Instantons and Electric-Magnetic Duality on Del Pezzo Surfaces

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Preface

This thesis is the product of two different directions of study this year. Under the supervision of Gil Cavalcanti and Stefan Vandoren a reading seminar was organized, in which Bram Bet, Reinier Storm, Ralph Klaasse and I presented and discussed the book “The Geometry of Four-Manifolds” written by Donaldson and Kronheimer [DK]. The idea of the seminar was to get a solid foundation in the modern study of the geometry of four-manifolds. After that we went onto projects of our own that were related to this field. The book is mostly concerned with the applications of Yang–Mills gauge theory to smooth four-manifolds, as developed by mathematicians such as Donaldson and Atiyah. A central theme that returns in our projects is the description of the moduli spaces of instantons that are associated to gauge theories. The moduli space of anti-self-dual $SU(2)$ connections on $S^2 \times S^2$, a class of instantons, is my project. Chapter 3 gives an exposition of the theory regarding this moduli space.

The thesis also contains a part in which the theory of four-manifolds is applied to Abelian gauge theories. Bram Bet and I got the assignment from Stefan Vandoren to study electric-magnetic duality in Abelian gauge theories on four-manifolds, with a focus on del Pezzo surfaces. The starting point for me was to study Abelian gauge theory on $S^2 \times S^2$, whereas Bram focused on $\mathbb{CP}^2$. However, the geometry of these two del Pezzo surfaces is highly intertwined and as a result our projects have become very much alike.

Although the part written on electric-magnetic duality in Chapter 4 is self-contained, it does have connections with other parts of physics and mathematics. Due to the scope and size of the thesis, these connections will only be dwelt upon shortly and much of these matters will leave room for possible further research.

Acknowledgements

First of all, I would like to thank my supervisors Stefan Vandoren and Gil Cavalcanti. Thanks to their combined wisdom I was able to navigate through the vast field of interplay between physics and mathematics. They shared a great deal of their knowledge about doing research and were very patient in redirecting my very ambitious goals into tangible results.

Second, I want to thank Bram Bet, Reinier Storm and Ralph Klaasse, with whom I managed to tackle the difficult theory from [DK] throughout the seminar presentations and numerous discussions. I would like to thank Bram in particular for the fruitful collaboration on some parts of the thesis and giving me useful feedback. We were ‘in the same boat’ as they would say in Dutch and we kept each other focused on our projects.

Lastly, I would like to thank friends, family and especially Charlotte for their support this year. Charlotte, you were there for me, when actually you needed me the most in this very tough and sad time. A part of the hard work that I have put in this thesis is therefore dedicated to you.

Contents

Chapter 1

Introduction

Gauge theory in four dimensions has many striking features that throughout the last decades have helped to make enormous progress in both mathematics and physics. This thesis is centered on two subjects in the interplay between gauge theory and four-dimensional geometry: instantons and electric-magnetic duality. The first subject concerns instantons in Yang–Mills theory and it has proved to be a powerful tool in the study of smooth four-manifolds. Electric-magnetic duality is present in Abelian gauge theories, where identities related to four-manifolds induce useful symmetries. Del Pezzo surfaces are a class of four-manifolds which will illustrate many of the interesting characteristics of these two subjects. In this introduction some more background information is given on instantons, electric-magnetic duality, four-manifolds and del Pezzo surfaces. This chapter is concluded with an outline of the thesis.

1.1 Gauge theory and instantons

Gauge theory describes a field theory for which the Lagrangian is invariant under so-called gauge transformations. This invariance corresponds to redundant degrees of freedom in the associated physical theory. To calculate physical quantities one can choose a preferred gauge of the fields. The gauge transformations act on the chosen gauge and the new field configuration will result in equivalent physical predictions. These local transformations take values in a specific representation of a Lie group, called the gauge group. Gauge fields are added to the theory to ensure gauge invariance. For this invariance the gauge fields need to satisfy certain transformation rules.

Gauge theory has been intensively studied by particle physicists during the second half of the twentieth century. This has resulted in the Standard Model which describes the quantum field theories of the strong force, the weak force and electromagnetism in a unified framework. This is a quantum field theory with gauge group $SU(3) \times SU(2) \times U(1)$.

Instantons were initially defined to be solutions to the equations of motions in Euclidean space with a finite non-zero Euclidean action [VN]. The Euclidean action is obtained from an action corresponding to a field theory on Minkowksi space, when one performs a Wick rotation. The solutions to the equations of motion form saddle points of the Euclidean action. In non-Abelian gauge theories on Minkowski space these instantons were encountered when the vacuum structure of the theory was studied. The most notable is Yang–Mills theory, for which the (non-Abelian) gauge group is $SU(N)$.

To quantize the associated quantum field theories, one uses perturbations of the fields around one of the vacua. These vacua correspond to minima of the potential. They can define topologically different backgrounds for the fields, in the sense that under continuous field transformations one is not able to go from one vacuum into the other. This obstruction is usually formed by a potential barrier. Between the vacua there still exist tunnelling effects, and the fields that mediate the tunnelling are the instantons. In order to describe the full quantum theory on Minkowski space, one needs to take into account these effects.

The background for gauge theories is usually formed by flat space-time, i.e. Minkowski space. According to Einstein's theory of general relativity, this four-dimensional background can become curved in the presence of matter. Due to the curvature the space can globally look very different than flat space-time. The properties of such spaces are captured in the framework of manifolds and depending on the curvature one can find obstructions for their topology. The manifolds that serve as models for curved space-time are called Lorentzian four-manifolds. The prefix Lorentzian indicates that we put a Lorentzian metric on it, a metric of signature $(1,3)$ or equivalently $(3,1)$. In this way the manifolds have the same causal structure as Minkowski space, which is needed to extend the relation between 'time' and 'space' as in special relativity to curved space-times.

Background spaces for the Euclidean action and the associated instantons are obtained by performing a Wick rotation on the time coordinate of a Lorentzian manifold. Such a transformation turns the metric into a Riemannian metric, which is positive-definite. These manifolds are known as four-dimensional Riemannian manifolds.

The saddle points of the Euclidean action in Yang–Mills theory satisfy the Yang–Mills equations, which are non-linear partial differential equations. The solutions for which the Euclidean action, also called the Yang–Mills functional, is non-zero and finite are known as *Yang–Mills instantons*. The observed correspondence between the Yang–Mills instantons and the structure of the underlying Riemannian four-manifolds has led to a revolution in mathematics. A special class of these manifolds are closed, meaning they are compact and have no boundary. The four-sphere is an example of a closed manifold that can be obtained as the Wick rotation of a Lorentzian manifold, namely four-dimensional de Sitter Space (Example 3.6, [Bet]). This study of Yang–Mills theory has been extended to closed manifolds that are not necessarily a Wick rotated Lorentzian four-manifold.

The instantons in non-Abelian gauge theories on $\mathbb{R}^4$ were the first examples that were studied, since they were of relevance for gauge theory on Minkowski space. A property of these instantons is that they form fields localized in space and time and this property motivated their name. The condition that the Euclidean action should be finite implies that the solutions should approach pure gauge at spatial infinity [VN]. Gauge fields satisfying this condition are topologically characterized by a quantity known as the instanton number. When we consider closed backgrounds, the Euclidean action is always finite, but the gauge fields can still be characterized by the instanton number.

For Abelian gauge theories on $\mathbb{R}^4$ such interesting structure for the saddle points is absent. In order to see interesting effects we have to extend the topology of the manifold, which is done when studying electric-magnetic duality.

1.2 Gauge theory on smooth four-manifolds

Several groundbreaking results in four-manifold theory came about in the late 70s and the 80s. The intersection form of four-manifolds played a central role in those breakthroughs. An early result by Milnor [Mil] in 1958 already stated that the oriented homotopy type of a simply-connected, closed, oriented four-manifold is determined by its intersection form. Building on Milnor's work Freedman deduced [Fre] that the homeomorphism type of such a manifold is determined by the intersection form and a $\mathbb{Z}_2$ invariant, called the Kirby-Siebenmann invariant.

Parallel to the development of gauge theory by physicists, gauge theory was also picked up by mathematicians to analyse smooth four-manifolds. The usual techniques to study the homeomorphism type of four-manifolds fell too short to study their diffeomorphism type and gauge theory was able to provide the tools necessary. From a mathematical point of view, the fields in a gauge theory are sections of a vector bundle over the background manifold. Mathematicians recognized the choice of gauge as a choice of bundle trivialization. Associated to this trivialization is a so-called structure group G which coincides with the gauge group. The gauge transformations are then formed by a class of automorphisms of the vector bundle and the gauge fields are naturally identified with connections on the bundle.

Atiyah was one of the first to study instantons in Yang–Mills theory on smooth four-manifolds. The class of Yang–Mills instantons he focused on were the self-dual (SD) and anti-self-dual (ASD) connections. In collaboration with others he classified all these instantons modulo gauge transformations on $\mathbb{R}^4$ or equivalently on the four-sphere S^4, culminating in the ADHM construction [ADHM]. By looking at the moduli space of ASD connections on $SU(2)$ bundles, Donaldson was able to tell apart different smooth structures on four-manifolds. Using this space he also constructed new polynomial invariants associated to smooth four-manifolds, nowadays called the Donaldson invariants [Don1, DK]. For his seminal work in this area Donaldson has been awarded the Fields Medal.

For Kähler manifolds, mathematicians later found a correspondence between the moduli spaces of ASD connections and stable holomorphic bundles. Through this correspondence one can use algebro-geometric methods to this describe this class of Yang–Mills instantons. See the notes of Chapter 6 in [DK] for a short history regarding this correspondence.

Supersymmetric $N = 2$ $SU(2)$ gauge theory is a special type of four-dimensional gauge theory that has lately seen a rise in interest in both the mathematics and physics community. Witten showed [Wit1] (with a technical addition by Baulieu and Singer [BS]) that certain type of correlation functions in a twisted version of this theory were equivalent to the Donaldson invariants. The low-energy physics of $N = 2$ supersymmetric $SU(2)$ gauge is calculated through Seiberg–Witten gauge theory [SW1, SW2]. A set of field equations that show up in this theory can be used to determine new invariants associated to four-manifolds. In analogy with Donaldson theory one uses the moduli space of the solutions to these equations. Witten conjectured that these invariants are actually related to the Donaldson invariants [Wit2]. The introduction of the Seiberg–Witten invariants, as they are called these days, caused a (second) revolution in mathematics, since they were technically easier to work with. A mathematically rigourous discussion of Seiberg–Witten theory can be found in [Mor, Don2].

1.3 Electric-magnetic duality

Our starting point for *electric-magnetic duality*, also known as *electromagnetic duality*, is Maxwell theory in vacuum and on a flat space-time, which is an Abelian gauge theory with gauge group $U(1)$. Soon after the discovery of Maxwell's equations it was noticed that the Minkowski energy and momentum densities associated to electromagnetism are invariant under 'duality rotations' of the electric and magnetic fields. In the formalism of tensor calculus one can write the electromagnetic four-potential as A_μ. In terms of the electromagnetic tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ Maxwell's equations in vacuum become

$$\partial^\mu F_{\mu\nu} = 0, \quad \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} = 0,$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally anti-symmetric tensor. The second identity is implied by the way $F^{\mu\nu}$ is defined.

If we want to introduce curved space-time backgrounds, the more general setup is to write the vector-potential A^μ as a 1-form

$$A = A_\mu dx^\mu.$$

The electromagnetic tensor $F_{\mu\nu}$ is then seen to be part of the exterior derivative of A given by

$$F = dA = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu,$$

also called the *two-form field strength*. The curved space-time background is given by a Lorentzian four-manifold with metric tensor $g_{\mu\nu}$. An important object which incorporates the non-trival curved background is the Hodge dual of F:

$$\star F = \frac{1}{4} \sqrt{-\tilde{g}} g^{\mu\rho} g^{\nu\sigma} \epsilon_{\alpha\beta\rho\sigma} F_{\mu\nu} dx^\alpha \wedge dx^\beta, \tag{1.1}$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ and $\tilde{g} = \det g_{\mu\nu}$. Maxwell's equations on a curved background can now be elegantly written as

$$d \star F = 0, \quad dF = 0.$$

The identity $dF = 0$ is always true, because of the way F is defined. Mathematically speaking F is the curvature of a connection A and $dF = 0$ is then known as the *Bianchi identity*.

The action we will consider is given by

$$S[A] = \frac{1}{g^2} \int_M F \wedge \star F + \frac{\theta}{8\pi^2} \int_M F \wedge F, \tag{1.2}$$

where θ is called the theta parameter and g the gauge coupling constant. The Euler-Lagrange equations for this action result in Maxwell's equations. In terms of the fields $F_{\mu\nu}$ (1.2) can be written as

$$S[A] = \int_M d^4x \sqrt{-\tilde{g}} \left(\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \right).$$

We prefer to write the action in terms of differential forms, since the geometric structure behind this theory is then easier to study.

Define the 'electric' field to be $F_D = \frac{4\pi}{g^2} F + \frac{\theta}{2\pi} \star F$. The 'duality rotations' which leave invariant the energy-momentum tensor (and the equations of motion) associated to this action are given by an action of $SL(2, \mathbb{R})$ on the 'electric' and magnetic field [OA]:

$$F_D \mapsto aF_D + bF, \quad F \mapsto cF_D + dF, \quad ad - bc = 1. \tag{1.3}$$

If one combines the constants θ and g in the 'complex' coupling constant $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$, the effect of the transformations in (1.3) is a new theory with complex coupling constant

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

The action of the group $SL(2, \mathbb{R})$ on τ is generated by $\tau \mapsto -1/\tau$ and $\tau \mapsto \tau + b$ with $b \in \mathbb{R}$.

In order to study the quantum theory associated to Maxwell theory in vacuum, one usually calculates path integrals over all possible gauge fields A on the curved background. A path integral that often is calculated (after setting $\hbar = 1$) is given by

$$\int [dA] \exp(iS[A]) = \int [dA] \exp \left(i \frac{1}{g^2} \int_M F \wedge \star F + i \frac{\theta}{8\pi^2} \int_M F \wedge F \right). \tag{1.4}$$

The term $\frac{\theta}{8\pi^2} \int_M F \wedge F$ does not affect the physics, since it does not change the Euler-Lagrange equations nor the value of the energy-momentum tensor. However, if the background space-time M has sufficiently non-trivial topology the term $\frac{1}{4\pi^2} \int_M F \wedge F$ is quantized for a general gauge field. If this is the case the action gets a non-trivial quantum phase and covariance under $SL(2,\mathbb{R})$ is not directly clear.

The path integrals for Maxwell theory are usually calculated through the Euclidean path integral which has input the Euclidean Maxwell action $S_E[A]$. These path integrals are mathematically better understood and the results obtained from this path integral may be Wick rotated back. Maxwell theory in this framework is often referred to as *Euclidean Maxwell theory*. The outcome of this theory would give the same physical results as one would get by appropriate treatment of the (potentially divergent) Minkowskian path integral.

One can obtain the Euclidean Maxwell action from the Maxwell action (1.2) by performing a Wick rotation on the Lorentzian four-manifold. Under this procedure one of the forms dx^μ obtains a factor i and the operator $\star$ as in (1.1) turns into $i\star$, where $\star$ now is the Hodge star operator acting on 2-forms on a Riemannian manifold. What we get is that the integrand in the path integral (1.4) becomes

$$\exp(iS[A]) \mapsto \exp(-S_E[A]) = \exp\left(-\frac{1}{g^2}\int_M F\wedge\star F + i\frac{\theta}{8\pi^2}\int_M F\wedge F\right), \tag{1.5}$$

and the gauge fields are now defined on a four-dimensional Riemannian manifold. Notice that the second term is unaffected in its form by the Wick rotation.

The path integral we get is highly convergent and for Riemannian manifolds of the form $M = S^1\times N$ it corresponds to a quantity known as the partition function

$$\tilde{Z}(\tau) = \mathrm{Tr}\left(e^{-E(\tau)}\right).$$

Since the energy functional $E(\tau)$ is left invariant under the transformations in (1.3), this partition function is therefore expected to be $SL(2,\mathbb{R})$ covariant, meaning that the expression is invariant up to some factor. The quantity

$$Z(\tau) = \int [dA]\exp(-S_E[A]), \tag{1.6}$$

is called the *Maxwell partition function* and the background manifolds M we will consider are closed, connected, oriented and Riemannian. The key idea is that for closed and oriented manifolds, we can make use of Poincaré duality in the calculations. The manifold M does not have to be a Wick rotated Lorentzian four-manifold, so we have less restrictions on the topology of the four-manifold and this will result in interesting effects.

The flux of F through a closed surface Σ in M, defined as the quantity $\int_\Sigma F$, usually equals zero if M has too trivial topology. Non-trivial fluxes are for example absent in classical Maxwell theory on Euclidean $\mathbb{R}^4$. Fluxes can be non-trivial for more complicated manifolds, when a topological quantity known as the *second Betti number* is non-zero. Dirac [Dir] stipulated that these fluxes should be quantized, a result that has become known as the *Dirac quantization condition*. Such quantized fluxes cause the term $\frac{1}{4\pi^2}\int_M F\wedge F$ in the action to be quantized and the values of this term will be dependent only on the *topology* of M. The resulting quantum theory is then not $SL(2,\mathbb{R})$ covariant anymore, since the transformation $\tau\mapsto\tau+b$ only leaves the action invariant for even b in general, or integral b if $\frac{1}{4\pi^2}\int_M F\wedge F$ is always even. Therefore the action of $SL(2,\mathbb{R})$ gets broken down to $SL(2,\mathbb{Z})$ or a subgroup thereof.

The partition function becomes intricately related with the topology and geometry of the underlying four-manifolds. An important object which will determine the values of $\frac{1}{4\pi^2}\int_M F\wedge F$, is given by the intersection form we mentioned in the previous section. For specific topologies of M, the partition function is invariant under the full group $SL(2,\mathbb{Z})$ and we can speak of *exact* electric-magnetic duality. In more general cases, the partition function transforms as a *modular form* under the action of $SL(2, \mathbb{Z})$ or a subgroup thereof and we then speak of modular covariance or $SL(2, \mathbb{Z})$ covariance.

Verlinde [Ver] and Witten [Wit3] have studied the global properties of this duality. For Abelian gauge fields which couple to complex spinor fields something peculiar happens and one has to study the partition function in a more generalized context, which has been done by Olive and Alvarez [OA].

Although electric-magnetic duality in Euclidean Maxwell theory is an interesting phenomenon on its own to study, this type of duality seems to be a recurring theme in physics and mathematics. In a recent paper Gaiotto and Witten [KW] have made a connection between electric-magnetic duality and the geometric Langlands program. Also recently, a correspondence between knot theory and topological gauge theories in four dimensions has been found, in which electric-magnetic duality plays a role [GW].

Electric-magnetic duality has been extended to the concept of S-duality, also known as strong-weak duality, which is present in string theory and plays a crucial role in supersymmetric gauge theories. S-duality was first conjectured to be present as the so-called Montonen-Olive electric-magnetic duality in $N = 4$ supersymmetric Yang–Mills theory [MO, Oli]. The duality concerns the equivalence of two regimes of a given quantum field theory. This equivalence is formed by a transformation *within* the quantum field theory that transforms a set of fields and vacua with coupling constant g into a set of dual fields and vacua with gauge coupling constant $1/g$. This means there is a duality between the weak coupling regime and the strong coupling regime of the same theory. The motivation behind finding these dualities is that QCD is still poorly understood for low energies, where the coupling is strong, in contrast to high energies, where the coupling is weak. Switching the 'electric' and magnetic fields in Euclidean Maxwell theory with θ set to zero in (1.5), gives us such a duality. Seiberg [Sei] extended this type of duality and showed it was present in $N = 1$ supersymmetric Yang–Mills gauge theory on 4-manifolds. Vafa and Witten subsequently showed [VW] S-duality is present in $N = 4$ supersymmetric Yang–Mills theory on a variety of four-manifolds. This duality is also present in Seiberg–Witten gauge theory [SW1, SW2]. When the gauge group is $SU(2)$, this S-duality is an embedded form of the electric-magnetic duality that is present in Euclidean Maxwell theory [Wit3]. This is a domain where the theory regarding electric-magnetic duality in Abelian gauge theories and Yang–Mills instantons meet.

1.4 Del Pezzo surfaces

Specific four-manifolds that are central within this thesis are del Pezzo surfaces. The Italian mathematician Pasquale del Pezzo - after whom the surfaces are named - studied these surfaces at the end of the 19th century and have been recurrently in the focus of mathematical research ever since. Only recently did these surfaces become interesting to physicists, where they are studied in the context of superstring theory. A large class of superstring theories requires ten space-time dimensions. To reduce this number to the four dimensions we observe, one compactifies the other six dimensions: they are rolled up onto very small six-dimensional manifolds. Often this six-dimensional manifold is conjectured to be a Calabi–Yau threefold, which can contain embedded del Pezzo surfaces. Such occurences of del Pezzo surfaces in superstring theory and its extension to 11 dimensions, M-theory, have been studied in [CKM, HMV] for example. By dimensional reduction one is sometimes able to obtain a Yang–Mills theory on a del Pezzo surface. Hence in this setting it is interesting to study Yang–Mills instantons and electric-magnetic duality on these manifolds. In the domain of string theory, one has also established a correspondence between toroidal compactifications of M-theory and del Pezzo surfaces, which is called the 'mysterious duality' [INV].

Del Pezzo surfaces are obtained by blowing up points in $\mathbb{CP}^1 \times \mathbb{CP}^1$, a product of two Riemann spheres, or $\mathbb{CP}^2$, the complex projective plane. Blowing-up is a geometric procedure from which one is able to construct new manifolds out of old ones. In four dimensions, it can be described as replacing a point by a copy of $\mathbb{CP}^1$. Del Pezzo surfaces have a Kähler structure, so that we can use methods from algebraic geometry to model the ASD moduli space. One result from algebraic and complex geometry is that blowing up $\mathbb{CP}^1 \times \mathbb{CP}^1$ in one point is the same as blowing up $\mathbb{CP}^2$ in two points. This explicit structure will play a role in the description of the moduli spaces.

Del Pezzo surfaces will also cover the full spectrum of the theory regarding electric-magnetic duality in Euclidean Maxwell theory. We will calculate the Maxwell partition functions for del Pezzo surfaces that are endowed with more generic metrics. The blow-ups of $\mathbb{CP}^1 \times \mathbb{CP}^1 \simeq S^2 \times S^2$ and $\mathbb{CP}^2$ will then be treated in terms of connected sums of manifolds. The resulting partition functions still reflect the topological correspondences between the blown-up surfaces and the dependence on the metric is reduced to a number of parameters satisfying certain conditions.

1.5 Outline

Chapter 2 will cover some basic facts about four-manifolds. The geometric properties of four-manifolds can be characterized through the homology and cohomology groups and the intersection form. Specific to four-manifolds is the notion of (anti-)self-duality which is fundamental for the description of the ASD moduli space and will also return when we discuss the Maxwell partition function. Del Pezzo surfaces will have some additional structure and in Chapter 2 we will also give an exposition of the algebraic and complex geometry related to del Pezzo surfaces. At some points we refer the reader to the Appendix on complex projective space.

Chapter 3 covers the Yang–Mills instantons formed by ASD connections. We recall some general theory about vector bundles and connections and cover some central results from [DK] regarding ASD moduli spaces. For Kähler surfaces we will relate the $SU(2)$ ASD moduli spaces to the moduli spaces of stable holomorphic vector bundles of rank 2. Based on results in [Sob], this correspondence will be combined with the constructions for del Pezzo surfaces to describe the $SU(2)$ ASD moduli space on $S^2 \times S^2$.

In Chapter 4 we will discuss electric-magnetic duality in Euclidean Maxwell theory on closed, connected, oriented, Riemannian four-manifolds. The associated Maxwell partition function is computed in the semiclassical approximation. Due to the Dirac quantization condition for fluxes the result becomes a sum over a discrete set which is dependent on the topology of the manifold. The intersection form is the topological invariant that will play the most significant role. The modular covariance of this theory can then be easily studied, since part of the partition function becomes a theta function. The theory developed will be applied to del Pezzo surfaces. Central to this chapter are the papers by Verlinde [Ver], Witten [Wit3] and Olive and Alvarez [OA].

Besides four-manifold theory, there is no direct overlap between Yang–Mills instantons and electric-magnetic duality in the way it is presented in this thesis. As a result Chapter 3 can be read separetely from Chapter 4 and only some results from Sections 3.1, 3.2 and 3.4.3 are used in Chapter 4.

In the Outlook we will cover some connections of what we presented with other parts of physics and mathematics. One connection is the duality of Euclidean Maxwell theory with toroidal models in string theory. Another main connection is formed by Seiberg–Witten theory, where both instantons and electric-magnetic duality are encountered.

Due to the scope of the thesis a part of the theory is covered in a somewhat condensed form. Whenever it is not possible to cover all details or there are more interesting aspects to the theory, references to more comprehensive works are given. I would like to point out that only a few results presented are original. The computations in Section 4.6 are joint work with Bram Bet under the supervision of Gil Cavalcanti and Stefan Vandoren. At some points I adapted and extended some original arguments from the literature. I hope the reader will enjoy many of the elegant features of gauge theory that are covered. The reader is encouraged to read Bram Bet's thesis [Bet] as well. Our theses supplement each other well at some points, especially in Chapter 4.

Chapter 2

Four-Manifolds

This chapter starts with a section covering some general theory about four-manifolds as a preliminary for Chapters 3 and 4. We will follow selected parts from Chapter 1 of [DK] and [AO] and assume the reader is familiar with basic algebraic topology and differential geometry. The second section will cover the geometry of del Pezzo surfaces. These surfaces will illustrate the theory from the first section and some related results will reappear in Sections 3.5, 4.5 and 4.6.

