}\bar{\tau}^{b^-/2}\Theta(\tau) \\ \Theta(\tau+1) &= \Theta(\tau) \qquad \text{if Q is even, and} \\ \Theta(\tau+2) &= \Theta(\tau) \qquad \text{otherwise.} \end{aligned} \tag{3.48}$$

Also note that $\Delta(\tau) = \mathrm{Im}(\tau)^{\frac{b_1-1}{2}}$ satisfies

$$\begin{aligned} \Delta(\tau+1) &= \Delta(\tau) \\ \Delta(-1/\tau) &= \mathrm{Im}(-1/\tau)^{\frac{b_1-1}{2}} = (\frac{1}{\tau\bar{\tau}}\mathrm{Im}(\tau))^{\frac{b_1-1}{2}} = \tau^{\frac{1-b_1}{2}}\bar{\tau}^{\frac{1-b_1}{2}}\Delta(\tau). \end{aligned} \tag{3.49}$$

Definition 3.4. A complex function F of τ transforms as a modular form of weight (u, v) for a subgroup Γ of $SL(2,\mathbb{Z})$ if

$$F\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^u(c\bar{\tau}+d)^v F(\tau), \tag{3.50}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Above, we see the actions of the elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{3.51}$$

Since S and T generate the group $SL(2,\mathbb{Z})$, we conclude that, if the intersection matrix Q is even, the partition function transforms (up to a phase) as a modular form for $SL(2,\mathbb{Z})$ of weights

$$(u, v) = \frac{1}{2}(1 - b_1 + b^+, 1 - b_1 + b^-) = \frac{1}{4}(\chi + \sigma, \chi - \sigma), \tag{3.52}$$

where $\chi = \sum_i (-1)^i b_i$ is the Euler characteristic.

Also, S and T^2 generate a subgroup $\Gamma_\theta \subset SL(2,\mathbb{Z})$ which is called the Hecke subgroup. We conclude that if Q is odd, the partition function is a modular form for Γ_θ with weights as in (3.52).

4. Spinor Fields

Recall that in the case of a spinor field on a manifold M, the fluxes were quantized with a possible half shift $\frac{w_i}{2}$ with $w_i = I(\Sigma_i, \Sigma_i) \mod 2 = Q_{ii} \mod 2$ and the classical solutions could be written as

$$F = (m_i + w_i)F^i. \tag{3.53}$$

Hence, repeating the above calculations but using the above classical saddle point results in a partition function of the form

$$Z(\tau) = \Delta(\tau)\Theta(\tau)_w, \tag{3.54}$$

where

$$\Theta(\tau)_w = \sum_{m_i\in\mathbb{Z}} e^{i\pi(m+w/2)^T\Omega(\tau)(m+w/2)}. \tag{3.55}$$

Remark 3.5. In the case that the intersection form is even, $Q_{ii} = 0 \mod 2$, the partition function (3.55) is equal to the partition function in the scalar field case. So in this case we can immediately conclude that the partition function transforms as a modular form under $SL(2,\mathbb{Z})$.

4.1. Theta functions. To investigate the transformation properties of the partition function (3.55), let us reconsider the Poisson resummation formula (3.41). Suppose that the n_j are coordinates of a point l in a lattice Λ with basis e_i and that the m_j are the coordinates of a point l^* in the dual lattice with basis f_j. See appendix C. We write

$$l = n_j e_j, \quad l^* = m_j f_j, \text{ and let } X = x_j e_j, \tag{3.56}$$

and express the matrix Ω as

$$\hat{\Omega} = f_j \Omega_{jk} f_k^T, \ (\hat{\Omega}^{-1}) = e_j (\Omega^{-1})_{jk} e_k^T, \tag{3.57}$$

then (3.41) becomes

$$\sum_{l \in \Lambda} e^{i\pi (X+l)\cdot\hat{\Omega}\cdot(X+l)} = \frac{1}{\sqrt{\det(-i\Omega)}} \sum_{l^* \in \Lambda^*} e^{2\pi i l^* \cdot X} e^{-i\pi l^* \cdot \hat{\Omega}^{-1} \cdot l^*}. \tag{3.58}$$

Now suppose the lattice is integral and that $Z(\Lambda) = \Lambda^*/\Lambda = \{0, \lambda_1, ..., \lambda_{|Z(\Lambda)|-1}\}$ such that

$$\Lambda^* = \Lambda \cup (\lambda_1 + \Lambda) \cup ... \cup (\lambda_{|Z(\Lambda)|-1} + \Lambda), \tag{3.59}$$

and we have $\lambda_0 = 0$.If we choose $X = \lambda_\alpha$, then $e^{2\pi i l^* \cdot X} = e^{2\pi i l^* \cdot \lambda_\alpha} = e^{2\pi i \lambda_\alpha \cdot \lambda_\beta}$ if $l^* \in \lambda_\beta + \Lambda$ (since $\lambda_\alpha \cdot l \in \mathbb{Z}$ by definition of Λ^*). We can then rearrange (3.58) to

$$\begin{aligned} \sum_{l \in \Lambda} e^{i\pi (X+l)\cdot\hat{\Omega}\cdot(X+l)} &= \sum_{l \in \lambda_\alpha + \Lambda} e^{i\pi l \cdot \hat{\Omega} \cdot l} \\ &= \frac{1}{\sqrt{\det(-i\Omega)}} \sum_{\beta=0} e^{2\pi i \lambda_\alpha \cdot \lambda_\beta} \sum_{l \in \lambda_\beta + \Lambda} e^{i\pi l \cdot \hat{\Omega}^{-1} \cdot l}. \end{aligned} \tag{3.60}$$

Now let $\Omega = \Omega(\tau) = \tau_1 Q + i\tau_2 G$, with $Q = e_i \cdot e_j$ as before, but now not necessarily unimodular, but $|\det Q| = |Z(\Lambda)|$. In this case, we can write

$$\hat{\Omega}(\tau) = \tau_1 + i\tau_2 \hat{G}, \text{ where } \hat{G} = f_i G_{ij} f_j^T. \tag{3.61}$$

Then $\hat{G}^2 = 1$, which follows from (3.30). We can then easily see that

$$\hat{\Omega}(1/\tau) = \frac{1}{\tau\bar{\tau}} (\tau_1 I - i\tau_2 \hat{G}) = \hat{\Omega}^{-1}(\tau). \tag{3.62}$$

Let us define the theta functions

$$\Theta_\alpha(\tau) = \sum_{l \in \lambda_\alpha + \Lambda} e^{i\pi l \dot{\Omega} \cdot l}, \text{ for } \alpha = 0, 1, ..., |Z(\Lambda)| - 1. \tag{3.63}$$

We can rewrite (3.58) in terms of these theta functions as

$$\Theta_\alpha(\tau) = \tau^{-b^+/2} \bar{\tau}^{-b^-/2} \frac{e^{2\pi i \sigma/8}}{|Z(\Lambda)|} \sum_\beta e^{2\pi \lambda_\alpha \cdot \lambda_\beta} \Theta_\beta(-1/\tau). \tag{3.64}$$

The factor $|\det Q|^{-1/2} = |Z(\Lambda)|^{-1/2}$ is put in because now Q might not be unimodular. We recognise (3.64) as the S-transformation. The T-transformation is;

$$\begin{aligned} \Theta_\alpha(\tau + 1) &= e^{\pi i \lambda_\alpha^2} \theta_\alpha(\tau) \text{ if } \Lambda \text{ is even}, \\ \Theta_\alpha(\tau + 2) &= e^{2\pi i \lambda_\alpha^2} \theta_\alpha(\tau) \text{ if } \Lambda \text{ otherwise}, \end{aligned} \tag{3.65}$$

which can easily be seen form the fact that $l \cdot l = (\lambda_\alpha + l') \cdot (\lambda_\alpha + l') = \lambda_\alpha^2 + 2\lambda_\alpha \cdot l' + l' \cdot l'$ and this last term is even depending on Λ being even or not. We see that the action of $SL(2;\mathbb{Z})$, or Γ_θ, transforms the different theta functions into each other.

For an element $B \in SL(2,\mathbb{Z})$, we define the action on a function F of τ as

$$A_{u,v}(B)F(\tau) = (c\tau+d)^{-u}(c\bar{\tau}+d)^{-v}F\left(\frac{a\tau+b}{c\tau+d}\right). \tag{3.66}$$

Definition 3.6. An N-tuple of functions $(F_1(\tau),...,F_N(\tau))$ forms a *vector modular form* for some subgroup Γ of $SL(2,\mathbb{Z})$ if there exist matrices $M_{ki}(B)$ such that

$$A_{u,v}(B)F_i(\tau) = \sum_{k=1}^{N} F_k M_{ki}(B), \tag{3.67}$$

for all $B \in \Gamma$. (u,v) are called the weights of this vector modular form F.

From the transformations (3.64), (3.65) and (3.66), we see that the theta functions Θ_α form a vector modular form of weights $(u,v) = (b^+/2, b^-/2)$ with matrices

$$\begin{aligned} M_{\alpha\beta}(S) &= \frac{e^{-2\pi i\sigma/8}}{\sqrt{|Z(\Lambda)|}}e^{-2\pi i\lambda_\alpha\cdot\lambda_\beta} \\ M_{\alpha\beta}(T) &= e^{\pi i\lambda_\alpha^2}\delta_{\alpha\beta} \\ M_{\alpha\beta}(T^2) &= e^{2\pi i\lambda_\alpha^2}\delta_{\alpha\beta}. \end{aligned} \tag{3.68}$$

Note that these matrices are independent of the choice of representatives λ_α: a different representative $\lambda'_\alpha = \lambda_\alpha + x, x \in \Lambda$ results in the same matrix, since $x \cdot \lambda_\beta \in \mathbb{Z}$ for all β.

4.2. Theta functions on cohomology lattices. If the lattice Λ is unimodular, $Z(\Lambda) = 1$ and the construction above gives us only 1 theta function. The matrices D above are then just phase factors.

If the lattice Λ is odd, there is an sublattice Λ_{even} such that $|Z(\Lambda_{even})| = 4$. Let c be a characteristic element of Λ, then we can decompose $\Lambda_{total} = \Lambda^*_{even}$ as

$$\Lambda_{total} = \Lambda_{even} \cup \Lambda_{odd} \cup (\Lambda_{even} + c/2) \cup (\lambda_{odd} + c/2), \tag{3.69}$$

and we choose as representatives for these cosets:

$$0, \quad \lambda_v, \lambda_s = \frac{c}{2}, \lambda_t = \lambda_v + \frac{c}{2}, \tag{3.70}$$

where $\lambda_v \in \Lambda_{odd}$. Following the construction above, this gives four theta functions Θ_α, $\alpha = 0, v, s, t$. We can now explicitly calculate the matrices given by (3.68) and (3.69). As an example:

$$\lambda_s \cdot \lambda_t = \frac{c}{2} \cdot (\lambda_v + \frac{c}{2}) = \frac{1}{2}(\lambda_v^2 + 2\mathbb{Z}) + \frac{1}{4}c^2 = \frac{1}{2}(2k+1) + \frac{1}{4}(\sigma + 8\mathbb{Z}), \tag{3.71}$$

where we use that c is characteristic, $\lambda_v \in \Lambda_{odd}$ and $c^2 = \sigma \mod 8$ (Appendix C.). Hence:

$$e^{-2\pi i\lambda_v\cdot\lambda_t} = -1 \cdot e^{-2\pi i\sigma/4}. \tag{3.72}$$

The total result is:

$$M_{\alpha\beta}(S) = \frac{\varphi}{2}e^{-2\pi i\lambda_\alpha\cdot\lambda_\beta} = \frac{\varphi}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \varphi^2 & -\varphi^2 \\ 1 & -1 & -\varphi^2 & \varphi^2 \end{pmatrix} \tag{3.73}$$

and

$$M_{\alpha\beta}(T) = \mathrm{diag}(1, -1, \varphi^{-1}, \varphi^{-1}), \tag{3.74}$$

where $\varphi = e^{-2\pi i\sigma/8}$.

Now with the help of these matrices, we see that

$$A(S)(T)(\Theta_s - \Theta_t) = \varphi^3(\Theta_s - \Theta_t), \quad A(T)(\Theta_s - \Theta_t) = \varphi^{-1}(\Theta_s - \Theta_t). \tag{3.75}$$

The other linear independent theta functions are given by Θ_0, Θ_v and $\Theta_s + \Theta_t$. We can also consider the following combinations:

$$\Theta_1 := \Theta_0 + \Theta_v = \sum_{l\in\Lambda_{even}} e^{\pi i l\cdot\Omega\cdot l} + \sum_{l\in\Lambda_{odd}} e^{\pi i l\cdot\Omega\cdot l} = \sum_{l\in\Lambda} e^{\pi i l\cdot\Omega\cdot l}, \tag{3.76}$$

$$\Theta_2 := A(T)\Theta_1 = \Theta_0 - \Theta_v = \sum_{l\in\Lambda_{even}} e^{\pi i l\cdot\Omega\cdot l} - \sum_{l\in\Lambda_{odd}} e^{\pi i l\cdot\Omega\cdot l}, \tag{3.77}$$

$$\Theta_3 := \varphi^{-1}A(S)\Theta_2 = (\Theta_s + \Theta_t) = \sum_{l\in c/2+\Lambda} e^{\pi i l\cdot\Omega\cdot l} = \sum_{l\in\Lambda} e^{\pi i(l+c/2)\cdot\Omega\cdot(l+c/2)}. \tag{3.78}$$

Note that Θ_1 is precisely the theta function associated to the original lattice Λ.

These 3 theta functions form an alternative basis for the 3-dimensional subspace of theta functions orthogonal to $\Theta_s - \Theta_t$. In this basis, the transformation matrices are

$$M(S) = \varphi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, M(T) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \varphi^{-1} \end{pmatrix}. \tag{3.79}$$

Now consider the partition function $Z(\tau)$ defined above for a manifold with odd intersection form. It is given as a sum over the free cohomology lattice $F^2(M;\mathbb{Z})$, which an unimodular lattice and now assumed to be odd. The partition function associated to the case of coupling to a scalar field is given by (3.33) and is precisely

$$Z_1(\tau) = Z_{scalar}(\tau) = \Delta(\tau)\Theta_1(\tau). \tag{3.80}$$

We can also define a partition function by

$$Z_2(\tau) = \Delta(\tau)\Theta_2(\tau), \tag{3.81}$$

which equals Z_{scalar} but with the angle θ replaced by $\theta + 2\pi$. The third partition function is the one relevant for the case of coupling to a spinor field:

$$Z_3(\tau) = Z_{spinor}(\tau) = \Delta(\tau)\Theta_3. \tag{3.82}$$

Recall that the element $w \in F^2(X;\mathbb{Z})$ formed a characteristic element for this free cohomology lattice, so we can identify w with c above and we conclude that Z_3 is precisely the partition function (3.55).

Taking into acount that $\Delta(\tau)$ transforms as a modular form of weight $\frac{1}{2}(1-b_1, 1-b_1)$, we conclude:

For a manifold with odd cohomology lattice, the partition functions Z_i defined above transform as a vector modular form of weigths $\frac{1}{2}(1-b_1+b^+, 1-b_1+b^-) = \frac{1}{4}(\chi+\sigma, \chi-\sigma)$ under the action of $SL(2,\mathbb{Z})$, with transformation matrices given in (3.79).

Finally, we point out that the product of the 3 partition functions,

$$Z_{prod}(\tau) = Z_1(\tau)\cdot Z_2(\tau)\cdot Z_3(\tau) = \Delta(\tau)^3\Theta_1(\tau)\Theta_2(\tau)\Theta_3(\tau), \tag{3.83}$$

transforms as a modular form itself, with weights $\frac{3}{4}(\chi+\sigma, \chi-\sigma)$. The transformation matrices are in this case just phase factors:

$$M(S) = \varphi^3, \quad M(T) = \varphi^{-1}. \tag{3.84}$$

This composite partition function corresponds to a physical system in which there are three gauge fields. Two gauge fields couple to a scalar field and are described by a Maxwell action with electromagnetic parameter τ and $\tau+1$ respectively, while the third gauge field, also described by a Maxwell action with parameter τ, couples to a spinor field. The $SL(2,\mathbb{Z})$ action on the partition function then permutes these fields.

In the next chapter, we will use the formalism constructed above to calculate the partition function on a type of four-manifolds, called *del Pezzo surfaces*.

5. Two physical examples

So far, we have considered the space-time manifolds to be compact and endowed with a Riemannian metric. These properties allow us to use Poincaré duality and the Hodge theorem. However, the space-time manifolds that appear in physics (as solution to the Einstein equations of general relativity), are usually non-compact and have a Minkowskian metric.

In some cases we can extend the discussion above to more 'physical' space-times. First, we can always Wick rotate the metric to obtain a Riemannian metric on the manifold. Also, one can use various extensions of the above theorems for non-compact spaces. However, sometimes the topology of the space-times is nice enough to avoid these extensions. Let us consider some examples inspired by general relativity.

Example 3.7. Consider four-dimensional de Sitter space dS_4. It is a maximally symmetric solution to the Einstein field equations in vacuum, with a positive cosmological constant Λ. The metric can be written as

$$ds^2 = -d\tau^2 + L^2\cosh^2(\tau/L)d\Omega_3^2, \tag{3.85}$$

where $L^2 = \Lambda/3$. This space-time has the topology of (is homeomorphic to) $\mathbb{R}\times S^3$. We can Wick rotate the metric to obtain a Euclidean metric:

$$ds^2 = d\tau_E^2 + L^2\cos^2(\tau_E/L)d\Omega_3^2. \tag{3.86}$$

The Euclidean time $\tau_E = i\tau$ is now periodic with period $\beta = 2\pi L$ (often identified with inverse temperature). We see that the Wick rotated version of dS_4 is homeomorphic to S^4. The homology and cohomology of S^4 are very simple:

$$H_k(S^4;\mathbb{Z}) = H^k(S^4;\mathbb{Z}) = \begin{cases}\mathbb{Z}, & \text{if } k=0,4,\\ 0 & \text{otherwise.}\end{cases} \tag{3.87}$$

Now consider electromagnetic fields on dS_4. Let F be a solution to the equations of motion and the Bianchi identity, $dF = d^*F = 0$. Since H^2 vanishes, F must be an exact two-form: $F = dA$ for some globally defined one-form. Now, the L^2-norm of F is

$$\|F\|^2 = \int_{S^4}(F,F)d\mu = \int_{S^4}(A,d^*F) = 0, \tag{3.88}$$

so we conclude that $F=0$.

Therefore, there is only one classical solution, $F=0$, and the action vanishes. The expansion of the path integral around classical saddle point results in the expansion around this trivial solution, and only the determinant factor remains:

$$Z_{S^4}(\tau) = \Delta(\tau) = C\tau_2^{-1/2} = C\frac{g}{\sqrt{4\pi}}, \tag{3.89}$$

where we use that $b_1 = 0$. We see that this partition function is a modular form of weight $(1/2,1/2)$, precisely as prescribed above, since $\chi = 2$ and $\sigma = 0$.

In the case of the Sitter space, the vanishing of the second (co)homology leads to a trivial result for the partition function. Let us consider an example with a slightly more non-trivial topology: that of the Schwarzschild black hole.