2.1 Cohomology, homology and intersection forms

Let M be a closed (compact without boundary), oriented n-manifold and denote $H_i(M,\mathbb{Z})$ and $H^i(M,\mathbb{Z})$ for the i-th homology and cohomology groups with integral coefficients, respectively. The i-th Betti number b^i denotes the rank of the i-th homology group. Poincaré duality gives an isomorphism between homology and cohomology in complementary dimensions, i and $n-i$. In combination with the universal coefficient, it is seen that b^i is also the rank of $H^i(M,\mathbb{Z})$, $H^{n-i}(M,\mathbb{Z})$ and $H_{n-i}(M,\mathbb{Z})$.

A simply connected manifold M has the property that its first homology group $H_1(M,\mathbb{Z})$ vanishes. If this is the case and M is four-dimensional, the universal coefficient theorem and Poincaré duality then greatly reduce the homology and cohomology groups of interest:

- $H^1(M,\mathbb{Z}) = H^3(M,\mathbb{Z}) = H_1(M,\mathbb{Z}) = H_3(M,\mathbb{Z}) = 0$;
- $H_2(M,\mathbb{Z}) \simeq H^2(M,\mathbb{Z}) \simeq \mathrm{Hom}(H_2(M,\mathbb{Z}),\mathbb{Z}) \simeq \mathbb{Z}^{b^2}$, i.e. they are infinite free Abelian discrete groups of rank b^2. The isomorphism with $\mathbb{Z}^{b^2}$ is obtained by picking a set of generators for $H_2(M,\mathbb{Z})$ and $H^2(M,\mathbb{Z})$;
- $H^0(M,\mathbb{Z}) \simeq H_0(M,\mathbb{Z}) \simeq H_4(M,\mathbb{Z}) \simeq H^4(M,\mathbb{Z}) = \mathbb{Z}$.

2.1.1 The intersection form

Before we define the intersection form, we first introduce the intersection number, which can be defined on any oriented smooth manifold.

Definition 2.1. Let V, W be two smooth submanifolds of an oriented smooth n-manifold M that are of complementary dimension and intersect each other transversely. The intersection number $I(V,W)$ is defined as

$$I(V,W) = \sum_{x\in V\cap W} \epsilon(x),$$

where $\epsilon(x) \in \{\pm 1\}$ is $+1$ if $T_xV \oplus T_xW$ matches the orientation of T_xM and -1 otherwise.

We fix M to be a closed, oriented, simply-connected four-manifold. The Poincaré duality isomorphism and the pairing between homology and cohomology result in a symmetric, bilinear form

$$Q : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \to Z; \quad Q(\alpha,\beta) = \langle \alpha \smile \beta, [M]\rangle \in \mathbb{Z}, \tag{2.1}$$

where $\smile$ denotes the cup product on cohomology, $[M]$ is the fundamental homology class of M and the brackets indicate the pairing between homology and cohomology. Using Poincaré duality on the entries of Q, we obtain a symmetric, bilinear form

$$I : H_2(M,\mathbb{Z}) \times H_2(M,\mathbb{Z}) \to \mathbb{Z}. \tag{2.2}$$

If M is smooth, then for Σ, $\Sigma' \in H_2(M,\mathbb{Z})$ the number $I(\Sigma,\Sigma')$ is given by the intersection number of the 2-dimensional oriented surfaces that represent Σ and Σ', which can be chosen to intersect transversely. Due to the properties of the intersection number, $I(\Sigma,\Sigma')$ is then indeed symmetric in Σ and Σ' and the value of $I(\Sigma,\Sigma')$ is independent of the representative chosen. As a consequence it is also possible to define the self-intersection number $I(\Sigma,\Sigma)$.

For smooth M the Poincaré dual of a homology 2-cycle Σ can be realized as the de Rham cohomology class of a closed 2-form ω. This 2-form has the property that $\int_\Sigma \alpha = \int_M \omega \wedge \alpha$ for all 2-forms α. Given two $\Sigma, \Sigma' \in H_2(M,\mathbb{Z})$ and its Poincaré dual forms ω, ω', we have the identity $I(\Sigma,\Sigma') = \int_M \omega \wedge \omega'$. To show this we first notice that integration of $\omega \wedge \omega'$ over M reduces to integration over neighbhourhoods of the points in $\Sigma \cap \Sigma'$ (the representatives are denoted by the same symbol). The integrals over these neighbourhoods then equal ± 1, depending on whether $T\Sigma \oplus T\Sigma'$ matches the orientations of TM at the points of intersection. The sum of these integrals gives us $I(\Sigma,\Sigma')$. A more explicit calculation can be found on pages 2 and 3 of [DK]. Whence, by using de Rham cocycles as representatives for $H^2(M,\mathbb{Z})$ the form I becomes Poincaré dual to the following symmetric, bilinear form

$$Q : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \to Z; \quad Q([\alpha],[\beta]) = \int_M \alpha \wedge \beta \in \mathbb{Z}, \tag{2.3}$$

which coincides with the right-hand side of (2.1). We conclude that in the presence of a smooth structrure, the expressions (2.1) and (2.2) can be computed in more geometric terms.

The isomorphism $H_2(M,\mathbb{Z}) \simeq H^2(M,\mathbb{Z}) \simeq \mathrm{Hom}(H_2(M,\mathbb{Z}),\mathbb{Z})$ is realized through the assignment $I(\Sigma)(\cdot) = I(\Sigma,\cdot)$. This means that the form I is *unimodular*. Similarly, Q induces an isomorphism between $H^2(M,\mathbb{Z})$ and $\mathrm{Hom}(H^2(M,\mathbb{Z}),\mathbb{Z}) \simeq H_2(M,\mathbb{Z})$, so that Q is a unimodular form as well.

Definition 2.2. The unimodular forms Q and I in (2.1) and (2.2) are both called the *intersection form* of M. When M is smooth, Q can also be given by (2.3) and I in terms of the intersection number of the surfaces that represent the homology 2-cycles (see Definition 2.1). The form Q can be extended to the full de Rham cohomology group $H^2(M,\mathbb{R})$ by applying the right-hand side of (2.3) to the de Rham cocycles. The result is a real, non-degenerate, bilinear, symmetric form on $H^2(M,\mathbb{R})$.

Both Q and I can be represented by symmetric non-degenerate matrices with integer entries after picking a basis for $H^2(M,\mathbb{Z})$ and $H_2(M,\mathbb{Z})$. These matrices must have determinant equal to ± 1, since their inverses must have integer entries as well by unimodularity of the forms. Symmetric matrices having integer entries and determinant equal to ± 1 are called *unimodular matrices.*

Remark 2.3. The values of I and Q are independent of the representatives chosen, i.e. $I(\Sigma + \partial\Pi, \Sigma' + \partial\Pi') = I(\Sigma,\Sigma')$ and $Q(\alpha + \partial\gamma, \alpha' + \partial\gamma') = Q(\alpha,\alpha')$. Therefore cycles and cocycles and their respective homology and cohomology classes are sometimes denoted by the same symbol. Whenever the underlying manifold of the intersection form needs to be distinguished, we will denote it by a subscript, i.e. $I = I_M$ and $Q = Q_M$.

The intersection form is a topological invariant associated to four-manifolds for which certain algebraic properties are valid. Discrete Abelian groups of rank n isomorphic to $\mathbb{Z}^n$ and endowed with a unimodular form can be classified algebraically up to isomorphism. We will cover this classification in Section 4.3.1, when we see these groups as lattices. An important property of an integral, symmetric, bilinear form on such a group is its type: odd or even.

Definition 2.4. Let Λ be a discrete Abelian group isomorphic to $\mathbb{Z}^n$. Then an integral, symmetric, bilinear form Q on Λ is *even* if $Q(x,x)$ is even for all $x \in \Lambda$ and *odd* otherwise.

Freedman showed [Fre] that *any* unimodular form can be obtained as the intersection form of some topological four-manifold. In some cases a certain intersection form is an obstruction for a smooth structure on the manifold. For a summary of results regarding smooth four-manifolds and their possible intersection forms see Chapters 8 and 9 of [DK].

2.1.2 Non-simply connectedness

When M is not necessarily simply connected, the intersection forms I and Q are still well-defined, but the manifold can have homology and cohomology classes that have torsion. This means there are elements $[\Sigma] \in H_i(M,\mathbb{Z})$ and $[\alpha] \in H^i(M,\mathbb{Z})$ which have finite order: there are numbers N, N', such that the cycle $N\Sigma$ and cocycle $N'\alpha$ are respectively a boundary and coboundary, i.e $N\Sigma = \partial\Pi$ and $N'\alpha = d\beta$ for some $(i+1)$-chain Π and $(i-1)$-cochain β. Let us denote the group of torsion elements in $H_i(M,\mathbb{Z})$ and $H^i(M,\mathbb{Z})$ by $T_i(M,\mathbb{Z})$ and $T^i(M,\mathbb{Z})$, respectively. One sees that for $\Sigma \in T_2(M,\mathbb{Z})$ we have

$$I(\Sigma,\tilde{\Sigma}) = \frac{1}{N} I(N\Sigma,\tilde{\Sigma}) = I(0,\tilde{\Sigma}) = 0.$$

Similarly, $Q(\alpha,\beta) = 0$ if α or β lies in $T^2(M,\mathbb{Z})$. If these torsion groups are non-trivial, I and Q are not unimodular anymore, since they have become degenerate. However, Poincaré duality and the universal coefficient theorem imply that the quotients

$$F_2(M,\mathbb{Z}) = H_2(M,\mathbb{Z})/T_2(M,\mathbb{Z}), \quad F^2(M,\mathbb{Z}) = H^2(M,\mathbb{Z})/T^2(M,\mathbb{Z}), \tag{2.4}$$

both are isomorphic to $\mathbb{Z}^{b^2}$ and that Q and I restricted to these quotients are unimodular. Furthermore the isomorphism $H^2(M,\mathbb{Z}) \simeq \mathrm{Hom}(H_2(M,\mathbb{Z}),\mathbb{Z})$ gets replaced with

$$F^2(M,\mathbb{Z}) \simeq \mathrm{Hom}(H_2(M,\mathbb{Z}),\mathbb{Z}) \simeq \mathrm{Hom}(F_2(M,\mathbb{Z}),\mathbb{Z}). \tag{2.5}$$

Examples of closed, oriented, non-simply connected four-manifolds are $\mathbb{RP}^2 \times \mathbb{RP}^2$, where $\mathbb{RP}^2$ is real projective 2-space, or $T^2 \times T^2$, the direct product of two tori.

For completeness we calculate the non-trivial torsion groups for a closed, oriented, connected four-manifold M. Connectedness gives $H_0(M,\mathbb{Z}) \simeq \mathbb{Z}$ and in combination with orientability and closedness we get $H_4(M,\mathbb{Z}) \simeq \mathbb{Z}$. Poincaré duality then tells us that M always has

$$H^0(M,\mathbb{Z}) \simeq H_0(M,\mathbb{Z}) \simeq H_4(M,\mathbb{Z}) \simeq H^4(M,\mathbb{Z}) = \mathbb{Z}.$$

The universal coefficient theorem implies

$$T^i(M,\mathbb{Z}) \quad \simeq \quad T_{i-1}(M,\mathbb{Z}), \tag{2.6}$$

so that $T^1(M,\mathbb{Z}) = T_0(M,\mathbb{Z}) = 0$ and $T_3(M,\mathbb{Z}) = T^4(M,\mathbb{Z}) = 0$. Due to Poincaré duality and (2.6) the only non-trivial torsion groups satisfy

$$T_2(M,\mathbb{Z}) \simeq T^2(M,\mathbb{Z}) \simeq T_1(M,\mathbb{Z}) \simeq T^3(M,\mathbb{Z}).$$

If M is simply-connected, all torsion groups vanish, since then $T_1(M,\mathbb{Z}) \subset H_1(M,\mathbb{Z}) = 0$.

2.1.3 (Anti-)Self-duality and harmonic forms

We fix our attention to oriented, Riemannian n-manifolds. On such manifolds, the Hodge star operator $\star : \Omega^i(M) \to \Omega^{n-i}(M)$ is defined through

$$\alpha \wedge \star\beta = (\alpha, \beta) d\mu, \tag{2.7}$$

where $(\cdot,\cdot)$ is the Euclidean inner-product on the fibres of $\Lambda^i T^*M$ and $d\mu$ is the Riemannian volume form. Using $\star$ we obtain the following inner-product on forms

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star\beta. \tag{2.8}$$

There is an adjoint map $d^* : \Omega^i(M) \to \Omega^{i-1}(M)$ defined using this inner-product:

$$\langle d\alpha, \beta \rangle = \langle \alpha, d^*\beta \rangle, \tag{2.9}$$

where it is assumed that at least one of the two forms has compact support. One can compute that $d^* = \pm \star d \star$.

An i-form α is *harmonic* if $d\alpha = d^*\alpha = 0$. We denote all such forms by

$$\mathcal{H}^i(M) = \{\alpha \in \Omega^i(M) \,|\, d\alpha = d^*\alpha = 0\}. \tag{2.10}$$

If M is closed, we can apply Hodge theory. The following decomposition is then valid for i-forms

$$\Omega^i(M) = \mathcal{H}^i \oplus d^*(\Omega^{i+1}(M)) \oplus d(\Omega^{i-1}(M)).$$

The first and third term form the closed forms, since $dd^*\gamma = 0$ implies $d^*\gamma = 0$. From this we conclude that $H^i(M,\mathbb{R}) = \ker(d : \Omega^i(M) \to \Omega^{i+1}(M))/d(\Omega^{i-1}(M)) \simeq \mathcal{H}^i$, i.e. each cohomology class has a unique harmonic representative.

In four dimensions $\star^2|_{\Omega^2(M)} = \mathrm{id}$. Accordingly, $\Omega^2(M) = \Gamma(\Lambda^2)$ can be split as

$$\Omega^2(M) = \Omega^+(M) \oplus \Omega^-(M) = \Gamma(\Lambda^+) \oplus \Gamma(\Lambda^-), \tag{2.11}$$

where $\Lambda^\pm$ is the ± 1 eigenspace of $\star$, i.e for $\alpha \in \Lambda^\pm$ one has

$$\alpha \wedge \alpha = \pm |\alpha|^2 d\mu.$$

For $\alpha^\pm \in \Lambda^\pm$, we have

$$\alpha^+ \wedge \star\alpha^- = -\alpha^+ \wedge \alpha^- = -\star\alpha^+ \wedge \alpha^- = -\alpha^+ \wedge \star\alpha^-,$$

so that α^+ and α^- are orthogonal for the Euclidean inner product $(\cdot,\cdot)$ in (2.7). This implies that the splitting of $\Lambda^2 TM = \Lambda^+ \oplus \Lambda^-$ is actually an orthogonal decomposition.

One important observation is that in four dimensions the action of $\star$ on 2-forms (hence the splitting in (2.11)) only depends on the conformal class of the Riemannian metric.

Proposition 2.5. *Let M be an oriented, Riemannian four-manifold. Then the action of the Hodge star operator on 2-forms only depends on the conformal class of the metric.*

Proof. What we need to proof is that for a positive smooth function c on M, the Riemannian metric cg will give the same value on the right-hand side of (2.7) when $\alpha, \beta \in \Omega^2(M)$. The Riemannian volume form for the new metric becomes $c^4 d\mu$. The elements in the cotangent bundle get rescaled by a factor $1/c$, so the inner product (α, β) gets rescaled by a factor $1/c^4$. The different powers of c cancel each other and therefore the right-hand side of (2.7) remains invariant. ☐

For closed and four-dimensional M, the cohomology classes in $H^2(M, \mathbb{R})$ can be uniquely represented by harmonic 2-forms and we have $d^* = - \star d \star$ and $\star^2 = \text{id}$ in two degrees. This shows $\star$ preserves $\mathcal{H}^2(M)$ and hence the splitting in (2.11) passes to cohomology:

$$H^2(M, \mathbb{R}) \simeq \mathcal{H}^2(M) = \mathcal{H}^+ \oplus \mathcal{H}^-, \tag{2.12}$$

where $\mathcal{H}^+$ and $\mathcal{H}^-$ are the self-dual and anti-self-dual harmonic 2-forms, respectively. The spaces $\mathcal{H}^+$ and $\mathcal{H}^-$ are seen to correspond with maximal positive and negative subspaces of the form Q, when Q is applied to the unique harmonic representatives of $H^2(M, \mathbb{R})$ through the right-hand side of (2.3).

Remark 2.6. The second cohomology group with real coefficients $H^2(M, \mathbb{R})$ can be realized as $H^2(M, \mathbb{Z}) \otimes \mathbb{R}$. Tensoring with $\mathbb{R}$ has the effect that all torsion cocycles vanish, so that the quotient $F^2(M, \mathbb{Z})$ forms a discrete subgroup of $H^2(M, \mathbb{R})$. As a consequence of the isomorphism $H^2(M, \mathbb{R}) \simeq \mathcal{H}^2(M)$, the generators of $F^2(M, \mathbb{Z})$ can be uniquely represented by harmonic forms. Furthermore, the intersection form Q extended to $H^2(M, \mathbb{R})$ is non-degenerate even if $H^2(M, \mathbb{Z})$ has torsion elements.

The splitting in (2.12) depends on the conformal class of the metric due to Proposition 2.5. The numbers $b^+ = \dim \mathcal{H}^+$ and $b^- = \dim \mathcal{H}^-$ are topological invariants however, since they correspond to dimensions of the maximal positive and negative subspaces of Q. The number

$$\sigma(M) = b^+ - b^- \tag{2.13}$$

is known as the *Hirzebruch signature* of M.

2.1.4 Connected sums

Let M and N be oriented, closed n-manifolds. A connected sum in its simplest definition is deleting two (small) n-balls in M and N and 'glue' the resulting boundary $(n-1)$-spheres. How one glues together these spheres depends on which properties of the manifolds we want to preserve. To preserve the orientation of both manifolds, one must identify the spheres through an orientation reversing map.

Often one also wants to preserve the smooth structure of both manifolds if they are present. In addition, one then takes two neighbourhoods of the boundary $(n-1)$-spheres, forming annuli diffeomorphic to $(0,1) \times S^{2n-1}$. These annuli are identified with each other through an orientation preserving diffeomorphism, where the 'radial' coordinate $x \in (0,1)$ gets inverted and the 'spheres' get identified through an orientation reversing map. This means the outer sphere $(\simeq \{1\} \times S^{n-1})$ in one annulus gets identified with the 'reversed' inner sphere $(\simeq \{0\} \times S^{n-1})$ of the other annulus and vice versa. After the two annuli are identified with each other, the resulting part of the manifold is denoted as the *neck* of the connected sum. The identifications are illustrated in Figure 2.1.

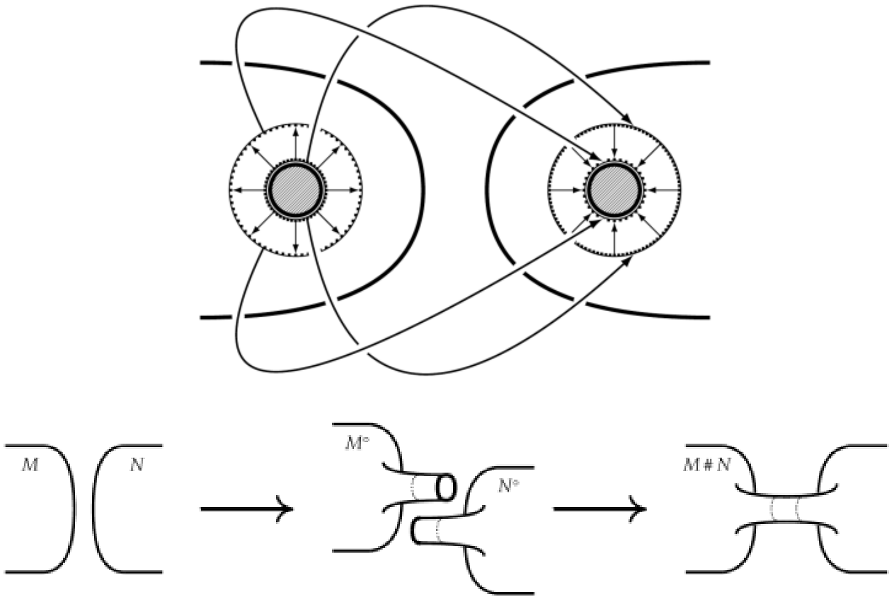

Figure 2.1: Smooth connected sum ([Sco], page 119).

The connected sum $M\#N$ is closed and oriented as well. We therefore have $H_0(M\#N,\mathbb{Z}) \simeq H_n(M\#N,\mathbb{Z}) \simeq \mathbb{Z}$. One can show that the rest of the homology splits using Mayer–Vietoris sequences (see [Hat], page 258), which in combination with Poincaré duality gives the following theorem.

Theorem 2.7. *Let $M\#N$ be the connected sum of oriented, closed n-manifolds M and N. Then we have:*

$$\begin{aligned} H_0(M\#N,\mathbb{Z}) \simeq H_n(M\#N,\mathbb{Z}) \simeq H^0(M\#N,\mathbb{Z}) \simeq H^n(M\#N,\mathbb{Z}) \simeq \mathbb{Z}, \\ H_i(M\#N,\mathbb{Z}) = H_i(M,\mathbb{Z}) \oplus H_i(N,\mathbb{Z}) \quad \text{for } 0<i<n, \\ H^i(M\#N,\mathbb{Z}) = H^i(M,\mathbb{Z}) \oplus H^i(N,\mathbb{Z}) \quad \text{for } 0<i<n. \end{aligned}$$

Another property of the connected sum $M\#N$ is that it is simply connected if M and N are, which is consistent with the theorem above.

In the construction of the connected sum, the representatives of the homology cycles in M can be chosen to not intersect those in N. Now suppose M and N are four-dimensional and smooth. In the connected sum $M\#N$, the intersection numbers between the two-cycles in M and between the two-cycles in N remain the same. Therefore, the intersection forms I and Q on $M\#N$ split as a direct sum of the two original intersection forms on M and N. This also holds when M and N are not smooth ([Sco], page 118).

Proposition 2.8. *Let M, N be closed, oriented four-manifolds and $M\#N$ their connected sum. Then the following is true for the intersection forms I and Q as in Definition 2.2:*

$$I_{M\#N} = I_M \oplus I_N, \quad Q_{M\#N} = Q_M \oplus Q_N.$$

2.2 Del Pezzo surfaces

In this section we will give a brief description of the algebraic and complex geometry needed to describe del Pezzo surfaces. More information on $\mathbb{CP}^n$, complex projective n-space, can be found in Appendix A, where we also set notation.

Del Pezzo surfaces are formally defined as specific abstract algebraic surfaces. When realized as complex surfaces, they form complex manifolds of dimension 2, which is the class of four-manifolds we are interested in.

Definition 2.9 ([Man]). A *del Pezzo surface* is a smooth projective complex surface with ample anticanonical bundle K^{-1}. The *degree* of a del Pezzo surface is the self-intersection number $([K],[K])$ of its canonical class.

The canonical bundle K over a complex surface M is defined as the top exterior power of the holomorphic cotangent bundle ([GH], page 146):

$$K = \Lambda^{2,0}T^*M,$$

which is a line bundle (Section 3.2 covers line bundles). The anticanonical bundle K^{-1} is the inverse of K, meaning $K \otimes K^{-1}$ is the trivial line bundle. The canonical class $[K] \in H^2(M,\mathbb{Z})$ is the first Chern class of K and the self-intersection number $([K],[K])$ is given by $Q([K],[K])$, where Q is the intersection form as in Definition 2.2. Since K^{-1} is ample for a del Pezzo surface, it can be embedded in $\mathbb{CP}^N$ for some N.

Remark 2.10. Definition 2.9 can be generalized to any algebraically closed field k instead of $\mathbb{C}$. This will be the definition of a del Pezzo surfaces in terms of an abstract algebraic surface.

Pasquale del Pezzo proved that the degree d of a del Pezzo surface satisfies $1 \leq d \leq 9$. There is also an alternative description of del Pezzo surfaces given in terms of blow-ups of $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$.

A blow-up of a complex manifold can heuristically be described as the surface one gets when one replaces a point by the set of all complex lines passing through that point (see Figure 2.2). For complex manifolds of dimension 2, this means that we replace a point by $\mathbb{CP}^1$. The following theorem [Var] will give us a description of del Pezzo surfaces in terms of blow-ups.

Theorem 2.11. *Let M be a del Pezzo surface of degree d. Then either M is biholomorphic to the blow-up of $\mathbb{CP}^2$ at $9-d$ points in general position in $\mathbb{CP}^2$, or $d = 8$ and M is biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$.*

Remark 2.12. The condition that a collection of points is in general position in $\mathbb{CP}^2$ means:

1) No three points lie in the same line;

2) No six points lie in the same quadric;

3) There is no singular cubic passing through all the 8 points and which has one of the points as its singularity.

This condition is implied by the fact that K^{-1} is ample for a del Pezzo surface.

In Section 2.2.2 blow-ups are covered in more detail. There it will be shown that the blow-up of a complex manifold M of dimension n is diffeomorphic to the smooth connected sum $M\#\overline{\mathbb{CP}}^n$, where $\overline{\mathbb{CP}}^n$ is the manifold $\mathbb{CP}^n$ with reversed orientation.

The complex manifold $\mathbb{CP}^1 \times \mathbb{CP}^1$ is diffeomorphic to $S^2 \times S^2$, but when $S^2 \times S^2$ is realized in such a way it is endowed with the product metric on two round spheres. Del Pezzo surfaces can hence be classified up to diffeomorphism as:

- $dP_0 = \mathbb{CP}^2$;
- $dP_1 = \mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ and $d\tilde{P}_1 = S^2 \times S^2$;

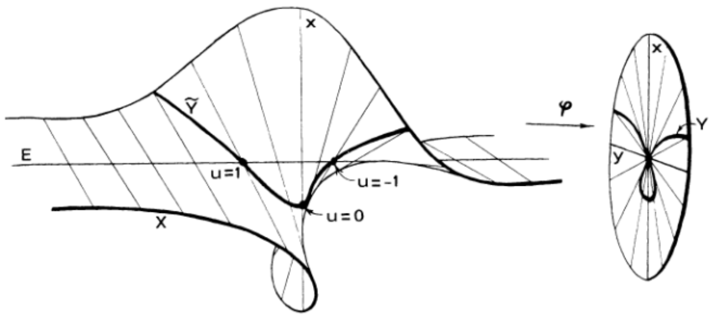

Figure 2.2: A depiction of a blow-up ([Har]), page 29).

- $dP_l = \mathbb{CP}^2\#l\overline{\mathbb{CP}}^2$, with $2 \leq l \leq 8$.

In Section 2.2.3 we will proof that the blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ at one point is biholomorphic to the blow-up of $\mathbb{CP}^2$ at two points. As a consequence the higher blow-ups of these two surfaces are biholomorphic to each other as well. This implies that $dP_l = \mathbb{CP}^2\#l\overline{\mathbb{CP}}^2$ can also be realized (up to diffeomorphism) as $S^2 \times S^2\#(l-1)\overline{\mathbb{CP}}^2$.

We will start with the introduction of algebraic varieties. Several geometrical properties of del Pezzo surfaces are better understood in that framework, since intersections of algebraic varieties are well-behaved. The main reference of what follows is [GH]. I assume the reader is familiar with basic notions from complex geometry.

2.2.1 Algebraic varieties

A large class of known complex manifolds can be obtained by studying zero-sets of holomorphic functions $f : \mathbb{C}^n \to \mathbb{C}$. The prime examples of holomorphic functions are the complex polynomials. A great deal of ring theory is devoted to the study of polynomial functions. The algebraic relations for polynomial functions gets reflected in the geometric properties of their zero-sets. Subsets of $\mathbb{C}^n$ that can be obtained in this way are known as *algebraic sets.*

Del Pezzo surfaces can be embedded in complex projective space, so it is more natural to consider subsets of $\mathbb{CP}^n$ cut out by polynomials. To define subsets of $\mathbb{CP}^n$ as the zero locus of a polynomial, we notice that these are only well defined for homogeneous polynomials. The zero set of a homogeneous polynomial $F \in \mathbb{C}[Z_0, \ldots, Z_n]$ in $\mathbb{CP}^n$ is denoted as

$$Z(F) = \{F = 0\} = \{(Z_0 : \ldots : Z_n) \in \mathbb{CP}^n \,|\, F(Z) = 0\}. \tag{2.14}$$

We can extend these definitions to zero-loci of subsets S of homogeneous polynomials:

$$Z(S) = \{(Z_0 : \ldots : Z_n) \in \mathbb{CP}^n \,|\, F(Z) = 0, \forall F \in S\}. \tag{2.15}$$

Definition 2.13 ([GH], page 166)**.** A subset of $\mathbb{CP}^n$ that can be realized as the zero-locus of a finite family of homogeneous polynomials is called an *algebraic variety.* An algebraic variety in $\mathbb{CP}^n$ has dimension k if the number of homogeneous polynomials defining it is $n - k$ and the polynomials are coprime. Algebraic varieties of dimension 2 are called *algebraic surfaces.*

To see the detailed structure of algebraic varieties, one uses the holomorphic coordinate charts (φ_i, U_i) for $\mathbb{CP}^n$ given by the sets

$$U_i = \{Z_i \neq 0\} \tag{2.16}$$

and maps

$$\varphi_i : \mathbb{CP}^n \cap U_i \to \mathbb{C}^n \, ; \quad (Z_0 : \ldots : Z_n) \mapsto (Z_0/Z_i, \ldots, Z_{i-1}/Z_i, Z_{i+1}/Z_i, \ldots, Z_n/Z_i), \qquad (2.17)$$

with inverse $\varphi_i^{-1}(z_1, \ldots, z_n) = (z_1, \ldots z_{i-1} : 1 : z_i : \ldots : z_n)$. The coordinates $z_j = Z_j/Z_i$ on U_i are also called *affine coordinates*.

Using the charts in (2.17) one can describe the zero-set $Z(F)$ of a homogeneous polynomial in terms of affine coordinates $(z_0, \ldots, z_{i-1}, z_{i+1}, z_n) \in \mathbb{C}^n$, where $z_j = Z_j/Z_i$:

$$Z(F) \cap U_i \simeq \{F(z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n) = 0\} = \{f(z_0, \ldots, z_n) = 0\}.$$

A point $p \in Z(F) \cap U_i$ is *regular* if $\frac{\partial f}{\partial z_j}(\varphi_i(p)) \neq 0$ for some j. By an application of the implicit function theorem a local neigboorhoud of p is then a complex manifold of dimension $n-1$. A point p is singular if it is not regular in some chart. If none of the points $p \in Z(F)$ are singular, then $Z(F)$ is a complex submanifold of $\mathbb{CP}^n$ of dimension $n-1$.

Now on U_i we have $\frac{\partial f}{\partial z_j} = \frac{\partial F}{\partial Z_j} Z_i$ and for homogeneous polynomials of degree d we have the Euler rule:

$$\sum_{j=0}^{n} \frac{\partial F}{\partial X_j} \cdot X_j = d \cdot F.$$

These identities show that the set of singular points in $Z(F)$ is given by $\bigcap_{j=0}^n Z\left(\frac{\partial F}{\partial X_j}\right)$. Hence, $Z(F)$ is a closed complex submanifold of $\mathbb{CP}^n$ of dimension $n-1$ if and only if

$$\bigcap_{j=0}^{n} Z\left(\frac{\partial F}{\partial X_j}\right) = \emptyset. \qquad (2.18)$$

We extend this discussion to a collection $S = \{F_1, \ldots, F_k\}$ of (coprime) homogeneous polynomials. Then the condition of a point $p \in Z(S)$ being singular is equivalent to

$$\operatorname{rank}\left(\frac{\partial F_i}{\partial X_j}(p)\right)_{i,j} < k, \qquad (2.19)$$

where $\left(\frac{\partial F_i}{\partial X_j}(p)\right)_{i,j}$ is the $k \times (n+1)$ matrix of partial derivatives $\frac{\partial F_i}{\partial X_j}(p)$. If none of the points in $Z(S)$ is singular, then $Z(S)$ can be realized as a closed complex $(n-k)$-submanifold of $\mathbb{CP}^n$. If this is the case, we say that the algebraic variety $Z(S)$ is *smooth* or *non-singular*. A consequence of Chow's theorem ([GH], page 167) is that the converse is also true, any closed complex submanifold of $\mathbb{CP}^n$ is a smooth algebraic variety. This shows that the del Pezzo surfaces can all be realized as smooth algebraic varieties.

Intersection numbers and degrees

Intersections of algebraic sets and varieties have very useful properties and form one of the main subjects of study in the field of algebraic geometry. Let us consider for example a closed complex manifold M and two complex submanifolds $V, W \subset M$ that are of complimentary dimension and intersect each other transversely. The intersection number $I(V, W)$ as in Definition 2.1 is then always positive. In fact, if V and W are defined as the smooth zero-set $\subset \mathbb{C}^n$ (or $\subset \mathbb{CP}^n$) of complex (homegeneous) polynomials, then the number $I(V, W)$ is equal to the number of simultaneous solutions to the polynomials defining V and W. This observation is in fact one of the corner stones of algebraic geometry.

One can extend the definition of the intersection number to account for intersections that are not transverse. In this definition one takes into account the multiplicity of intersection points and basically the result counts the number of transverse intersection points one gets under small perturbations of the manifolds V, W.

Based on this property of the intersection number for *smooth* algebraic varieties, we introduce the notion of degree.

Definition 2.14 ([GH], page 172)**.** Let V be an algebraic variety of dimension k in $\mathbb{CP}^n$. The *degree* of V is defined as the number of points (counted with multiplicity) in the intersection of V with a generic complex projective $(n-k)$-plane $\mathbb{CP}^{n-k}$ in $\mathbb{CP}^n$.

The degree is multiplicative with respect to intersections. For instance, let us take two algebraic varieties $V, W \subset \mathbb{CP}^4$ of dimension 3 and 2 and having degree 4 and 5, respectively. Then their intersection will generally be a non-empty 1-dimensional algebraic variety, since $Z(S) \cap Z(S') = Z(S \cup S')$. This intersection will then intersect a generic complex projective 3-plane in $4 \times 5 = 20$ points (counted with multiplicity).

Another property of the degree is that if $V \subset \mathbb{CP}^n$ is a hypersurface, i.e. the zero locus of a homogeneous polynomial F, then the degree of V is equal to the degree of F. One special case of the properties of the degree is also known as Bezout's theorem.

Theorem 2.15 (Bezout)**.** *Let f_1 and f_2 in $\mathbb{C}[x, y, z]$ be homogeneous irreducible polynomials of degree d_1 and d_2, respectively. Assume $Z(f_1) \neq Z(f_2)$. Then $\#(Z(f_1) \cap Z(f_2)) = d_1 d_2$ counted with multiplicity.*

2.2.2 Blow-ups

For a general subset $U \subset \mathbb{C}^n$ containing 0 we look at the following set,

$$\tilde{U} = \{(z, [w]) \in U \times \mathbb{CP}^{n-1} \mid z \in [w]\} = \{(z, [w]) \in U \times \mathbb{CP}^{n-1} \mid z_i w_j = z_j w_i\}.$$

One sees that $\tilde{U} - \{(0, [w])\} \simeq U - \{0\}$, since every point $z \in U - \{0\}$ determines $[w] \in \mathbb{CP}^{n-1}$ uniquely. By taking the closure on both sides we see that we get a holomorphic map $\pi : \tilde{U} \to U$ extending the identity on $U - \{0\}$. The set $\tilde{U}$ is called the *blow-up* of U at 0 and $\pi^{-1}(0) = \{0\} \times \mathbb{CP}^{n-1} \simeq \mathbb{CP}^{n-1}$. It can be seen as a subset of the tautological bundle L over $\mathbb{CP}^{n-1}$, which is a complex manifold of dimension n given by

$$L = \{(z, [w]) \in \mathbb{C}^n \times \mathbb{CP}^{n-1} \mid z \in [w]\}. \tag{2.20}$$

Now let M be a complex n-manifold. For an open set $U' \subset M$ centered around a point $p \in M$ we can find a holomorphic coordinate chart $\phi : U' \to U \subset \mathbb{C}^n$, such that $\phi(p) = (z_1(p), \ldots, z_n(p)) = 0$. Extending this chart to the blow-up of U at 0, we obtain a set $\tilde{U}'$ and a holomorphic map $\pi' : \tilde{U}' \to U'$ such that:

- $(\pi')^{-1}(p) \simeq \mathbb{P}(T_p U') \simeq \mathbb{P}(T_p M) \simeq \mathbb{CP}^{n-1}$, which is the set of complex lines in the tangent space at $p \in U'$;
- $\pi' : \tilde{U}' - (\pi')^{-1}(p) \to U' - \{p\}$ is biholomorphic.

The *blow-up* $\tilde{M}$ of M at p is then obtained by replacing U' with the set $\tilde{U}'$. What we obtain is a complex manifold $\tilde{M}$ with a natural holomorphic projection map $\pi : \tilde{M} \to M$ extending the identity on $M - \{p\}$. The set $E := \pi^{-1}(p) \simeq \mathbb{P}(T_p M) \simeq \mathbb{CP}^{n-1}$ is called the *exceptional divisor* of the blow-up. A complex manifold M is said to be the *blow-down* of another complex manifold N if N can be realized as the blow-up of M. We have the following proposition as a corollary of the Kodaira embedding theorem ([GH], pages 191-192):

Proposition 2.16. *If M is a smooth algebraic variety, then the blow-up $\tilde{M}$ of M at a point p is again a smooth algebraic variety.*

Remark 2.17. In Section 2.2.3 we will show how $\mathbb{CP}^1 \times \mathbb{CP}^1$ can be realized as a smooth quadric in $\mathbb{CP}^3$. Theorem 2.11 and Proposition 2.16 then show again that del Pezzo surfaces can be realized as smooth algebraic varieties.

Let V be a subset of M, then the *proper transform* of $\tilde{V} \subset \tilde{M}$ is given by

$$\tilde{V} = \overline{\pi^{-1}(V - \{p\})} = \overline{\pi^{-1}(V) - E}.$$

Again $\pi : \tilde{V} - E \to V - \{p\}$ is a bijection and a biholomorpism if $V - \{p\}$ is embedded in M as a complex submanifold. If V does not intersect p, $\tilde{V} = V$. If $p \in V$, and V is smooth, then one easily sees that the proper transform $\tilde{V}$ is just the blow-up of V at p. If V is singular and p is one of its singularities, $\tilde{V}$ may have a milder singularity at some points in $\tilde{V} \cap E$. In Figure 2.2 one can see the proper transform of a curve that intersects itself in the blown-up point.

Blow-ups as connected sum

When the blow-up $\tilde{M}$ of M at a point p is considered as a smooth manifold, it can be shown that $\tilde{M}$ is diffeomorphic to the smooth connected sum $M\#\overline{\mathbb{CP}}^n$, which was defined in Section 2.1.4. In the discussion below we follow parts of Section 2.4 of [Bal].

One thing we first need to prove is that $\overline{\mathbb{CP}^n}$ minus a point is diffeomorphic to the tautological line bundle L over $\mathbb{CP}^{n-1}$ (see (2.20)). We can assume without loss of generality that the point we delete in $\overline{\mathbb{CP}^n}$ is $(1 : 0 : \ldots : 0)$. The orientation of $\mathbb{CP}^n$ is reversed by taking the complex conjugate of z_0. The set we then get is given by

$$\overline{\mathbb{CP}}^n - \{(1 : 0 : \ldots : 0)\} \quad = \quad \{(\overline{z_0} : z_1 : \ldots : z_n) \,|\, z_i = 0 \text{ for not all } i \neq 0\}.$$

The map $\psi : \overline{\mathbb{CP}}^n - \{(1 : 0 : \ldots : 0)\} \to L$ defined by

$$\psi((\overline{z_0} : z_1 : \ldots : z_n)) = \left(\frac{z_0}{|z|^2}(z_1, \ldots, z_n), (z_1 : \ldots : z_n) \right) \text{ with } |z|^2 = \sum_{i=1}^{n} \overline{z_i} z_i, \tag{2.21}$$

is then well-defined and a diffeomorphism.

By deleting a closed ball in $\overline{\mathbb{CP}}^n$ we obtain the set

$$V_\epsilon = \{(\overline{z_0} : z_1 : \ldots : z_n) \,|\, |\overline{z_0}| < \epsilon |z|\}, \text{ where } |z|^2 = \sum_{i=1}^{n} \overline{z_i} z_i.$$

The image of V_ϵ under ψ is the open set

$$\tilde{U}_\epsilon = \{(z', [w]) \in L \,|\, |z'| < \epsilon\}.$$

Now $\tilde{U}_\epsilon$ can be used as the open set $\tilde{U}$ in our description of the blow-up of $0 \in \mathbb{C}^n$. We see that $\tilde{U}_\epsilon$ is diffeomorphic (even biholomorphic) to an open set $\tilde{U}'_\epsilon$ of the exceptional divisor $E = \pi^{-1}(p)$ of $\tilde{M}$. Let U'_ϵ be the corresponding neigbhourhood of p in M, for which we can make the identification

$$(\dagger) \quad U'_\epsilon - \{p\} \simeq \tilde{U}'_\epsilon - E \simeq V_\epsilon - \psi^{-1}(\{0\} \times \mathbb{CP}^{n-1}).$$

Subsequently we pick a smaller closed neighbourhood $U''_\epsilon \subset U'_\epsilon$.

The connected sum $M\#\overline{\mathbb{CP}}^n$ can now be realized as the union $(M - U''_\epsilon) \cup_{(\dagger)} V_\epsilon$, where the subscript $(\dagger)$ indicates $z' \in U'_\epsilon - U''_\epsilon$ is identified with $z \in V_\epsilon$ through $(\dagger)$. The annuli in the smooth connected sum (see Figure 2.1) are now given by $U'_\epsilon - U''_\epsilon$ and a small neighbourhood of the boundary of V_ϵ, which are both diffeomorphic to $(0, 1) \times S^{2n-1}$. The orientation preserving diffeomorphism through which the two annuli are identified is now just induced by $(\dagger)$. Since $V_\epsilon \simeq \tilde{U}'_\epsilon$, this connected sum is easily seen to be diffeomorphic to $\tilde{M}$.

Homology and cohomology for blown-up manifolds

Let us denote the i-th homology group with integral coefficients of a manifold N in short by $H_i(N)$. One can prove the following for the homology of blown-up complex manifolds.

Theorem 2.18 ([GH], page 473)). *Let M be a complex manifold of dimension n and let $\tilde{M}$ denote its blow-up at a point p. The i-th homology group of $\tilde{M}$ is given by $H_i(\tilde{M}) = H_i(M) \oplus H_i(E) \simeq H_i(M) \oplus H_i(\mathbb{CP}^{n-1})$ for $i > 0$.*

Proof. Let U' and $\tilde{U}'$ be the open sets as in the definition of the blow-up of M, which we will denote by U and $\tilde{U}$ for simplicity. We denote $M^* = M - \{p\}$, $\tilde{M}^* = \pi^{-1}M = \tilde{M} - E$, $U^* = U - \{p\}$ and $\tilde{U}^* = \pi^{-1}(U^*) = \tilde{U} - E$. Now the Mayer–Vietoris sequences of $M = M^* \cup U$ and $\tilde{M} = \tilde{M}^* \cup \tilde{U}$ are exact and are related by the pushforward of π:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_i(\tilde{U}^*) & \xrightarrow{i} & H_i(\tilde{U}) \oplus H_i(\tilde{M}^*) & \xrightarrow{j} & H_i(\tilde{M}) & \xrightarrow{h} & H_{i-1}(\tilde{U}^*) & \longrightarrow \cdots \\
& \pi_* \downarrow & & \pi_* \downarrow & & \pi_* \downarrow & & \pi_* \downarrow & \\
\cdots \longrightarrow & H_i(U^*) & \xrightarrow{i'} & H_i(U) \oplus H_i(M^*) & \xrightarrow{j'} & H_i(M) & \xrightarrow{h'} & H_{i-1}(U^*) & \longrightarrow \cdots
\end{array}$$

The sets U and $\tilde{U}$ are homotopic to p and E, respectively. Furthermore, π is a diffeomorphism from $\tilde{U}^*$ to U^* and from $\tilde{M}^*$ to M^*, so the diagram above simplifies to

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_i(\tilde{U}^*) & \xrightarrow{i} & H_i(E) \oplus H_i(M^*) & \xrightarrow{j} & H_i(\tilde{M}) & \xrightarrow{h} & H_{i-1}(\tilde{U}^*) & \longrightarrow \cdots \\
& \pi_* \downarrow \simeq & & (0,\mathrm{id}) \downarrow & & \pi_* \downarrow & & \pi_* \downarrow \simeq & \\
\cdots \longrightarrow & H_i(U^*) & \xrightarrow{i'} & 0 \oplus H_i(M^*) & \xrightarrow{j'} & H_i(M) & \xrightarrow{h'} & H_{i-1}(U^*) & \longrightarrow \cdots
\end{array}$$

We replace $H_i(\tilde{M})$ by $H_i(\tilde{M})/j(H_i(E) \oplus 0)$, $H_i(E)$ by the trivial group and i by its composition with the projection onto the second factor. The map $(0, \mathrm{id})$ then becomes an isomorphism in this new exact sequence and we can apply the 5-lemma, which gives us $H_i(\tilde{M})/j(H_i(E) \oplus 0) \simeq H_i(M)$ for $i > 0$.

If we can prove $j(e, 0) = 0 \Rightarrow e = 0$, then $j(H_i(E) \oplus 0) \simeq H_i(E)$ and

$$H_i(\tilde{M}) = H_i(M) \oplus H_i(E) \qquad \text{for } i > 0.$$

Since we know $\pi_* : H_i(\tilde{M}) \to H_i(M)$ is surjective, the exact sequence above gives us the following implications

$$\begin{aligned}
&j(e, 0) = 0 \Rightarrow (e, 0) = i(u) \Rightarrow 0 = i'(\pi_*(u)) \Rightarrow \pi_*(u) = h'(m) = h'(\pi_*(\tilde{m})) = \pi_*(h(\tilde{m})) \\
&\Rightarrow u = h(\tilde{m}) \Rightarrow (e, 0) = i(u) = i(h(\tilde{m})) = 0,
\end{aligned}$$

which proves our claim. □

Remark 2.19. According to Theorem 2.7, we have $H_i(\tilde{M}) = H_i(M \# \overline{\mathbb{CP}}^n) = H_i(M) \oplus H_i(\overline{\mathbb{CP}}^n)$ for $0 < i < n$ and $H_n(\tilde{M}) \simeq \mathbb{Z}$. Using $H_i(\overline{\mathbb{CP}}^n) = H_i(\mathbb{CP}^n) = H_i(\mathbb{CP}^{n-1})$ for $0 < i < n$ and $H_n(\mathbb{CP}^{n-1}) = 0$, we could have obtained Theorem 2.18 otherwise. In fact the proof of the more general statement in Theorem 2.7 is very similar to the proof above, although at some points it is a little bit more involved.

The blow-up of a complex surface M is diffeomorphic to $M \# \overline{\mathbb{CP}}^2$, so that Proposition 2.8 gives us $I_{\tilde{M}} = I_M \oplus I_{\overline{\mathbb{CP}}^2}$. We know $H_2(E) = \mathbb{Z}E = H_2(\overline{\mathbb{CP}}^2)$ and in Example 4.23 it will be calculated that we can pick a basis of $H^2(\overline{\mathbb{CP}}^2)$ such that the intersection form $Q_{\overline{\mathbb{CP}}^2}$ is represented by the matrix $-(1)$. This implies $I_{\overline{\mathbb{CP}}^2}(E, E) = -1$. Summarizing, we get the following proposition.

Proposition 2.20. *Let $\tilde{M}$ be the blow-up of a closed complex manifold M of dimension 2 at p and let $E \simeq \mathbb{CP}^1$ be the corresponding exceptional divisor. Then we have $I(E, \Sigma) = 0$ for all $\Sigma \in H_2(M)$ and $I(E, E) = -1$, where I is the intersection form of $\tilde{M}$ as in Definition 2.2.*

2.2.3 Quadrics in $\mathbb{CP}^3$

The discussion below concerning the Segre embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1$ and quadrics in $\mathbb{CP}^3$ is an adaptation of the section "The Quadric Surface" in Chapter 4 of [GH].

A quadric in $\mathbb{CP}^3$ is given by the zero-locus of a homogeneous polynomial of degree 2, also called a quadratic form. Such a homogeneous polynomial F is in direct correspondence with a symmetric matrix $Q = (q_{ij})$ of rank 3, i.e.:

$$F = Z \cdot QZ = \sum q_{ij} Z_i Z_j. \tag{2.22}$$

Let S be the zero-locus of the polynomial, then S is a smooth submanifold of $\mathbb{CP}^3$ if and only if Q is non-degenerate as can be seen from (2.18). Since all non-degenerate symmetric quadratic forms on $\mathbb{C}^4$ are isomorphic, any two smooth quadric surfaces in $\mathbb{CP}^3$ are biholomorphic when seen as complex manifolds. A quadric that is smooth will also be called *non-singular*.

Remark 2.21. The space of quadrics in $\mathbb{CP}^3$ is parametrized by a copy of $\mathbb{CP}^9$, hence is a smooth algebraic variety of dimension 9.

The Segre embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1$

Using the Segre embedding (see Appendix A) we get an embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1$ in $\mathbb{CP}^3$:

$$\Psi : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^3; \quad ((s_0 : s_1), (t_0 : t_1)) \mapsto (s_0 t_0 : s_0 t_1 : s_1 t_0 : s_1 t_1). \tag{2.23}$$

Its image is given by the quadric $\{Z_0 Z_3 - Z_1 Z_2 = 0\}$, which we call S_0. The symmetric matrix Q as in (2.22) is non-degenerate. From the discussion above regarding quadratic forms we deduce that *any* smooth quadric surface in $\mathbb{CP}^3$ is biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$. To make this biholomorphism explicit, we first must study the geometry of quadrics and the image of the Segre embedding more closely.

We claim that for all $s, t \in \mathbb{CP}^1$, the images of $\mathbb{CP}^1 \times \{t\}$ and $\{s\} \times \mathbb{CP}^1$ are lines in $\mathbb{CP}^3$ that intersect each other in $\Psi(s, t)$. A line in $\mathbb{CP}^3$ is given by the intersection of two hyperplanes going through the origin and corresponds to a copy of $\mathbb{CP}^1$ sitting inside $\mathbb{CP}^3$ (see Appendix A). It is easy to see that the image of $\{s\} \times \mathbb{CP}^1$ equals

$$\begin{aligned} \Psi(\{s\} \times \mathbb{CP}^1) &= \{(s_0 t_0' : s_0 t_1' : s_1 t_0' : s_1 : t_1') \in \mathbb{CP}^3 \,|\, (t_0 : t_1) \in \mathbb{CP}^1\} \\ &= \{s_1 Z_0 - s_1 Z_2 = 0\} \cap \{s_1 Z_1 - s_0 Z_3 = 0\}, \end{aligned}$$

which is a line. Similarly we have

$$\Psi(\mathbb{CP}^1 \times \{t\}) = \{t_1 Z_0 - t_0 Z_1 = 0\} \cap \{t_1 Z_2 - t_0 Z_3 = 0\}.$$

By injectivity of Ψ we see that the two lines meet in $\Psi(s, t)$.