EXAMPLE 3.8. Consider the Schwarzschild black hole in four dimensions. It is a solution to the Einstein field equations with zero cosmological constant. After Wick rotating, we can write the metric as

$$ds^2 = (1 - \frac{2m}{r})d\tau^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2 d\Omega_2^2, \tag{3.90}$$

in a coordinate chart outside the horizon $r > 2m$ and the imaginary time $\tau \in [0, 8\pi m)$ [**10**]. Also, $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element on S^2 in spherical coordinates. This manifold is known as the Euclidean Schwarzschild manifold, here denoted M, and is homeomorphic to $\mathbb{R}^2 \times S^2$. The homology of this manifold can be calculated using the Künneth formula or by using that this space is homotopy equivalent to S^2:

$$H_k(\mathbb{R}^2 \times S^2; \mathbb{Z}) \cong H_k(S^2; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.91}$$

So the second homology of the Schwarzschild black hole has one generator and we can think of this embedded S^2 as being the horizon of the black hole. To express the partition function as a sum, we should find an harmonic form ω dual to S^2:

$$\int_{S^2} F = 1. \tag{3.92}$$

Since the manifold is not compact, Hodge theorem is not applicable. However, we can find the harmonic form explicitly. One can check that the closed two-form

$$\omega = \frac{1}{2}(\frac{1}{r^2} d\tau \wedge dr + \sin\theta d\theta \wedge d\phi), \tag{3.93}$$

is self-dual, hence harmonic. It satisfies

$$\frac{1}{2\pi} \int_{S^2} \omega|_{S^2} = \frac{1}{4\pi} \int_{S^2} \sin\theta d\theta \wedge d\phi = \frac{1}{4\pi} \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi = 1, \tag{3.94}$$

so that $\frac{1}{2\pi}\omega$ is indeed dual to S^2 and the cohomology class of $\frac{1}{2\pi}\omega$ is the generator of $H^2(\mathbb{R}^2 \times S^2; \mathbb{Z}) \cong H^2(S^2; \mathbb{Z}) = \mathbb{Z}$. Also, ω satisfies

$$\frac{1}{4\pi^2} \int_M \omega \wedge \omega = 2. \tag{3.95}$$

Therefore, we find that the intersection matrix is given by $Q = (2)$.

Now let F be any solution to the equations of motion, $dF = d * F = 0$. If there are charged scalar fields on the Schwarzschild manifold, there will be a quantization of the flux: we can cover the S^2 with a finite open cover and repeat the arguments of the previous chapter. We deduce that

$$\frac{1}{2\pi} \int_{S^2} F \in \mathbb{Z}. \tag{3.96}$$

Therefore, F is an integer multiple of the self-dual harmonic form ω. This expression is also valid for the spinor case since the intersection form is even.

Using this information, we can expand the path integral as a sum over classical solutions and we find that the partition function for the Schwarzschild black hole becomes:

$$Z(\tau) = \Delta(\tau) \sum_{m \in \mathbb{Z}} e^{2\pi i \tau m^2} = C \frac{g}{\sqrt{4\pi}} \sum_{m \in \mathbb{Z}} e^{2\pi i \tau m^2}. \tag{3.97}$$

REMARK 3.9. It is important to note that the constant C, although denoted the same for both examples, depends on the geometry of the specific space-time. In the non-compact case, this constant may be a somewhat ill-defined, as it contains the determinant of the Laplacian, which can be an infinite quantity in a non-compact space-time.

CHAPTER 4

The Maxwell partition function on del Pezzo surfaces

In this chapter, we will use the Dirac quantization condition and the expansion method to express the partition function as a lattice sum, to explicitly calculate the partition function on a type of manifolds called *del Pezzo surfaces*.

Before we can calculate the partition function on the del Pezzo surfaces, we need to have an understanding of their topology. We will see that they can be described in terms of a *blow up* construction.

We start by describing in detail the complex projective plane $\mathbb{CP}^2$, which will be the basic object from which to construct the del Pezzo surfaces.

1. The complex projective plane $\mathbb{CP}^2$

The *complex projective space* is the space of all lines in $\mathbb{C}^n$. It can be constructed from $\mathbb{C}^n \setminus \{0\}$ by identifying $(z_0, ..., z_n) \sim (\lambda z_0, ..., \lambda z_n)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. We denote the equivalence class of $(z_0, ..., z_n)$ by $[z_0, ..., z_n]$. The collection of charts (U_i, ϕ_i) where $U_i \subset \mathbb{CP}^n$ is the set of all points $[z_0, ..., z_n]$ with $z_i \neq 0$ and

$$\phi_i : U_i \to \mathbb{C}^n, \quad \phi_i([z_0, ..., z_n]) = (\frac{z_0}{z_i}, ..., \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, ...\frac{z_n}{z_i}) \tag{4.1}$$

form an atlas for $\mathbb{CP}^n$.

To compute its cohomology, we give $\mathbb{CP}^n$ a cell complex (CW-) structure:
we have an inclusion $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ by

$$[z_0, ...z_{n-1}] \mapsto [z_0, ..., z_{n-1}, 0]. \tag{4.2}$$

Now we can represent a point in $\mathbb{CP}^n$ by $(z_0, ..., z_{n-1}, t)$, where $t = \sqrt{1 - z_i \bar{z}_i}$. Then we can define a map

$$D^{2n} \to \mathbb{CP}^n, \quad (z_0, ..., z_{n-1}) \mapsto [z_0, ..., z_{n-1}, t], \tag{4.3}$$

such that the boundary S^{2n-1} of D^{2n} maps to $\mathbb{CP}^{n-1}$. This gives $\mathbb{CP}^n$ a CW-structure with 1 cell in each even dimension.
We put this in the cellular chain complex

$$... \to H_{n+1}(X_{n+1}, X_n) \to H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2}) \to , \tag{4.4}$$

where each $H_n(X_n, X_{n-1})$ is freely generated with a generator for each cell in dimension n and X_n denotes the n-skeleton of $\mathbb{CP}^n$. Now the homology of this complex is isomorphic to that of $\mathbb{CP}^n$, so that we have

$$H_k(\mathbb{CP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } k \text{ even} \\ 0 \text{ if } k \text{ odd.} \end{cases} \tag{4.5}$$

From the construction of the CW-complex and the definition of the cellular chain complex we immediately get that $H_2(X;\mathbb{Z})$ is generated by the 2-cell of $\mathbb{CP}^1$ embedded in $\mathbb{CP}^n$:

$$\mathbb{C} \supset D^2 \to \mathbb{CP}^n, \quad z \mapsto [z, \sqrt{1-|z|^2}, ..., 0]. \tag{4.6}$$

If we use Poincaré duality (since $\mathbb{CP}^n$ is compact, oriented and without boundary) we get

$$H^k(\mathbb{CP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } k \text{ even} \\ 0 \text{ if } k \text{ odd.} \end{cases} \tag{4.7}$$

To describe $H^2(\mathbb{CP}^n;\mathbb{Z})$ we have a nice representative for the generator: the Fubini-Study form. $\mathbb{CP}^n$ is Kähler, so we can write the Kähler form as $\omega = i\partial\bar{\partial}K_i$ on U_i, for some complex function K_i on U_i. The Fubini-Study form is defined by $K_i = \log 1 + |z_\alpha|^2$ where $z_\alpha = \frac{w_\alpha}{w_i}$ for $[w_0, ..., w_n] \in U_i$.

We find that

$$\omega = i\partial\bar{\partial}K_0 = \frac{i}{(1+|z_\alpha|^2)^2}\left((1+|z_\alpha|^2)\delta_{\alpha\bar{\beta}} - \bar{z_\alpha}z_{\bar{\beta}}\right) dz^\alpha \wedge dz^{\bar{\beta}} := i g_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}. \tag{4.8}$$

This form is obviously closed: $\partial\partial\bar{\partial}\omega = 0$ and $\bar{\partial}\partial\bar{\partial}\omega = -\partial\bar{\partial}\bar{\partial}\omega = 0$, hence $d\omega = 0$. However, to construct the partition function, we need to expand the 2-form F into a basis of harmonic forms. Below, we explicitely check that ω is harmonic.

We can choose the metric on $\mathbb{CP}^n$ to be the Fubini-Study metric: $g = 2g_{\alpha\bar{\beta}}dz^\alpha \otimes dz^{\bar{\beta}}$, where $g_{\alpha\bar{\beta}}$ is as above. Now we want to compute $*\omega$.

Locally, we can write a general (p,q)-form on an m-dimensional complex manifold as

$$\psi = \psi_{\boldsymbol{\alpha}_p\bar{\boldsymbol{\beta}}_q} dz^{\boldsymbol{\alpha}_p} \wedge dz^{\bar{\boldsymbol{\beta}}_q}, \tag{4.9}$$

where we define

$$\begin{aligned} \boldsymbol{\alpha}_p &= (\alpha_1, ..., \alpha_p),\, \alpha_1 < \alpha_2 < ... < \alpha_p,\, 1 \le \alpha_i \le n, \\ \boldsymbol{\alpha}_{n-p} &= (\alpha_{p+1}, ..., \alpha_n),\, \alpha_{p+1} < ... < \alpha_n,\, 1 \le \alpha_i \le n, \end{aligned} \tag{4.10}$$

such that $(\alpha_1, ..., \alpha_n)$ is a permutation of $(1, ..., n)$. We define the same for $\boldsymbol{\beta}_q$ and $\boldsymbol{\beta}_{n-q}$. Now we denote $g^{\bar{\alpha}\beta} = (g_{\alpha\bar{\beta}})^{-1}$ and

$$\psi^{\bar{\boldsymbol{\alpha}}_p\boldsymbol{\beta}_q} = g^{\bar{\alpha}_1\lambda_1}...g^{\bar{\mu}_q\beta_q}\psi_{\lambda_1...\lambda_p\bar{\mu}_1...\bar{\mu}_q}. \tag{4.11}$$

Also, define $g_{\boldsymbol{\alpha}_p\boldsymbol{\alpha}_{n-p}\boldsymbol{\beta}_q\boldsymbol{\beta}_{n-q}} = g_{\alpha_1...\alpha_n\bar{\beta}_1...\bar{\beta}_n} = \det(g_{\alpha_i\bar{\beta}_k})$.

Now we define the $(n-q, n-p)$-form [**18**]

$$*\,\psi = (i)^n(-1)^{\frac{1}{2}n(n-1)+pn}\, g_{\boldsymbol{\alpha}_q\boldsymbol{\alpha}_{n-q}\boldsymbol{\beta}_p\boldsymbol{\beta}_{n-p}}\, \psi^{\bar{\boldsymbol{\beta}}_p\boldsymbol{\alpha}_q} dz^{\boldsymbol{\alpha}_{n-q}} \wedge dz^{\bar{\boldsymbol{\beta}}_{n-p}}. \tag{4.12}$$

In our case of $\mathbb{CP}^2$ we have that in the chart U_0:

$$g_{\alpha\bar{\beta}} = \frac{1}{(1+|z_\alpha|^2)^2}\begin{pmatrix} 1+|z_2|^2 & -\bar{z}_1 z_2 \\ -z_1\bar{z}_2 & 1+|z_1|^2 \end{pmatrix}, \tag{4.13}$$

such that $\det(g_{\alpha\bar{\beta}}) = \frac{1}{(1+|z_i|^2)^3}$, and

$$g^{\bar{\alpha}\beta} = (1+|z_\alpha|^2)\begin{pmatrix} 1+|z_1|^2 & \bar{z}_1 z_2 \\ z_1\bar{z}_2 & 1+|z_2|^2 \end{pmatrix}, \tag{4.14}$$

such that

$$\begin{aligned}(4.15)\quad *w &= ii^2(-1)^3(\det(g)g^{11}dz^2\wedge\overline{dz^2}+(-\det(g))g^{21}dz^2\wedge\overline{dz^1}+\\ &\quad(-\det(g))g^{12}dz^1\wedge\overline{dz^2}+\det(g)g^{22}dz^1\wedge\overline{dz^1})\\ &= \frac{i}{(1+|z_\alpha|^2)^2}((1+|z_1|^2)dz^2\wedge\overline{dz^2}-z_1\bar{z}_2dz^2\wedge\overline{dz^1}-\bar{z}_1z_2dz^1\wedge\overline{dz^2}+(1+|z_2|^2)dz^1\wedge\overline{dz^1})\\ &= g_{\alpha\bar{\beta}}dz^\alpha\wedge dz^{\bar{\beta}}=\omega,\end{aligned}$$

where we used that $g_{1212}=-g_{2112}=-g_{1221}=g_{2121}=\det(g)=(1+|z_\alpha|^2)^{-3}$. In the other charts U_1 and U_2 the calculation in identical. We conclude that ω is globally selfdual.
Note that if we would have taken the manifold $\overline{\mathbb{CP}^2}$, which has the opposite orientation of $\mathbb{CP}^2$, which is induced from an opposite orientation of $\mathbb{C}^2$, then $*$ becomes $-*$ such that ω will be anti-selfdual.
Since the adjoint of d satisfies $d^*=\pm*d*$, we get that $d^*\omega=0$ such that ω is harmonic: $\Delta\omega=(d+d^*)^2\omega=0$.

The Fubini-Study form satisfies $\int_{\mathbb{CP}^1}\omega=1$, where this $\mathbb{CP}^1$ denotes the embedded copy of $\mathbb{CP}^1$ in $\mathbb{CP}^2$. This implies that ω is dual to $\mathbb{CP}^1$. The intersection matrix is given by

$$Q(\omega,\omega)=\int_{\mathbb{CP}^2}\omega\wedge\omega=\int_{\mathbb{CP}^2}\omega\wedge*\omega=\int_{\mathbb{CP}^2}d\mu(\omega,\omega)=1, \tag{4.16}$$

where we assume that ω is normalized. Following the notation of the previous chapter, $\Sigma=\mathbb{CP}^1$ is a representative for the generator (a basis of 1 element) $H_2(M;\mathbb{Z})=F_2(M;\mathbb{Z})=\mathbb{Z}$ and $F=\omega$ is a harmonic representative for the dual generator of $H^2(M;\mathbb{Z})=F^2(M;\mathbb{Z})=\mathbb{Z}$.

We conclude the following topological data for $\mathbb{CP}^2$:

- The (co)homology lattice $H_2=F_2=H^2=F^2=\mathbb{Z}$.
- The Euler characteristic $\chi=3$, which we can conclude from the CW-structure.
- The intersection form $Q=1$, hence $b^+=1, b^-=0$ and $\sigma=1$.

Remark 4.1. If we had taken $\overline{\mathbb{CP}}^2$ instead of $\mathbb{CP}^2$, we would obtain $Q=-1$, $G=1$, since

$$\int_{\overline{\mathbb{CP}}^2}\omega\wedge\omega=-\int_{\overline{\mathbb{CP}}^2}\omega\wedge*\omega=-1. \tag{4.17}$$

1.1. Electromagnetic duality on $\mathbb{CP}^2$.

We can explicitely check the formalism developed in the previous chapter. As concluded above, the cohomology lattice Λ of $\mathbb{CP}^2$ is equal to $\mathbb{Z}$. The partition function in the presence of scalar fields is given by

$$Z_{\mathbb{CP}^2,sc}(\tau)=\Delta(\tau)\sum_{n\in Z}e^{\pi in^2\tau}=\Delta(\tau)\ \Theta_{\mathbb{CP}^2,sc}(\tau). \tag{4.18}$$

Note that $\Theta_{\mathbb{CP}^2,sc}$ is of the form Θ_1, defined in the previous chapter. We compute the T- and S-transformation:

$$\Theta_{\mathbb{CP}^2,sc}(\tau+1)\ =\ \sum_{n\in\mathbb{Z}}e^{\pi in^2\tau}e^{\pi in^2}=\sum_{n\in2\mathbb{Z}}e^{\pi in^2\tau}-\sum_{n\in1+2\mathbb{Z}}e^{\pi in^2\tau}=\Theta_2(\tau), \tag{4.19}$$

$$\Theta_{\mathbb{CP}^2,sc}(\tau+2)\ =\ \Theta_2(\tau+1)=\sum_{n\in\mathbb{Z}}e^{\pi in^2\tau}e^{2\pi in^2}=\Theta_1(\tau)=\Theta_{\mathbb{CP}^2,sc}(\tau). \tag{4.20}$$

For the S-transformation we can use another form of the Poisson Resummation Formula [**6**]:

$$\sum_{n\in\mathbb{Z}}\exp(-\pi an^2+bn)=\frac{1}{\sqrt{a}}\sum_{k\in\mathbb{Z}}\exp(-\frac{\pi}{a}(k+\frac{b}{2\pi i})^2). \tag{4.21}$$

Using $a = -i\tau, b = 0$, this gives us

$$\begin{aligned}\tau^{-1/2}\ \Theta_{\mathbb{CP}^2,sc}(-1/\tau) &= \tau^{-1/2}\sum_{n\in\mathbb{Z}} e^{\pi i n^2\cdot(-1/\tau)} = \tau^{-1/2}\sqrt{-i\tau}\sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau} \\ &= \sqrt{-i}\sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau} = e^{-\pi i/4}\sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau} = \kappa\ \Theta_{\mathbb{CP}^2,sc}(\tau).\end{aligned} \tag{4.22}$$

Note that $1 \in \mathbb{Z}$ is a characteristic element for Λ, as $x \equiv x^2 \bmod 2$. This is to be expected, since $Q = 1$, hence $w(\Sigma) = I(\mathbb{CP}^1, \mathbb{CP}^1) = 1$. Λ is of course an odd lattice, an we can define the even sublattice $2\mathbb{Z}$. Then Λ_{total} splits into cosets with representatives (following notation of the previous chapter):

$$\lambda_0 = 0,\ \lambda_v = 1,\ \lambda_s = 1/2 \text{ and } \lambda_t = 3/2. \tag{4.23}$$

In the presence of spinor fields, the flux (through $\mathbb{CP}^1$) is fractionally quantized and the Maxwell partition function is then:

$$Z_{\mathbb{CP}^2,sp}(\tau) = \Delta(\tau)\sum_{n\in\mathbb{Z}} e^{\pi i(n+\frac{1}{2})^2\tau} = C\Delta(\tau)\ \Theta_{\mathbb{CP}^2,sp}(\tau). \tag{4.24}$$

First note that

$$\Theta_{\mathbb{CP}^2,sp}(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+\frac{1}{2})^2\tau} = \sum_{n\in(1/2+2\mathbb{Z})} e^{\pi i(n+\frac{1}{2})^2\tau} + \sum_{n\in(3/2+2\mathbb{Z})} e^{\pi i(n+\frac{1}{2})^2\tau} = \Theta_3(\tau). \tag{4.25}$$

We compute:

$$\begin{aligned}\Theta_{\mathbb{CP}^2,sp}(\tau+1) &= \sum_{n\in\mathbb{Z}} e^{\pi i(n+\frac{1}{2})^2\tau} e^{\pi i(n+\frac{1}{2})^2} = \sum n \in Z e^{\pi i(n+\frac{1}{2})^2} e^{\pi i(n^2+n+\frac{1}{4})} \\ &= \sum_{n\in\mathbb{Z}} (-1)^{n^2+n} e^{\pi i(n+\frac{1}{2})^2\tau} e^{2\pi i/8} = \kappa^{-1}\Theta_3(\tau),\end{aligned} \tag{4.26}$$

$$\Theta_{\mathbb{CP}^2,sp}(\tau+2) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+\frac{1}{2})^2\tau} e^{2\pi i(n+\frac{1}{2})^2} = (e^{2\pi i/8})^{-2}\Theta_3(\tau) = \kappa^{-2}\Theta_3(\tau), \tag{4.27}$$

where $\kappa = e^{-2\pi i\sigma/8}$. For the S-transformation we use again (4.21) with $a = -i\tau, b = \pi i$:

$$\begin{aligned}\tau^{-1/2}\Theta_{\mathbb{CP}^2,sp}(-1/\tau) &= \tau^{-1/2}\sum_{n\in\mathbb{Z}} e^{\pi i(n+\frac{1}{2})^2\cdot -1/\tau} = \tau^{-1/2}\sqrt{-i\tau}\sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau+\pi i n} \\ &= \sqrt{-i}\sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau+\pi i n^2} = e^{-\pi i/4}\sum_{n\in\mathbb{Z}} e^{\pi i n^2(\tau+1)} = \kappa\Theta_2(\tau).\end{aligned} \tag{4.28}$$

We see that these expressions precisely match with the descriptions of the previous chapter.