Injectivity tells us furthermore that the lines $\Psi(\{s\} \times \mathbb{CP}^1)$ and $\Psi(\{s'\} \times \mathbb{CP}^1)$ are disjoint for $s \neq s'$. Such lines are called A-lines. The same holds for the images of $\mathbb{CP}^1 \times \{t\}$, which we call B-lines. The rules for these lines in S_0 are summarized as follows:

- Every A-line meets every B-line;
- Every two A-lines are disjoint;

- Every two B-lines are disjoint.

In fact, we can show that these are *all* the lines in S_0. Let L be a line on S_0 going through $\Psi(s,t)$ and $\Psi(s',t')$. If we can deduce that then either $s = s'$ or $t = t'$, L must necessarily be either an A-line or B-line, respectively.

The line L between $\Psi(s,t)$ and $\Psi(s',t')$ is given by

$$\{\Psi(s,t) + \lambda(\Psi(s',t') - \Psi(s,t)) \in \mathbb{CP}^3 \mid \lambda \in \mathbb{C}\}.$$

If L must lie on S_0, then the equation $Z_0Z_3 - Z_1Z_2 = 0$ implies that for all $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} 0 &= \lambda(1-\lambda)(s_0t_0s_1't_1' + s_0't_0's_1t_1 - s_0t_1s_1't_0' - s_0't_1's_1t_0) \\ &= \lambda(1-\lambda)(s_0s_1' - s_0's_1)(t_0t_1' - t_1t_0'). \end{aligned}$$

This shows that either $s_0s_1' = s_0's_1$ or $t_0t_1' = t_1t_0'$, i.e. either $s = s'$ or $t = t'$.

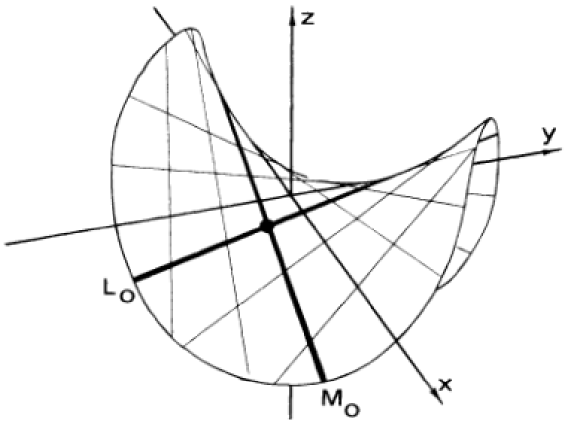

Figure 2.3: A quadric in $\mathbb{CP}^3$ ([Har]), page 14).

Non-singular quadrics

Let S be a non-singular quadric in $\mathbb{CP}^3$. Given a point $p \in S$, then its tangent space T_pS is a hyperplane in $\mathbb{CP}^3$, which is cut out by a homogeneous polynomial of degree 1. Let us recall the properties of the degree discussed after Definition 2.14. We see that $T_pS \cap S$ is an algebraic variety of degree 2 and complex dimension 1. Take $q \in T_pS \cap S$ and assume the line $\overline{pq}$ does not lie in $T_pS \cap S$. By the properties of the degree, $\overline{pq}$ either intersects S transversely in p and q, in p with multiplicity two or in q with multiplicity two. None of these options are possible, since $\overline{pq}$ is tangent to S at p and meets q. Hence $\overline{pq}$ lies in $T_pS \cap S$ and we conclude that $T_pS \cap S$ is a union of lines. Since $T_pS \cap S$ has degree 2, any generic plane in $\mathbb{CP}^3$ intersects it in two points, meaning it is a union of two lines (counted with multiplicity). These are exactly the lines on S that go through p.

There is still the option that the two lines do not meet transversely at p, i.e. $T_pS \cap S$ is just one line L. In this case T_pS would be everywhere tangent to S along L and so no other line on S can meet L (that line would then have to lie in T_pS). However, L must meet any general tangent space T_qS (any line and plane intersect in $\mathbb{CP}^3$) and this cannot be outside of S. Therefore, L intersects $T_qS \cap S$, a union of lines, contradicting our earlier assumption.

Now we can draw the analogy with the A- and B-lines in the image of the Segre embedding. We start with one line $L_0 \subset S$. We call a line L on S an A'-line if it is equal to or disjoint from L_0 and a B'-line if it meets L_0 in one point. Take two lines $L, L' \neq L_0$ on S meeting in a point. The plane they span in $\mathbb{CP}^3$ must meet L_0 in a point $p \in S$. Since L, L' are the only points in the plane that lie on S, p must be a point of either L or L'. It cannot be both, since no more than two lines can go through p. We see that one of the two lines L, L' is an A'-line and the other a B'-line. Conversely, let $L \neq L_0$ be an A'-line and L' a B'-line, then by similar reasoning the plane spanned by L_0 and L' must meet L in one point, which by definition of an A'-line cannot be L_0.

We see that through any point in S there passes an A'-line and a B'-line and any two such lines meet if and only if they are of different type, i.e. the properties of these families of lines match those of the A- and B-lines we defined on $\Psi(\mathbb{CP}^1 \times \mathbb{CP}^1)$. From this we can construct the explicit biholomorphism $S \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$. By mapping each A'-line and B'-line on S to an A-line and B-line on $\Psi(\mathbb{CP}^1 \times \mathbb{CP}^1)$, the two families of lines on S are parametrized by a point in $\mathbb{CP}^1$. Any point $q \in S$ is then mapped to the point $\Psi(s, t)$ of intersection of the A-line and B-line, corresponding to the A'-line and B'-line intersecting in q.

Blow-ups of quadrics in $\mathbb{CP}^3$

Now that we know that every non-singular quadric $S \subset \mathbb{CP}^3$ is biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ and we have a description of all the lines that lie on such a quadric, we would like to establish the claim that blowing up $\mathbb{CP}^1 \times \mathbb{CP}^1$ is the same as blowing up $\mathbb{CP}^2$ at two points. Equivalently, this means that any non-singular quadric S blown-up at one point is biholomorphic to a generic hyperplane $H \subset \mathbb{CP}^3$ blown-up at two points ($\mathbb{CP}^2$ can be embedded as a general hyperplane in $\mathbb{CP}^3$).

Take H any hyperplane in $\mathbb{CP}^3$ and S a smooth quadric in $\mathbb{CP}^3$. Let $p \in S$ be such that H is not tangent to S at p. We can then define a projection map $\pi_p : S - \{p\} \to H$, by letting $\pi_p(s)$ be the point in H that intersects the line $\overline{ps}$. This point exists, since any line and plane should meet at least once in $\mathbb{CP}^3$, and it is unique since H is not tangent to p. By blowing-up S at p we can extend the map π_p. The result is a holomorphic map $\tilde{\pi}_p : \tilde{S} \to H$, which on the exceptional divisor $E \subset \tilde{S}$ is defined as follows. For $r \in E \simeq \mathbb{P}(T_pS)$, let $\overline{r} \subset T_pS$ be the line it defines in $\mathbb{CP}^3$. The point $\tilde{\pi}_p(r)$ is then just the point of intersection of H with $\tilde{r}$.

Now by a familiar line of reasoning every line in $\mathbb{CP}^3$ must meet S in two points or must be tangent to it. If we take $q \in H$, then the line $\overline{pq}$ either intersects $S - \{p\}$ in one point or lies in the tangent space T_pS, which means it defines an element in E. From this we see that the map $\tilde{\pi}_p : \tilde{S} \to H$ is surjective. All lines in the tangent space have different points of intersection with H and two points $s, s' \in S$ are mapped to the same point if and only if they lie on the same line through p, which can only happen if $s, s' \in S \cap T_p(S)$. Therefore injectivity of $\tilde{\pi}_p$ fails only on the proper transforms of the A'-line and B'-line passing through p. Denote these lines L_1 and L_2 and their corresponding images q_1 and q_2. The inverse images of q_1 and q_2 under $\tilde{\pi}_p$ are then the proper transforms $\tilde{L}_1, \tilde{L}_2$.

We see $\tilde{\pi}_p : \tilde{S} - \tilde{L}_1 - \tilde{L}_2 \to H - \{q_1, q_2\}$ is a biholomorphism. The proper transforms $\tilde{L}_1, \tilde{L}_2$ are biholomorphic to $\mathbb{CP}^1$, since $L_1, L_2 \subset$ were both embedded as a copy of $\mathbb{CP}^1$ in $\mathbb{CP}^3$. If $\tilde{L}_1, \tilde{L}_2$ are mapped to the exceptional divisors E_1, E_2 associated to the blow-up $\tilde{H}$ of H in q_1 and q_2, we obtain a biholomorphism $\psi : \tilde{S} \to \tilde{H}$ that extends $\tilde{\pi}_p$.

Notice that under these identifications any line in S that meets L_1 (a B'-line) corresponds to a unique line through q_1 and vice versa. Similarly, there is such a correspondence between the A'-lines and lines through q_2. The line $L_0 = \overline{q_1 q_2} \subset H$ runs through both q_1 and q_2 and on S there are no lines unequal to L_1, L_2 going through both lines. This means that $\tilde{\pi}_q^{-1}(L_0)$ must be the exceptional divisor E.

The relations for the blow-up $\sigma : \tilde{S} \to S$ at p and the blow-up $\tau : \tilde{H} \to H$ at q_1 and q_2, can be summarized through the following commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow[\simeq]{\psi} & \tilde{H} \\ {\scriptstyle\sigma}\downarrow & \searrow^{\tilde{\pi}_p} & \downarrow{\scriptstyle\tau} \\ S & \xrightarrow[\pi_p]{} & H \simeq \mathbb{CP}^2, \end{array} \tag{2.24}$$

with the relations $E = \tilde{\pi}_p^{-1}(L_0) = \psi^{-1}(\tilde{L}_0) = \sigma^{-1}(p)$ and

$$\tilde{L}_i = \sigma^{-1}(L_i) = \tilde{\pi}_p^{-1}(q_i) = \psi^{-1}(E_i) = \psi(\tau^{-1}(q_i)) \text{ for } i = 1, 2.$$

Remark 2.22. Extending this discussion to higher del Pezzo surfaces, one needs to take into account the condition that the blown-up points are in general position (see Remark 2.12). If one wants to study all the algebraic properties of del Pezzo surfaces, one needs more tools from the field of algebraic geometry.

The surfaces $\mathbb{CP}^1 \times \mathbb{CP}^1$ and dP_l with $0 \leq l \leq 3$ can also be described as a torus fibration over a convex polytope. An algebraic variety realized in such a way is called a toric variety and the associated convex polytope is called a toric diagram. The blow-ups of del Pezzo surfaces then correspond to manipulations of the toric diagram by means of cutting the corners. Using these diagrams one can show in another way that the blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ is biholomorphic to the blow-up of $\mathbb{CP}^2$ in 2 points. A description of del Pezzo surfaces using toric diagrams can be found in [INV].

Chapter 3

Yang–Mills instantons on four-manifolds

This chapter will consist of the theory describing the moduli space of anti-self-dual connections in four-dimensional Yang–Mills theory. These connections correspond to a class of instantons and are particulary interesting when studied on $SU(2)$ bundles. Following Chapters 2, 4 and 6 of [DK] we will start with basic theory on vector bundles and quickly move on to the ASD moduli space and its correspondence with stable bundles on Kähler surfaces. The ASD moduli space for $SU(2)$ bundles over $S^2 \times S^2$ is discussed using theory from [Sob] and Section 2.2.3. Due to the scope of the thesis, we will state a lot of results without proof. Most of the results from [DK] that we present here, were covered in the reading seminar organized by Gil Cavalcanti and Stefan Vandoren. All the statements and their proofs can be found back in the two main references above.

3.1 Bundles and connections

Let E be a smooth vector bundle of rank n over a smooth manifold M and let $\mathbb{A}$ be underlying field of the vector bundle. For this bundle we have a smooth surjection $\pi : E \to M$ and a cover $\{U_\alpha\}$ of M with diffeomorphisms

$$\varphi_\alpha : U_\alpha \times \mathbb{A}^n \to \pi^{-1}(U), \tag{3.1}$$

satisfying $\pi \circ \varphi_\alpha = \mathrm{id}|_{U_\alpha}$. The map φ_α is called a *trivialization* over U_α. On overlapping neighbourhoods one has

$$\varphi_\beta^{-1} \circ \varphi_\alpha : (U_\alpha \cap U_\beta) \times \mathbb{A}^n \to (U_\alpha \cap U_\beta) \times \mathbb{A}^n \,;\, (x, v) \mapsto (x, g_{\alpha\beta}(x)v), \tag{3.2}$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{A})$ is smooth and is called a *transition function.* The collection of transition functions $g_{\alpha\beta}$ associated to the cover $\{U_\alpha\}$ satisfy the cocycle condition:

$$g_{\alpha\alpha} = \mathrm{id} \quad \text{and} \quad g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = \mathrm{id}, \tag{3.3}$$

whenever the intersection $U_\alpha \cap U_\beta \cap U_\gamma$ is non-empty.

Let G be a Lie group. A vector bundle E over M is said to have *structure group* G, if there is a covering $\{U_\alpha\}$ of M and trivializations over that cover such that the transition functions $g_{\alpha\beta}$ take values in a representation $\rho : G \to GL(n, \mathbb{A})$ of the Lie group G on $\mathbb{A}^n$. If E has structure group G, we will often name it a G *bundle* and we will denote the transition functions accordingly as $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$.

For real vector bundles of rank n we can consider G to be a subgroup of $GL(n,\mathbb{R})$, such as $SO(n)$, $Sp(n)$ or $SL(n,\mathbb{R})$. For complex vector bundles of rank n we can consider subgroups of $GL(n,\mathbb{C})$, like $U(n)$ and $SU(n)$. Recall that associated to the Lie group G there is a Lie algebra $\mathfrak{g}$. From E we can construct a new bundle $\mathfrak{g}_E$ having structure group G, with fibers isomorphic to $\mathfrak{g}$. The group G acts on the fibers in the adjoint representation, meaning $\mathfrak{g}_E$ is a subbundle of $\operatorname{End} E = E \otimes E^*$. Let us denote two spaces of bundle valued forms which will often appear in this chapter:

$$\Omega^p_M(E) = \Gamma(\Lambda^p T^*X \otimes E), \quad \Omega^p_M(\mathfrak{g}_E) = \Gamma(\Lambda^p T^*X \otimes \mathfrak{g}_E). \tag{3.4}$$

Central to the construction of Yang–Mills instantons are connections on E.

Definition 3.1. A *connection* on a smooth vector bundle $E \to M$ is a linear map

$$\nabla_A : \Omega^0_M(E) \to \Omega^1_M(E), \tag{3.5}$$

satisfying the Leibniz rule:

$$\nabla_A(f \cdot s) = f\nabla_A s + df \otimes s, \quad \forall s \in \Omega^0_M(E) = \Gamma(E) \text{ and } \forall f \in C^\infty(M).$$

When there is no confusion, we will denote ∇_A as A. The space of all connections is denoted as $\mathscr{A}$.

The sum $\nabla_A + a$ is again a connection for $a \in \Omega^1_M(\text{End}_E)$, where a acts on E by the linear extension to $\Omega^1_M(\text{End}E)$ of the contraction

$$\Omega^0_M(E) \times \Omega^1_M(\text{End}E) \to \Omega^1_M(E)\,;\ (s, \alpha \otimes a) \mapsto \alpha \otimes a(s).$$

By an application of the Leibniz rule one can show that the difference $\nabla_A - \nabla_{A'}$ of two connections is an element of $\Omega^1(\text{End}E)$.

If E has structure group G, we have associated local trivializations $\tau : U \times \mathbb{A}^n \to E|_U$ which are compatible with the structure group. We put some extra structure on the connections by demanding that for any such local trivialization $\nabla_A|_U = \tau^*(\nabla_A)$ is given by

$$\nabla_A|_U = d + A^\tau, \tag{3.6}$$

where A^τ is a $\mathfrak{g}$-valued 1-form over U called the *connection form* and d is the *product connection* on the trivial bundle $\mathbb{A}^n \times U \to U$. The product connection is given by letting the exterior derivative d act on the components of sections in this bundle.

Convention 3.2. Whenever E has structure group G, we assume the local trivializations are compatible with the structure group, i.e. the transition functions take values in G. A connection A on a G bundle satisfies (3.6) in such a local trivialization.

As a consequence of this convention, the difference $\nabla_A - \nabla_{A'}$ of two connections is now an element of $\Omega^1(\mathfrak{g}_E)$. In conclusion, the space $\mathscr{A}$ of all connections on a G bundle E is an infinite-dimensional affine space, modeled on $\Omega^1_M(\mathfrak{g}_E)$.

The form $\nabla_A|_U$ takes in (3.6) is obviously dependent on the trivialization chosen over U, sometimes also referred to as a *choice of gauge*. What we would like to know is how the definitions above change under a different choice of trivilization and how to achieve different triviliziations.

A different trivialization is achieved by choosing a different set of sections spanning E over U which we used to construct the isomorphism $U \times \mathbb{A}^n \simeq E|_U$. The transformation relating these sections must be compatible with the structure group. This transformation must therefore be a section of the bundle $U \times G \to U$. We can glue together such sections over M to obtain an automorphism $u : E \to E$, which covers the identity on M, i.e. it acts fiberwise. Futhermore u should preserve the structure of the fibers, so that $u(x) : E_x \to E_x$ must be an element in G.

Definition 3.3. The set of automorphisms $u : E \to E$ covering the identity and preserving the structure on the fibers forms a group $\mathscr{G}$ called the *gauge group* of E.[1]

By applying the exponential map $\exp : \mathfrak{g} \to G$ pointwise to sections of $\mathfrak{g}_E$, we obtain a map $\exp : \Omega^0(\mathfrak{g}_E) \to \mathscr{G}$. The gauge group $\mathscr{G}$ acts on connections $A \in \mathscr{A}$ through the rule

$$\nabla_{u(A)} s = u \nabla_A (u^{-1} s) \qquad \text{for } s \in \Gamma(E). \tag{3.7}$$

Writing out the right-hand side gives

$$\nabla_{u(A)} = \nabla_A - (\nabla_A u) u^{-1} \quad \text{or noted otherwise} \quad u(A) = A - (\nabla_A u) u^{-1}, \tag{3.8}$$

where ∇_A acts on u by seeing it as a section of $\operatorname{End} E = E \otimes E^*$. This equivalence of (3.7) and (3.8) shows that we can view $u(A)$ either as a new connection on E or as the effect of changing the local trivializations of E.

A change of trivialization τ of E over U can be described by an automorphism u of the trivial bundle $U \times \mathbb{C}^n \to U$ taking values in G. If τ is a trivialization of E over U, then $(u\tau)$ is a new trivialization and $\nabla_A|_U = (u\tau)^*(\nabla_A)$ takes the form

$$\nabla_A = d + A^{u\tau},$$

where

$$\begin{aligned} A^{u\tau} = u(A^\tau) &= A^\tau - \{(d + [A^\tau,])u\} u^{-1} \\ &= A^\tau - \{du + A^\tau u - u A^\tau\} u^{-1} \\ &= u A^\tau u^{-1} - (du) u^{-1}. \end{aligned} \tag{3.9}$$

By recognizing the new connection as $\nabla_A|_U = u(d + A^\tau)u^{-1}$, we would have obtained this result directly from (3.7). The reason why we demanded the connections A on a G bundle should satisfy (3.6), is that $A^{u\tau}$ in (3.9) now again is a $\mathfrak{g}$-valued 1-form.

3.1.1 Exterior derivatives and curvature

In the definition of the connection $\nabla_A : \Omega^0_M(E) \to \Omega^1_M(E)$ we used the exterior derivative $d : \Omega^0(M) \to \Omega^1(M)$. The ordinary de Rham complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M),$$

can be extended such that we obtain exterior derivatives

$$d_A : \Omega^p_M(E) \to \Omega^{p+1}_M(E), \tag{3.10}$$

which are uniquely determined by the properties:

i) $d_A = \nabla_A$ on $\Omega^0_M(E)$;

ii) $d_A(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^p \omega \wedge d_A \theta$, for $\omega \in \Omega^p(M), \theta \in \Omega^q_M(E)$.

The operators ∇_A and d_A will be used interchangably from here on. Using this construction we can also extend the operator ∇_A as defined on $\operatorname{End} E$ and the subbundle $\mathfrak{g}_E$.

[1] This definition 3.3 should not be confused with the definition of the gauge group in physics, which in this formalism corresponds to the structure group G.

Unlike the exterior derivative, d_A does not necessarily square to zero. However, using the Leibniz rule one has $d_A d_A(fs) = f d_A d_A s$ for $f \in C^\infty(M)$ and $s \in \Omega^0_M(E)$. Therefore $d_A d_A$ acts as multiplication on sections s with a $\mathfrak{g}$-valued 2-form. This 2-form $F_A \in \Omega^2_M(\mathfrak{g}_E)$ is called the *curvature*, which is then defined as

$$d_A d_A s = F_A s. \tag{3.11}$$

Under the addition of $a \in \Omega^1_M(\mathfrak{g}_E)$ we have

$$F_{A+a} = F_A + d_A a + a \wedge a. \tag{3.12}$$

In a local trivialization $\tau : U \times \mathbb{C}^n \to E|_U$ the curvature is expressed as

$$F^\tau_A = dA^\tau + A^\tau \wedge A^\tau, \tag{3.13}$$

where the wedge product in the second term is a combination of the wedge product between 1-forms and multiplication in $\mathfrak{g}_E$. Under gauge transformations of the connection A we get

$$F_{u(A)} = u F_A u^{-1}. \tag{3.14}$$

One sees that $\mathscr{G}$ preserves connections with curvature zero, which we call *flat* connections. The last identity we note is the *Bianchi identity*

$$d_A F_A = 0. \tag{3.15}$$

Now suppose M is an oriented, Riemanninan four-manifold. Recall from (2.11) that we had the splitting $\Omega^2(M) = \Omega^+(M) \oplus \Omega^-(M)$. This induces a splitting of the curvature into a self-dual part and anti-self-dual part

$$F_A = F^+_A + F^-_A \in \Omega^+_M(\mathfrak{g}_E) \oplus \Omega^-_M(\mathfrak{g}_E), \tag{3.16}$$

where $\Omega^\pm_M(\mathfrak{g}_E) = \Gamma(\Lambda^\pm \otimes \mathfrak{g}_E)$.

Definition 3.4. A connection A on a G bundle over an oriented, Riemannian four-manifold is *anti-self-dual* (ASD) if $F^+_A = 0$ and *self-dual* (SD) if $F^-_A = 0$.

Remark 3.5. Self-duality and anti-self-duality are a conformally invariant property of the connection due to Proposition 2.5. By (3.14) these conditions are preserved under gauge transformations. Furthermore, self-dual and anti-self-dual connections get interchanged if we reverse the orientation of M.

3.1.2 Adjoints, inner-products and norms

Let E be a vector bundle over an oriented, Riemannian manifold M with a Hermitian or Euclidean metric on the fibers. This means that the representation of the structure group in the definition of the transition functions lies in a subgroup of respectively $U(n)$ or $SO(n)$. On $\Omega^p_M(E)$ and $\Omega^p_M(\mathfrak{g}_E)$ we then have the inner products

$$\langle \phi, \varphi \rangle = \int_M (\phi, \varphi) d\mu \quad \text{for } \phi, \varphi \in \Omega^p_M(E), \tag{3.17}$$

and

$$\langle a, b \rangle = \int_M (a, b) d\mu \quad \text{for } a, b \in \Omega^p_M(\mathfrak{g}_E), \tag{3.18}$$

where $d\mu$ is the Riemannian volume form on M and $(\cdot,\cdot)$ denotes the inner-product on the fibers of the bundles $\Lambda^p T^*M \otimes E$ and $\Lambda^p T^*M \otimes \mathfrak{g}_E$. We have formal adjoints $d_A^* : \Omega_M^p(E) \to \Omega_M^{p-1}(E)$ and $d_A^* : \Omega_M^p(\mathfrak{g}_E) \to \Omega_M^{p-1}(\mathfrak{g}_E)$, defined through

$$\langle d_A\phi, \varphi\rangle = \langle \phi, d_A^*\varphi\rangle \quad \text{and} \quad \langle d_A a, b\rangle = \langle a, d_A^* b\rangle.$$

In this definition of d_A^* at least one of the forms should have compact support. We also define the following L^2-norms:

$$\|\varphi\| = \langle \varphi, \varphi\rangle^{1/2}, \quad \varphi \in \Omega_M^p(E), \tag{3.19}$$

$$\|a\| = \langle a, a\rangle^{1/2}, \quad a \in \Omega_M^p(\mathfrak{g}_E). \tag{3.20}$$

Remark 3.6. With respect to the L^2-norms (3.19) and (3.20), the splitting in (3.16) is orthogonal, since the splitting $\Lambda^2 TM = \Lambda^+ \oplus \Lambda^-$ is orthogonal with respect to the inner product on the fibers of $\Lambda^2 TM$.

3.2 Chern–Weil theory for vector bundles

A Hermitian vector bundle over a smooth manifold M has structure group $U(r)$. For $r = 1$ we can construct such a $U(1)$ bundle $L \to M$, which is called a Hermitian *line* bundle over M. Let A be a connection on L. According to the theory in the previous section the curvature F_A is a purely-imaginary 2-form since $\mathfrak{g} = i\mathbb{R}$ for a $U(1)$ bundle, which we write as $-2\pi i\phi$. According to (3.13), the curvature can be written in a local trivialization as $F = dA$, where A now is a purely imaginary local 1-form. The Bianchi identity (3.15) now tells us that $dF = 0$ in any local trivialization, hence the form ϕ is globally closed and defines a de Rham cohomology class $[\phi] \in H^2(M, \mathbb{R})$.