Remark 4.2. For $\overline{\mathbb{CP}}^2$, $Q = -1, G = 1$ and we can define the partition functions

$$Z_{\overline{\mathbb{CP}}^2,sc}(\tau) = \Delta(\tau)\ \Theta_{\overline{\mathbb{CP}}^2,sc}(\tau) = \Delta(\tau)\sum_{m\in\mathbb{Z}} e^{\pi i m^2(-\tau_1+i\tau_2)}, \tag{4.29}$$

$$Z_{\overline{\mathbb{CP}}^2,sp}(\tau) = \Delta(\tau)\ \Theta_{\overline{\mathbb{CP}}^2,sp}(\tau) = \Delta(\tau)\sum_{m\in\mathbb{Z}} e^{\pi i(m-\frac{1}{2})^2(-\tau_1+i\tau_2)}. \tag{4.30}$$

2. Blowing up a point

When we construct the del Pezzo surfaces, we need a mathematical construction called *blowing up*.

Let M be a complex manifold of (complex) dimension n and let $z = (z_1, ..., z_n)$ be holomorphic coordinates in a neighbourhood $p \in U \subset M$. Let

$$\tilde{U} = \{(z, l) : z \in l\} \subset U \times \mathbb{CP}^{n-1}. \tag{4.31}$$

Definition 4.3. The *blow-up* $\tilde{M}$ of M at p is the complex manifold obtained from M by glueing $\tilde{U}$ to $M \setminus \{p\}$ along the isomorphism

$$\tilde{U} \setminus \{z = 0\} \cong U \setminus \{p\}, \quad (z, l) \mapsto z. \tag{4.32}$$

The projection map $\pi : \tilde{M} \to M$ extends the identity on $M \setminus \{p\}$. The inverse image $E := \pi^{-1}(\{p\}) \cong \mathbb{CP}^{n-1}$ is called the *exceptional divisor* of the blowup $\tilde{M} \to M$. We can visualize the blow up of a point as *making all the lines through p disjoint*, see Figure 2.

We can calculate the homology of $\tilde{M}$ by using the *Mayer-Vietoris sequence*. Let $M^* = M \setminus \{p\}$, $\tilde{M}^* = \pi^{-1}(M^*) = \tilde{M} \setminus E$, $U^* = U \setminus \{p\}$ and $\tilde{U}^* = \pi^{-1}U^* = \tilde{U} \setminus E$, such that $M^* \cap U = U^*$ and $\tilde{M}^* \cap \tilde{U} = \tilde{U}^*$. We compare the Mayer-Vietoris sequences for $M^* \cup U$ and $\tilde{M}^* \cup \tilde{U}$:

$$\begin{tikzcd}
H_i(\tilde{U}^*) \arrow[r] \arrow[d, "\pi_*"] & H_i(\tilde{U}) \oplus H_i(\tilde{M}^*) \arrow[r] \arrow[d, "\pi_*"] & H_i(\tilde{M}) \arrow[r] \arrow[d, "\pi_*"] & H_{i+1}(\tilde{U}^*) \arrow[d, "\pi_*"] \\
H_i(U^*) \arrow[r] & H_i(U) \oplus H_i(M^*) \arrow[r] & H_i(M) \arrow[r] & H_{i+1}(U)
\end{tikzcd}$$

Now π is an isomorphism $\tilde{U}^* \cong U^*$ and $\tilde{M}^* \cong M^*$, hence it induces isomorphisms π_* between $H_i(\tilde{U}^*)$ and $H_i(U^*)$, and between $H_i(\tilde{M}^*)$ and H_i. Also, if we choose U to be a small ball around p, the contraction $U \to \{p\}$ by $z \mapsto tz$ induces a contraction $\tilde{U} \to E$ via π. Then, for $i > 0$, $H_i(U) = 0$ and $H_i(\tilde{U}) \cong H_i(E)$. We conclude that

$$H_i(\tilde{M}) = H_i(M) \oplus H_i(E), \ i > 0. \tag{4.33}$$

We can also describe the blow-up $\tilde{M}$ of M at a point in terms of a *connected sum*. The connected sum of two manifolds is obtained by removing a small disc from both manifolds and identifying the boundary spheres. Let U be a coordinate chart at p as above, and let $V_0 = \{[w_0, w] | w_0 \neq 0\} \subset \overline{\mathbb{CP}}^n$ be the coordinate chart at $q = [1, 0, , 0]$ with coordinates w/w_0. We can form the connected sum $M \# \overline{\mathbb{CP}}^n$ by identifying the two open neighborhoods $U \cong V_0$.

Now, let $\tilde{U} \subset U \times \mathbb{CP}^{n-1}$ as above and let us define $\tilde{U}_\epsilon = \{(z, l) \in \tilde{U} : |z| < \epsilon\}$ and

$$V_\epsilon = \{[w_0, w] \in \overline{\mathbb{CP}}^n : |w_0| < \epsilon |w|\}. \tag{4.34}$$

Then, the map $f : M \# \overline{\mathbb{CP}}^n \to \tilde{M}$,

$$f(x) = \begin{cases} x & \text{if } x \in M \setminus \{p\}, \\ (\frac{w_0 w}{|w|^2}, [w]) \in \tilde{U}_\epsilon & \text{if } x = [w_0, w] \in V_\epsilon \end{cases} \tag{4.35}$$

is well-defined and defines an orientation preserving diffeomorphism.

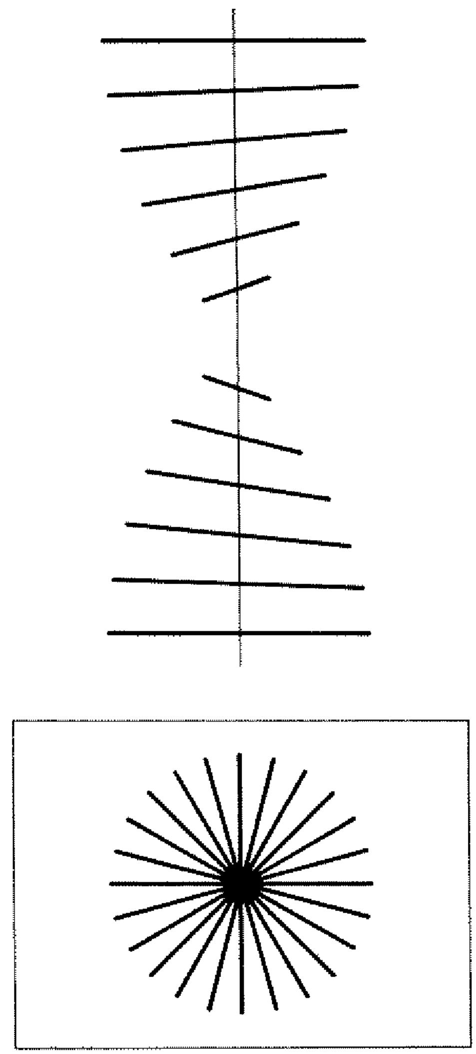

Figure 1. The blow-up of a point. [**11**]

In the case of a four-manifold M, or $n = 2$, the structure of the intersection form becomes immediately clear: a blow-up results in a connected sum with $\overline{\mathbb{CP}}^2$. This attached $\overline{\mathbb{CP}}^2$ contains a 2-cycle E, which we can choose to lie outside the glued neighborhoods, such that E does not intersect the 2-cycles of M and has self-intersection -1. The resulting intersection matrix is of the form

$$Q_{\tilde{M}} = \begin{pmatrix} Q_M & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.36}$$

3. Del Pezzo surfaces

Now we turn to the del Pezzo surfaces. The classical definition of a del Pezzo surface is the following [**8**].

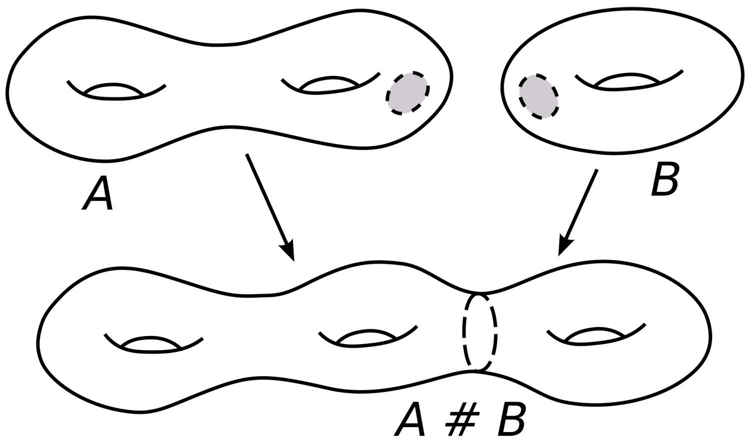

Figure 2. The connected sum of two tori. [**1**]

Definition 4.4. A *del Pezzo surface* is a nondegenerate irreducible surface of degree d in $\mathbb{CP}^d$ that is not a cone and not isomorphic to a surface of degree d in $\mathbb{CP}^{d+1}$.

This definition requires a lot of algebraic geometry and is not of much use for our purposes. However, the following theorem [**22**] provides us with a better understanding of the topology of del Pezzo surfaces.

Theorem 4.5. *Let X be a del Pezzo surface of degree d. Then either X is isomorphic to the blow-up of $\mathbb{CP}^2$ at $9-d$ points in general position in $\mathbb{CP}^2$, or $d=8$ and X is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1 \cong S^2 \times S^2$.*

We say a collection of points in $\mathbb{CP}^2$ is said to be in *general position* if no 3 points lie on a line, no 6 points lie on a a conic and no 8 points lie on a singular cubic, with one of the point at the singularity. We will not proof the theorem.

Let $\mathbb{B}_k$ denote the del Pezzo surface of degree $9-k$ obtained by blowing up k points in general position. We start with $\mathbb{B}_0 = \mathbb{CP}^2$. As we have seen above, has homology $H_2(\mathbb{CP}^2;\mathbb{Z}) = \mathbb{Z}$ with generator $C := \mathbb{CP}^1 \subset \mathbb{CP}^2$. This has self intersection $C \cdot C = 1$.

Now to obtain $\mathbb{B}_k$, we blow up $k \leq 8$ points in general position, which results in k exceptional divisors $E_1, ..., E_k$. As described above, the homology of the iterated blow-up is given by

$$H_2(\mathbb{B}_k;\mathbb{Z}) = \mathbb{Z}\ C \oplus \mathbb{Z}\ E_1 \oplus ... \oplus \mathbb{Z}\ E_k. \tag{4.37}$$

The intersection numbers are given by

$$C \cdot C = 1,\ C \cdot E_i = 0,\ E_i \cdot E_j = -\delta_{ij}, \quad 1 \leq i, j \leq k. \tag{4.38}$$

We conclude that the intersection matrix for the del Pezzo surface $\mathbb{B}_k$ is given by

$$Q^{-1} = \mathrm{diag}(1, -1, -1, ..., -1) = Q, \tag{4.39}$$

where the last equality is obivious since Q^{-1} squares to the identity. We conclude that $\mathbb{B}_k$ has $b^+ = 1$, $b^- = k$ and hence $\sigma = 1-k$. Finally, note that the intersection matrix of each $\mathbb{B}_k$ is odd, so that we have to consider fractionally quantized fluxes in the presence of fermionic field on these manifolds.

For the 'other' del Pezzo surface of degree 8, $\mathbb{CP}^1 \times \mathbb{CP}^1 \cong S^2 \times S^2$, we note that $H_2(S^2 \times S^2;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$, where the generators are the individual spheres $S^2 \times \{p\}$ and $\{p\} \times S^2$. These spheres have zero self intersection and intersect each other transversely in one point (p, p). Therefore, the intersection matrix is given by

$$Q^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = Q. \tag{4.40}$$

REMARK 4.6. As we have seen, each time we blow up a point in $\mathbb{CP}^2$, we add a generator to the second homology. Since $H_1(\mathbb{CP}^2; \mathbb{Z}) = 0$, we see that $H_1(\mathbb{B}_k; \mathbb{Z}) = 0$. Also, $H_1(S^2 \times S^2; \mathbb{Z}) = 0$. We conclude that $b_1 = 0$ for all del Pezzo surfaces. Therefore, the determinant factor $\Delta(\tau)$ is equal to $\tau_2^{-1/2} = C \frac{g}{\sqrt{4\pi}}$ for all these del Pezzo surfaces.

4. The partition function for del Pezzo surfaces

Now that the topology of the del Pezzo surfaces is known, we can return to the original question of calculating the partition function on these surfaces. As we have seen in the previous chapter, the partition function is given as a lattice sum of the exponentiated action. The action contains a term involving the intersection matrix Q, and term involving the symmetric and positive definite matrix G. For a choice of harmonic representative F^i for the cohomology, the matrix G was defined by $*F^i = G^{ij}(Q^{-1})_{jk}F^k$. We see that G depends on a choice of harmonic basis for $H^2(\mathbb{B}_k)$ and on the choice of a metric on $\mathbb{B}_k$, as the Hodge star operator depends on the metric. However, since $*^2 = 1$, we obtain the relation $(GQ^{-1})^2 = 1$. This relation allows us to express the matrix G in terms of some parameters (moduli), which result in parameters for the partition function $Z(\tau)$. We start by some simple cases.

4.1. The surface $\mathbb{B}_1$. The del Pezzo surface $\mathbb{B}_1$ has intersection matrix

$$Q = Q^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.41}$$

We write down the general expression for a symmetric 2×2 matrix G,

$$G = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \tag{4.42}$$

and solve

$$(GQ^{-1})^2 = \begin{pmatrix} a & b \\ -b & -c \end{pmatrix}^2 = \begin{pmatrix} a^2 - b^2 & b(c-a) \\ b(a-c) & -b^2 + c^2 \end{pmatrix} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.43}$$

From the off-diagonal terms we get $c = a$ or $b = 0$. Setting $c = a$, we obtain $a^2 - b^2 = 1$. This can be parametrized $a = \pm\cosh\alpha, b = \pm\sinh\alpha$. However, the plus-sign in front of the cosh must be chosen to ensure positive definiteness of G and we obtain

$$G = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}. \tag{4.44}$$

If we set $b = 0$, we solve $a^2 = c^2 = 1$ by $a^2 = \pm 1, c^2 = \pm 1$. Again, $a = c = 1$ must be chosen to make G positive definite and we obtain the solution above with $\alpha = 0$.

Remember that the action in the presence of scalar fields was given by $\frac{i}{\hbar}S(\tau, m) = i\pi m^T \Omega(\tau) m$, where $\Omega(\tau) = \tau_1 Q + i\tau G$. Plugging in our parametrization for G we obtain

$$\begin{aligned} iS_{sc}(\tau, m_1, m_2) &= \pi i \; (m_1, m_2) \begin{pmatrix} \tau_1 + i\tau_2\cosh\alpha & i\tau_2\sinh\alpha \\ i\tau_2\sinh\alpha & -\tau_1 + i\tau_2\cosh\alpha \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \\ &= \pi i \left[(m_1^2 - m_2^2)\tau_1 + i\tau_2\cosh\alpha(m_1^2 + m_2^2) + 2im_1m_2\tau_2\sinh\alpha \right]. \end{aligned} \tag{4.45}$$

Thus we obtain the partition function

$$Z_{\mathbb{B}_1,sc}(\tau,\alpha) = \Delta(\tau) \sum_{m_1,m_2\in\mathbb{Z}} e^{\pi i\left[(m_1^2-m_2^2)\tau_1+i\tau_2\cosh\alpha(m_1^2+m_2^2)+2im_1m_2\tau_2\sinh\alpha\right]}. \tag{4.46}$$

For the special case $\alpha = 0$, the partition function factorizes as

$$\begin{aligned} Z_{\mathbb{B}_1,sc}(\tau,0) &= C\Delta(\tau)\left(\sum_{m\in\mathbb{Z}} e^{\pi i m^2(\tau_1+i\tau_2)}\right)\left(\sum_{m\in Z} e^{\pi i m^2(-\tau_1+i\tau_2)}\right) \\ &= \Delta(\tau)\Theta_{\mathbb{CP}^2,sc}(\tau)\Theta_{\overline{\mathbb{CP}^2},sc}(\tau). \end{aligned} \tag{4.47}$$

We see that setting α to zero results in a partition function with factors into the different partition functions for $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$.