Furthermore, (3.12) tells us that a second $A' = A + a$ with $a \in \Omega^2(\mathfrak{g}_L)$ has curvature $F' = F + d_A a$. The term $d_A a$ now corresponds to a globally exact form and hence $[\phi'] = [\phi]$. This cohomology class therefore only depends on the bundle L and Chern–Weil theory tells us that this cohomology class equals the first Chern class $c_1(L)$ of this bundle. This is a topological invariant associated to the bundle L and all line bundles up to isomorphism are classified by this invariant. In fact more is known: $c_1(L)$ lies in the integral cohomology $H^2(M, \mathbb{Z})$ of M (see pages 63-64 of [MK] for a proof). For general complex vector bundles E, one can extend the previous construction to the cohomology class of $\frac{i}{2\pi}\mathrm{Tr}(F_A)$, which then equals the first Chern class $c_1(E)$ of this bundle.

Extending $\mathrm{Tr}(F_A)$ to higher forms and their cohomology classes, we can consider the 4-form $\mathrm{Tr}(F_A^2)$ for a connection A on a Hermitian bundle E, also called a *unitary connection*. The relation

$$\mathrm{Tr}(F_{A+a}^2) - \mathrm{Tr}(F_A^2) = d\left(\mathrm{Tr}(a \wedge d_A a + \frac{2}{3} a \wedge a \wedge a)\right),$$

shows the cohomology class only depends on the isomorphism class of E. Using Chern–Weil theory one can relate the cohomology class to the characteristic classes of E:

$$\frac{1}{8\pi^2}[\mathrm{Tr}(F_A^2)] = (c_2(E) - \frac{1}{2}c_1(E)^2) \in H^4(M, \mathbb{Z}). \tag{3.21}$$

For $G = SU(r)$ the generators of the Lie-algebra are anti-Hermitian traceless matrices, hence $c_1(E) = \mathrm{Tr}(F_A)/2\pi = 0$. So $\frac{1}{8\pi^2}\mathrm{Tr}(F_A^2)$ represents the cohomology class $c_2(E)$. If M is a closed, oriented, Riemannian four-manifold, $H^4(M, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. Then we can say $c_2(E) = k$ for some $k \in \mathbb{Z}$ and this number is defined as the *topological charge* of the connection A. In the literature this number is often also called the instanton number or winding number.

Yang–Mills theory is concerned with connections A on a $SU(r)$ bundle over oriented, Riemannian four-manifolds M. A central object in this theory is the Yang–Mills functional, which is given by the square of the L^2-norm of the curvature (see equation (3.20)):

$$S(A) = \|F_A\|^2 = \int_M |F_A|^2 d\mu = \int_M |F_A^-|^2 d\mu + \int_M |F_A^+|^2 d\mu, \tag{3.22}$$

where the last identity follows from the fact that the SD/ASD splitting as in (3.16) is orthogonal according to Remark 3.6.

Definition 3.7. A saddle point of the Yang–Mills functional (3.22) for which the functional is finite is called a *Yang–Mills instanton*.

A general saddle point of the Yang–Mills functional satisfies the corresponding Euler-Langrange equations. These equations can be obtained by looking at the variation $\delta S = S(A+a) - S(A)$, where $a \in \Omega^1_M(E)$, and then impose the condition that all terms linear in a should vanish. Writing out δS gives us

$$\begin{aligned} S(A+a) - S(A) &= 2\int_M (F_A, d_A a) d\mu + O(\|a\|^2) \\ &= 2\int_M (d_A^* F_A, a) d\mu + O(\|a\|^2), \end{aligned}$$

where d_A^* is the adjoint of d_A. The Euler-Lagrange equations then are seen to be given by

$$d_A^* F_A = 0. \tag{3.23}$$

According the properties for d_A^*, the equation above is equivalent to $\star d_A \star F_A = 0$. Using the Bianchi identity in (3.15) it is seen that the SD/ASD connections are saddle points of the Yang–Mills functional.

Now let us assume M is closed. Then (3.22) is finite, so that the SD/ASD connections correspond to a class of Yang–Mills instantons. If $c_2(E) > 0$, the ASD connections are in fact minima of the Yang–Mills functional as can be explained by the following observation. For $G = SU(r)$ the fibers of $\mathfrak{g}_E$ are formed the Lie-algebra $\mathfrak{su}(n)$ and the norm on this vector space is given by $|\xi^2| = -\mathrm{Tr}(\xi^2)$. Furthermore, for the fibers of $\Lambda^2 T^* M$ we have

$$\alpha \wedge \alpha = (|\alpha^+|^2 - |\alpha^-|^2) d\mu,$$

where α^+ is the self-dual and α^- the anti-self-dual part of $\alpha = \alpha^+ + \alpha^- \in \Lambda^2 T^* M$. Now we combine these two facts and (3.21) to obtain the following identity

$$8\pi^2 c_2(E) = \int_M \mathrm{Tr}(F_A \wedge F_A) = \int_M (|F_A^-|^2 - |F_A^+|^2) d\mu \in 8\pi^2 \mathbb{Z}. \tag{3.24}$$

Plugging this in (3.22) we get

$$\|F_A\|^2 = 8\pi^2 c_2(E) + 2\int_M |F_A^+|^2 d\mu.$$

Hence $\|F_A\|^2$ is minimized with minimal value $8\pi^2 c_2(E)$ if and only if $F_A^+ = 0$, i.e when A is an ASD connection.

Remark 3.8. Notice that by (3.24), SD connections are not possible on $SU(r)$ bundles E for which $c_2(E) > 0$. Similarly ASD connections do not exist if $c_2(E) < 0$ and in that case the SD connections form minima of the Yang–Mills functional. If $c_2(E) = 0$ the only SD and ASD connections are the flat connections and conversely flat connections only exist when $c_2(E) = 0$. Proposition 2.2.3 of [DK] states that the gauge equivalence classes of flat-connections are in one-to-one correspondence with conjugacy classes of representations $\pi_1(M) \to SU(r)$. This means that if M is simply connected, all flat connections are gauge equivalent to the product connection, i.e. we can find a coordinate chart of M and corresponding trivilizations τ on each coordinate patch, such that A^τ as in (3.6) equals zero.

3.3 The ASD moduli space

In this section we assume E is a G bundle over a closed, oriented, Riemannian four-manifold M. Let $\mathscr{A}$ be the space of connections on E and $\mathscr{G}$ be the space of gauge transformations as in Definition 3.3. Define $\mathscr{B}$ to be the quotient space

$$\mathscr{B} = \mathscr{A}/\mathscr{G}, \tag{3.25}$$

with quotient topology and where $g \in \mathscr{G}$ acts on a connection A through (3.8). Denote $[A] \in B$ for the gauge equivalence class of A. Using the L^2 norm on $\Omega^1(\mathfrak{g}_E)$ (see (3.20)), we have the L^2 metric $\|A - B\|$ and the expression

$$d([A],[B]) = \inf_{g \in \mathscr{G}} \|A - g(B)\|$$

defines a metric on $\mathscr{B}$ ([DK], Lemma 4.2.4).

Let $A \in \mathscr{A}$ and $\epsilon > 0$, then we define the *local slice* of A by

$$T_{A,\epsilon} = \{a \in \Omega^1(\mathfrak{g}_E) \,|\, d_A^* a = 0, \, \|a\|_{L^2} \leq \epsilon\}. \tag{3.26}$$

The isotropy group Γ_A of A is defined as

$$\Gamma_A = \{u \in \mathscr{G} \,|\, u(A) = A\}. \tag{3.27}$$

Proposition 4.2.9 from [DK] tells us that for ϵ small enough a neigbourhood of $[A] \in \mathscr{B}$ is homeomorphic to $T_{A,\epsilon}/\Gamma_A$.

3.3.1 Reducible and irreducible connections

The condition $u(A) = A$ in (3.27) means $d_A u = 0$, which is a partial differential equation for $u : E \to E$. The solution to this equation is uniquely determined by its value in G at a point $x \in M$. An important remark here is that some values don't have a solution to this equation. This shows that in fact $\Gamma_A \subset G \subset \operatorname{Aut} E_x$, so Γ_A is finite-dimensional unlike $\mathscr{G}$. The group Γ_A also has a more geometric interpretation in terms of the holonomy associated to the connection A.

For a path $\alpha : [0,1] \to M$ the holonomy T_α is defined as follows. We consider the connection $\alpha^*(A)$ on the bundle $\alpha^*(E) \to [0,1]$ and look at constant covariant sections s of this bundle:

$$\alpha^*(\nabla_A)(s(\alpha(t))) = 0.$$

Such a section is uniquely determined by its value at 0, i.e a vector $x \in E_{\alpha(0)}$. The vector $T_\alpha(x)$ is defined as the value of this section at 1, which is an element of $(\alpha^*(E))_1 = E_{\gamma(1)}$. In sum, we obtain a map

$$T_\alpha : E_{\alpha(0)} \to E_{\alpha(1)}.$$

For closed loops γ based at x the holonomy map T_γ must be an element of $G \subset \operatorname{Aut} E_x$, since the connection A was modeled on $\Omega^1(\mathfrak{g}_E)$. This shows that the holonomy of loops based at x induced by A forms a subgroup of G, which we call H_A.

Lemma 4.2.8 of [DK] states that under the correspondences made, Γ_A equals the centralizer of H_A in G. We say that a connection is *reducible* if H_A is a proper subgroup of G and *irreducible* if that is not the case. For $G = SU(2)$ this means that if $\Gamma_A = C(G)$, we have $H_A = G$. This first condition indicates that the isotropy group is minimal for irreducible connections and this is how we define the notion of irreducibility in the general sense. Write $\mathscr{A}^*$ for all irreducible connections, i.e

$$\mathscr{A}^* = \{A \in \mathscr{A} \,|\, \Gamma_A = C(G)\},$$

and $\mathscr{B}^* \subset \mathscr{B}$ for the quotient $\mathscr{A}^*/\mathscr{G}$.

For $SU(2)$ bundles over simply connected M, reducible connections with $\Gamma_A = S^1$ (or equivalently $H_A = S^1$) correspond to a splitting $E = L \oplus L^{-1}$, where L is a Hermitian line bundle unequal to the trivial line bundle. A necessary and sufficient condition for such a spliting is $c_2(E) = -Q(c_1(L), c_1(L))$ where Q is the intersection form and $c_1(L) \neq 0$. A connection on E reduces to one on L and this connection is ASD precisely when $c_1(L)$ is represented by an anti-self-dual 2-form. For simply connected M this connection is unique modulo gauge transformations. The following statement summarizes this result.

Proposition 3.9 ([DK], Proposition 4.2.15)**.** *Let E be an $SU(2)$ bundle over a closed, simply connected, oriented, Riemannian four-manifold M with $c_2(E) > 0$. The gauge equivalence classes of reducible ASD-connections on E, with holonomy group S^1, are in one-to-one corresponce with pairs $\pm c$, where c is a non-zero class in $H^2(M, \mathbb{Z})$ represented by an anti-self-dual 2-form and satisfies $c_2(E) = -Q(c, c)$, where Q is the intersection form.*

Remark 3.10. We take pairs, since the spltting $E = L \oplus L^{-1}$ is symmetric and $c_1(L^{-1}) = -c_1(L)$. For simply connected manifolds, the only other possible reducible connections on $SU(2)$ bundles are the flat connections, which have the trivial group as holonomy group (or equivalently $\Gamma_A = G$) and are all gauge equivalent to the product connection. However, these connections only occur when $c_2(E) = 0$ and all ASD connections are then flat (see Remark 3.8). For $c_2(E) < 0$ there are no ASD connections, but Proposition 3.9 then applies to *SD* connections.

3.3.2 Local models using elliptic theory

We write $(\cdot)^+$ for the projection $\Omega^2(\mathfrak{g}_E) \to \Omega^+(\mathfrak{g}_E)$. The operator

$$d_A^+ : \Omega^1_M(\mathfrak{g}_E) \to \Omega^+(\mathfrak{g}_E)\,;\; d_A^+ a = (d_A a)^+, \tag{3.28}$$

appears in a sequence of bundles and differentials, known as the *instanton complex*,

$$0 \longrightarrow \Omega^0_M(\mathfrak{g}_E) \xrightarrow{d_A} \Omega^1_M(\mathfrak{g}_E) \xrightarrow{d_A^+} \Omega^+_M(\mathfrak{g}_E) \longrightarrow 0. \tag{3.29}$$

This is an elliptic complex whenever A is an ASD connection, since $d_A^+ \circ d_A = F_A^+ = 0$ and the operator

$$\delta_A = d_A^* \oplus d_A^+ : \Omega^1_M(\mathfrak{g}_E) \to \Omega^0_M(\mathfrak{g}_E) \oplus \Omega^+_M(\mathfrak{g}_E), \tag{3.30}$$

is elliptic for any connection A ([DK], page 137).

We assume A is ASD in what follows. By using Hodge theory for the elliptic complex in (3.29) we get decompositions

$$\Omega^1(\mathfrak{g}_E) = \operatorname{Im} d_A \oplus \ker d_A^*, \quad \Omega^0(\mathfrak{g}_E) = \ker d_A \oplus \operatorname{Im} d_A^*, \quad \Omega^+(\mathfrak{g}_E) = \ker d_A^* \oplus \operatorname{Im} d_A^+. \tag{3.31}$$

The cohomology groups H^0_A, H^1_A and H^2_A associated to the complex are finite, where $H^0_A = \ker d_A$ is the Lie algebra of Γ_A and there are natural isomorphisms

$$H^1_A = \frac{\ker d^+_A}{\operatorname{Im} d_A} \simeq \ker d^+_A \cap \ker d^*_A = \ker \delta_A, \tag{3.32}$$

$$H^2_A = \operatorname{coker} d^+_A \simeq \ker d^*_A \subset \Omega^+_M(\mathfrak{g}_E). \tag{3.33}$$

Let us denote $\mathcal{M} = \mathcal{M}_E \subset \mathscr{B}$ for the space of gauge equivalence classes of anti-self-dual connections on a G bundle E, in short called the *ASD moduli space*. Sometimes it is denoted as $\mathcal{M}_E(g)$ to express the dependence of this space on the metric put on M. The claim is that $\mathcal{M}_E$ is finite-dimensional and a local model of this space can be given in terms of H^1_A, H^2_A and Γ_A.

Proposition 3.11 (Proposition 4.2.23, [DK]). *If A is an ASD connection over M, a neighbourhood of $[A]$ in $\mathcal{M}_E$ is modeled on a quotient $f^{-1}(0)/\Gamma_A$, where*

$$f : H^1_A \to H^2_A$$

is a Γ_A-equivariant map.

In the proposition above H^1_A represents the linearization of the ASD equations modulo $\mathscr{G}$ and the map f corresponds to a specific realization of this linearized quotient. From (3.31) it can be read off that $\operatorname{coker} d^*_A \simeq \ker d_A \subset \Omega^0_M(\mathfrak{g}_E)$, so that the index of δ_A equals minus the Euler-characteristic of the instanton complex:

$$\operatorname{ind} \delta_A = \dim H^1_A - \dim H^0_A - \dim H^2_A.$$

The number $s := \operatorname{ind} \delta_A$ is called the *formal dimension* of the moduli space. When we consider f as in Proposition 3.11, the set of points in $f^{-1}(0)$ which are both regular points for f and represent free Γ_A orbits, form a manifold of dimension s. The Atiyah-Singer index theorem relates s to topological data associated to E and M:

$$s = \operatorname{ind} \delta_A = a(G)\kappa(E) - \dim G(1 - b^1(M) + b^+(M)), \tag{3.34}$$

where $a(G)$ and $\kappa(E)$ are integers dependent on the structure group G and the bundle E. For $SU(r)$ bundles $\kappa(E) = c_2(E)$ and in the special case $G = SU(2)$ we get:

$$s = 8c_2(E) - 3(1 - b^1(M) + b^+(M)). \tag{3.35}$$

A specific construction of f as in Proposition 3.11, corresponds to a linearization of the zero-locus $Z(\psi)$ of the map

$$\psi : T_{A,\epsilon} \to \Omega^+(\mathfrak{g}_E); \quad \psi(a) = F^+(A + a) = d^+_A a + (a \wedge a)^+. \tag{3.36}$$

A neighbourhood of $[A] \in \mathcal{M}_E$ is then homeomorphic to the quotient $Z(\psi)/\Gamma_A$. The operator δ_A is elliptic, hence Fredholm, i.e. the kernel and cokernel of δ_A are finite-dimensional. This implies that the derivative of ψ given by

$$d^+_A : \ker d^*_A \to \Omega^+_M(\mathfrak{g}_E), \tag{3.37}$$

is a Fredholm operator, which turns ψ in a smooth Fredholm map. For this Fredholm map ψ one can construct a Γ_A-equivariant map f from $\ker(d^+_A|_{\ker d^*_A}) = \ker \delta_A$ to $\operatorname{coker}(d^+_A|_{\ker d^*_A}) = \operatorname{coker} d^+_A$. The zero-set of f is then a finite-dimensional model for $Z(\psi) \subset T_{A,\epsilon}$, so that a neighbourhood of $[A] \in \mathcal{M}_E$ is modeled on $f^{-1}(0)/\Gamma_A$.

An irreducible ASD connection is *regular* if $H_A^2 = 0$ and the moduli space $\mathcal{M}_E(g)$ is regular if all its irreducible points are regular. Notice that all regular points in $\mathcal{M}_E^* = \mathcal{M}_E \cap \mathscr{B}^*$ form a smooth manifold of dimension s, and its tangent space at $[A]$ is given by H_A^1.

When M is simply connected, some extra structure is present for $SU(2)$ bundles. A combination of results in Section 4.3.3 of [DK] give the following proposition regarding the moduli space $\mathcal{M}_E$.

Proposition 3.12. *Let M be a closed, simply connected, oriented, Riemannian four-manifold with $b^+(M) > 0$ and fix $l > 0$. Then for a dense set of metrics g on M, the moduli space $\mathcal{M}_E(g)$ for any $SU(2)$ bundle with $0 < c_2(E) \leq l$ contains no reducible connections and is regular.*

In order to have a reducible ASD connection on a $SU(2)$ bundle with $0 < c_2(E) \leq l$, we need $\mathcal{H}^-(M)$ to contain a form α such that $Q([\alpha],[\alpha]) = -c_2(E)$ and $[\alpha] \in H^2(M,\mathbb{Z})$ (see Proposition 3.9). This is a closed condition on how $\mathcal{H}^-(M)$ is realized. One of the results contained in Proposition 3.12 is that we can deform the conformal class of the metric, such that $\mathcal{H}^-(M)$ no langer satisfies this condition.

Remark 3.13. In the discussion of the ASD moduli space we ignored the fact that the L^2-norm used at several induces a too weak topology. One in general also needs to have control over higher derivatives of the connections, curvatures and gauge transformations. The space in which we need to put these connections, curvatures and gauge transformation are the higher Sobolev spaces L^2_{l-1}, L^2_{l-2} and L^2_l, respectively (with $l > 2$). This means that in a local trivialization of a G bundle E, the associated transition functions are L^2_l maps taking values in G and the connection forms as in (3.6) are given by L^2_{l-1} matrix valued functions. As a consequence the curvature matrix as in (3.13) lies in L^2_{l-2}. One of the results in [DK] is that the resulting moduli spaces $\mathcal{M}_E$ are homeomorphic to each other regardless of the Sobolev space we initially chose. Since we are primarily interested in $\mathcal{M}_E$, we can neglect this analytical subtleties.

3.3.3 Donaldson–Uhlenbeck compactification

For $SU(2)$ bundles E over M, the formal dimension (3.35) of the ASD moduli space depends on the charge $c_2(E)$. Therefore we label these ASD moduli spaces accordingly as $\mathcal{M}_k$, where $k = c_2(E)$. According to Remark 3.8, $k \geq 0$ and if M is simply connected $\mathcal{M}_0$ is just formed by the gauge equivalence class of the product connection.

An *ideal* ASD connection of Chern class k is given by a pair

$$([A],(x_1,\ldots,x_l)) \in \mathcal{M}_{k-l} \times s^l(M),$$

where $0 \leq l \leq k$ and $(x_1,\ldots,x_l)$ is an unordered l-tuple of points in M. Associated to such an ideal ASD connection is a measure, called the curvature density:

$$\mathrm{Tr}(F_A^2) + 8\pi^2 \sum_{r=1}^{l} \delta_{x_r} = |F_A|^2 d\mu + 8\pi^2 \sum_{r=1}^{l} \delta_{x_r}.$$

Integration of this measure over M gives $8\pi^2 k$ (see (3.24)).

The Donaldson–Uhlenbeck compactification of the ASD moduli space $\mathcal{M}_k$ is given by the closure of $\mathcal{M}_k$ in the set of all ideal ASD connections of Chern class k:

$$I\mathcal{M}_k = \mathcal{M}_k \cup \mathcal{M}_{k-1} \times M \cup \mathcal{M}_{k-2} \times s^2(M) \cup \ldots \cup \mathcal{M}_0 \times s^k(M). \tag{3.38}$$

The topology in which we take the closure is given by *weak convergence*. A sequence $[A_\alpha]$ in $\mathcal{M}_k$ converges weakly to an ideal ASD connection $([A],(x_1,\ldots,x_l))$ if:

i) For any $f \in C^0(M)$ we have

$$\int_M f|F_{A_\alpha}|^2 d\mu \longrightarrow \int_M f|F_A|^2 d\mu + 8\pi^2 \sum_{r=1}^{l} f(x_r);$$

ii) For each α there is a bundle map

$$\rho_\alpha : E'|_{X-\{x_1,\ldots,x_l\}} \to E|_{X-\{x_1,\ldots,x_l\}},$$

such that $\rho_\alpha^*(A_\alpha)$ converges to A (in C^∞ on compact sets of $X - \{x_1, \ldots, x_r\}$) and where E' is the bundle associated to $\mathcal{M}_{k-l}$ with $c_2(E') = k - l$.

In this formalism we have a strong result for the closure $\bar{\mathcal{M}}_k$ of $\mathcal{M}_k$.

Theorem 3.14 ([DK], Theorem 4.4.3)**.** *The space $\bar{\mathcal{M}}_k$ is compact.*

3.4 Correspondence with stable holomorphic bundles over Kähler surfaces

When Kähler structures are present on a closed, oriented, Riemannian four-manifold M, one can make a correspondence between the $SU(2)$ ASD moduli space $\mathcal{M}_E$ on one side and the moduli space of stable holomorphic bundles topologically equivalent to E on the other side. In what follows we will give a brief description regarding this correspondence following Chapter 6 of [DK].

Definition 3.15. A *Hermitian manifold* is a complex manifold with a Hermitian metric h on the complexified tangent space. Such a metric is given by a smooth section $h \in \Gamma(T^{1,0}M \otimes T^{0,1}M)^*$ satisfying $h_p(\eta, \overline{\zeta}) = \overline{h_p(\zeta, \overline{\eta})}$ $\forall \eta, \zeta \in T_p^{1,0}M$ and $h_p(\zeta, \overline{\zeta}) \geq 0$ for $\forall \zeta \in T_p^{1,0}M$ with equaltity if and only if $\zeta = 0$. Associated to h is the $(1,1)$-form $\omega = \frac{i}{2}(h - \overline{h})$, where h is now the image under the projection $(T^{1,0}M \otimes T^{0,1}M)^* \to \Lambda^{1,1}T^*M$

A specific class of Hermitian manifolds that have extra structure are Kähler manifolds.

Definition 3.16. A *Kähler manifold* K is a Hermitian manifold K with closed ω.

A Kähler manifold is the same as a symplectic manifold (K, ω) with an integrable almost complex structure J [2] that is compatible with the symplectic form. This last condition means $g(u, v) = \omega(u, Jv)$ defines a Riemannian metric.

The original metric h is called the *Kähler metric* and the associated $(1,1)$-form ω the *Kähler form.* The Riemannian volume form $d\mu$ on K (determined by the Riemannian metric g induced by ω) is given by the canonical volume form $\mathrm{vol}_K = \frac{\omega^n}{n!}$ with $n = \dim_{\mathbb{C}}(K)$.

3.4.1 Holomorphic structures

Recall that on a complex manifold Z, the de Rham complex $(\Omega^*(Z), d)$ splits into a double complex $(\Omega^{p,q}(Z), \partial, \overline{\partial})$ with $d = \partial + \overline{\partial}$ and

$$\partial : \Omega^{p,q}(Z) \to \Omega^{p+1,q}(Z) \quad \overline{\partial} : \Omega^{p,q}(Z) \to \Omega^{p,q+1}(Z).$$

A complex valued function f is holomorphic if and only if $\overline{\partial} f = 0$.

Let E be a complex vector bundle over a general complex manifold Z. A holomorphic structure $\mathscr{E}$ is defined by the following equivalent definitions:

[2] This condition is equivalent to K being a complex manifold. One can show that such a structure induces holomorphic coordinate charts.

- There is a holomorphic projection map $\pi : E \to Z$ where each fibre $\mathscr{E}_z = \pi^{-1}(z)$ is a complex vector space;
- We can find a covering $\{U_\alpha\}$ of Z and trivializations of E over the cover, such that the corresponding transition functions $g_{\alpha\beta} : Z_\alpha \cap Z_\beta \to GL(n, \mathbb{C})$ are holomorphic;
- There exists a preferred collection $\mathscr{E}$ of local holomorphic sections, denoted as $\mathscr{O}(\mathscr{E})$, which forms a sheaf of modules over the structure sheaf $\mathscr{O}_Z$ of local holomorphic functions on Z.

The symbols $\mathscr{E}$ and E will be used interchangably from here on to denote the vector bundle. Given a holomorphic structure, we can construct a linear map $\partial_{\mathscr{E}} : \Omega^0_Z(E) \to \Omega^{0,1}_Z(E)$, uniquely determined by:

i) $\overline{\partial}_{\mathscr{E}}(fs) = (\overline{\partial} f) \otimes s + f(\overline{\partial}_{\mathscr{E}} s)$ for all $f \in C^\infty(M)$ and $s \in \Omega^0_Z(E)$, similar to the Leibniz rule in Definition 3.1;

ii) $(\overline{\partial}_{\mathscr{E}} s)$ vanishes on open $U \subset Z$ if and only if s is holomorphic over U, i.e. $s \in \mathscr{O}(\mathscr{E})$.