Now we turn to the presence of a spinor field on $\mathbb{B}_1$. The homology group $H_2(\mathbb{B}_1;\mathbb{Z})$ has 2 generators $\Sigma_1 = C = \mathbb{CP}^1 \subset \mathbb{CP}^2$ and $\Sigma_2 = E_1 \subset \overline{\mathbb{CP}}^2$, and from the intersection matrix we see that Σ_1 has self-intersection 1, Σ_2 has self-intersection -1. Therefore, the fluxes through these cycles will be quantized with a shift of $\pm 1/2$ respectively. The sign is irrelevant, since it can be changed by a simple basis transformation (a translation by 1) of the lattice. The action becomes:

$$\begin{aligned} iS_{sp}(\tau,\alpha,m_1,m_2) &= \pi i\,\left(m_1+\frac{1}{2}, m_2+\frac{1}{2}\right)\begin{pmatrix} \tau_1+i\tau_2\cosh\alpha & i\tau_2\sinh\alpha \\ i\tau_2\sinh\alpha & -\tau_1+i\tau_2\cosh\alpha \end{pmatrix}\begin{pmatrix} m_1+\frac{1}{2} \\ m_2+\frac{1}{2} \end{pmatrix} \\ &= \pi i\left[(m_1(m_1+1)-m_2(m_2+1))\tau_1+i\tau_2\cosh\alpha(m_1(m_1+1)+m_2(m_2+1)+1/2)\right. \\ &\quad \left.+2i(m_1+1/2)(m_2+1/2)\tau_2\sinh\alpha\right]. \end{aligned}$$

The partition function is then given by

$$Z_{\mathbb{B}_1,sp}(\tau,\alpha) = \Delta(\tau)\sum_{m_i\in Z} e^{\pi i S_{sp}(\tau,\alpha,m_1,m_2)}. \tag{4.48}$$

Again, if we set α equal to zero, the partition function factorizes:

$$Z_{\mathbb{B}_1,sp}(\tau,0) = \Delta(\tau)\Theta_{\mathbb{CP}^2,sp}(\tau)\Theta_{\overline{\mathbb{CP}}^2,sp}(\tau). \tag{4.49}$$

4.2. The surface $\mathbb{B}_2$. Now we turn to the next del Pezzo surface, $\mathbb{B}_2$. It has intersection matrix

$$Q = Q^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.50}$$

Let

$$G = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}, \tag{4.51}$$

and we solve

$$(GQ^{-1})^2 = \begin{pmatrix} a^2-b^2-c^2 & ab+bd+ce & -ac+be+cf \\ ab-bd-ce & -b^2+d^2+e^2 & -bc+de+ef \\ ac-be-cf & -bc+de+ef & -c^2+e^2+f^2 \end{pmatrix} = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.52}$$

The solutions to these equations can be parameterized by two parameters:

$$\begin{pmatrix} -1+\cosh\alpha+\cosh\beta & \pm\sqrt{(-1+\cosh\alpha)(\cosh\alpha+\cosh\beta)} & \pm\sqrt{(-1+\cosh\beta)(\cosh\alpha+\cosh\beta)} \\ \pm\sqrt{(-1+\cosh\alpha)(\cosh\alpha+\cosh\beta)} & \cosh\alpha & \pm\sqrt{(-1+\cosh\alpha)(-1+\cosh\beta)} \\ \pm\sqrt{(-1+\cosh\beta)(\cosh\alpha+\cosh\beta)} & \pm\sqrt{(-1+\cosh\alpha)(-1+\cosh\beta)} & \cosh\beta \end{pmatrix}. \tag{4.53}$$

We see that setting one of the parameters to zero makes the harmonic form dual to E_i orthogonal to the harmonic form ω, dual to C:

$$G_{\alpha=0} = \begin{pmatrix} \cosh\beta & 0 & \sinh\beta \\ 0 & 1 & 0 \\ \sinh\beta & 0 & \cosh\beta \end{pmatrix}, \tag{4.54}$$

$$G_{\beta=0} = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 \\ \sinh\alpha & \cosh\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.55}$$

$$G_{\alpha=\beta=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.56}$$

As before, in the presence of scalar fields on $\mathbb{B}_2$, the fluxes are quantized and the action is given by

$$S_{sc}(\tau,\alpha,\beta,m) = (m_1,m_2,m_3)\cdot(\tau_1 Q + i\tau_2 G(\alpha,\beta))\cdot(m_1,m_2,m_3). \tag{4.57}$$

In the presence of spinor fields, the fluxes will be fractionally quantized according to the self intersection of the generators of $H_2(\mathbb{B}_2;\mathbb{Z})$. We find that the action is given by

$$S_{sp}(\tau,\alpha,\beta,m) = (m_1+\frac{1}{2},m_2+\frac{1}{2},m_3+\frac{1}{2})\cdot(\tau_1 Q+i\tau_2 G(\alpha,\beta))\cdot(m_1+\frac{1}{2},m_2+\frac{1}{2},m_3+\frac{1}{2}). \tag{4.58}$$

In either case, we find that the partition function becomes

$$Z_{\mathbb{B}_2,sc/sp}(\tau,\alpha,\beta) = \Delta(\tau)\sum_{m\in\mathbb{Z}^3} e^{\pi i S_{sc/sp}(\tau,\alpha,\beta,m)}. \tag{4.59}$$

Now from the parameterization in (4.54), we see that setting one of the parameters to zero factorizes the partition function, with factors equal to the partition function of $\mathbb{B}_1$ and of $\mathbb{CP}^2$ with $\bar{\tau}$:

$$Z_{\mathbb{B}_2,sc/sp}(\tau,\alpha,0) = \Delta(\tau)\ \Theta_{\mathbb{B}_1,sc/sp}(\tau,\alpha)\ \Theta_{\overline{\mathbb{CP}}^2,sc/sp}(\tau). \tag{4.60}$$

4.3. The del Pezzo surface $\mathbb{B}_k$. This procedure is easily generalized to the other del Pezzo surfaces $\mathbb{B}_k$, for $1\le k\le 9$. $\mathbb{B}_k$ has intersection matrix $Q=Q^{-1}=\mathrm{diag}(1,-1,...,-1)$. Then G is a symmetric $(k+1)\times(k+1)$ matrix and we solve $(GQ^{-1})^2=1$. As noted in Remark 3.3, the metric G is an element of $SO(b^+,b^-)$, in this case precisely the group of Lorentz transformations $SO(1,k)$. As described in Remark 3.3, we quotient this group by the group $SO(k)$ to find the distinct partition functions. The Lorentz group is generated by two types of transformations, the boosts and the rotations. The group $SO(k)$ corresponds to these rotations, so we conclude that the possible matrices G are precisely the Lorentz boosts. Therefore, we can parameterize G as a general Lorentz boost in $1+k$ dimensions:

$$(G_{ij})(\alpha_1,...,\alpha_k) = \begin{pmatrix} \frac{1}{\sqrt{1-\alpha^2}} & \frac{-1}{\sqrt{1-\alpha^2}}\alpha_i \\ \frac{-1}{\sqrt{1-\alpha^2}}\alpha_j & \delta_{ij}+(\frac{1}{\sqrt{1-\alpha^2}}-1)\frac{\alpha_i\alpha_j}{\alpha^2} \end{pmatrix}, \tag{4.61}$$

where $\alpha^2 = \sum_i \alpha_i^2$ and $\alpha_i \in (-1,1)$ (in a Lorentz boost $\alpha_i = \frac{v_i}{c}$, where v is the velocity of the moving frame). This is consistent with the previous parameterizations of $\mathbb{B}_1$ and $\mathbb{B}_2$ by taking $\alpha_i = \tanh \beta_i$.

In the presence of scalar of spinor fields, the fluxes will be quantized or fractionally quantized. We have seen that by blowing-up, we obtain an extra generator for the homology, with self intersection -1. We conclude that the actions for the general $\mathbb{B}_k$ become

$$S_{sc}(\tau,\alpha_1,...,\alpha_k,m) = m^T(\tau_1 Q + i\tau_2 G(\alpha_1,...,\alpha_k))m, \tag{4.62}$$
$$S_{sp}(\tau,\alpha_1,...,\alpha_k,m) = (m+w/2)^T(\tau_1 Q + i\tau_2 G(\alpha_1,...\alpha_k)(m+w/2), \tag{4.63}$$

where in the spinor case, the vector w is given by $(1,1,1,...,1)$. The result is a partition function that depends on k moduli:

$$Z_{\mathbb{B}_k,sc/sp}(\tau,\alpha_1,...,\alpha_k) = \Delta(\tau)\sum_{m\in\mathbb{Z}^k} e^{\pi i S_{sc/sp}(\tau,\alpha_1,...,\alpha_k,m)}. \tag{4.64}$$

Also, by setting α_i to zero we set each $(i+1,i+1)$-th entry to 1 and the $(i+1,j)$-th and $(j,i+1)$-th entries to zero for all $j=1,,k+1$. The partition function then factorizes as

$$Z_{\mathbb{B}_k,sc/sp}(\tau,\alpha_{j\neq i},\alpha_i=0) = Z_{\mathbb{B}_{k-1},sc/sp}(\tau,\alpha_{j\neq i})\ \Theta_{\overline{\mathbb{CP}}^2,sc/sp}(\tau). \tag{4.65}$$

Setting all α_j to zero results in the factorization

$$Z_{\mathbb{B}_k,sc/sp}(\tau,\alpha_j=0) = \Delta(\tau)\ \Theta_{\mathbb{CP}^2,sc/sp}(\tau)\ \left(\Theta_{\overline{\mathbb{CP}}^2,sc/sp}(\tau)\right)^k. \tag{4.66}$$

4.4. The del Pezzo surface $S^2\times S^2$. The only del Pezzo surface we have not yet considered is $\mathbb{CP}^1\times\mathbb{CP}^1 \cong S^2\times S^2$. As mentioned above, $S^2\times S^2$ has intersection form

$$Q^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = Q. \tag{4.67}$$

Note that this intersection form is even, hence there will be no shift in the Dirac quantization condition and the partition function in the presence of a scalar field will be equal to the one in the presence of a spinor field. To find the metric G, we solve again:

$$(GQ^{-1})^2 = \begin{pmatrix} b^2+ac & 2ab \\ 2bc & b^2+ac \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.68}$$

This is solved by $b=0, a=\frac{1}{c}$ or $a=c=0, b=\pm 1$. Positive definiteness of G rules out the second solution, and forces $a>0$ in the first, so we obtain

$$G(\alpha) = \begin{pmatrix} r^2 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}. \tag{4.69}$$

We can interpret the parameter r as the ratio of the two spheres in $S^2\times S^2$. The result for the partition function is

(4.70)

$$Z_{S^2\times S^2,sc/sp}(\tau,r) = \Delta(\tau)\sum_{m\in Z^2} e^{\pi i[2\tau_1 m_1 m_2 + i\tau_2(r^2m_1^2+\frac{1}{r^2}m_2^2)} = \Delta(\tau)\sum_{m\in Z^2} e^{\pi i[\frac{\tau}{2}(rm_1+\frac{m_2}{r})^2 - \frac{\bar\tau}{2}(rm_1-\frac{m_2}{r})^2]}.$$

4.5. Blowing up $S^2 \times S^2$. When we blow up a point in $S^2 \times S^2$, we obtain the manifold $S^2 \times S^2 \# \overline{\mathbb{CP}}^2$, which is again a del Pezzo surface. In fact, there is a diffeomorphism $S^2 \times S^2 \# \overline{\mathbb{CP}}^2 \cong \mathbb{B}_2$. [**9**] Since these manifolds are diffeomorphic, they must have isomorphic intersection forms.

Definition 4.7. Given a lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z} \oplus ... \oplus \mathbb{Z}$, we say that two bilinear forms $\psi, \psi' : \Lambda \times \Lambda \to \mathbb{Z}$ are isomorphic if there exist a matrix $P \in GL(n, \mathbb{Z})$ such that

$$[\psi]' = P^T [\psi] P \tag{4.71}$$

where $[\cdot]$ denotes the matrix corresponding to the bilinear form.

If we blow up a point in $S^2 \times S^2$, we obtain the intersection form

$$Q_{S^2 \times S^2 \# \overline{\mathbb{CP}}^2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.72}$$

This must be isomorphic to $Q_{\mathbb{B}_2} = \mathrm{diag}(1, -1, -1)$, so we must find a matrix P such that $P^T Q_{\mathbb{B}_2} P = Q_{S^2 \times S^2 \# \overline{\mathbb{CP}}^2}$. After a calculation, we find that the matrix

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}, \tag{4.73}$$

does the job. We see that $\det P = 1$, hence $P \in GL(3, \mathbb{Z})$, as was required.

For brevity, let us denote $S = S^2 \times S^2 \# \overline{\mathbb{CP}}^2$ and Q_S its intersection matrix, such that $P^T Q_{\mathbb{B}_2} P = Q_S$. Also, let $G_S(\alpha, \beta)$ be the metric on the harmonic forms associated to S, that is, it is a solution to $(G_S Q_S^{-1})^2 = 1$. Now $Q_{\mathbb{B}_2}, Q_S$ are both symmetric and satisfy $Q_{\mathbb{B}_2}^2 = Q_S^2 = 1$. Therefore,

$$Q_{\mathbb{B}_2} = Q_{\mathbb{B}_2}^{-1} = ((P^{-1})^T Q_S P^{-1})^{-1} = P Q_S P^T, \tag{4.74}$$

and

$$1 = (G_{\mathbb{B}_2} Q_{\mathbb{B}_2}^{-1})^2 = G_{\mathbb{B}_2} P Q_S P^T G_{\mathbb{B}_2} P Q_S P^T \to (P^T G_{\mathbb{B}_2} P \, Q_S)^2 = (P^T G_{\mathbb{B}_2} P \, Q_S^{-1})^2. \tag{4.75}$$

We see that $P^T G_{\mathbb{B}_2} P$ solves $(G_S Q_S^{-1})^2 = 1$, so we have $P^T G_{\mathbb{B}_2}(\alpha, \beta) P = G_S(\alpha', \beta')$ where the parameters are different. We can solve the parameters α, β in terms of α', β', but this produces, except for some special cases, a complicated relation which does not have a direct geometrical interpretation (yet). We conclude that the actions on these manifolds can be related by

$$\begin{aligned} S_S(\tau, m, \alpha', \beta') &= \pi i m^T (\tau_1 Q_S + i\tau_2 G_S(\alpha', \beta') m \\ &= \pi i m^T P^T (\tau_1 Q_B + i\tau_2 G_B(\alpha, \beta) P m \\ &= S_B(\tau, m', \alpha, \beta), \end{aligned} \tag{4.76}$$

where $m' = Pm$.

In the presence of a scalar field on the manifold, we have that

$$\begin{aligned} Z_{S,sc}(\tau, \alpha', \beta') &= \Delta(\tau) \sum_{m \in \mathbb{Z}^3} e^{\pi i m^T (\tau_1 Q_S + i\tau_2 G_S(\alpha', \beta'))m} \\ &= \Delta(\tau) \sum_{m' \in P\mathbb{Z}^3} e^{\pi i m'^T (\tau_1 Q_{\mathbb{B}_2} + i G_{\mathbb{B}_2}(\alpha, \beta))m'} = Z_{\mathbb{B}_2, sc}(\tau, \alpha, \beta). \end{aligned} \tag{4.77}$$

Here we used that $P \in GL(3, \mathbb{Z})$, hence P is an isomorphism $P\mathbb{Z}^3 = \mathbb{Z}^3$. If there is a spinor field on the manifold, we have that the fluxes are fractionally quantized according to the intersection form, which leads to $F = (m_i + \frac{w_i}{2})F^i$. From the intersection form of both manifolds, we see that $w_S = (0,0,1)$ mod 2, while $w_{\mathbb{B}_2} = (1,1,1)$ mod 2. However, we have that

$$P(0,0,1)^T = (1,1,1)^T = (1,1,1)^T. \tag{4.78}$$

Therefore, the partition functions on S and $\mathbb{B}_2$ are related by

$$\begin{aligned} Z_{S,sp}(\tau,\alpha',\beta') &= \Delta(\tau) \sum_{m\in\mathbb{Z}^3} e^{\pi i (m+\frac{w_S}{2})^T(\tau_1 Q_S + i\tau_2 G_S(\alpha',\beta'))(m+\frac{w_S}{2})} \\ &= \Delta(\tau) \sum_{m'\in P\mathbb{Z}^3=\mathbb{Z}^3} e^{\pi i (m'+\frac{Pw_S}{2})^T(\tau_1 Q_{\mathbb{B}_2} + i\tau_2 G_{\mathbb{B}_2}(\alpha,\beta))(m'+\frac{Pw_S}{2})} \\ &= \Delta(\tau) \sum_{m'\in\mathbb{Z}^3} e^{\pi i (m'+\frac{w_{\mathbb{B}_2}}{2})^T(\tau_1 Q_{\mathbb{B}_2} + i\tau_2 G_{\mathbb{B}_2}(\alpha,\beta))(m'+\frac{w_{\mathbb{B}_2}}{2})} = Z_{\mathbb{B}_2,sp}(\tau,\alpha,\beta). \end{aligned} \tag{4.79}$$

We conclude that $S^2 \times S^2 \# \overline{\mathbb{CP}}^2$ and $\mathbb{B}_2$, have the same partition function, albeit for a different choice of parameters α, β.

CHAPTER 5

The moduli space of instantons on four-manifolds

The partition function is a central object in quantum field theory. In Yang-Mills theory, the partition function has the form

$$Z[A] = \int \mathcal{D}A e^{-S[A]}. \tag{5.1}$$

In the previous chapters, we have seen that in the abelian $U(1)$ gauge theory, the partition function could be expressed as a sum over the integral cohomology classes of solutions F to the classical equations of motion. Equivalently, the partition function could be expressed as a sum over a lattice of (anti)-self-dual harmonic forms F.

Under gauge transformations, the $U(1)$ connections transform as $A \to A - d\chi$, such that the curvature F remains unchanged. Therefore, the cohomology class of F can be identified with a gauge equivalence class of connections under the gauge group $U(1)$. We can then interpret the partition function as a sum over gauge-equivalence classes of (anti)-self-dual connections, that is, an integral over a zero-dimensional *moduli space of instantons.*

One can ask if this can be generalized to higher gauge groups. Certainly, we have seen in Chapter 1 that the instanton solutions minimize the Yang-Mills action, therefore one might expect that a saddle point expansion of the path integral will lead to a sum over the instanton number of an integral over the moduli space for fixed instanton number. However, since the $SU(N)$ gauge groups are non-abelian, the action contains terms of higher than quadratic order and the saddle point expansion will not be exact.

For some theories this can be done. For instance, in [**4**], there is described how a localization technique, the Mathai-Quillen formalism, is used to localize the path integral of a topological field theory to a finite dimensional integral: the example used is the instanton moduli space. Witten describes in [**24**] how for a supersymmetric version of Yang-Mills theory, the path integral *can* be expanded around instanton solutions. In [**20**], Witten and Vafa show that for $N = 4$ supersymmetric Yang-Mills theory, the partition function can be expressed as theta function of the form $\sum_k \chi(M_k) q^k$, where $\chi(M_k)$ is the Euler characteristic of the moduli space of instanton number k, and $q = \exp 2\pi i \tau$.

For these descriptions, a good understanding of the structure and topology of the instanton moduli space is essential.

Also from a mathematical viewpoint these moduli spaces are interesting. This moduli space can, under some assumptions, be described as a finite dimensional smooth manifold. The pairing of certain cohomology classes of the moduli produces diffeomorphism invariants of the underlying four-manifold, the *Donaldson invariants.*

In this chapter, we give an overview of results for the description moduli space of instantons, and the route to defining the Donaldson invariants. For the most part, we will consider the moduli space of instantons of a general vector bundle over a compact Riemannian four-manifold, but sometimes we restrict to the case $SU(2)$ for simplicity. The results below are mainly based on the book by Donaldson [**9**] and on the seminar organized by Gil Cavalcanti on this topic.

Let E be a vector bundle over a compact oriented Riemannian four-manifold X, with structure group G. We define the moduli space

$$M_E = \{\text{gauge equivalence classes of anti-self-dual connections on } E\}, \tag{5.2}$$

so we identify connections which are in the same orbit of the gauge group $\mathscr{G}$.

This space has a topology induced from the affine space $\mathscr{A}$, the space of all connections on E. The moduli space M_E will turn out to be a finite dimensional, real analytic space and that in most cases, M_E can be assumed to be a smooth manifold, except for some singular points. Also, this moduli space can be compactified in a natural way.

1. Fixing a gauge

We start by considering connection one-forms A on the trivial $U(1)$ bundle over X. Any gauge transformation can be written as $u = \exp(i\chi)$ for a real-valued function χ on X and the connections transform under these gauge transformations as $A \to A - id\chi$. Given a connection A, we can choose χ such that $\tilde{A} = A - id\chi$ satisfies

$$d^*\tilde{A} = 0. \tag{5.3}$$

In classical electromagnetism, the gauge freedom can be fixed by setting $d^*A = 0$, or $\nabla \cdot \vec{A} = 0$, which is called the *Coulomb gauge*. In a Lorentz covariant formulation of electromagnetism, we can choose a gauge by setting $d^*A = 0$ or $\partial_\mu A^\mu = 0$, which is called the *Lorentz gauge*.

Inspired by this, we can make the following definition. Let A_0 be a connection on E and

$$\mathscr{H} = \{u(A) : u \in \mathscr{G}\} \subset \mathscr{A}, \tag{5.4}$$

be the gauge equivalence class of another connection A on E.

Definition 5.1. We say that $B \in \mathscr{H}$ is in *Coulomb gauge* relative to A_0 if

$$d^*_{A_0}(B - A_0) = 0. \tag{5.5}$$

If we consider a family of gauge transformations $u(t) = \exp(t\chi), \chi \in \Omega^0_X(\mathfrak{g}_E)$, we have

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \|e^{t\chi}(B) - A_0\|^2_{L^2} &= \left.\frac{d}{dt}\right|_{t=0} \|B - e^{-t\chi}(A_0)\|^2_{L^2} = 2\left.\frac{d}{dt}\right|_{t=0} \langle B - e^{-t\chi}(A_0), B - A_0\rangle \\ &= 2\left.\frac{d}{dt}\right|_{t=0} \langle B - e^{-t\chi}A_0e^{t\chi} - (de^{-t\chi})e^{t\chi}, B - A_0\rangle \\ &= 2\langle \chi A_0 - A_0\chi - d\chi, B - A_0\rangle = 2\langle d_{A_0}\chi, B - A_0\rangle = \\ &= 2\langle \chi, d^*_{A_0}(B - A_0)\rangle. \end{aligned} \tag{5.6}$$

So the Coulomb gauge relation (5.5) is precisely the Euler-Lagrange equation for the functional

$$B \to \|B - A_0\|^2_{L^2}. \tag{5.7}$$

and we see that when a connection B is in Coulomb gauge relative to A_0, their L^2 distance is extremized. On the other hand, the following proposition tells us that for a connection B, which is close to a given connection A in the sense that the difference is small in some norm, we can find a gauge equivalent connection which is in Coulomb gauge relative to A.

PROPOSITION 5.2. *For any connection A on E, there is a constant $c(A)$ such that if B is another connection on E and $a = B - A$ satisfies*

$$\|\nabla_A \nabla_A a\|_{L^2}^2 + \|a\|_{L^2}^2 < c(a), \tag{5.8}$$

then there is a gauge transformation $u \in \mathscr{G}$ such that $u(B)$ is in Coulomb gauge relative to A

Here, $\nabla_A \nabla_A$ denotes the second covariant derivative $\nabla_A \nabla_A : \Omega^1_X(\mathfrak{g}_E) \to \Gamma(T^*X \otimes^2 T^*X \otimes \mathfrak{g}_E)$, which is different from the curvature operator $d_A d_A$.

As a special case, consider the product connection θ on the trivial bundle. Another connection A is in Coulomb gauge relative to θ if $d^*A = 0$. For a general bundle, a trivialization τ represents a connection in Coulomb gauge if $d^*A^\tau = 0$.

2. The moduli space

Now we turn to the moduli space of instantons. The strategy for constructing the moduli space will be to first construct the space $\mathscr{B}$ of gauge equivalence classes of connections on E, and then describe the moduli space as a subset of this quotient space.

2.1. The quotient space.

DEFINITION 5.3. We define $\mathscr{B}$ as the quotient space

$$\mathscr{B} = \mathscr{A}/\mathscr{G}, \tag{5.9}$$

equipped with the quotient topology, and we denote the gauge equivalence class of a connection A by $[A]$.

The gauge transformations preserve the L^2 metric on $\mathscr{A}$, so this norm descends to the quotient space by

$$d([A],[B]) = \inf_{u \in \mathscr{G}} \|A - u(B)\|, \tag{5.10}$$

which defines a metric on $\mathscr{B}$.

For $A \in \mathscr{A}$ and $\epsilon > 0$, let us define

$$T_{A,\epsilon} = \{a \in \Omega^1(\mathfrak{g}_E) : d^*a = 0, \|a\| < \epsilon\}. \tag{5.11}$$

Now if $[B]$ is close to $[A]$, we can represent this by connections B close to A and Proposition 5.2 tells us that there is a connection $\tilde{B}$ in coulomb gauge relative to A. Therefore, any orbit close to the orbit of A meets $A + T_{A,\epsilon}$ so we can describe a neighbourhood of $[A]$ in $\mathscr{B}$ by a quotient of $T_{A,\epsilon}$.

DEFINITION 5.4. A connection A on a G-bundle E is *reducible* if for all $x \in X$, the holonomy group at x, $hol_{x,A}$, is a proper subgroup of Aut $E_x \cong G$.

If X is connected, the holonomy groups at different points are related by conjugation with parallel transport, hence we can restrict to a single fiber to define a conjugacy class of subgroups $H_A \subset G$.

We can define isotropy group Γ_A of A by

$$\Gamma_A = \{u \in \mathscr{G} : u(A) = A\}. \tag{5.12}$$

The two groups are related by the following lemma:

LEMMA 5.5. *For any connection A over a connected base space X, Γ_A is isomorphic to the centralizer of H_A in G, $C_G(H_A)$.*

REMARK 5.6. Note that Γ_A always contains the center of G, $C(G)$.

Γ_A is a closed subgroup of G, hence itself a Lie group and its elements are precisely the covariant constant sections of Aut E. Its Lie algebra is then $\ker d_A \subset \Omega^0_X(\mathfrak{g}_E)$. Then, Γ_A acts on $\Omega^1_X(\mathfrak{g}_E)$ and in particular on $T_{A,\epsilon}$. With this definition, the following proposition gives a local description of the quotient space $\mathscr{B}$:

PROPOSITION 5.7. *For small enough $\epsilon > 0$, the projection map $\mathscr{A} \to \mathscr{B}$ induces a homeomorphism h for the quotient $T_{A,\epsilon}/\Gamma_A$ to a neighborhood $[A]$ in $\mathscr{B}$. For $a \in T_{A,\epsilon}$, the isotropy group of a in Γ_A is naturally isomorphic to that of $h(a)$ in $\mathscr{G}$*

We write $\mathscr{A}^* \subset \mathscr{A}$ for the open subset of connections that have minimal isotropy group:

$$\{A \in \mathscr{A} : \Gamma_A = C(G)\}, \tag{5.13}$$

and we define $\mathscr{B}^* \subset \mathscr{B}$ by the quotient $\mathscr{A}^*/\mathscr{G}$. The proposition tells us that $\mathscr{B}^*$ is a smooth Hilbert manifold: it is modeled locally on the balls $T_{A,\epsilon}$, which are open subsets of the Hilbert space $\ker d_A^* \subset \Omega^1_X(\mathfrak{g}_E)$. However, at a point in $\mathscr{B} \setminus \mathscr{B}^*$, we have to quotient by a bigger group, so we have a singular point. There is a way of describing these singular points as follows.

We partition $\mathscr{B}$ into disjoint subspaces $\mathscr{B}^\Gamma$, labelled by the conjugacy class Γ of the isotropy groups Γ_A in G, of the connections in $\mathscr{B}^\Gamma$. For each connection A, we can decompose

$$\mathfrak{g}_E = V \oplus V^\perp, \tag{5.14}$$

where V us the set of elements in $\mathfrak{g}_E$ which are fixed by Γ_A and $V^\perp$ is the orthogonal complement. Since $\Gamma_A \cong C_G(H_A)$, we have that V is actually the Lie algebra of H_A. Now the subset $\mathscr{B}^\Gamma$ is a Hilbert manifold, modeled locally on the space

$$\ker d_A^* \cap \Omega^1_X(V). \tag{5.15}$$

Also, the structure of $\mathscr{B}$ normal to $\mathscr{B}^\Gamma$ is modeled on

$$\left(\ker d_A^* \cap \Omega^1_X(V^\perp)\right) \Big/ \Gamma_A. \tag{5.16}$$

Also, we have the property that if a point $[A]$ lies in the closure of $\mathscr{B}^\Gamma$, then Γ_A contains (a representative of the conjugacy class) Γ. This structure makes the quotient space $\mathscr{B}$ into a *stratified space*.

We call the open subset $\mathscr{B}^*$ the manifold of *irreducible connections*. Strictly speaking, this is not completely correct in general, since the reducibility is determined by the centralizer of the holonomy group. For example, if a connection on a $SU(n)$ bundle reduces to the subgroup $SO(n)$ (its holonomy group H_A is the subgroup $SO(n) \subset SU(n)$), this connection still represents a point in $\mathscr{B}^*$ if $n > 2$, since $C_{SU(n)}(SO(n)) = \mathbb{Z}/n = C(SU(n))$ in this case. However, for $SU(2)$ or $SO(3)$ bundles over a simply connected manifold X, the two notions coincide.

2.2. The moduli space. Now that we have established the space of gauge equivalence classes $\mathscr{B}$, we can describe the moduli space M_E as a subset of $\mathscr{B}$, consisting of solutions to the equation $F(A)^+ = 0$.

REMARK 5.8. To construct the space of equivalence classes of connections, we should work in the framework of Sobolev spaces L^2_l. The Sobolev embedding theorem tells us that L^2_l consists of continuous functions if $l > 2$. We can then define locally an L^2_l map $X \supset U \to G$ and we define an L^2_l G-bundle as bundle defined by a family of L^2_l transition functions. Connections on this bundle are given, in local trivialization, by L^2_{l-1} matrix valued functions, which have curvature in L^2_{l-2} (this follows from the Sobolev multiplication $L^2_{l-1} \times L^2_{l-1} \to L^2_{l-2}$.

The moduli space then depends by construction on the choice of Sobolev space, i.e. on l, and we denote it $M_E(l)$. However, one can prove that the inclusion $M_E(l) \hookrightarrow M_E(l+1)$ is a homeomorphism. Therefore, we do not refer to the choice of Sobolev space, as the resulting moduli spaces are homeomorphic.

We can now find try to find local models for M_E within the local models for $\mathscr{B}$ as follows. Let A be an ASD connection and define the map

$$\psi : T_{A,\epsilon} \to \Omega^+(\mathfrak{g}_X), \quad \psi(a) = F^+(A+a) = d_A^+ a + (a \wedge a)^+, \tag{5.17}$$

where the superscript + denote the projection onto the self-dual two-forms. Let $Z(\psi) \subset T_{A,\epsilon}$ be the zero set of ψ. The map h of Proposition 5.7 induces a homeomorphism from the quotient $Z(\psi)/\Gamma_A$ to a neighborhood of $[A]$ in M_E.

A bounded linear map of Banach spaces,

$$L : U \to V, \tag{5.18}$$

is called *Fredholm* if $\dim \ker L < \infty, \dim \operatorname{coker} L < \infty$. We can then write

$$U = U_0 \oplus F, \quad V = V_0 \oplus G, \tag{5.19}$$

where F, G finite dimensional and L is an isomorphism between U_0 and V_0.

Definition 5.9. The *index* of L is defined as

$$\text{ind } (L) = \dim \ker L - \dim \text{coker } L = \dim F - \dim G. \tag{5.20}$$

Now for a connected open neighbourhood $N \subset U$, a smooth map $\phi : N \to V$ is called Fredholm if the derivative,

$$(D\phi)_x : U \to V, \tag{5.21}$$

is a Fredholm operator. The index of this operator is independent of x and is called the index of ϕ.

We say that ϕ is *right equivalent* to a map L is they agree under composition on the right with a local diffeomorphism. The implicit function theorem in Banach spaces implies the following proposition.

Proposition 5.10. *A Fredholm map ϕ form a neighborhood of 0 is locally right equivalent to a map of the form*

$$\tilde{\phi} : U_0 \times F \to V_0 \times G, \quad \tilde{\phi}(\xi, \eta) = (L(\xi), \alpha(\xi, \eta)), \tag{5.22}$$

where L is a linear isomorphism form U_0 to V_0, F and G are finite-dimensional, $\dim F - \dim G =$ *ind ϕ and the derivative of α vanishes at 0.*

From this proposition we obtain a finite dimensional model for a neighborhood of 0 in the zero set $Z(\phi)$ of ϕ. Under a diffeomorphism of U, this zero set can be identified with the zero set of the map ϕ' above and since L is an isomorphism, this zero set can is precisely the zero set of the map

$$f : F \to G, \quad f(y) = \alpha(0, y). \tag{5.23}$$

We apply the above to our map ψ, whose zero set describes the moduli space. We have

$$\frac{d}{dt}\bigg|_{t=0} \psi(ta) = \frac{d}{dt}\bigg|_{t=0} td_A^+ + t^2(a \wedge a)^+ = d_A^+ a, \tag{5.24}$$

so the derivative of ψ at 0 is given by

$$d_A^+ : \ker d_A^* \to \Omega^+(\mathfrak{g}_E). \tag{5.25}$$

Now the operator $\delta_A := d_A^* \oplus d_A^+ : \Omega^1(\mathfrak{g}_E) \to \Omega^0(\mathfrak{g}_E) \oplus \Omega^+(\mathfrak{g}_E)$ is an elliptic operator. In particular, an elliptic operator is Fredholm, so this implies that the restriction of d_A^+ to $\ker d_A^*$ is Fredholm. Therefore, ψ is a smooth Fredholm map with index

$$\begin{aligned} \text{ind } (\psi) &= \text{ind } d_A^+|_{\ker d_A^*} = \dim\ker d_A^+|_{\ker d_A^*} - \dim\text{coker } d_A^+|_{\ker d_A^*} \\ &= \dim(\ker d_A^+ \cap \ker d_A^*) - \dim \Omega^+/\text{im } d_A^+ \\ &= \dim\ker(\delta_A) - (\dim \Omega^+/\text{im } d_A^+ \dim \Omega^0/\text{im } d_A^*) + \dim \Omega^0/\text{im } d_A^* \\ &= \dim\ker \delta_A - \dim\text{coker } \delta_A + \dim\ker(d_A|_{\Omega^0}) \\ &= \text{ind } \delta_A + \dim \Gamma_A, \end{aligned} \tag{5.26}$$

where we use that $\ker d_A \oplus \text{coker } d_A^*$ and $\text{coker } \delta_A = \Omega^+/\text{im } d_A^+ \oplus \Omega^0/\text{im } d_A^*$.

The index of δ_A is given by the following formula, which follows from the Atiyah-Singer index theorem:

$$\text{ind } \delta_A = a(G)\kappa(G) - \dim G(1 - b_1(X) + b^+(X)), \tag{5.27}$$

where $a(G)$ is an integer depending on G and $\kappa = c_2$ for $SU(n)$ bundles and $-\frac{1}{4}p_1$ for $SO(n)$ bundles. For example, in the case of an $SU(2)$ bundles, we have

$$\text{ind } \delta_A = 8c_2(E) - 3(1 - b_1(X) + b^+(X)). \tag{5.28}$$

If we apply the Fredholm decomposition to the map d_A^+,

$$U = \ker d_A^* = U_0 \oplus (\ker d_A^+ \cap \ker d_A^*) = U_0 \oplus \ker \delta_A, \quad V = \Omega^+(\mathfrak{g}_E) = V_0 \oplus \text{coker } d_A^+, \tag{5.29}$$

we can linearize our map ψ, using Proposition 5.10, to get:

Proposition 5.11. *If A is an ASD connection over X, a neighborhood of $[A]$ in M is modeled on a quotient $f^{-1}(0)/\Gamma_A$, where*

$$f : \ker \delta_A \to \textit{coker } d_A^+ \tag{5.30}$$

is a Γ_A-equivariant map.

The index $s = \text{ind } \delta_A$ is sometimes referred to as the *virtual* dimension of the moduli space. This comes from the fact that points in the zero set of f which are both regular points of f and represent free Γ_A orbits form a smooth manifold of dimension

$$\dim\ker \delta_A - \dim\text{coker } d_A^+ - \dim \Gamma_A = \text{ind } \delta_A. \tag{5.31}$$

The construction of the map ψ and its linearization f, depended on our 'choice of gauge', namely, we considered a local slice $T_{A,\epsilon}$ in Coulomb gauge relative to A. We can take a more invariant viewpoint by considering the *deformation complex*:

$$\Omega^0_X(\mathfrak{g}_E) \xrightarrow{d_A} \Omega^1_X(\mathfrak{g}_E) \xrightarrow{d_A^+} \Omega^+_X(\mathfrak{g}_E). \tag{5.32}$$

If A is ASD, then we have $d_A^+ \circ d_A = F^+(A) = 0$, so this is indeed a complex and we obtain three cohomology groups H_A^0, H_A^1, H_A^2. From the Hodge theory for the elliptic operator δ_A, we have decompositions $\Omega_X^0(\mathfrak{g}_E) = \ker d_A \oplus \mathrm{im}\ d_A^*, \Omega_X^1 = \mathrm{im}\ d_A \oplus \ker d_A^*$, such that

$$H_A^1 = \frac{\ker d_A^+}{\mathrm{im}\ d_A} = \ker d_A^+ \cap \ker d_A^* = \ker \delta_A, \tag{5.33}$$

$$H_A^2 = \mathrm{coker}\ d_A^+ = \ker d_A^* \subset \Omega^+(\mathfrak{g}_E) \tag{5.34}$$

and H_A^0 is the Lie algebra of Γ_A. As above, the space H_A^1 represents the linearization of the ASD equations, modulo the gauge group $\mathscr{G}$. In this setting, we obtain a local model for the moduli space as a map

$$f : H_A^1 \to H_A^2. \tag{5.35}$$

The index s is then given by minus the Euler characteristic of the complex:

$$s = \dim H_A^1 - \dim H_A^0 - \dim H_A^2. \tag{5.36}$$

In most cases, H_A^2 will be zero, which can be proved using a Weitzenböck formula. Also, we have for structure groups $SU(2)$ or $SO(3)$ that $\dim \Gamma_A = 0$, so in this case the index s is indeed the dimension of the moduli space.

3. Smoothness of the moduli space

The ASD relation depends in principle on the metric, as the Hodge star does. To be precise, it depends on the conformal class $[g]$ of the metric. We write $M_E(g)$ to denote the moduli space of anti-self-dual connections defined by $[g]$. To make a more precise statement about the smoothness of the moduli space, we describe it as the zero set of a section and use transversality results to conclude smoothness.

The free $\mathscr{G}/C(G)$ action on $\mathscr{A}^*$ makes the quotient $\mathscr{A}^* \to \mathscr{B}^*$ into a principal $\mathscr{G}/C(G)$-bundle. Also, $\mathscr{G}/C(G)$ has an action on $\Omega_X^+(\mathfrak{g}_E)$, so we can form the associated bundle

$$\mathscr{E} = \mathscr{A}^* \times_{\mathscr{G}/C(G)} \Omega_X^+(\mathfrak{g}_E) \to \mathscr{B}^*. \tag{5.37}$$

The map $F^+ : \mathscr{A}^* \to \Omega_X^+(\mathfrak{g}_E)$ translates into a section Ψ_g of $\mathscr{E}$, by $\Psi_g([A]) = F^+(A)$. This is a Fredholm section of index s. The part of the moduli consisting of irreducible connections, denoted $M_E^*(g)$, is then the set of solutions to the family of equations

$$\Psi_g([A]) = 0. \tag{5.38}$$

If we have a smooth map $F : P \to Q$ between finite dimensional manifolds of dimensions p and q, we say that a point $x \in P$ is regular if DF_x is surjective and point $y \in Q$ is regular if DF_x surjective for all $x \in F^{-1}(y)$. If y is a regular value, then $F^{-1}(y)$ is a smooth manifold of dimension $p - q$. *Sard's theorem* states that there are many regular values: they form a dense set in Q.