By property i) the operator $\overline{\partial}_{\mathscr{E}}$ is local and in a local holomorphic trivialization the sections of $\mathscr{E}$ are therefore represented by vector valued functions. The operator $\overline{\partial}_{\mathscr{E}}$ just works on these sections by the ordinary $\overline{\partial}$ operator. This is independent of the holomorphic trivialization chosen, since $\overline{\partial}(g \cdot s) = g\overline{\partial}(s)$ for holomorphic $g : Z \to GL(n, \mathbb{C})$.

The splitting of $d = \partial + \overline{\partial}$ and $\Omega^k = \bigoplus_{p+q=k} \Omega^{p,q}$, induces a splitting

$$d_A = \partial_A \oplus \overline{\partial}_A : \Omega^0_Z(E) \to \Omega^{1,0}_Z(E) \oplus \Omega^{0,1}_Z(E). \tag{3.39}$$

The map $\overline{\partial}_A$ forms a so-called *partial connection* on E, which is a map $\overline{\partial}_\alpha : \Omega^0_Z(E) \to \Omega^{0,1}_Z(E)$ satisfying the same first condition as for the operator $\overline{\partial}_{\mathscr{E}}$ above.

A general partial connection $\overline{\partial}_\alpha$ is *integrable* if $\overline{\partial}_\alpha$ defines a holomorphic structure on E. This means that for each $z \in Z$ there exists a neighbourhood U over which there is a trivialization of E such that $\overline{\partial}_\alpha = \overline{\partial} + \alpha^\tau = \overline{\partial}$. Now integrability has a necessary and sufficient condition given by the integrability theorem (see Section 2.2.2 of [DK] for a proof).

Theorem 3.17 (Integrability theorem)**.** *A partial connection $\overline{\partial}_\alpha$ on a smooth complex vector bundle E over a complex manifold Z is integrable if and only if $\overline{\partial}_\alpha^2 = 0$.*

If E is Hermitian, we have the following result relating the map $\overline{\partial}_A$ associated to a connections as in (3.39) and a partial connection.

Proposition 3.18. *Let E be a Hermitian complex vector bundle over a complex manifold Z. Then for each partial connection $\overline{\partial}_\alpha$ on E, there is a unique unitary connection A, such that $\overline{\partial}_A = \overline{\partial}_\alpha$.*

Proof. In a local trivialization we have $\overline{\partial}_\alpha = \overline{\partial} + \alpha^\tau$. The connection form $A^\tau = \alpha^\tau - (\alpha)^\tau$ is then anti-Hermitian and has $\overline{\partial}_{A^\tau} = \overline{\partial} + \alpha^\tau$. Gluing the connection forms A^τ over Z gives us a unique unitary connection A satisfying $\overline{\partial}_A = \overline{\partial}_\alpha$. □

A connection A is called *compatible* with a holomorphic structure $\mathscr{E}$ if $\overline{\partial}_A = \overline{\partial}_{\mathscr{E}}$, where $\overline{\partial}_A$ is as in (3.39). This means the partial connection given by $\overline{\partial}_A$ is integrable, which by the integrability theorem is equivalent to $\overline{\partial}_A^2 = 0$.

The complex structure on Z gives us the decomposition

$$\Omega^2 = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2} \tag{3.40}$$

and accordingly the curvature $F_A \in \Omega^2_Z(\mathfrak{g}_E)$ can be written as

$$F_A = F_A^{2,0} + F_A^{1,1} + F_A^{0,2}. \tag{3.41}$$

Now the compatibility condition of A can be given in terms of its curvature.

Proposition 3.19. *A unitary connection on a Hermitian complex vector bundle over Z is compatible with a holomorphic structure if and only if it has curvature of type $(1,1)$. In this case the connection is uniquely determined by the metric and holomorphic structure.*

Proof. To be compatible with a holomorphic structure, we must have $F_A^{0,2} = \overline{\partial}_A^2 = 0$. Then also $F_A^{2,0} = -(F_A^{2,0})^* = 0$. The uniticity comes from the fact that the holomorphic structure defines a unique partial connection for which we have Proposition 3.18. □

If we take Z to be a complex surface, we also have the decomposition as in (2.11) besides the decomposition as in (3.40). If in addition, Z is a Hermitian manifold with associated Hermitian form ω, then the two are related as follows ([DK], Lemma 2.1.57):

$$\Omega^+ = \Omega^{2,0} \oplus \Omega^{0,2} \oplus \Omega^0 \cdot \omega \tag{3.42}$$
$$\Omega^- = \Omega_0^{1,1}, \tag{3.43}$$

where $\Omega_0^{1,1}$ is the space of $(1,1)$-forms orthogonal to ω. Let $\hat{F}_A = (F_A, \omega) \in \Omega^0(\mathfrak{g}_E)$ be the inner-product of F_A with ω. The previous statements combine into the following proposition.

Proposition 3.20. *If A is an ASD connection on a Hermitian complex vector bundle E over a Hermitian manifold Z, then $\overline{\partial}_A$ defines a holomorphic structure on E. Conversely if $\mathscr{E}$ is a holomorphic structure on E and A is a compatible unitary connection, then A is ASD if and only if $\hat{F}_A = 0$.*

3.4.2 Stable holomorphic vector bundles of rank 2

In the following we assume M to be a closed Kähler manifold of complex dimension 2 and ω the corresponding Kähler form. Let $\mathscr{E}$ be a rank two holomorphic vector bundle over M with $\Lambda^2\mathscr{E}$ trivial, i.e. we have a holomorphic $SL(2,\mathbb{C})$ bundle.

For any line bundle L over M we define the *degree* of L to be:

$$\deg(L) = Q(c_1(L), [\omega]), \tag{3.44}$$

where $[\omega] \in H^2(M,\mathbb{R})$ is the cohomology class of ω and Q is the intersection form on M extended to $H^2(M,\mathbb{R})$ (see Definition 2.2).

Definition 3.21. A holomorphic $SL(2,\mathbb{C})$ bundle $\mathscr{E}$ is *stable* if for each holomorphic line bundle $\mathscr{U}$ over M for which there is a non-trivial holomorphic map $\mathscr{E} \to \mathscr{U}$, we have $\deg \mathscr{U} > 0$.

Now let E be a Hermitian complex vector bundle. Define $\mathscr{A}^{1,1}$ to be the space of unitary connections on E with curvature of type $(1,1)$. Each $A \in \mathscr{A}^{1,1}$ defines a holomorphic structure by Proposition 3.19, which we denote by $\mathscr{E}_A$.

The natural question to ask is when two holomorpic structures $\mathscr{E}_{A_1}$ and $\mathscr{E}_{A_2}$ are isomorphic. This is the case when there is a bundle automorphism g of E (covering the identity), such that $\overline{\partial}_{A_1} = g\overline{\partial}_{A_2}g^{-1}$, since there should be an isomorphism between the local holomorphic sections in $\mathscr{O}(\mathscr{E}_{A_1})$ and those in $\mathscr{O}(\mathscr{E}_{A_2})$. This is the case when $A_1 = g(A_2)$ as in (3.8) and g lies in the gauge group $\mathscr{G}$, since $d_{A_1} = d_{g(A_2)} = g d_{A_2} g^{-1}$ and $d_A = \partial_A + \overline{\partial}_A$.

Now define the group $\mathscr{G}^c$ to be the group of all bundle automorphisms of E covering the identity, which we call the *complex gauge group*. We extend the action of $\mathscr{G}$ on $\mathscr{A}$ to an action of $\mathscr{G}^c$ on $\mathscr{A}$ as follows:

$$\begin{aligned}\overline{\partial}_{g(A)} &= g\overline{\partial}_A g^{-1} = \overline{\partial}_A - (\overline{\partial}_A g)g^{-1} \\ \partial_{g(A)} &= \tilde{g}\partial_A\tilde{g}^{-1} = \partial_A + ((\overline{\partial}_A g)g^{-1})^* = (\overline{\partial}_{g(A)})^*,\end{aligned}$$

where $\tilde{g} = (g^*)^{-1}$. This is indeed seen to coincide with the definition of $g(A)$ as in (3.8) if $g \in \mathscr{G}$. Under these tranformations the curvature of $g(A)$ is of type (1,1), if that of A is of type (1,1) as well. We can see this by expanding $(\partial_{g(A)} + \overline{\partial}_{g(A)})^2$ and subsequently identify all the components. This means that $\mathscr{G}^c$ preserves $\mathscr{A}^{1,1}$.

The discussion shows that a $\mathscr{G}^c$ orbit defines an equivalence class of isomorphic holomorphic structures, i.e. of unitary connections compatible with the same holomorphic structure. If a bundle is stable, then a bundle with an isomorphic holomorphic structure is also stable as one can readily check from Definition 3.21. *Stable* $\mathscr{G}^c$ orbits are defined as orbits of connections A such that $\mathscr{E}_A$ is stable.

According to (3.43), $A \in \mathscr{A}^{1,1}$ is ASD if and only if $\hat{F}_A = 0$. Theorem 6.1.5 of [DK] gives a correspondence between the ASD connections on a $SU(2)$ bundle over M and the stable holomorphic vector bundles $\mathscr{E}$ they define.

Theorem 3.22.

i) Any $\mathscr{G}^c$ orbit contains at most one $\mathscr{G}$ orbit of solutions to the equation $\hat{F}_A = 0$.

ii) A $\mathscr{G}^c$ orbit contains a solution to $\hat{F}_A = 0$ if and only if it is either a stable orbit or the orbit of a decomposable holomorphic structure $\mathscr{E} = \mathscr{U} \oplus \mathscr{U}^{-1}$, where $\deg \mathscr{U} = 0$. In the first case the solution A is an irreducible connection and in the second case it is a reducible solution compatible with the holomorphic splitting.

Note that by Theorem 3.22 the moduli space $\mathcal{M}_E^*$ of irreducible ASD connections on a $SU(2)$ bundle E is naturally identified with the $\mathscr{G}^c$ orbits of stable holomorphic $SL(2,\mathbb{C})$ bundles $\mathscr{E}$ topologically equivalent to E. In that case all $A \in \mathscr{A}^{1,1}$ are unitary and by Proposition 3.20 and Theorem 3.22 there is a correspondence $\mathscr{G} \cdot \mathscr{A} \simeq \mathscr{G}^c \cdot \mathscr{E}$ for

$$[A] \in \mathcal{M}_E^* = \{A \in \mathscr{A}^{1,1} \,|\, \hat{F}_A = 0,\ A \text{ irreducible}\}/\mathscr{G},$$

where $\mathscr{E} = \mathscr{E}_A$ is the holomorphic stable $SL(2,\mathbb{C})$ bundle uniquely determined by A. As a corollary we have:

Corollary 3.23 ([DK], Corollary 6.1.6)**.** *If E is an $SU(2)$ bundle over a compact Kähler manifold M, the moduli space $\mathcal{M}_E^*$ of irreducible ASD connections is naturally identified, as a set, with the set of $\mathscr{G}^c$ equivalence classes of stable holomorphic $SL(2,\mathbb{C})$ bundles $\mathscr{E}$ which are topologically equivalent to E.*

Remark 3.24. Notice that in the definitions up till now we have not explicitly used the fact that the Kähler form is closed. However, we must mention that the definition of stability will come to its full right when considered on Kähler manifolds. Using the so-called Kähler identities the notion of degree as in (3.44) will have an effect on the values of $\hat{F}_A = (F_A, \omega)$. The correspondences mentioned in Theorem 3.22 can then be proven.

3.4.3 Kähler structure on del Pezzo surfaces

On $\mathbb{CP}^n$ the Fubini–Study metric is defined on each of the sets $U_k = \{Z_k \neq 0\}$ separately. Recall (see (2.17)) that on U_k we can pick affine coordinates $(z_0, \dots, z_{k-1}, z_{k+1}, \dots z_n) \in \mathbb{C}^n$, with $z_i = Z_i/Z_k$. On this chart the following Hermitian $(1,1)$-form defining the metric is called the *Fubini–Study metric*:

$$\omega = \frac{(\sum_i dz_i \wedge d\bar{z}_i)}{1 + \sum_i |z_i|^2} - \frac{(\sum_i \bar{z}_i dz_i) \wedge (\sum_j z_j d\bar{z}_j)}{(1 + \sum_i |z_i|^2)^2}. \tag{3.45}$$

To see that this form is globally well-defined, we notice that $\omega = \partial\bar{\partial}\log(1 + \sum_i |z_i|^2) = \partial\bar{\partial}K$. Now on $U_l = \{Z_k \neq 0\}$, ω is given by $\omega = \partial\bar{\partial}K' = \partial\bar{\partial}\log(1 + \sum_i |z_i'|^2)$, where $z_i' = Z_i/Z_l$ are the affine coordinates associated to U_l. We have $\log(1 + \sum_i |z_i|^2) = \log(|z_l|^2(1 + \sum_i |z_i'|^2))$, so that $K - K' = \log(|z_l|^2) = -\log(|z_k'|^2)$. We see $\partial\bar{\partial}(K - K') = 0$, i.e. the definitions of ω agree on the overlap $U_k \cap U_l$. The form ω is closed as well:

$$d\omega = (\partial + \bar{\partial})\partial\bar{\partial}K = (\partial^2\bar{\partial} - \partial\bar{\partial}^2)K = 0.$$

It is seen that $\mathbb{CP}^n$ endowed with the Fubini–Study metric is a Kähler manifold. As a consequence all smooth algebraic varieties are Kähler manifolds, since they can be realized as complex submanifolds of $\mathbb{CP}^n$ and the Kähler form restricted to these submanifolds is again Kähler. Due to Proposition 2.16 all blow-ups of a smooth algebraic variety are again smooth algebraic varieties and therefore have a Kähler structure. This applies to the del Pezzo surfaces, since we could realize them as closed submanifolds of complex projective space and as blow-ups of $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ (recall from Section 2.2.3 that it can be embedded as a quadric in $\mathbb{CP}^3$). The specific Kähler structure can be computed if we know how the del Pezzo surfaces are embedded in complex projective space.

One can show that the first Chern class of the tautological line bundle L over $\mathbb{CP}^n$ (see (2.20)) equals $-[\omega]/(2\pi i)$ and that

$$\frac{1}{(2\pi i)^n} \int_{\mathbb{CP}^n} \omega^n = 1. \tag{3.46}$$

Furthermore, it can be shown that ω is self-dual ([Bet], Section 4.1) on $\mathbb{CP}^2$ and hence harmonic.

Every other line bundle over $\mathbb{CP}^2$ (and $\mathbb{CP}^n$ in general) is a power of L ([GH], page 145), hence every harmonic curvature F on $\mathbb{CP}^2$ must be an integral multiple of $2\pi i\omega$. The Dirac quantization condition in Section 4.1 then agrees with this for $\mathbb{CP}^2$. On $\overline{\mathbb{CP}}^2$, we can still define the form ω and it remains closed. Since the orientation is reversed, ω is now anti-self-dual and hence harmonic as well.

3.5 The $SU(2)$ ASD moduli space for $S^2 \times S^2$

One of the implications of the Kähler structure on del Pezzo surfaces is that we can use Theorem 3.22 and its Corollary 3.23 to find models for the ASD moduli spaces using algebro-geometric methods. The moduli spaces of $\mathscr{G}^c$ equivalence classes of stable holomorphic $SL(2,\mathbb{C})$ bundles have been studied for $\mathbb{CP}^2$ in [OSS] (Section II.4) and [Bar]. With this model at hand, we can construct a model for the ASD moduli space for $S^2 \times S^2 \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$, when realized as a non-singular quadric $S \subset \mathbb{CP}^3$. The references we will follow are page 128 of [DK] and [Sob]. In [Sob] a connection between stable holomorphic $SL(2,\mathbb{C})$ bundles over S and over $\mathbb{CP}^2$ is made, where the intricate relation between the two surfaces as covered in Section 2.2.3 plays a role. The notions from algebraic geometry used in what follows can be read back in Section 2.2.1.

The space $\mathbb{CP}^2$ can be embedded as a general plane $H \subset \mathbb{CP}^3$. The quadric $S \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$ has two rulings by lines (the A'- and B'-lines) biholomorphic to one of the $\mathbb{CP}^1$ and parametrized by the other factor $\mathbb{CP}^1$. Lines in the same ruling do not meet and lines lying in different rulings meet in a point $q \in S$, with the property that $T_qS \cap S$ is the union of the two lines. If H is not tangent to $p \in S$, we have a projection map $\pi_p : S - \{p\} \to H$ with the property that the lines L_1 and L_2 in $T_pS \cap S = L_1 \cup L_2$, are mapped to points q_1 and q_2, respectively. These constructions fit in the commutative diagram in (2.24), which describes the equivalence of the blow-up of S at p and the blow-up of H at the points q_1 and q_2.

We restrict to stable holomorphic $SL(2,\mathbb{C})$ bundles $\mathscr{E}$, with $c_1(\mathscr{E}) = 0$ and $c_2(\mathscr{E}) = k$, for which the moduli space of $\mathscr{G}^c$ equivalence classes is denoted as $\mathcal{M}(k)(S)$. The corresponding moduli space for $\mathbb{CP}^2$ is denoted as $\mathcal{M}(k)(\mathbb{CP}^2)$. According to (3.35) the ASD moduli spaces to which these spaces should correspond, both have formal dimension $8k-6$ (or complex formal dimension $4k-3$), which is the case for $\mathcal{M}(k)(S)$ ([Sob], §1) and $\mathcal{M}(k)(\mathbb{CP}^2)$ [Bar]. In fact more is true, $\mathcal{M}(k)(S)$ is a smooth space of this dimension if it is non-empty.

A bundle $\mathscr{E}$ is called *projectable from a point* $q \in S$ if $\mathscr{E}|_{L_1}$ and $\mathscr{E}|_{L_2}$ are trivial where L_1 and L_2 are the A'- and B'-lines going through q, i.e. $T_qS \cap S = L_1 \cup L_2$. We say that a bundle is *projectable* if it is from some point q. There is a direct correspondence ([Sob], Proposition 2.1) through the projection map $\pi_q : S \to \mathbb{CP}^2$ between stable bundles projectable from q and stable bundles over $\mathbb{CP}^2$ trivial over the line $L_0 = \overline{q_1q_2}$, where $q_1 = \pi_q(L_1)$ and $q_2 = \pi_q(L_2)$. This correspondence gives an isomorphism $\mathcal{M}_q(k) \simeq \mathcal{M}_{L_0}(k)$ of the respective $\mathscr{G}^c$ equivalence classes of bundles.

The obstruction for a bundle $\mathscr{E}$ to be projectable is that we cannot find an A'- and B'-line on S, over which $\mathscr{E}$ is trivial. Lemma 2.5 of [Sob] states that there is a small deformation of $\mathscr{E}$ such that this obstruction disappears. Therefore, the set of equivalence classes of projectable bundles given by $\mathcal{M}^0(k) = \cup_{q \in S}\mathcal{M}_q(k)$ is an open set of $\mathcal{M}(k)(S)$.[3] Now there is already one implication: if $\mathcal{M}(k)(\mathbb{CP}^2)$ is empty, $\mathcal{M}(k)(S)$ is as well. This is for example the case when $k = 1$ [Bar].

A projectable bundle $\mathscr{E}$ over S has a set of lines in S on which $\mathscr{E}$ is *not* trivial, which are called the *jumping lines* of S. Proposition 2.2 of [Sob] states that the jumping lines consist of k A'-lines and k B'-lines counted with multiplicity.

A conic on S can be realized by intersecting S with a plane in $\mathbb{CP}^3$, which can be non-degenerate (a curve of degree 2) or degenerate (a double line, which is the intersection of S with a tangent plane). A plane is called *a jumping plane* of a projectable bundle $\mathscr{E}$, if the conic C it defines satisfies either one of the following properties:

i) C is non-degenerate and has the property that $\mathscr{E}|_C$ is non-trivial;

ii) C is degenerate, i.e. $C = L_1 \cap L_2$, and at least one of the two lines is a jumping line.

A generic non-degenerate conic on E usually does *not* have this property. Let us denote the set of all jumping planes as $J(\mathscr{E})$, which is a subset of the dual of $\mathbb{CP}^3$, denoted as $(\mathbb{CP}^3)^*$.

Any point $z \in (\mathbb{CP}^3)^*$ defines a plane in $\mathbb{CP}^3$ and vice versa. Any non-singular quadric $Q \subset \mathbb{CP}^3$ induces an isomorphism $\mathbb{CP}^3 \simeq_Q (\mathbb{CP}^3)^*$ through the symmetric non-singular matrix as in (2.22). This isomorphism maps a point $q \in Q$ to its tangent plane $T_q \in (\mathbb{CP}^3)^*$. Similarly, one has the dual $(\mathbb{CP}^2)^*$ of $\mathbb{CP}^2$, i.e. a point $z \in (\mathbb{CP}^2)^*$ defines a line in $\mathbb{CP}^2$.

Using the correspondence between projectable bundles $\mathscr{E}$ over S and bundles over $\mathbb{CP}^2$ that are trivial over a line, one can show that the set of jumping planes that intersect S in q corresponds to the set of jumping lines for the bundle $\pi_q\mathscr{E}$ (lines on which the associated bundle

[3] These spaces can be realized as abstract varieties and the topology that is put on these spaces is the Zariski topology (see [Har]).

over $\mathbb{CP}^2$ is non-trivial). This set of jumping lines for $\pi_q\mathscr{E}$ forms a curve in $(\mathbb{CP}^2)^*$ of degree $c_2(E) = k$. The set of all planes intersecting $q \in S$, denoted as E_q^*, forms a plane in $(\mathbb{CP}^3)^*$. This shows $J(\mathscr{E})$ is a surface in $(\mathbb{CP}^3)^*$ of degree k, since the intersection with the plane E_q^* in $(\mathbb{CP}^3)^*$ is a curve of degree k and for any generic plane P in $(\mathbb{CP}^3)^*$ there exists $q \in S$, such that $P = E_q^*$.

Under the isomorphism $\mathbb{CP}^3 \simeq_S (\mathbb{CP}^3)^*$ the surface $J(\mathscr{E})$ gets mapped to a surface of degree k in $\mathbb{CP}^3$, which we also call $J(\mathscr{E})$. The main statement ([Sob], Theorem 3.4) is then that $J(\mathscr{E}) \cap S$ consist of the $2k$ jumping lines (counted with multiplicity), i.e. k A'-lines and k B'-lines.

The condition of projectability and the condition of L being a jumping line for $\mathscr{E}$ is invariant under the action of $\mathscr{G}^c$ on $\mathscr{E}$. Hence the set $S(\mathscr{E})$ of jumping planes is indeed well-defined for a $\mathscr{G}^c$ equivalence of projectable bundles in $\mathcal{M}^0(k)(S)$. Summarizing, we have a model for the open subset $\mathcal{M}^0(k)(S)$ of $\mathcal{M}(k)(S)$: to each $\mathscr{G}^c$ equivalence class $x \in \mathcal{M}^0(k)(S)$ there is a collection of surfaces of degree k that intersect S in the jumping lines associated to a bundle $\mathscr{E}$ which represents x.

The moduli space for $c_2(E) = 2$

Let E be an $SU(2)$ bundle over $S \simeq S^2 \times S^2$ with $c_2(E) = 2$. Using the model above for $\mathcal{M}^0(2)(S)$, every $\mathscr{G}^c$ equivalence class of a projectable bundle $\mathscr{E}$ over S (topologically equivalent to E) can be identified with a non-singular quadric which intersects S in 2×2 lines (counted with multiplicity), 2 in each system of A'- and B'-lines. Using Theorem 3.22, a subset of the moduli space $\mathcal{M}_2(E)$ of irreducible ASD connections can be modeled by this set of quadrics, which we will call $\mathcal{M}^0$.

Given one such quadric S', one can show that each quadric in the pencil $S'+tS$ ($t \in \mathbb{C}$) meets S in four lines as well. This implies that part of $\mathcal{M}_2(E)$ is a subset of a complex projective cone K, i.e. an algebraic variety ruled by projective lines through a common vertex. This singular variety of complex dimension 5 is embedded in the space of quadrics in $\mathbb{CP}^3$, which is given by $\mathbb{CP}^9$ (see Remark 2.21). This agrees with the formal (real) dimension of $\mathcal{M}_2(E)$, which is 10. When we extend $\mathcal{M}^0$ to all non-singular quadrics in the cone K, we obtain a set of quadrics $\mathcal{M}$ that corresponds to $\mathcal{M}_2(E)$ [DK]. This set has a distinguised quadric, namely S itself.

The complement $K \setminus \mathcal{M}$ consists of singular quadrics intersecting S in four lines, which must necessarily be pairs of planes tangent to S. The pair of planes is then determined by the pair of points at which these planes are tangent to S. This shows $K \setminus \mathcal{M} \simeq s^2(S)$, which is in agreement with the Donaldson–Uhlenbeck compactification of $\mathcal{M}_2(E)$. Since $\mathcal{M}(1)(S)$ is empty, $\mathcal{M}_1(E)$ is empty and $\mathcal{M}_0(E)$ is just a point (the product connection) by simply connectedness of S. According to (3.38) we would then have $\bar{\mathcal{M}}_2(E) \subset \mathcal{M}_2(E) \cup s^2(S)$.

For generic metrics on $S^2 \times S^2$ the moduli space $\mathcal{M}_2(E)$ does not contain reducible points according to Proposition 3.12. However, we embed $S^2 \times S^2$ as a quadric S in $\mathbb{CP}^3$. The Kähler structure on $S^2 \times S^2$ then results in the product metric on two round spheres of equal radii. In $H^2(S^2 \times S^2, \mathbb{Z})$, the cohomology class $c = [a] = [\alpha_1 - \alpha_2]$ then has as representative (see (4.90)) an anti-self-dual harmonic form satisfying $-Q(c, c) = -\int_M a \wedge a = 2 = c_2(E)$. According to Proposition 3.9 this results in a reducible point in $\mathcal{M}_2(E)$. Under the correspondence $\mathcal{M} \simeq \mathcal{M}_2(E)$, this point corresponds to the quadric S itself [DK].

Using the Kähler structure on S, we can give a local model for the moduli space around this gauge equivalence class of a reducible ASD connection A. Theorem 3.22 tells us that $[A] \in \mathcal{M}_2(E)$ corresponds to a $\mathscr{G}^c$ orbit of a decomposable holomorphic structure $\mathscr{E} = \mathscr{U} \oplus \mathscr{U}^{-1}$ and that A is compatible with this holomorphic splitting. For such $\mathscr{E}$ and A, one can show that $H^1(A) = \mathbb{C}^6$, $H^2_A = H^0_A = \mathbb{R}$ and a local model as in Proposition 3.11 is given by $f^{-1}(0)/S^1$, where

$$f : \mathbb{C}^6 \to \mathbb{R}; f(z) = \sum_{i=1}^{3} |z_i|^2 - \sum_{i=4}^{6} |z_i|^2.$$

In the quotient, $e^{i\theta} \in S^1$ acts as $e^{2i\theta}$ on z_1, z_2, z_3 and as $e^{-2i\theta}$ on z_4, z_5, z_6. More details regarding the calculations of the identities above can be found on page 140 and Section 6.4 of [DK].

Chapter 4

Electric-magnetic duality in Euclidean Maxwell theory

Throughout this chapter we will consider the Abelian gauge theory we discussed in the introduction, namely source-free Maxwell theory on a closed, connected, oriented, Riemannian four-manifold M. One of the crucial ingredients needed to calculate the partition function is the intersection form as in Definition 2.2.

Starting point for this chapter will be the papers by Verlinde and Witten [Ver, Wit3], which explore electric-magnetic duality in Abelian gauge theories where the gauge field couples to a background scalar field on M. The Maxwell partition function for this theory is calculated using the semiclassical approximation. The result becomes a sum over a discrete set since the saddle points of the action are harmonic 2-forms forming an integral lattice. This integrality is implied by the Dirac quantization condition, which tells us that the magnetic fluxes for these 2-forms are integrally quantized. However, the partition function can change for theories in which the gauge fields couple to background spinor fields on M. Under some circumstances, the Dirac quantization condition for the magnetic fluxes is then shifted by a half-integer.

The partition functions in both cases become functions of the complex coupling constant $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$ and we can study its modular behaviour under $SL(2,\mathbb{Z})$ transformations of τ. In the scalar case, the modular properties of this function can be computed using Poisson resummation. To encompass the spinor case as well, the Maxwell partition function will be looked at in a more generalized form in Section 4.3. The modular transformation properties will then be studied in terms of theta functions, which are related to the topology of M.

There are a number of parameters in the partition function dependent on the detailed geometry of the manifold. The moduli of these parameters will form a space that classifies all distinct partition functions on M under variations of the metric. The construction of this space will be covered in Section 4.4. Another result of the discussion there, is an algorithm to determine these missing parameters using algebraic relations.

In Sections 4.5 and 4.6, the results we obtained will be applied to the del Pezzo surfaces dP_l and $S^2 \times S^2$ we covered in Section 2.2. For $S^2 \times S^2$ we can calculate the partition function using an explicit metric, without having to resort to algebraic relations.

4.1 The Maxwell partition function

Let A_i be a $U(1)$ gauge field on the manifold M and write it as a 1-form $A = A_i dx^i$. If the gauge field couples to a complex scalar field $\phi(x)$ on M, this form A is a connection on a complex line bundle L (see Section 3.2) and ϕ a section of this bundle. Under gauge transformations of the field ϕ we have

$$\phi(x) \mapsto e^{i\chi(x)}\phi(x), \quad A \mapsto A - d\chi, \tag{4.1}$$

which is just (3.8). One can check that this leaves the term $\sum_i |(\frac{\partial}{\partial x^i} - iA_i)\phi(x)|^2$ invariant, which is a term one gets in the Langrangian of this theory. We assume however that the scalar field is a fixed background field: it does not show up in the path integrals defining the partition function. Up till Section 4.3 only these couplings are taken into consideration and no couplings to spinor fields.

The *Euclidean Maxwell action* for such a gauge field field A is given by

$$S[A] = \frac{1}{g^2}\int_M F \wedge \star F - i\frac{\theta}{8\pi^2}\int_M F \wedge F, \tag{4.2}$$

where the curvature $F = dA$ is the 2-form field strength of A, $\star$ is the Hodge star operator as defined in (2.7) and the parameters g and θ are the gauge coupling constant and theta parameter, respectively.

The Dirac quantization condition for fluxes on four-manifolds [AO] states that for any non-trivial homology 2-cycle Σ and 2-form field strength F

$$\int_\Sigma F = 2\pi m, \quad m \in \mathbb{Z}. \tag{4.3}$$

This condition is due to the cocycle condition that holds for transition functions associated to a trivialization of the bundle L . These functions can be seen as gauge transformations of the field ϕ and A must transform accordingly as in equation (4.1). The cocycle condition induces consistency conditions for A that result into equation (4.3).

The Maxwell partition function corrsponding to the action in equation (4.2) is given by

$$Z(\tau) = \frac{1}{|\mathscr{G}|}\int [dA] \exp(-S[A]), \tag{4.4}$$

where we integrate over all $U(1)$ gauge fields and divide by the volume of the group of gauge transformations, which corresponds to the infinite-dimensional group $\mathscr{G}$ we considered in Definition 3.3. We expect the partition function to be dependent on g and θ and these parameters are combined into the 'complex' coupling constant

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{4.5}$$

One can calculate the partition function in the semiclassical approximation. The partition function $Z(\tau)$ then factorizes as a sum over all classical saddle points times a product of determinants denoted as $\Delta(\tau)$, which can be determined by a Faddeev–Popov regularization procedure. The result is

$$Z(\tau) = \Delta(\tau) \sum_{saddle\, points} \exp(-S[A_{cl}]).$$

One exclusive feature of $U(1)$ gauge theory is that these determinants can be computed exactly by performing Gaussian integrals. These integrals are taken over the quadratic fluctuations of the gauge fields around the saddle points. For general gauge groups, the localization techniques one uses in this factorization only work in a few special cases.

The factor $\Delta(\tau)$ needs exact geometric details about M. In [Wit3] it is argued that

$$\Delta(\tau) = Z_0(\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)},$$

where b^1 is the first Betti number and Z_0 is a quantity dependent on the geometry of M, but independent of τ. The number b^1 only vanishes when M is simply connected.

The classical saddle point contribution can be computed using only a few geometric and topological parameters. We combine this contribution with the factor $(\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)}$ to get the τ dependent part of the Maxwell partition function:

$$Z_{cl}(g,\theta) = Z_{cl}(\tau) = (\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)} \sum_{saddle\,points} \exp(-S[A_{cl}]). \tag{4.6}$$

The saddle points of the action (4.2) are solutions the (Euclidean) equations of Maxwell:

$$dF = d\star F = 0.$$

This can be computed in a similar way as in the derivation of (3.23). The adjoint map d^* as in (2.9) is given by $d^* = -\star d \star$ in degree two, hence we conclude that the saddle point are harmonic forms as in (2.10). We always have $dF = 0$ by the Bianchi-identity (3.15), hence the term $\frac{\theta}{8\pi^2}\int F\wedge F$ in the action does not change the equations of motion.

The harmonic forms span a vector space of dimension b^2 and so all the saddle points can be decomposed as $F = \sum_{I=1}^{b^2} f^I\alpha_I$, where the α_I form a basis for the harmonic forms. We can choose the α_I such that their cohomology classes generate $F^2(M,\mathbb{Z})$ (see (2.4) and Remark 2.6), since we still have the freedom in how to normalize these basis vectors. The identity in (2.5) implies we can then find homology cycles Σ_I dual to the α_I, i.e.

$$\int_{\Sigma_I} \alpha_J = \delta^I_J. \tag{4.7}$$

An application of equation (4.3) to these Σ_I, tells us that every harmonic form F corresponding to a saddle point can be written as

$$F = 2\pi\sum_I m^I\alpha_I, \quad m^I\in\mathbb{Z}. \tag{4.8}$$

Remark 4.1. Since F is the curvature of a connection A on a line bundle L, Chern–Weil theory tells us that $\frac{1}{2\pi}F$ is a representative of the first Chern class $c_1(L)$ [1] of this line bundle. The equation above tells us in particular that $c_1(L)$ lies in the quotient group $F^2(M,\mathbb{Z})$, since the cohomology classes of the α_I were generators of this space. This statement was already made at the beginning of Section 3.2. In fact, the proof of this statement uses the same type of argument as the proof of the Dirac quantization condition (4.3) given in [Alv].

After plugging the expansion (4.8) of F into equation (4.2), the action evaluated for a classical saddle point takes the following form:

$$S[m_I] = \sum_{I,J}\frac{4\pi^2}{g^2}m^I G_{IJ}m^J - \sum_{I,J}\frac{i}{2}\theta m^I Q_{IJ}m^J, \tag{4.9}$$

where

$$G_{IJ} = \int_M \alpha_I\wedge\star\alpha_J, \quad Q_{IJ} = \int_M \alpha_I\wedge\alpha_J. \tag{4.10}$$

Equation (4.6) then takes the form

$$Z_{cl}(\tau) = (\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)}\sum_{m^I}\exp\left(\sum_{I,J}\frac{4\pi^2}{g^2}m^I G_{IJ}m^J + \sum_{I,J}\frac{i}{2}\theta m^I Q_{IJ}m^J\right). \tag{4.11}$$

[1] In Section 3.2 we have set the cohomology class of $\frac{iF}{2\pi}$ equal to $c_1(L)$. The reason we do not include i is due to the fact that it is already included in the definition of the action of $U(1)$ as in (4.1).

Dependence on the intersection form and metric

The α^I were chosen, such that their cohomology classes generate $F^2(M,\mathbb{Z})$. The symmetric matrix Q_{IJ} in (4.10), then corresponds to the quantity $Q([\alpha_I],[\alpha_J])$ as in (2.3). The intersection form Q (see Definition 2.2) restricted to $F^2(M,\mathbb{Z})$ is unimodular, hence the matrix Q^{IJ} is unimodular (symmetric, integer-valued and determinant equal to one).

The inverse of the matrix Q_{IJ} is equal to $Q^{IJ} = I(\Sigma_I, \Sigma_J)$, where I is the intersection form as in Definition 2.2 and Σ_I as in (4.7). The elements in $H_2(M,\mathbb{Z})$ with torsion give zero on the left-hand side of equation (4.3):

$$\int_\Sigma F = \frac{1}{N}\int_{N\Sigma} F = \frac{1}{N}\int_{\partial\Gamma} F = \frac{1}{N}\int_\Gamma dF = 0.$$

Therefore the Σ_I lie in the quotient group $F_2(M,\mathbb{Z})$ and I restricted to this space is unimodular, so that the matrix Q^{IJ} is unimodular as well.

To see why Q^{IJ} is the inverse of Q_{IJ}, we notice that Q^{IJ} can also be expressed as

$$Q^{IJ} = \int \tilde{\alpha}^I \wedge \tilde{\alpha}^J,$$

where $\tilde{\alpha}^I$ is the Poincaré dual of Σ_I, i.e.

$$\int_{\Sigma_I} \beta = \int_M \tilde{\alpha}^I \wedge \beta \text{ for all 2-forms } \beta. \tag{4.12}$$

We can express $\tilde{\alpha}^I$ as $\tilde{\alpha}^I = A^{IJ}\alpha_J$. One can then check that

$$Q^{IJ} = \int \tilde{\alpha}^I \wedge \tilde{\alpha}^J = \sum_K \int \tilde{\alpha}^I \wedge (A^{JK}\alpha_K) = \sum_K A^{JK} \int_{\Sigma_I} \alpha_K = A^{IJ},$$

which implies $\tilde{\alpha}^I = Q^{IJ}\alpha_J$. Using this expression we compute:

$$\sum_K Q^{IK} Q_{KJ} = \sum_K \int Q^{IK}\alpha_K \wedge \alpha_J = \int \tilde{\alpha}^I \wedge \alpha_J = \delta^I_J.$$

The symmetric matrix G_{IJ} is positive-definite and determined by the action of the Hodge star operator on harmonic 2-forms. Due to Proposition 2.5 this action depends on the conformal class of the Riemannian metric g on M. Similar reasoning as before shows we can write $\star\alpha_I$ as

$$\star\, \alpha_I = \sum_J G_{IJ}\tilde{\alpha}^J = \sum_{J,K} G_{IJ} Q^{JK} \alpha_K. \tag{4.13}$$

Since $\star^2|_{\Omega^2(M)} = 1$, we deduce that

$$\sum_{J,K,L} G_{IJ} Q^{JK} G_{KL} Q^{LN} = \delta_I^N \tag{4.14}$$

Remark 4.2. An important observation is that when $b^2 = 0$, there are no non-trivial harmonic forms and Z_{cl} reduces to $(\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)}$, which only depends on g and not on θ. For general field strengths $F = dA$ we have

$$\frac{i\theta}{8\pi^2}\int_M F \wedge F = \frac{i\theta}{2} Q(c_1(L), c_1(L)),$$

which implies that the theta term in the action only depends on the topology of M and the first Chern class of the bundle L on which A is a connection. This confirms our statement in Section 1.3 saying that the theta term in the action is *topological*. As a result the partition function in (4.11) only depends on θ through the intersection form Q.

The partition function in terms of (anti-)self-dual harmonic forms

Recall (see (2.12)) that every $\alpha \in \mathcal{H}^2(M) = \mathcal{H}^+ \oplus \mathcal{H}^-$ can be split into a self-dual part and anti-self-dual part, i.e. $\alpha = \alpha^+ + \alpha^-$, where $\star\alpha^\pm = \pm\alpha^\pm$. With respect to the inner-product as in (2.8) this splitting is orthogonal. Writing $\alpha_I = e_I^+ + e_I^-$, $F = 2\pi \sum_I m^I e_I^+ + 2\pi \sum_I m^I e_I^-$ and $(\alpha)^2 = \langle \alpha, \alpha \rangle$, equation (4.2) becomes:

$$\begin{aligned}
S[A] &= \frac{1}{g^2}\int_M F\wedge\star F - i\frac{\theta}{8\pi^2}\int_M F\wedge F \\
&= \frac{1}{g^2}4\pi^2\left(\int_M (\sum_I m^I e_I^+)\wedge(\star\sum_I m^I e_I^+) + \int_M (\sum_I m^I e_I^-)\wedge(\star\sum_I m^I e_I^-)\right) \\
&\quad -i\frac{\theta}{8\pi^2}4\pi^2\left(\int_M (\sum_I m^I e_I^+)\wedge(\star\sum_I m^I e_I^+) - \int_M (\sum_I m^I e_I^-)\wedge(\star\sum_I m^I e_I^-)\right) \\
&= \left(\frac{4\pi^2}{g^2} - i\frac{\theta}{2}\right)(\sum_I m^I e_I^+)^2 + \left(\frac{4\pi^2}{g^2} + i\frac{\theta}{2}\right)(\sum_I m^I e_I^-)^2 \\
&= -i\pi\tau(p^+)^2 + i\pi\overline{\tau}(p^-)^2, \qquad (4.15)
\end{aligned}$$

where $p^\pm = \sum_I m^I e_I^\pm$.

Equation (4.11) can now be rewritten as

$$Z_{cl}(\tau,\overline{\tau}) = (\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)} \sum_{(p^+,p^-)\in\Gamma_{b^+,b^-}} \exp(i\pi\tau(p^+)^2 - i\pi\overline{\tau}(p^-)^2), \qquad (4.16)$$

where Γ_{b^+,b^-} is a self-dual Lorentzian lattice of signature $(\dim\mathcal{H}^+, \dim\mathcal{H}^-) = (b^+, b^-)$ given by

$$\Gamma_{b^+,b^-} = \bigoplus_I \mathbb{Z}(e_I^+ \oplus e_I^-). \qquad (4.17)$$

Self-duality of the lattice $\Gamma_{b^+,b^-} \subset \mathcal{H}^2(M)$ means that the set of y' in $\mathcal{H}^2(M)$ such that $\int_M y'\wedge y \in \mathbb{Z}$ for $y \in \Gamma_{b^+,b^-}$ is equal to Γ_{b^+,b^-} itself. More details about lattices are covered in Section 4.3.1.

If we write $e_I^\pm = \sum_i^{b^\pm}(e_I^\pm)^i f_i$ where $\langle f_i, f_j\rangle = \delta_{ij}$, then $(p^\pm)^2 = \sum_{I,J} m^I m^J \sum_i^{b^\pm}(e_I^\pm)^i(e_J^\pm)^i$. Note that $S[m_I]$ as in equation (4.9) can also be written as

$$S[m^I] = -i\pi\tau\sum_{I,J} m^I m^J \frac{1}{2}(G_{IJ} + Q_{IJ}) + i\pi\overline{\tau}\sum_{I,J} m^I m^J\frac{1}{2}(G_{IJ} - Q_{IJ}). \qquad (4.18)$$

Hence comparing this to equation (4.15) we obtain

$$\frac{1}{2}(G_{IJ} \pm Q_{IJ}) = \sum_i^{b^\pm}(e_I^\pm)^i(e_J^\pm)^i. \qquad (4.19)$$

4.2 Modular properties of the Maxwell partition function

In this section the properties of the partition function $Z_{cl}(\tau)$ in (4.11) will be analyzed under modular transformations of the parameter $\tau \in \mathbb{C}$:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \text{with} \quad ad - bc = 1, \quad a,b,c,d \in \mathbb{Z}.$$

If we denote the above transformation by $g \cdot \tau$, with

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4.20}$$

then by succesively applying these transformations one can check that this forms an action of the group $SL(2,\mathbb{Z})$ on the parameter τ. Furthermore, we can calculate that under these transformation the sign of the imaginary part gets preserved, which is what we want since $\mathrm{Im}(\tau) = 4\pi/g^2$ needs to stay positive. To be more explicit:

$$\mathrm{Im}(g \cdot z) = \frac{1}{|cz+d|^2}\mathrm{Im}(z).$$

Hence we can restrict τ to the upper-half plane of $\mathbb{C}$, denoted by $\mathbb{H}$. The elements $\{\pm 1\} \in SL(2,\mathbb{Z})$ act the same on $\mathbb{H}$. The group that faithfully acts on τ, is therefore given by $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm 1\}$. In [Ser] it is shown that $SL(2,\mathbb{Z})$ (and therefore $PSL(2,\mathbb{Z})$) are generated by the elements:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{4.21}$$

Notice that $S^2 = (ST)^3 = 1$ in $PSL(2,\mathbb{Z})$, whereas in $SL(2,\mathbb{Z})$ we have $S^2 = (ST)^3 = -1$. Although $PSL(2,\mathbb{Z})$ acts faithfully on $\mathbb{H}$, there is a sign ambiguity if it acts on the partition function. To see how $SL(2,\mathbb{Z})$ (or a subgroup thereof) acts on the partition function, it usually suffices to compute the action of the generators S and T above (or the generators of the subgroup). The subgroup of our interest is the Hecke subgroup Γ_θ, which is generated by S and T^2. It is a subgroup of index three, since one can show

$$SL(2,\mathbb{Z}) = \Gamma_\theta \cup T\Gamma_\theta \cup ST\Gamma_\theta. \tag{4.22}$$

The following generalization of Proposition 1 in Section VII.2.1 of [Ser] motivates the story above. All the proofs of the statements above can also be found back in Chapter VII of this book.

Proposition 4.3. *Let $f(\tau)$ be a function in the variables τ and $\overline{\tau}$, where τ lies in $\mathbb{H}$. Then the following two statements are equivalent.*

i) $f(\tau+1) = f(\tau)$ and $f(-1/\tau) = \tau^u \overline{\tau}^v f(\tau)$;

ii) $f(g\cdot\tau) = (c\tau+d)^u(c\overline{\tau}+d)^v f(\tau)$, where $g \in SL(2,\mathbb{Z})$ is as in equation (4.20).

. Statement ii) is the definition of f being a modular form of weight (u,v).

Proof. Case $i)$ is a special case of case $ii)$, hence we only need to check that $i)$ gives $ii)$. We have $\frac{d(g\cdot\tau)}{d\tau} = \frac{1}{(c\tau+d)^2}$ and $\frac{d(g\cdot\overline{\tau})}{d\overline{\tau}} = \frac{1}{(c\overline{\tau}+d)^2}$. Case $ii)$ is now equivalent to:

$$\begin{aligned} \frac{f(g\cdot\tau)}{f(\tau)} &= \left(\frac{d(g\cdot\tau)}{d\tau}\right)^{-u/2}\left(\frac{d(g\cdot\overline{\tau})}{d\overline{\tau}}\right)^{-v/2} \\ f(g\cdot\tau)(d(g\cdot\tau))^{u/2}(d(g\overline{\tau}))^{v/2} &= f(\tau)(d\tau)^{u/2}(d\overline{\tau})^{v/2}. \end{aligned}$$

This is the same as $f(\tau)(d\tau)^{u/2}(d\overline{\tau})^{v/2}$ being invariant under the action of $SL(2,\mathbb{Z})$. Since this is true for the generators S and T, as one can easily calculate, this must must be true for all elements in $SL(2,\mathbb{Z})$. For even u,v the form $(d\tau)^{u/2}(d\overline{\tau})^{v/2}$ is well-defined. For other u,v we must check whether this form is still analytically well-defined, since there could be a sign ambiguity. This issue is resolved by studying the action of the metaplectic cover of $SL(2,\mathbb{R})$ on τ and restrict it to the inverse image of $SL(2,\mathbb{Z})$. In Chapter 1 of [Bru] more details can be found. □

Remark 4.4. The proof of Proposition 4.3 shows as well that if in the first condition $f(\tau+1) = f(\tau)$ is replaced by $f(\tau+2) = f(\tau)$, condition *ii)* holds but then for the subgroup Γ_θ generated by S and T^2. The function f is then still called a modular form of weight (u, v). Notice that if we would have considered only the group $PSL(2,\mathbb{Z})$ case *ii)* would contain a sign ambiguity.

To compute the action of $S \in SL(2,\mathbb{Z})$ on the partition function, we need the Poisson resummation formula given by

$$\sum_m f(m) = \sum_n \int dx \exp(2\pi i n x) f(x), \tag{4.23}$$

and its multi-dimensional analogue

$$\sum_{m\in\mathbb{Z}^l} f(m) = \sum_{n\in\mathbb{Z}^l} \int d^l x \exp(2\pi i n \cdot x) f(x). \tag{4.24}$$

These identities will be proven in a more general form in Proposition 4.13.

To see the modular covariance under S and T (or T^2) of the partition function in (4.11), we write it in the following form:

$$Z(\tau) = (\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)} \sum_{m\in\mathbb{Z}^{b^2}} \exp(i\pi m\cdot\Omega(\tau)\cdot m) = (\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)}\Theta(\tau), \tag{4.25}$$

where $\Omega(\tau) = \tau_1 Q_{IJ} + i\tau_2 G_{IJ} = \tau_1 Q + i\tau_2 G$, $\tau_1 = \mathrm{Re}(\tau) = \frac{\theta}{2\pi}$ and $\tau_2 = \mathrm{Im}(\tau) = \frac{4\pi}{g^2}$. For the modular transformation properties we have the following theorem [OA].

Theorem 4.5. *Let M be a closed, oriented, Riemannian four-manifold with Q its unimodular intersection form. Then for $Z(\tau)$ as in equation (4.25), the following is true:*

- $Z(\tau) = Z(\tau+1)$ *if Q is even, otherwise* $Z(\tau) = Z(\tau+2)$*;*
- $Z(-1/\tau) = e^{-2\pi i(b^+-b^-)/8}\tau^{\frac{1}{2}(1-b^1+b^+)}\overline{\tau}^{\frac{1}{2}(1-b^1+b^-)}Z(\tau)$.

Hence up to a phase factor $Z(\tau)$ is a modular form of weight $(u,v) = \frac{1}{2}(1-b^1+b^+, 1-b^1+b^-)$, where the group $SL(2,\mathbb{Z})$ is restricted to Γ_θ when Q is odd.

Proof. We first note that $\Omega(\tau)$ has right-inverse (and therefore also left-inverse) $-Q^{-1}\Omega(-1/\tau)Q^{-1}$:

$$\begin{aligned}
-\Omega(\tau)Q^{-1}\Omega(-1/\tau)Q^{-1} &= \frac{-1}{\tau\overline{\tau}}(\tau_1 Q + i\tau_2 G)Q^{-1}(-\tau_1 Q + i\tau_2 G)Q^{-1} \\
&= \frac{-1}{\tau\overline{\tau}}(\tau_1 Q + i\tau_2 G)(-\tau_1 Q^{-1} + i\tau_2 G^{-1}) \\
&= \frac{-1}{\tau\overline{\tau}}(\tau_1 Q + i\tau_2 G)(-\tau_1 Q^{-1} + i\tau_2 G^{-1}) \\
&= \frac{-1}{\tau\overline{\tau}}\left(-\tau_1^2 I - \tau_2^2 I - i\tau_1\tau_2(QG^{-1} - GQ^{-1})\right) \\
&= I,
\end{aligned}$$

where we used equation (4.14), i.e. $(GQ^{-1})^2 = I$. Now we apply the Poisson resummation formula (4.24) on $\Theta(\tau) = \Theta(\Omega(\tau))$:

$$\begin{aligned}
\Theta(\Omega(\tau)) &= \sum_{m\in\mathbb{Z}^{b2}} \exp(i\pi m\cdot\Omega(\tau)\cdot m) \\
&= \sum_{l\in\mathbb{Z}^{b2}} \int d^{b^2}x \exp(2\pi i l\cdot x)\exp(i\pi x\cdot\Omega(\tau)\cdot x) \\
&= \sum_{l\in\mathbb{Z}^{b2}} \int d^{b^2}x \exp(i\pi(x+\Omega^{-1}\cdot l)\cdot\Omega\cdot(x+\Omega^{-1}\cdot l))\exp(-i\pi l\cdot\Omega^{-1}\cdot l) \\
&= \frac{1}{\sqrt{\det(-i\Omega)}}\sum_{l\in\mathbb{Z}^{b2}} \exp(-i\pi l\cdot\Omega^{-1}\cdot l) \\
&= \frac{1}{\sqrt{\det(-i\Omega(\tau))}}\Theta(-\Omega(\tau)^{-1}),
\end{aligned}$$

where in the second-last step we used an identity for Gaussian integrals.