There is an analog of this for Fredholm maps between Banach spaces. If $F : U \to V$ is a smooth Fredholm map between paracompact Banach spaces, and $y \in V$ is a regular value, then $F^{-1}(y)$ is a smooth manifold of dimension ind (F). The *Smale-Sard theorem* then tells us that these regular values form a dense set in U.

Recall that two maps $f : P \to Q, h : R \to Q$ are called transverse if $TQ_{fp} = \mathrm{span}(\mathrm{im}\ Df_p, \mathrm{im}\ Dh_r)$ for all points $f(p) = h(r)$. If we have a vector bundle $V \to P$, a section Ψ is transverse to the zero section if the map $D\Psi_x : TP_x \to V_x$ is surjective for all x such that $\Psi(x) = 0$.

The Smale Sard theorem implies the following transversality result:

Proposition 5.12. *Given a Fredholm map $F : U \to V$ between paracompact spaces and $h : R \to V$ a smooth map from a finite-dimensional manifold R, there is a map $h' : R \to V$ arbitrarily close to h and transverse to F.*

We apply this to sections of vector bundles. Let $V \to P$ be a bundle of Banach spaces over a Banachh manifold, and Φ a Fredholm section of V, that is, a section represented by Fredholm maps in local trivializations. We want to perturb Φ to obtain a section with a regular zero set. Consider the bundle $\bar{V} \to P \times S$, where S is an auxiliary Banach manifold. Let $\bar{\Phi}$ be a section of $\bar{V}$ which extends Φ and is Fredholm which is Fredholm as a function on P in local trivialization. Then we have a section Φ_s of V for each $s \in S$. Proposition (5.12) then implies:

Proposition 5.13. *If the zero set $Z \subset P \times S$ is regular then there is a dense set of $s \in S$ for which the zero sets of Φ_s are regular.*

We apply this to the section Ψ above. For each conformal class $[g]$ we have a space $\Omega^+_{X,g}(\mathfrak{g}_E)$ of self-dual forms defined by g. The gauge group $\mathscr{G}$ acts on this space, we obtain a quotient bundle $\underline{\mathscr{E}} \to \mathscr{B}^* \times \mathscr{C}$, where $\mathscr{C}$ denotes the space of conformal classes $[g]$.The self dual part of the curvature defines a section $\underline{\Psi}$ of $\underline{\mathscr{E}}$. The following result is due to Freed and Uhlenbeck:

Theorem 5.14. *For any $SU(2)$ or $SO(3)$ bundle E over X, the zero set of $\underline{\Psi}$ in $\mathscr{B}^* \times \mathscr{C}$ is regular.*

We write $\mathfrak{M}^*$ for the zero set of $\underline{\Psi}$. The theorem tells us that $\mathfrak{M}^*$ is a smooth Banach manifold. The individual moduli spaces $M^*(g)$ are the fibres of the projection $\mathfrak{M}^* \to \mathscr{C}$ and proposition (5.13) implies:

Corollary 5.15. *There is a dense subset $\mathscr{C}' \subset \mathscr{C}$ of conformal classes, such that for $[g] \in \mathscr{C}$ and for any $SU(2)$ of $SO(3)$ bundle E over X, the moduli space $M^*_E(g)$ is regular (as the zero set of the section determined by $[g]$).*

We conclude that for any given metric g on X, we can find a metric g' arbitrarily close to g such that the moduli space is, apart from some singular point formed by reducible connections, a smooth manifold.

4. Compactification of the moduli space

We have seen that the moduli space of anti-self-dual connections is locally a smooth manifold. This manifold need not be a compact manifold. However, the moduli space has a compactification in the following way.

As before, X is a compact oriented Riemannian four-manifold. Let us restrict to the case $G = SU(2)$ for simplicity and denote M_k for the moduli space of anti-self-dual connections with second Chern class

$$k = c_2(E) = \frac{1}{8\pi^2}\int_X |F(A)|^2 d\mu \geq 0. \tag{5.39}$$

Definition 5.16. An *ideal ASD connection* over X, of Chern class k , is a pair

$$([A], (x_1, ..., x_l)), \tag{5.40}$$

where $[A]$ is a point in M_{k-l} and $(x_1, ..., x_l)$ is a multiset of degree l of points of X. The curvature density of $([A], (x_1, ..., x_l)$ is the measure

$$|F(A)|^2 + 8\pi^2 \sum_{r=1}^{l} \delta_{x_r}. \tag{5.41}$$

Now let $A_\alpha, \alpha \in \mathbb{N}$ be a sequence of connections on an $SU(2)$ bundle P_k of Chern class k. We can define the following convergence:

Definition 5.17. We say that the gauge equivalence class $[A_\alpha]$ *converges weakly* to an ideal ASD connection $([A], (x_1, ..., x_l))$ if

(1) The action densities converge as measures, i.e. for any continuous function f on X,

$$\int_X f|F(A_\alpha)|^2 d\mu \longrightarrow \int_X f|F(A)|^2 + 8\pi^2 \sum_{r=1}^{l} f(x_r). \tag{5.42}$$

(2) There are bundle maps $\rho_\alpha : P_l|_{X\setminus\{x_1,...,x_l\}} \to P_k|_{X\setminus\{x_1,...,x_l\}}$ such that $\rho^*(A_\alpha)$ converges (in C^∞ on compact subsets of the punctured manifold) to A.

This notion of convergence can be extended to convergence of a sequence of ideal ASD connections by the same definition: $([B_\beta], (x_1, ..., x_l) \in M_{k-l} \times s^l(X)$ converges weakly to $([B], (y_1, ...y_m) \in M_{k-m}$ if

$$\int_X f|F(B_\beta)|^2 d\mu + \sum_{r=1}^{l} f(x_r) \longrightarrow \int_X f|F(B)|^2 d\mu + \sum_{r=1}^{m} f(y_r), \tag{5.43}$$

for any continuous function f on X.

We define the set of all ideal ASD connection with a fixed Chern class k:

$$IM_k = M_k \cup M_{k-1} \times X \cup M_{k-2} \times s^2(X) \cup ... \tag{5.44}$$

The notion of weak convergence of ideal connections equips IM_k with a topology. The moduli space M_k sits inside IM_k as an open subset.

Definition 5.18. $\bar{M}_k$ is the closure of M_k in the space of ideal connection IM_k.

With these definitions, we are able to state the main theorem:

Theorem 5.19. *Any infinite sequence in M_k has a weakly convergent sub-sequence, with a limit point in $\bar{M}_k$.*

The compactification of the moduli space follows from this theorem:

Corollary 5.20. *The space $\bar{M}_k$ is compact.*

5. Topology of the moduli space

Let $P \to X$ be a principal G-bundle over a compact, connected manifold X. We want to describe the cohomology of the moduli space, which sits inside the quotient space $\mathscr{B}$. In studying the topological properties of $\mathscr{B}$ and $\mathscr{B}^*$, some difficulties may arise due to the fact that the action of $\mathscr{G}$ on $\mathscr{A}$ is not free. It turns out to be convenient to work with framed connections.

Definition 5.21. Let (X, x_0) be a manifold with base point. A *framed connection* is a pair (A, ϕ), where A is a connection and ϕ an isomorphism $\phi : P_{x_0} \to G$.

The gauge group acts on the set of framed connections as

$$(A, \phi) \mapsto (u(A), \phi \circ u), \quad u \in \mathscr{G}, \tag{5.45}$$

and we define the space of equivalence classes

$$\tilde{\mathscr{B}} = (\mathscr{A} \times \mathrm{Hom}(P_{x_0}, G))/\mathscr{G}. \tag{5.46}$$

Alternatively, we can consider the framing ϕ fixed and define $\mathscr{G}_0$ its stabilizer:

$$\mathscr{G}_0 = \{u \in \mathscr{G} : u(x_0) = 1\}. \tag{5.47}$$

Then $\tilde{\mathscr{B}} = \mathscr{A}/\mathscr{G}_0$.

We have a map $\beta : \tilde{\mathscr{B}} \to \mathscr{B}$ by 'forgetting' the framing (or dividing out the remainder of the gauge group $\mathscr{G}/\mathscr{G}_0 \cong G$ in the second description). The fiber $\beta^{-1}([A])$ is isomorphic to G/Γ_A. Hence, $\beta : \tilde{\mathscr{B}}^* \to \mathscr{B}^*$ define a principal $G/C(G)$-bundle.

Next we define a family of framed connections:

Definition 5.22. A family of connections in P parametrized by a topological space T is a bundle $\underset{\sim}{P} \to T \times X$ such that $\underset{\sim}{P}|_{\{t\}\times X} = P_t \to \{t\} \times X$ is isomorphic to P for each $t \in T$ and on each P_t we have a connection A_t, forming a family $\underset{\sim}{A} = \{A_t\}$

A family of connections if *framed* is an isomorphism

$$\underline{\phi} : \underline{P}|_{T\times\{x_0\}} \to G \times T. \tag{5.48}$$

Then, for each $t \in T$, (A_t, ϕ_t) is a framed connection in P_t.

We obtain a an interesting family of connection by taking $T = \mathscr{A}$: the bundle $\underline{P} = \mathscr{A} \times P \to \mathscr{A} \times X$. This bundle carries a tautological family of connection: in the slice $\{A\} \times X$, we have a connection A. Then, given a framing $\phi : P_{x_0} \to G$, this extends to a family of framings for $\underline{P}$ by $\underline{\phi} : \underline{P}|_{\mathscr{A}\times x_0} \to \mathscr{A} \times G$.

$\mathscr{G}_0$ has a free action on $\mathscr{A} \times X$ and on $\underline{P}$, hence we can form the quotient bundles $\tilde{\mathbb{P}} = \underline{P}/G_0 \to \tilde{B} \times X$. Then $\tilde{\mathbb{P}}$ carries a family of connections $(\tilde{\mathbb{A}}, \underline{\phi})$, this is the *universal family of connections* in $P \to X$ parameterized by $\tilde{\mathscr{B}}$.

We can also construct a family of connections parameterized by $\mathscr{B}^*$. Let $\underline{P} \to \mathscr{A}^* \times X$ be the pullback bundle $\underline{P} = \mathscr{A}^* \times P$. This also carries a tautological family of connections. The action of $\mathscr{G}$ on the base $\mathscr{A}^* \times X$ is not free, unlesss $C(G)$ is trivial. $C(G)$ acts trivial on the base, but non-trivial on $\underline{P}$, there the quotient is a bundle with structure group $G^{ad} = G/C(G)$: we define

$$\mathbb{P}^{ad} \to \mathscr{B}^* \times X, \tag{5.49}$$

as the quotient $\underline{P}/\mathscr{G}$. For example, if $G = SU(2)$, $\mathbb{P}^{ad}$ is an $SO(3)$ bundle over $\mathscr{B}^* \times X$.

Now we can describe the cohomology of $\tilde{\mathscr{B}}$ and $\mathscr{B}^*$ in the case of an $SU(2)$ bundle over a simply connected manifold. For any G-bundle $P \to X$ we can construct cohomology classes in $\tilde{\mathscr{B}}_{X,P}$ using the slant-product pairing:

$$/ : H^d(\tilde{\mathscr{B}} \times X) \times H_i(X) \to H^{d-i}(\tilde{\mathscr{B}}). \tag{5.50}$$

If c is a characteristic class associated with G-bundles, there is a cohomology class $c(\tilde{\mathbb{P}}) \in H^d(\tilde{\mathscr{B}} \times X)$, $d = \deg(c)$, and we define the map

$$\tilde{\mu}_c : H_i(X) \to H^{d-i}(\tilde{\mathscr{B}}), \quad \tilde{\mu}_c(\alpha) = c(\tilde{\mathbb{P}})/\alpha. \tag{5.51}$$

Similarly, we can construct cohomology classes in $\mathscr{B}^*_{X,P}$: if c is a characteristic class for G^{ad}-bundles, we have

$$\mu_c : H_i(X) \to H^{d-i}(\mathscr{B}^*), \quad \mu_c(\alpha) = c(\mathbb{P}^{ad})/\alpha. \tag{5.52}$$

Definition 5.23. (1) For an $SU(2)$ bundle $P \to X$, the map $\tilde{\mu} : H_2(X;\mathbb{Z}) \to H^2(\tilde{\mathscr{B}}_{X,P};\mathbb{Z})$ is given by

$$\tilde{\mu}(\Sigma) = c_2(\tilde{\mathbb{P}})/[\Sigma]. \tag{5.53}$$

(2) The map $\mu : H_2(X;\mathbb{Q}) \to H^2(\mathscr{B}^*_{X,P};\mathbb{Q})$ is given by

$$\mu(\Sigma) = -\frac{1}{4}p_1(\mathbb{P}^{ad})/[\Sigma]. \tag{5.54}$$

With these definitions at hand, the following proposition describes the cohomology of $\tilde{\mathscr{B}}$.

Proposition 5.24. *Let P be an $SU(2)$-bundle over a simply-connected four-manifold X, and let $\Sigma_1, ..., \Sigma_b$ be a basis for $H_2(X;\mathbb{Z})$. Then the rational cohomology ring $H^*(\tilde{\mathscr{B}}_{X,P};\mathbb{Q})$ is polynomial algebra on the generator $\tilde{\mu}(\Sigma_1), ..., \tilde{\mu}(\Sigma_b)$:*

$$H^*(\tilde{\mathscr{B}};\mathbb{Q}) = \mathbb{Q}[\tilde{\mu}(\Sigma_1), ..., \tilde{\mu}(\Sigma_b)]. \tag{5.55}$$

A similar result can be given for the cohomology of $\mathscr{B}^*_{X,P}$, using the base-point fibration β. There is a four-dimensional class $\nu = -\frac{1}{4}p_1(\beta) \in H^4(\mathscr{B}^*;\mathbb{Q})$, where $p_1(\beta)$ is the Pontryagin class of the $SO(3)$ bundle $\beta : \tilde{\mathscr{B}}^* \to \mathscr{B}^*$. The cohomology is then given by:

Proposition 5.25. *The rational cohomology ring $H^*(\mathscr{B}^*_{X,P};\mathbb{Q})$ is a polynomial algebra on the four-dimensional generator ν and the two-dimensional generators $\mu(\Sigma_i)$:*

$$H^*(\mathscr{B}^*;\mathbb{Q}) = \mathbb{Q}[\nu, \mu(\Sigma_1), ..., \mu(\Sigma_b)]. \tag{5.56}$$

6. Donaldson invariants

We have seen that the moduli space M can be described as the zero-set of a section Ψ_g of a bundle $\mathscr{E}$ of Banach spaces over $\mathscr{B}^*$. As a finite dimensional analogy, consider a vector bundle V over a compact finite-dimensional manifold B and a section s of V. Of s vanishes transversely (the image $s(B)$ intersects the zero section transversely), the zero-set $Z(s)$ is a smooth submanifold of B whose fundamental class represents the Poicaré dual of the Euler class of V:

$$[Z(s)] \in H_d(B), \quad d = \dim B - \text{rank}V. \tag{5.57}$$

We can try to apply the same to the bundle $\mathscr{E}$ for the section Ψ_g and associate a fundamental homology class $[M]$ to the moduli space which sits inside $\mathscr{B}^*$. As we have seen, under some assumptions on the metric, the moduli space is a smooth sub manifold of $\mathscr{B}^*$.

Using this homology class, it is possible to define invariants of the manifold by pairing it with certain cohomology classes in $H^*(\mathscr{B}^*)$. A difficulty in this strategy is the fact that the moduli space may be non-compact, therefore, we can not define a fundamental class $[M]$ properly. However, one can still form a well-defined pairing $\langle \beta, [M] \rangle$ for certain cohomology classes $\beta \in H^*(\mathscr{B}^*)$ whose restriction to M are compactly supported.

This can be done for an $SU(2)$ bundle $E \to X$. Let $c_2(E) = k$ and M_k be the moduli space of instantons on E. Suppose that the moduli space has even dimension:

$$\dim M_k = 2d, \tag{5.58}$$

where $d = 4k - \frac{3}{2}(b^+(X) + 1)$, which follows from (5.28). This requires b^+ to be odd. Also, we require $b^+ > 1$ to avoid reducible connections.

Let $[\Sigma_1], ..., [\Sigma_d] \in H_2(X;\mathbb{Z})$ and let $\mu(\Sigma_i) \in H^2(\mathscr{B}^*)$ be there image under the map μ in the cohomology of $\mathscr{B}^*$. The cup product $\mu(\Sigma_1) \cup ... \cup \mu(\Sigma_d)$ had degree$2d$, so we can try to evaluate it on M_k:

$$q = \langle \mu(\Sigma_1) \cup ... \cup \mu(\Sigma_d), [M_k] \rangle \in \mathbb{Z}. \tag{5.59}$$

One can show that q depends only on the homology classes of the Σ_i and that it behaves natural under oriented diffeomorphisms $f : X \to Y$. Therefore, it is an invariant of the oriented diffeomorphism type of X. These invariants are called *Donaldson invariants*.

REMARK 5.26. In [**24**], Witten shows how the path integral for a supersymmetric Yang-Mills theory can be expanded around instanton solutions. The path integral then produces topological invariants which can be identified with Donaldson invariants.

7. Example

As an example, we give the result for the moduli space of $SU(2)$ instantons with Chern class 1 on $\mathbb{R}^4$. This result also applies to the four-sphere S^4, which is conformally flat and can be interpreted as the conformal compactification of $\mathbb{R}^4$.

EXAMPLE 5.27. Let E be an $SU(2)$ bundle over $X = \mathbb{R}^4$, with Chern class $c_2(E) = 1$. The instantons can be constructed using the ADHM construction [**9**], or making a suitable ansatz to solve the anti-self-dual equations [**21**]. The result is

$$A_{y,\lambda}(x) = \frac{1}{\lambda^2 + |x-y|^2}(\theta_1\tau_1 + \theta_2\tau_2 + \theta_3\tau_3), \tag{5.60}$$

where $\tau_i = \sigma_i/i$ (σ_i the Pauli matrices) which form a basis for $\mathfrak{su}(2)$ and

$$\begin{aligned}
\theta_1 &= (x_1 - y_1)dx_2 - (x_2 - y_2)dx_1 - (x_3 - y_3)dx_4 + (x_4 - y_4)dx_3 \\
\theta_2 &= (x_1 - y_1)dx_3 - (x_3 - y_3)dx_1 - (x_4 - y_4)dx_2 + (x_2 - y_2)dx_4 \\
\theta_3 &= (x_1 - y_1)dx_4 - (x_4 - y_4)dx_1 - (x_2 - y_2)dx_3 + (x_3 - y_3)dx_2.
\end{aligned}$$

The parameters y and λ denote the position and size of the instanton. We see that the moduli space is $M = \mathbb{R}^4 \times \mathbb{R}_+$. We can calculate the curvature, using

$$F = F_{ij}\, dx_i \wedge dx_j, \quad F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j], \tag{5.61}$$

and the result is

$$F(A_{y,\lambda}) = \frac{\lambda^2}{(\lambda^2 + |x-y|^2)^2}(d\theta_1\tau_1 + d\theta_2\tau_2 + d\theta_3\tau_3). \tag{5.62}$$

The curvature density is then

$$\mathrm{Tr}(F(A_{y,\lambda})^2) = 48\left(\frac{\lambda^2}{(\lambda^2 + |x-y|^2)^2}\right)^2 d\mu = |F(A_{y,\lambda})|^2 d\mu, \tag{5.63}$$

where we use that $\mathrm{Tr}(\tau_i\tau_j)d\theta_i \wedge d\theta_j = -2\delta_{ij}\, d\theta_i \wedge d\theta_j = 48d\mu$, since $d\theta_1^2 = d\theta_2^2 = d\theta_3^2 = -8d\mu$. We see that as $\lambda \to 0$, the density $|F(A_{y,\lambda})|^2$ converges as a distribution to a delta distribution at y, with mass

$$\lim_{\lambda\to 0} 48 \int_{\mathbb{R}^4} d^4x \left(\frac{\lambda^2}{(\lambda^2 + |x-y|^2)^2}\right)^2 = 8\pi^2. \tag{5.64}$$

Hence, the connection $A_{y,\lambda}$ converges weakly to the ideal connection $([\Theta], y)$, where Θ is the product connection.

Conclusion and Discussion

We formulated electromagnetism on a general compact Riemannian four-manifold M in the framework of Yang-Mills theory. By using this language of bundles and connections, the Dirac quantization condition was deduced. The presence of a charged boson on the manifold, described by a scalar field, was regarded as a section of a line bundle and the vector potential as a connection on this line bundle. The flux through a two-cyle, or the integral over the curvature over this two-cycle, takes only integer values, as it is the evaluation of the first Chern class on the two-cycle. The presence of a charged fermion, described by a spinor, imposes that the flux through some two-cycles is half-integer quantized, depending on the second Stiefel-Whitney class, or equivalently, on the parity of the intersection form.

Electromagnetic duality of the Maxwell theory on M was studied by investigating the effect of $SL(2,\mathbb{Z})$ transformations on the partition function. Using the Dirac quantization, the partition function of the electromagnetic theory on the manifold was expressed as a sum over classical solutions. The resulting partition function is a theta function. Using the properties of this theta function, we established the modular transformation properties of the partition function under $SL(2,\mathbb{Z})$ or a subgroup Γ_θ of order 3. In the presence of charged bosons, the partition function transforms as a modular form under $SL(2,\mathbb{Z})$ (or Γ, depending on the parity of the intersection), with weights given by topological invariants of M. In the presence of fermions, we can define three different theta functions which together transform as a vector modular form under $SL(2,\mathbb{Z})$, the partition function being one of them. The product of these 3 partition functions does transform as a modular form.

The Maxwell partition functions were explicitly calculated for del Pezzo surfaces. We constructed the metric on two-forms G in terms of a number of parameters. Via this construction we have explicitly expressed the partition function as a theta function depending on these parameters. By setting the parameters to zero, the partition function factorizes as a product of theta functions over $\mathbb{Z}$.

The moduli space of Yang-Mills instantons was described. We have seen that under some assumptions on the manifold, the moduli space is a finite dimensional smooth manifold. By pairing of certain cohomology classes of this moduli space, diffeomorphism invariants can be defined.

The Dirac quantization and electromagnetic duality were studied on compact and Riemannian four-manifolds. However, the space-time manifolds appearing in general relativity are Minkowskian and usually non-compact. We have encountered two examples of these in the end of Chapter 3. It is interesting to extend the treatment of Dirac quantization and electromagnetic duality to these types of manifolds. The non-compactness can be a difficulty in this program, as the Poincaré duality and Hodge theorem will not apply and one might need to use extensions of these. Also, calculation of the determinant Δ, which contains the formal determinant of the Laplacian, has some subtleties for a non-compact space-time.

It is interesting to investigate if the techniques used in the $U(1)$ setting will also apply for higher gauge groups. As mentioned in Chapter 5, due to the non-abelian nature of $SU(N)$ gauge groups, a saddle point expansion method will not be exact. Still, one might try to find similar $SL(2,\mathbb{Z})$ transformation properties for the resulting partition function.

For the $N = 4$ supersymmetric Yang-Mills theory, there is such a $SL(2,\mathbb{Z})$ duality present, as shown by Vafa and Witten in [**20**]. The partition function can be written as a theta function of the form $\sum_k \chi(M_k)q^k$, $q = \exp 2\pi i\tau$, which transforms under $SL(2,\mathbb{Z})$ transformations of τ, similar to the electromagnetic duality.

APPENDIX A

Principal G-bundles

In this appendix, we briefly discuss Principal G-bundles.

1. Principal G-Bundles

Let X be a paracompact Hausdorff space and G be a topological group.

Definition A.1. A *Principal G-bundle* is a fiber bundle $P \to X$ together with a continuous right action of G which preserves the fibers and acts freely and transitively on them. We say that P has structure group G.

So, the fibers are exactly the orbit of G. Also, each point in X has a neighbourhood U and a homeomorphism $h_U : \pi^{-1}(U) \xrightarrow{\cong} U \times G$ which is of the form $h_U(p) = (\pi(p), \gamma(p)$ (where $\gamma : \pi^{-1}(\{x\}) \to G$ is a homeomorphism between the fibers and G), and which satisfies $h_U(pg) = (\pi(p), \gamma(p)g))$. So locally, the bundle looks like:

$$\begin{tikzcd} U \times G \arrow[d] \\ U \end{tikzcd}$$

and G acts by multiplication on the right. Two principal G-bundles are *equivalent* if there is a homeomorphism $H : P \to P'$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "H"] \arrow[d, "\pi"'] & P' \arrow[dl, "\pi"] \\ X & \end{tikzcd}$$

commutes and is G-equivariant: $H(pg) = H(p)g$ for all $g \in G$. We denote the equivalence classes of principal G-bundles by $\mathrm{Prin}_G(X)$.

Example A.2. Let $\pi : P \to X$ be a 2-sheeted covering space of X and $G = \mathbb{Z}_2$. This is a principal $\mathbb{Z}_2$-bundle, where the group $\mathbb{Z}_2$ acts by interchanging the sheets. Conversely, any principal $\mathbb{Z}_2$-bundle is a 2-sheeted covering space and we have that $\mathrm{Prin}_{\mathbb{Z}_2}(X) \cong \mathrm{Cov}_2(X)$, where $\mathrm{Cov}_2(X)$ is the set of equivalence classes of 2-sheeted covering spaces.

Example A.3. Let E be a real n-dimensional vector bundle over X and let $P_{GL}(E)$ be the associated *frame bundle*: it is a bundle whose fiber over $x \in X$ is the set of all bases for the vector space E_x. This is a principal $GL(n)$-bundle, where $GL(n)$ is the group of invertible $n \times n$ real matrices. Let $g = ((a_{ij})) \in GL(n)$ and given a basis $p = (v_1, ..., v_n)$ of E_x, we define the action of g by

$$pg := (v'_1, ..., v'_n) \tag{A.1}$$
$$v'_j = \sum_k v_k a_{kj}.$$

This action is obviously free and transitive.

If the vector bundle E has additional structure, we obtain a principal bundle with smaller structure group. For instance, if E is equipped with a Riemannian metric, we can consider the bundle $P_O(E)$ of orthonormal frames. The construction above makes this into a principal $O(n)$-bundle. Also, if X is oriented we obtain the principal $SO(n)$-bundle $P_{SO}(E)$ of oriented orthonormal frames.

Conversely, given a linear representation of G on $\mathbb{C}^n$ or $\mathbb{R}^n$, we obtain an associated vector bundle E over X. For the classical groups (automorphisms of a vector space that preserve some linear algebraic structure), the concepts of principal and vector bundles are completely equivalent.

Recall that a general fiber bundle $B \xrightarrow{\pi} X$ with fiber F and structure group $G \subseteq \mathrm{Homeo}(F)$ is given by the following data.

- We have an open cover $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ of X and over each neighbourhood U_α there is a *local trivialization* $\pi^{-1}(U_\alpha) \xrightarrow{h_\alpha} U_\alpha \times F$ such that $\mathrm{pr} \circ h_\alpha = \pi$ (where pr is projection onto U_α).
- The change of trivialization over $U_\alpha \cap U_\beta =: U_{\alpha\beta}$ is of the form $U_{\alpha\beta} \times F \xrightarrow{h_\alpha \circ h_\beta^{-1}} U_{\alpha\beta} \times F$, where $h_\alpha \circ h_\beta^{-1}(x, f) = (x, g_{\alpha\beta}(x)(f))$ and $g_{\alpha\beta} : U_{\alpha\beta} \to G$ are continuous functions called the *transition functions*. They satisfy

$$\begin{cases} g_{\alpha\alpha} \equiv 1 \text{ on } U_\alpha \\ g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} \equiv 1 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma := U_{\alpha\beta\gamma}. \end{cases} \tag{A.2}$$

The latter condition is called *the cocycle condition*

The bundle can then be reconstructed from the above data by gluing the local products $U_\alpha \times F$ together with the transition maps.

We observe that any principal G-bundle can be represented by transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ multiplying G on the left. Since a principal G-bundle is a fiber bundle, it is given by an open cover $\{U_\alpha\}$ and a family of transition functions $G_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Homeo}(G)$. Also, since the bundle is principal, these functions must pointwise commute with right G-multiplication If then we define $g_{\alpha\beta}(x) = G_{\alpha\beta}(x)(1)$, we have $G_{\alpha\beta}(x)(g) = G_{\alpha\beta}(x)(1)g = g_{\alpha\beta}(x)g$.

Concluding, we see that every principal G-bundle on X is given by a pair $(\mathscr{U}, \{g_{\alpha\beta}\})$ where $\mathscr{U} = \{U_\alpha\}_{\alpha\in A}$ is an open cover of X and where $g_{\alpha\beta} : U_{\alpha\beta} \to G$ are continuous functions that satisfy the cocycle conditon (A.2). Such a pair can be interpreted as a Čech 1-cocycle with coefficients in G. Two such bundles, constructed from cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ on $\mathscr{U}$ are equivalent if and only if there exist continuous maps $g_\alpha : U_\alpha \to G$ such that

$$g'_{\alpha\beta} = g_\alpha^{-1} \cdot g_{\alpha\beta} \cdot g_\beta, \tag{A.3}$$

on $U_{\alpha\beta}$ for all α, β. Therefore, we define two 1-cocycle $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ on $\mathscr{U}$ to be equivalent if and only if there exists a Čech 0-cochain $\{g_\alpha\}_{\alpha\in A}$ such that (A.3) holds. The set of equivalence classes is denoted by $H^1(\mathscr{U}; G)$, which represents the equivalence classes of principal G-bundles on X which can be trivialized over the open sets of $\mathscr{U}$.

Remark A.4. Let $\{f_{\alpha_0\ldots\alpha_k}\}$ be a Čech k-chain with values in G on $\mathscr{U}$. Then the *Čech differential* of $\{f_{\alpha_0\ldots\alpha_k}\}$ is the $k+1$-chain

$$(\delta^k f)_{\alpha_0\ldots\alpha_{k+1}} = \prod_{i=0}^{k+1}(f_{\alpha_0\ldots\alpha_{i-1}\alpha_{i+1}\ldots\alpha_{k+1}})^{(-1)^i} \tag{A.4}$$

in multiplicative notation, or

$$(\delta^k f)_{\alpha_0\ldots\alpha_{k+1}} = \sum_{i=0}^{k+1}(-1)^i(f_{\alpha_0\ldots\alpha_{i-1}\alpha_{i+1}\ldots\alpha_{k+1}}) \tag{A.5}$$

in additive notation.

We call f a *Čech cocycle* if $\delta^k f = 1$, and a *Čech coboundary* if $f = \delta^{k-1}g$ for some $k-1$-chain g. This definition justifies the above terminology: a Čech 1-chain $\{g_{\alpha\beta}\}$ defines a principal G-bundle if and only if $\{g_{\alpha\beta}\}$ is a cocycle. Also, two cocycles define isomorphic bundles if they differ by a coboundary, which we see from (A.3).

Suppose that $(\mathscr{V}, j)$ is a refinement of $\mathscr{U}$: $\mathscr{V}$ is an open cover of X and $j : \mathscr{V} \to \mathscr{U}$ is a map such that $V \subseteq j(V)$ for all $V \in \mathscr{V}$. Then by restriction of the cocycles we get a map $r_{\mathscr{V}\mathscr{U}} : H^1(\mathscr{U}; G) \to H^1(\mathscr{V}; G)$ which can be shown to be independent of the refinement function j. These satisfy $r_{\mathscr{W}\mathscr{U}} = r_{\mathscr{W}\mathscr{V}} \circ r_{\mathscr{V}\mathscr{U}}$ for refinements $\mathscr{W} \to \mathscr{V} \to \mathscr{U}$ such that we can take the direct limit

$$H^1(X; G) := \varinjlim H^1(\mathscr{U}; G). \tag{A.6}$$

This limit represents the equivalence classes of principal G-bundles on X:

$$\mathrm{Prin}_G(X) \cong H^1(X; G). \tag{A.7}$$

If G is abelian, $H^1(X; G)$ is the first Čech cohomology group of X with coefficients in G.

Remark A.5. If X is a smooth manifold and G a Lie group, we can require the maps $g_{\alpha\beta}, g_\alpha$ above to be smooth. We then obtain the set of equivalence classes of smooth principal G-bundles over X, which is denoted $H^1(X; G)_\infty$. However, the map $H^1(X; G)_\infty \to H^1(X; G)$ can be shown to be a bijection.

If G is not abelian, $H^1(X; G)$ is not a group, it is only a pointed set with the special element given by the trivial G-bundle. Still, if

$$1 \to K \xrightarrow{i} G \xrightarrow{j} G' \to 1 \tag{A.8}$$

is an exact sequence of topological groups, there is an exact sequence of pointed sets [**16**]

$$\begin{aligned} \{*\} \quad &\to H^0(X; K) \xrightarrow{i_*} H^0(X; G) \xrightarrow{j_*} H^0(X; G') \\ \to \quad &H^1(X; K) \xrightarrow{i_*} H^1(X; G) \xrightarrow{j_*} H^1(X; G'). \end{aligned} \tag{A.9}$$

$H^0(X; G)$ is the set of 0-cocycles and is identified with the space of continuous maps $X \to G$. The maps i_* and j_* are coefficient homomorphisms induced by the maps i, j above.

Remark A.6. If the group K is abelian, then $H^2(X; K)$ is defined such that we can extend the exact sequence to

$$\ldots \to H^1(X; K) \xrightarrow{i_*} H^1(X; G) \xrightarrow{j_*} H^1(X; G') \xrightarrow{\delta} H^2(X; K). \tag{A.10}$$

EXAMPLE A.7. For the sequence

$$0 \to \mathbb{Z}_2 \to Spin(n) \xrightarrow{\xi_0} SO(n) \to 0, \tag{A.11}$$

the induced coboundary map

$$w_2 = \delta : H^1(X; SO(n)) \to H^2(X; \mathbb{Z}_2), \tag{A.12}$$

is the is the second Stiefel-Whitney class. By the exactness of the sequences above, we see that $w_2(P) = 0$ if and only if P is in the image of j_*, i.e. if and only if P carries a spin structure.

Let $(\mathscr{U}, g_{\alpha\beta})$ be a cocycle representing P, where each $U_{\alpha\beta}$ is contractible. We can lift each $g_{\alpha\beta}$ to a map $\bar{g}_{\alpha\beta} : U_{\alpha\beta} \to Spin(n)$ and define

$$w_{\alpha\beta\gamma} = \bar{g}_{\alpha\beta}\, \bar{g}_{\beta\gamma}\, \bar{g}_{\gamma\alpha}, \tag{A.13}$$

on $U_{\alpha\beta\gamma}$. Since $\xi_0(w_{\alpha\beta\gamma}) \equiv 1$, $w_{\alpha\beta\gamma}$ maps to $\mathbb{Z}_2$. This $\mathbb{Z}_2$-cocycle $w_{\alpha\beta\gamma}$ represents $w_2(P)$.

EXAMPLE A.8. Let us consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{\sigma} S^1 \to 1, \tag{A.14}$$

where σ is the map $x \mapsto \exp(2\pi i x)$. All the groups in this sequence are abelian, so we can extend the sequence (A.9). Now in Čech cohomology, $H^i(X; \mathbb{R}) = 0$ for all $i > 0$. Therefore, from (A.10) we get an isomorphism

$$c_1 = \delta : H^1(X; S^1) \xrightarrow{\cong} H^2(X; \mathbb{Z}), \tag{A.15}$$

which is called the first *Chern class*. We see that the equivalence classes of principal $U(1)$-bundles correspond 1-to-1 with elements of $H^2(X; \mathbb{Z})$.

APPENDIX B

Spin Structures

In this appendix we briefly discuss the notion of a spin structure.

1. Orientations on vector bundles.

Let $\pi : E \to X$ be a real vector bundle of rank n over a manifold X. We assume E to be *Riemannian*: it has a positive definite inner product on the fibers which depends continuously on the basepoint. We also assume that E is *oriented*: there is an orientation continuously defined on the fibers. This structure does not always exist.

Consider the bundle of orientations $Or(E) = P_O(E)/SO(n)$, where 2 two orthonormal bases are identified if they have the same orientation: they can be transformed into each other by an orthogonal matrix with determinant $+1$. Then $Or(E)$ is a 2-sheeted covering space of X and E is orientable if and only if this coveringspace is trivial.

LEMMA B.1. *There is a natural isomorphism*

$$Cov_2(X) \cong H^1(X;\mathbb{Z}_2). \tag{B.1}$$

PROOF. This is an immediate consequence of the isomorphism $H^1(X;G) \cong \mathrm{Prin}_G(X)$ and the fact that $\mathrm{Cov}_2(X) \cong \mathrm{Prin}_{\mathbb{Z}_2}(X)$, see the Appendix A. □

DEFINITION B.2. For each vector bundle E, we denote the image of the equivalence class of $Or(E)$ under this isomorphism by $w_1(E) \in H^1(X;\mathbb{Z}_2)$ called the *first Stiefel-Whitney class* of E.

We conclude:

THEOREM B.3. *A vector bundle $E \to X$ is orientable if and only if $w_1(E) = 0$. Furthermore if $w_1(E) = 0$, then the distinct orientations are in 1-to-1 correspondence with elements of $H^0(X;\mathbb{Z}_2)$.*

The second statement says that there are two possible orientations of E over each connected component of X.

There is an equivalent definition of $w_1(E)$. Suppose X is connected. Then from the fibration $O(n) \xrightarrow{i} P_O(E) \xrightarrow{\pi} X$ we get a long exact sequence in cohomology:

$$0 \to H^0(X;\mathbb{Z}_2) \to H^0(P_O(E);\mathbb{Z}_2) \to H^0(X;\mathbb{Z}_2) \xrightarrow{w_E} H^1(X;\mathbb{Z}_2). \tag{B.2}$$

We can then define $w_1(E) = w_E(g_1)$, where g_1 is the generator of $H^0(O(n);\mathbb{Z}_2)$. From the exactness of this sequence see that $w_1(E) = 0$ if and only if $P_O(E)$ has two connected components (as $O(n)$), that is if and only if E is orientable.

2. Spin structures on vector bundles

Now let E be an oriented n-dimensional vector bundle. We recall that, for $n \geq 3$, the group $Spin(n)$ can be defined by the following exact sequence:

$$0 \to \mathbb{Z}_2 \to Spin(n) \xrightarrow{\xi_0} SO(n) \to 1, \tag{B.3}$$

where ξ_0 is the universal covering homomorphism of $SO(n)$.

DEFINITION B.4. Suppose $n \geq 3$. A *spin structure* on E is a principal $Spin(n)$-bundle $P_{Spin}(E)$ together with a 2-sheeted covering

$$\xi : P_{Spin}(E) \to P_{SO}(E), \tag{B.4}$$

such that $\xi(pg) = \xi(p)\xi_0(g)$ for all $p \in P_{Spin}(E)$ and $g \in Spin(n)$.

Note that the diagram

$$\begin{tikzcd} P_{Spin}(E) \arrow[r, "\xi"] \arrow[d, "\pi'"'] & P_{SO}(E) \arrow[dl, "\pi"] \\ X & \end{tikzcd}$$

where π, π' are the bundle projections is commutative. If we restrict ξ to the fibers, we obtain the covering ξ_0 These fibrations fit together in the diagram

$$\begin{tikzcd} & Spin(n) \arrow[r, "\xi_0"] \arrow[d] & SO(n) \arrow[d] \\ \mathbb{Z}_2 \arrow[ur] \arrow[r] & P_{Spin}(E) \arrow[r, "\xi"] \arrow[d, "\pi'"'] & P_{SO}(E) \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

Conversely, suppose that $\xi : P_{Spin}(E) \to P_{SO}(E)$ is a 2-sheeted covering which is non-trivial on the fibers, that is such that the diagram

$$\begin{tikzcd} \mathbb{Z}_2 \arrow[r] \arrow[dr] & Spin(n) \arrow[r, "\xi_0"] \arrow[d] & SO(n) \arrow[d] \\ & P_{Spin}(E) \arrow[r, "\xi"] & P_{SO}(E) \end{tikzcd}$$

commutes. If we set $\pi' = \pi \circ \xi$ we make $P_{Spin}(E)$ into a fibre bundle over X. Lifting the action of $SO(n)$ on $P_{SO}(E)$ to a compatible action of $Spin(n)$ on $P_{Spin}(E)$ makes this into a principal $Spin(n)$-bundle. We conclude:

THEOREM B.5. *The spin structures on E are in 1-to-1 correspondence with 2-sheeted coverings of $P_{SO}(E)$ which are non-trivial on the fibers.*

Whit the help of lemma B.1 we can reformulate this as

COROLLARY B.6. *Suppose X is connected, then the spin structures on E are in 1-to-1 correspondence with elements of $H^1(X;\mathbb{Z}_2)$ whose restriction to the fiber of $P_{SO}(E)$ is non-zero.*

Now we turn to the question of the existence of such a spin structure. Associated to the fibration $SO(n) \xrightarrow{i} P_{SO}(E) \xrightarrow{\pi} X$ there is an exact sequence in cohomology:

$$0 \to H^1(X;\mathbb{Z}_2) \xrightarrow{\pi^*} H^1(P_{SO}(E);\mathbb{Z}_2) \xrightarrow{i^*} H^1(SO(n);\mathbb{Z}_2) \xrightarrow{w_E} H^2(X;\mathbb{Z}_2). \tag{B.5}$$

DEFINITION B.7. $w_2(E) = w_E(g_2) \in H^2(X;\mathbb{Z}_2)$, where g_2 is the generator of $H^1(SO(n);\mathbb{Z}_2)$ is the *second Stiefel-Whitney* class of E.

From the corollary and the exactness of the sequence (B.5) we conclude:

THEOREM B.8. *There exists a spin structure on E if and only if $w_2(E) = 0$. If $w_2(E)$, the distinct spin structures are in 1-to-1 correspondence with elements of $H^1(X;\mathbb{Z}_2)$.*

3. Spin Manifolds

DEFINITION B.9. A *spin manifold* is an oriented Riemannian manifold with a spin structure on its tangent bundle.

We define the Stiefel-Whitney classes $w_i(X)$ of a manifold X as the Stiefel-Whitney classes of the tangent bundle TX. Hence we have the following:

THEOREM B.10. *An oriented Riemannian manifold X admits a spin structure if and only if $w_2(X) = 0$. In this case, the spin structures on X are in 1-to-1 correspondence with elements of $H^1(X;\mathbb{Z}_2)$.*

APPENDIX C

Lattices

In this appendix we briefly recall the concept of integral lattices.

1. Integral lattices

DEFINITION C.1. A *lattice* in $\mathbb{R}^n$ is an additive subgroup $\Lambda \subset \mathbb{R}^n$ which is additively generated by some basis $e_1, ..., e_n$ for $\mathbb{R}^n$.

So elements of Λ are linear combinations $l = \sum_i n_i e_i$ of the basis vectors with coefficients $n_i \in \mathbb{Z}$. Choosing such a basis $e_1, ..., e_n$, we can form the *fundamental domain* P consisting of all $X = \sum_i x_i e_i$ with $0 \leq x_i < 1$. The volume of P is given by

$$\mathrm{vol}(P) = \int_P dx_1 ... dx_n = |\det(e_1, , e_n)|, \tag{C.1}$$

where $(e_1, ..., e_n$ is the matrix with rows e_i. This volume can be identified with the volume of the quotient torus $\mathbb{R}^n/\Lambda$, and denote it by $\mathrm{vol}(\Lambda)$.

DEFINITION C.2. We call a lattice $\Lambda \subset \mathbb{R}^n$ *unimodular* if $\mathrm{vol}(\Lambda) = 1$.

The vector space $\mathbb{R}^n$ can be endowed with a real, symmetric, nondegenerate scalar product, denoted $x \cdot y$. This scalar product need not be positive definite. Given such an (indefinite) inner product, we can define the signature:

DEFINITION C.3. Given a basis $e_1, ..., e_n$ for Λ and an inner product $\cdot$, we can define the *signature* σ of Λ as

$$\sigma(\Lambda) = b^+ - b^-, \tag{C.2}$$

where b^+ (b^-) is the number of basis elements e_i with $e_i \cdot b_i > 0$ (< 0).

The signature does not depend on the choice of basis, as it equals the difference in dimensions of the maximal positive and negative subspaces of $\mathbb{R}^n$ for $\cdot$.

DEFINITION C.4. We call an element $c \in \Lambda$ a *characteristic* element if $c \cdot x \equiv x \cdot x \mod 2$ for all $x \in \Lambda$.

It can easily be shown [**17**] that such a characteristic elements always exists. A characteristic element c satisfies [**17**]

$$c \cdot c = \sigma(\Lambda) \mod 8. \tag{C.3}$$

DEFINITION C.5. Given a lattice Λ and a scalar product $\cdot$, can define the *dual lattice* Λ^*:

$$\Lambda^* = \{x \in \mathbb{R}^n : y \cdot x \in \mathbb{Z}, \forall y \in \Lambda\}. \tag{C.4}$$

We have $\Lambda^{**} = \Lambda$. A standard basis for the dual lattice is the dual basis $f_1, ..., f_n$, which are defined by $e_i \cdot f_i = \delta_{ij}$.

We call Λ *integral*, if $\Lambda \subseteq \Lambda^*$ as a subgroup. If this is the case, the quotient

$$Z(\Lambda) = \Lambda^*/\Lambda, \tag{C.5}$$

is well-defined. By choosing representatives λ_α for each coset, we can write this quotient group as

$$Z(\Lambda) = \Lambda \cup (\lambda_1 + \Lambda) \cup ... \cup (\lambda_{|Z(\Lambda)|-1} + \Lambda), \tag{C.6}$$

where we set $\lambda_0 = 0$. Now the volumes of the lattices Λ and Λ^* are related by

$$\mathrm{vol}(\Lambda)\mathrm{vol}(\Lambda^*) = |\det(e_1, ..., e_n)|\,|\det(f_1, ..., f_n)| = |\det(e_i \cdot f_j)| = \det(\delta_{ij}) = 1, \tag{C.7}$$

hence $\mathrm{vol}(\Lambda) = \mathrm{vol}(\Lambda^*)^{-1}$. On the other hand, $|Z(\Lambda)|$ copies of the fundamental domain of Λ^* form a fundamental domain for Λ, hence

$$1 = \mathrm{vol}(\Lambda)\mathrm{vol}(\Lambda^*) = \mathrm{vol}(\Lambda)\mathrm{vol}(\Lambda)|Z(\Lambda)|^{-1}, \tag{C.8}$$

from which we conclude that $\mathrm{vol}(\Lambda) = \sqrt{|Z(\Lambda)|}$. Hence, Λ is unimodular if $|Z(\Lambda)| = 1$, or $\Lambda^* = \Lambda$.

An integral lattice Λ is said to be *even* if $x \cdot x \in 2\mathbb{Z}$ for all $x \in \Lambda$, otherwise Λ is *odd.* Note that

$$\Lambda_{total} = \Lambda \cup (\Lambda + \frac{c}{2}) \tag{C.9}$$

is a lattice. Also, we can splitt $\Lambda = \Lambda_{even} \cup \Lambda_{odd}$ where Λ_{even} (Λ_{odd}) consists of all $x \in \Lambda$ with $x \cdot x$ even (odd). So we can further split Λ_{total} as

$$\Lambda_{total} = \Lambda_{even} \cup \Lambda_{odd} \cup (\Lambda_{even} + \frac{c}{2}) \cup (\Lambda_{odd} + \frac{c}{2}). \tag{C.10}$$

This splitting will become relevant for assigning the theta functions to the unimodular cohomology lattice $F^2(M; \mathbb{Z})$ of our space-time manifold M.

APPENDIX D

Een samenvatting voor leken

Natuurkundige theorieën zijn soms invariant onder bepaalde symmetrieën: dit houdt in dat we de velden waar de theorie uit is opgebouwd een beetje kunnen veranderen, maar dat de fysische voorspellingen van de theorie niet veranderen. Zo'n symmetrie heet een ijk-symmetrie. We kunnen de ijk-symmetrie in de theorie beschrijven door het toevoegen van een ijk-veld, de theorie die we dan krijgen heet een *ijk-theorie*.

Yang-Mills is een voorbeeld van zo'n ijk-theorie. De symmetrieën van deze theorie zijn de transformaties van de symmetrie-groepen $U(N)$ of $SU(N)$. Deze theorie beschrijft verschillende interacties in het standaard model waarbij de ijk-velden de boodschapper-deeltjes voor de verschillende fundamentele krachten vormen. Voor $N = 1$ krijgen we de $U(1)$ ijk-theorie die elektromagnetisme beschrijft, het ijk-veld is in dit geval het foton, het boodschapper-deeltje van de elektromagnetische kracht. Voor $SU(2)$ krijgen we de zwakke kernkracht en voor $SU(3)$ de sterke kernkracht, met bijbehorende boodschapper-deeltjes respectievelijk de W en Z bosonen en de gluonen.

Wij mensen leven in een vier-dimensionale ruimtetijd, die wij als vlak ervaren. De algemene relativiteitstheorie van Einstein vertelt ons echter dat de ruimtetijd gekromd is, als gevolg van de aanwezigheid van materie en energie in deze ruimte. De kromming wordt pas duidelijk op kosmologische schaal, met als gevolg dat op menselijke schaal, deze kromming zo klein is dat wij er niets van merken. De wiskundige manier om zo'n ruimtetijd te beschrijven is in termen van een variëteit. Een variëteit is een object wat lokaal lijkt op de vlakke ruimte, maar dat globaal een ingewikkelder vorm heeft. Een standaard voorbeeld is het aardoppervlak: wanneer wij om ons heen kijken lijkt de aarde plat, maar vanuit de ruimte gezien is de aarde een bol.

Omdat Einstein voorspelt dat de kosmos een gekromde ruimte is, is het belangrijk dat de natuurkundige theorien beschreven kunnen worden op zo'n gekromde ruimte. Om dit te doen voor Yang-Mills, is het belangrijk de wiskundige structuur hieronder te analyseren. Het wiskundige begrip wat hierin centraal staat is dat van een vector-bundel. Een vector-bundel wijst op een consistente manier aan elk punt in een variëteit een vlakke ruimte (een vector-ruimte) toe. De verschillende manieren waarop dit gedaan kan worden zijn aan elkaar gerelateerd door een ijk-transformatie; de ijk-symmetrie is dan een symmetrie van de vector-ruimte. Een veld kan dan beschreven worden door een afbeelding die aan elk punt in de variëteit een punt in de vector ruimte toekent (een sectie). De ijk-velden worden dan beschreven door een connectie: een manier om de ruimtes boven verschillend punten te verbinden (parallel transport).

Wanneer we de Yang-Mills theorie gebruiken voor de symmetrie-groep $U(1)$ (rotaties in een 2-dimensionale ruimte), beschrijven we de theorie van elektromagnetisme. De klassieke variant van elektromagnetisme werd door James Clerk Maxwell geformuleerd in de Maxwell vergelijkingen, dit zijn de wetten waaraan elektrische en magnetische velden voldoen in de vlakke ruimte. Om de theorie op een gekromde ruimte te formuleren gebruiken we de formalismen van Yang-Mills theorie door het elektromagnetische ijkveld, de vector-potentiaal, als een connectie op te vatten. Op deze manier kunnen we bewijzen dat in de aanwezigheid van geladen deeltjes, de elektromagnetische velden alleen bepaalde waarden aan kunnen nemen. De flux, een maat voor hoeveelheid elektromagnetisch veld wat door een oppervlak gaat, door twee-dimensionale oppervlakken kan alleen een geheel getal zijn of juist alleen een geheel getal met een correctie van $1/2$, afhankelijk van het oppervlak en het type geladen deeltje. Dit staat bekend als de *Dirac quantizatie voorwaarde*.

De klassieke wetten van Maxwell in vacuüm, dat wil zeggen zonder de aanwezigheid van geladen deeltjes, vertonen een speciale symmetrie: wanneer we het elektrische en het magnetische veld omwisselen, krijgen we dezelfde vergelijkingen terug. Dit verschijnsel heet *elektromagnetische dualiteit*. Deze symmetrie kan zelfs vergroot worden: een willekeurige lineaire transformatie met *reële* coëfficiënten laat de vergelijkingen invariant. Het is belangrijk om hierbij op te merken dat deze dualiteit geen ijk-symmetrie is: alleen de bewegingsvergelijkingen zijn invariant. We kunnen ons afvragen hoeveel van deze symmetrie overblijft wanneer er wel geladen deeltjes aanwezig zijn. Om dit te beschrijven moeten we de overstap maken van klassieke natuurkunde naar de kwantum fysica. Een belangrijk object in de kwantum theorie is de zogenaamde partitiefunctie: een soort gewogen gemiddelde van alle mogelijke toestanden die de velden kunnen aannemen. De klassieke toestanden (die toestanden die voldoen aan Maxwells vergelijkingen) hebben in deze partitiesom het grootste gewicht. Met de hulp van de Dirac quantizatie voorwaarde kunnen we de partitiefunctie schrijven als een som over de klassiek toestanden. Vervolgens kunnen we onderzoeken hoeveel van de elektromagnetische dualiteit is overgebleven door te onderzoeken hoe de partitiefunctie verandert onder de uitwisseling van de elektrische en magnetische velden. Het blijkt dat de dualiteit verkleind is: alleen onder bepaalde transformatie met *gehele* coëfficiënten blijft de partitiefunctie (vrijwel) invariant: zij verandert volgens een bepaalde vaste regel (we noemen dit een modulaire vorm). De manier waarop de partitiefunctie verandert blijkt af te hangen van de vorm en structuur van de gekromde ruimtetijd.

Del Pezzo-oppervlakken zijn een familie van vier-dimensionale oppervlakken die interessant zijn vanuit snaartheorie. Volgens snaartheorie is de ruimtetijd tien-dimensionaal. De reden dat wij maar vier dimensies ervaren is omdat de overige 6 dimensies heel klein zijn: ze zijn opgerold in extreem kleine 6 dimensionale oppervlakken, Calabi-Yau variëteiten genaamd. Del Pezzo-oppervlakken kunnen beschreven worden als oppervlakken die zich in zo'n Calabi-Yau oppervlak bevinden. Daarom is het interessant om te weten hoe de elektromagnetische theorie op die oppervlakken eruit ziet. De del Pezzo-oppervlakken kunnen beschreven worden in termen van het opblazen van punten: we verwijderen een punt uit een del Pezzo-oppervlak en plakken daar een bol aan vast. Het resultaat: een nieuw del Pezzo-oppervlak. Met behulp van de Dirac quantizatie zijn we in staat om de elektromagnetische partitiefunctie uit te rekenen op de verschillende del Pezzo-oppervlakken. Hierin blijkt een bepaalde keuzevrijheid: de partitiefunctie hangt niet alleen af van de vorm (de topologie) van het oppervlak, maar ook van de keuze van een afstandsbegrip (een metriek). Deze keuze kunnen we uitdrukken door een aantal parameters, zo vinden we een partitiefunctie voor elke keuze van een metriek. Het blijkt dat we door een specifieke keuze van parameters de partitiefuncties van de verschillende del Pezzo-oppervlakken aan elkaar kunnen relateren.

Yang-Mills theorie is de ijk-theorie die hoort bij de symmetrie-groep $SU(N)$ (dit zijn een soort rotaties in een N-dimensional *complexe* vector ruimte). Deze groepen zijn niet *Abels*, dat betekent dat de volgorde van opvolgende transformaties van belang is voor de uitkomst, in tegenstelling tot de $U(1)$ transformaties; dit maakt de theorie een stuk complexer. De bewegingsvergelijkingen van de Yang-Mills theorieën hebben een speciaal soort oplossingen: instantonen. Deze oplossingen danken hun naam aan het feit dat ze gelokaliseerd zijn in ruimte en tijd.

Verschillende connecties (ijk-velden), kunnen aan elkaar gerelateerd worden door ijk - transformaties, we noemen dit ijk-equivalent. Wanneer we de ijk-equivalente connecties met elkaar identificeren, verkrijgen we een verzameling van ijk-equivalentieklassen van connecties, een oneindig dimensionaal object. Echter, de deelverzameling van ijk-equivalentieklassen van instantonen blijkt een eindig dimensionaal object te zijn, die onder bepaalde voorwaarden de structuur heeft van een glad oppervlak: een variëteit. Dit kan gebruikt worden om een bepaald getal (een invariant) toe te kennen aan de specifieke variëteit waarop we de Yang-Mills theorie beschrijven. Wiskundigen kunnen met deze invarianten vier-dimensionale oppervlakken classificeren.

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