To calculate $\sqrt{\det(-i\Omega(\tau))}$, we observe that $\det Q = (1)^{b^+}(-1)^{b^-}$ and $\det(-i\Omega(\tau)) = \det(-i\Omega(\tau)Q^{-1}Q) = \det(-i\tau_1 I + \tau_2 GQ^{-1}) = \det Q$. By equation (4.14) and the fact that $\langle GQ^{-1}x, x\rangle = \langle x, x\rangle$ if $x \in \mathcal{H}^+$ and $\langle GQ^{-1}x, x\rangle = -\langle x, x\rangle$ if $x \in \mathcal{H}^-$, GQ^{-1} has b^+ eigenvalues 1 and b^- eigenvalues -1. Therefore

$$\det(-i\tau_1 Id + \tau_2 GQ^{-1}) = (-i)^{(b^+ + b^-)}\tau^{b^+}\overline{\tau}^{b^-}.$$

Altogether we get:

$$\begin{aligned}
\Theta(Q^{-1}\Omega(-1/\tau)Q^{-1}) &= \Theta(-\Omega(\tau)^{-1}) \\
&= \Theta(-\Omega(\tau)^{-1}) \\
&= \sqrt{\det(-i\Omega)}\Theta(\tau) \\
&= e^{-2\pi i(b^+ + b^-)/8}e^{2\pi i(b^-)/4}\tau^{\frac{b^+}{2}}\overline{\tau}^{\frac{b^-}{2}}\Theta(\tau) \\
&= e^{-2\pi i(b^+ - b^-)/8}\tau^{\frac{b^+}{2}}\overline{\tau}^{\frac{b^-}{2}}\Theta(\tau).
\end{aligned}$$

Since Q^{-1} is a unimodular matrix, $Q^{-1} = (Q^{-1})^t$ is an automorphism of $\mathbb{Z}^{b^2}$ and hence $\Theta(Q^{-1}\Omega(-1/\tau)Q^{-1}) = \Theta(\Omega(-1/\tau))$. Furthermore, we have the following relation

$$\mathrm{Im}(-1/\tau)^{\frac{1}{2}(b^1-1)} = (\tau\overline{\tau})^{\frac{1}{2}(1-b^1)}\mathrm{Im}(\tau)^{\frac{1}{2}(b^1-1)},$$

so that

$$Z(-1/\tau) = e^{-2\pi i(b^+ - b^-)/8}\tau^{\frac{1}{2}(1-b^1+b^+)}\overline{\tau}^{\frac{1}{2}(1-b^1+b^-)}Z(\tau).$$

To see what happens under $\tau \mapsto \tau + 1$, we notice that $\Omega(\tau+1) = \Omega(\tau) + Q$, hence

$$Z(\tau+1) = (\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)}\sum_{m\in\mathbb{Z}^{b2}} \exp(i\pi m\cdot\Omega(\tau)\cdot m)\exp(i\pi m\cdot Q\cdot m),$$

The sum on the right-hand side is equal to $Z(\tau)$ if Q is even, otherwise we have only invariance under $\tau \mapsto \tau + 2$. □

Remark 4.6. The modular weights of the partition function can be expressed in terms of two other quantities of the manifold M, namely the Euler characteristic $\chi(M) = \sum_i (-1)^i b^i$ and the Hirzebruch signature $\sigma(M) = b^+ - b^-$. By Poincaré duality we have the following equality

$$\frac{1}{2}(1 - b^1 + b^+, 1 - b^1 + b^-) = \frac{1}{4}(\chi(M) + \sigma(M), \chi(M) - \sigma(M)). \tag{4.26}$$

Theorem 4.5 can also be found in [Wit3], but there τ and $\overline{\tau}$ are interchanged.

4.3 The generalized Maxwell partition function

When we consider a $U(1)$ gauge field coupling to a spinor field $\psi(x)$ on M, $U(1)$ gauge transformations take the following form

$$\psi(x) \mapsto e^{i\chi(x)}\psi(x), \quad A \mapsto A - d\chi. \tag{4.27}$$

Again we will we assume the spinor field is a fixed background field, but the addition of such a field to the theory changes the consistency conditions for the gauge field A that resulted in (4.3). One has to take into account the extra structure that holds for the spinor fields. The result [AO] is the adjusted Dirac quantization condition for the flux of $F = dA$ through a two-cycle Σ

$$\frac{1}{2\pi}\int_\Sigma F = m + I(\Sigma,\Sigma)/2, \quad m \in \mathbb{Z}. \tag{4.28}$$

If $I(\Sigma,\Sigma)$ is odd, we see that equation (4.3) gets shifted by $1/2$. In the calculation of the partition function we needed the expansion of the saddle points as in (4.8), so this shift will change the form of the partition function. The effect is that the discrete set $\mathbb{Z}^{b^2}$ in (4.25) is translated by an element that is not an element in $\mathbb{Z}^{b^2}$. We then should not directly expect invariance of the partition function under S. The relations in the proof of Theorem 4.5 relied on the fact that the Poisson resummation took the specific form as in equation (4.24).

When we derived equation (4.25), we chose a basis $\{\alpha_I\}$ for the harmonic forms, which were the unique harmonic representatives of the generators of $F^2(M,\mathbb{Z})$. The sum over the saddle points could then be replaced by a sum over $\mathbb{Z}^{b^2}$ through (4.8). The more general setup is by considering $F^2(M,\mathbb{Z})$ as a *lattice* in the real vector space $H^2(M,\mathbb{R}) \simeq \mathbb{R}^{b^2}$ for which the cohomology classes $[\alpha_I]$ provide a basis. The intersection form Q can be extended to $H^2(M,\mathbb{R})$ to give a non-degenerate scalar product (see Remark 2.6):

$$[\alpha]\cdot[\beta] = Q([\alpha],[\beta]) = \int_M \alpha\wedge\beta. \tag{4.29}$$

Let $\hat{\Omega}(\tau)$ be the linear transformation on $H^2(M,\mathbb{R})$, which in the basis $\{[\alpha_I]\}$ can be represented by the matrix $\Omega(\tau) = \frac{\theta}{2\pi}I + \frac{4\pi i}{g^2}Q^{-1}G$, with $Q = Q_{IJ}$ and $G = G_{IJ}$ given by (4.10). We can now rewrite the partition function in (4.25) as:

$$Z(\tau) = Z_{sc}(\tau) = (\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)} \sum_{m\in F^2(M,\mathbb{Z})} \exp(i\pi m\cdot\hat{\Omega}(\tau)m) = (\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)}\Theta(\tau), \tag{4.30}$$

which we call the *scalar partition function*. The form Q is unimodular on $F^2(M,\mathbb{Z})$ and this turns $F^2(M,\mathbb{Z})$ into what is known as a *unimodular lattice*. When we have to take into account (4.28) for gauge fields coupling to spinor fields, the sum in (4.30) must be taken over $F^2(M,\mathbb{Z})+w/2$, where $w = \sum_I Q^{II}[\alpha_I] \in F^2(M,\mathbb{Z})$. This element w forms a so-called *characteristic element* that is associated to the unimodular lattice. The resulting partition function

$$Z_{sp}(\tau) = (\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)} \sum_{m\in F^2(M,\mathbb{Z})+w/2} \exp(i\pi m\cdot\hat{\Omega}(\tau)m) = (\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)}\Theta_{w/2}(\tau), \tag{4.31}$$

is called the *spinor partition function*.

The scalar and spinor partition functions can be seen as special cases of a more generalized form of (4.25), where the discrete set over which we sum can still be varied. This generalized partition function is better understood in the framework of lattices and associated theta functions. With this theory at hand we are able to proof something similar as Theorem 4.5 for the spinor partition function.

From a mathematical point of view spinor fields are sections of a spinor bundle. In order to construct such a bundle one needs a so-called *Spin^c-structure*. For oriented, Riemannian four-manifolds such a structure is always present. When the manifold is also closed and connected, the nature of a Spin^c-structure depends on whether the intersection form Q is even or odd. The consistency conditions for A that result in (4.28) are induced by this Spin^c-structure. In the context of lattices we will give a proof of this adjusted Dirac quantization condition. For this proof we will only need some basic notions concerning spinor bundles and Spin^c-structures. Good references regarding the more general theory are [Mor, LM]. What follows in this section is an adaptation of Sections 4, 6 and 7 of [OA].

4.3.1 Integral lattices

Let V be a n-dimensional real vector space. A *lattice* Λ in V of dimension n is a discrete subgroup of V isomorphic to $\mathbb{Z}^n$ defined as

$$\Lambda = \left\{ \sum_{i=1}^{n} n^i e_i, \quad n^i \in \mathbb{Z} \right\}, \tag{4.32}$$

where the vectors $e_1, \ldots, e_n$ form a basis of V. The vector space can be endowed with a real, symmetric, non-degenerate scalar product, denoted by $x \cdot y$. Such a scalar product can be represented by a non-degenerate symmetric matrix $(e_i \cdot e_j)$ in the basis $\{e_i\}$. Let $b^\pm$ be the number of positive/negative eigenvlues of this matrix, then the signature η of Λ is defined as

$$\eta(\Lambda) = b^+ - b^-. \tag{4.33}$$

This scalar product is equivalent to a symmetric, non-degenerate, bilinear form on V and the numbers $b^\pm$ correspond to the dimensions of maximal positive and negative subspaces of this form.

The *dual* lattice of Λ is defined as

$$\Lambda^* = \{x \in V \,|\, y \cdot x \in \mathbb{Z} \quad \forall y \in \Lambda\}. \tag{4.34}$$

One sees immediately from the definition that $\Lambda^{**} = \Lambda$. The lattice Λ is *integral* when Λ forms a subgroup of Λ^*. Then it is possible to define the quotient group $Z(\Lambda)$:

$$Z(\Lambda) = \Lambda^*/\Lambda. \tag{4.35}$$

An integral lattice Λ is *unimodular* if $Z(\Lambda)$ is trivial or equivalently if $\Lambda^* = \Lambda$.

The fundamental domain of the lattice is formed by the quotient V/Λ. The volume of the fundamental domain is then defined by

$$V(\Lambda) = \mathrm{vol}(V/\Lambda) := \sqrt{|\det(e_i \cdot e_j)|}, \tag{4.36}$$

where $(e_i \cdot e_j)$ is short-hand notation for the matrix representing the scalar product.

Remark 4.7. Equation (4.36) coincides with the volume of $\mathbb{R}^n/\Lambda$ if Λ is the $\mathbb{Z}$-span of n linear independent vectors $e_i \in \mathbb{R}^n$. In that case, one namely has $\mathrm{vol}(\mathbb{R}^n/\Lambda) = |\det(e_1, \ldots, e_n)| = \sqrt{|\det(e_i \cdot e_j)|}$. By the definition in equation (4.36) we implicitly put a measure on the vector space V.

A standard basis for the dual basis is given by vectors $f_1, \ldots, f_n$ such that $f_i \cdot e_j = \delta_{ij}$. One then checks that

$$V(\Lambda^*)V(\Lambda) = \sqrt{|\det(f_i \cdot f_j)|}\sqrt{|\det(e_i \cdot e_j)|} = |\det(f_i \cdot e_j)| = \det(\delta_{ij}) = 1.$$

In combination with the fact that $V/\Lambda \simeq Z(\Lambda) \oplus V/\Lambda^*$, we have

$$V(\Lambda) = (V(\Lambda^*))^{-1} = \sqrt{|Z(\Lambda)|}. \tag{4.37}$$

We see that for unimodular lattices $\det(e_i \cdot e_j) = 1$, since $|Z(\Lambda)| = 1$. The matrix $(e_i \cdot e_j)$ is then unimodular, i.e. it is integer-valued and has determinant one. Conversely, if in a chosen basis of the lattice the matrix representing the scalar product is unimodular, then the lattice is unimodular.

Let Λ be an integral lattice. The lattice Λ is said to be *even* if $x \cdot x$ is even for all $x \in \Lambda$, otherwise Λ is *odd*. An element $c \in \Lambda$ is said to be a *characteristic* element if

$$c \cdot x + x \cdot x \in \mathbb{Z}, \quad \forall x \in \Lambda. \tag{4.38}$$

According to Lemma 5.2 in Chapter II of [MH] a unimodular lattice always possesses a characteristic element, which is then uniquely defined modulo 2 and satisfies

$$c \cdot c \equiv \eta(\Lambda) \mod 8. \tag{4.39}$$

For unimodular lattices this identity can be proven using Milgram's formula, which we will cover (see equation (4.66)) when discussing vector modular forms in Section 4.3.2. Equation (4.39) shows that an even lattice must have a signature that is divisible by 8, since $c = 0$ forms a characteristic element.

Classification of unimodular forms and lattices

A scalar product restricted to a lattice gives a symmetric bilinear form on the lattice. For integral lattices Λ this form is said to be *unimodular* if it induces a bijection between Λ and $\mathrm{Hom}(\Lambda, \mathbb{Z})$. From the definition we see that this is equivalent to the lattice being unimodular. These forms are odd or even if the corresponding lattices are odd or even. Since the intersection form Q is unimodular, $F^2(M, \mathbb{Z})$ forms a unimodular lattice in $H^2(M, \mathbb{R})$ if the scalar product is given by (4.29). The following definition will give us a notion through which we can classify lattices.

Definition 4.8. Given two lattices Λ, Λ' with a symmetric bilinear form Q and Q' on it. The pairs (Λ, Q) and (Λ', Q') are said to be *isomorphic* if there is a bijection $f : \Lambda \to \Lambda'$ such that $Q'(f(a), f(b)) = Q(a, b)$ for all a, b in Λ. We say that a form Q on Λ is *equivalent* to a matrix Q_{ij} if there is a choice of basis $\{e_1, \ldots, e_{\mathrm{rank}(\Lambda)}\}$ for the lattice Λ such that $Q(e_i, e_j) = Q_{ij}$. The pair (Λ, Q) is then seen to be isomorphic to $(\mathbb{Z}^{\mathrm{rank}(\Lambda)}, Q_{ij})$. When there can be no confusion, we will denote the matrix Q_{ij} and the form Q by the same symbol.

Hasse and Minkowski have given a classification of the isomorphism classes of unimodular forms on lattices that are indefinite (neither positive-definite or negative-definite). This classification has become known as the Hasse–Minkowski theorem and is given in terms of the unimodular matrices the forms are equivalent to. More details can be found in Chapter II of [MH] or Chapter V of [Ser].

Theorem 4.9 (Hasse–Minkowski)**.** *Let Λ be a lattice with an indefinite unimodular form on it.*

i) If the form is odd, then it is equivalent to $l(1) \oplus m(-1)$, with $\mathrm{rank}(\Lambda) = m + l$ and $\eta(\Lambda) = l - m$.

ii) If the form is even and $\eta(\Lambda) \geq 0$, then it is equivalent to

$$l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus mE_8,$$

where E_8 is the following matrix defining a positive-definite even form of rank 8:

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & & & & & \\ 0 & 2 & 0 & -1 & & & & \\ -1 & 0 & 2 & -1 & & & & \\ & -1 & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}.$$

Here $m + l = \text{rank}(\Lambda)$ and $\eta(\Lambda) = m$. For $\eta(\Lambda) < 0$ the same result holds, but then with E_8 replaced by $-E_8$ and $m = |\eta(\Lambda)|$.

In short, any indefinite unimodular form is defined by its rank, signature and its type (even or odd).

Example 4.10. The forms defined by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are of different type, hence are not equivalent to each other. Therefore two lattices ($\simeq \mathbb{Z}^2$) endowed with these two different forms are not isomorphic to each other.

The form given by the matrix

$$Q' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus m(-1) \tag{4.40}$$

is equivalent to $Q = (1) \oplus (m+1)(-1)$ according to Theorem 4.9, since Q' is odd. The matrix

$$B = \begin{pmatrix} 1 & 0 & 1 & \\ 1 & 1 & 0 & O \\ -1 & -1 & -1 & \\ & O & & I_{m-1} \end{pmatrix}, \tag{4.41}$$

represents an automorphism of the lattice (the matrix is integer-valued and $\det(B) = 1$) and has the property that $B^t Q' B = Q$. This matrix represents a different choice of basis for the lattice and hence realizes the equivalence of the two forms.

Remark 4.11. For definite unimodular forms the classification problem becomes more difficult. The odd positive-definite /negative-definite forms are isomorphic to $m(1)/m(-1)$, where $m = \text{rank}(\Lambda)$. The possibilities for the rank of definite even forms is restricted to multiples of 8 due to (4.39). The number of isomorphism classes of unimodular forms grows rapidly with the rank and the classification is generally considered as extraordinarily difficult. Chapter II of [MH] contains a discussion of this problem.

However, when these definite unimodular forms are realized as intersection forms of smooth, oriented, simply-connected, closed four-manifolds, there are strong theorems that put a restriction on what is possible. Rokhlin's theorem ([DK], Theorem 1.2.6) states that the signature of a smooth, closed, oriented, spin (see Definition 4.18) four-manifold is divisble by 16. As a consequence, such manifolds cannot have $\pm E_8$ as intersection form. Donaldson's diagonalisability theorem ([DK], Theorem 1.3.1) states the following: the only possible negative-definite forms realized as the intersection form of a smooth, oriented, simply connected, closed four-manifold are isomorphic to $m(-1)$. By reversing the orientation of X we see that the only possible positive-definite forms are then isomorphic to $m(1)$.

Constructions for odd unimodular lattices

If Λ is an odd unimodular lattice, we can construct some derived lattices using the characteristic element.

- Since $c/2 \notin \Lambda$, the union
$$\Lambda_{total} = \Lambda \cup (\Lambda + c/2), \tag{4.42}$$
defines a new lattice. We see $\Lambda_{total} \simeq \mathbb{Z}_2 \oplus \Lambda$ or equivalently $\Lambda_{total}/\Lambda \simeq \mathbb{Z}_2$. Similarly as in the derivation of (4.37) $V(\Lambda_{total}) = V(\Lambda)/2 = 1/2$.

- We can decompose Λ as
$$\Lambda = \Lambda_{even} \cup \Lambda_{odd}, \tag{4.43}$$
where $x \cdot x \in 2\mathbb{Z}$ for $x \in \Lambda_{even}$ and where $x \cdot x \in 2\mathbb{Z}+1$ for $x \in \Lambda_{odd}$. Since for any $v \in \Lambda_{odd}$, we can write $\Lambda_{odd} = \Lambda_{even} + v$, we have $\Lambda/\Lambda_{even} \simeq \mathbb{Z}_2$ and $V(\Lambda_{even}) = 2$. Furthermore $c \in \Lambda_{odd/even}$ if $\eta(\Lambda)$ is respectively even or odd by equation (4.39).

- The lattice Λ_{total} equals the dual lattice Λ^*_{even}. This can be seen by the fact that Λ is already integral and $c/2 \cdot y \equiv (1/2)y \cdot y \equiv 0 \mod 1$. Let ϕ be the map
$$x \mapsto x + c/2.$$
By making use of
$$(y + c) \cdot (y + c) \equiv c \cdot c - y \cdot y \mod 2,$$
we see that if $c \in \Lambda_{even}$
$$\phi^2(\Lambda_{even}) = \Lambda_{even}, \quad \phi^2(\Lambda_{odd}) = \Lambda_{odd},$$
and if $c \in \Lambda_{odd}$
$$\phi^2(\Lambda_{even}) = \Lambda_{odd}, \quad \phi^2(\Lambda_{odd}) = \Lambda_{even}.$$
We can write
$$\Lambda_{total} = \Lambda_{even} \cup (\Lambda_{even} + c/2) \cup \Lambda_{odd} \cup (\Lambda_{odd} + c/2) \tag{4.44}$$
and according to the equations above the coset structure is given by
$$Z(\Lambda_{even}) = \Lambda_{total}/\Lambda_{even} = \begin{cases} \mathbb{Z}_4 & \text{if } c \in \Lambda_{odd}, \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } c \in \Lambda_{even}. \end{cases} \tag{4.45}$$
When $c \in \Lambda_{even}$, the second $\mathbb{Z}_2$ action comes from the translation $x \mapsto x + \lambda_v$, where λ_v is any element in Λ_{odd}. Accordingly, we then obtain two new lattices
$$\Lambda' := \Lambda_{even} \cup (\Lambda_{odd} + c/2), \quad \Lambda'' := \Lambda_{even} \cup (\Lambda_{even} + c/2), \tag{4.46}$$
with $V(\Lambda') = V(\Lambda'') = V(\Lambda_{even})/2 = 1$. Hence, when $c \cdot c$ is a multiple of four, these lattices become integral hence unimodular and are odd or even depending on whether $c \cdot c/4$ is.

General Poisson resummation formula

Let Λ be an integral lattice in a real vector space V endowed with scalar product $x \cdot y$. The following lemma is then true [Sch].

Lemma 4.12. *If $g : V \to \mathbb{C}$ is smooth and Λ-periodic, then*

$$g(x) = \sum_{v \in \Lambda^*} c_v \exp(2\pi i v \cdot x), \tag{4.47}$$

where

$$c_v = \frac{1}{V(\Lambda)} \int_{V/\Lambda} g(y) \exp(-2\pi i y \cdot v) dy.^{2}$$

Let f be a smooth and fast-decaying function. Then $g(x) = \sum_{v\in\Lambda} f(x+v)$ is smooth and Λ-periodic and we can write

$$\sum_{v\in\Lambda} f(x+v) = \sum_{l\in\Lambda^*} c_l \exp(2\pi l \cdot x). \tag{4.48}$$

As an application of this lemma we obtain the next proposition.

Proposition 4.13. *Let $f : V \to \mathbb{C}$ be the function defined by $f(x) = \exp(i\pi x \cdot \hat{\Omega} x)$, where $\hat{\Omega}$ acts as a linear transformation on $x \in V$, represented by a complex symmetric invertible matrix $\tilde{\Omega}$ in a given basis $\{e_i\}$ of Λ. If the imaginary part of $\tilde{\Omega}$ is positive-definite with respect to the given scalar product, i.e. when $x \cdot \mathrm{Im}(\hat{\Omega}x) > 0$ for all $x \in V - \{0\}$, then*

$$\sum_{v\in\Lambda} f(x+v) = \sum_{v\in\Lambda} \exp(i\pi(x+v)\cdot\hat{\Omega}(x+v)) = \frac{(-i)^{b^-}}{V(\Lambda)\sqrt{\det(-i\tilde{\Omega})}} \sum_{l\in\Lambda*} \exp(-i\pi l\cdot\hat{\Omega}^{-1}l + 2\pi i x\cdot l), \tag{4.49}$$

where b^- is the number of negative eigenvalues of the matrix $(e_i \cdot e_j)$ representing the scalar product.

Proof. The function f satisfies the conditions of Lemma 4.12, so using equation (4.48) we get:

$$\begin{aligned}
\sum_{v\in\Lambda} \exp(i\pi(x+v)\cdot\hat{\Omega}(x+v)) &= \sum_{v\in\Lambda}\sum_{l\in\Lambda^*} \frac{1}{V(\Lambda)} \int_{V/\Lambda} dy \exp(i\pi(y+v)\cdot\hat{\Omega}(y+v)) \\
&\quad \times \exp(-2\pi i y\cdot l + 2\pi i x\cdot l) \\
&= \sum_{l\in\Lambda^*} \frac{1}{V(\Lambda)} \int_V dy \exp(i\pi y\cdot\hat{\Omega}y) \exp(-2\pi i y\cdot l + 2\pi i x\cdot l) \\
&= \sum_{l\in\Lambda^*} \frac{1}{V(\Lambda)} \int_V dy \exp(i\pi(y-\hat{\Omega}^{-1}l)\cdot\hat{\Omega}(y-\hat{\Omega}^{-1}l)) \\
&\quad \times \exp(-i\pi l\cdot\hat{\Omega}^{-1}l + 2\pi i x\cdot l) \\
&= \frac{1}{V(\Lambda)\sqrt{\det(-i\tilde{\Omega})}} \frac{V(\Lambda)}{\sqrt{\det(e_i\cdot e_j)}} \sum_{l\in\Lambda*} \exp(-i\pi l\cdot\hat{\Omega}^{-1}l + 2\pi i x\cdot l) \\
&= \frac{(-i)^{b^-}}{V(\Lambda)\sqrt{\det(-i\tilde{\Omega})}} \sum_{l\in\Lambda*} \exp(-i\pi l\cdot\hat{\Omega}^{-1}l + 2\pi i x\cdot l),
\end{aligned}$$

where we used the fact that $a\cdot\hat{\Omega}^{-1}b = ((\hat{\Omega}^{-1})^t a)\cdot b = (\hat{\Omega}^{-1}a)\cdot b$, since $\hat{\Omega}$ could be represented by a symmetric matrix. □

[2] The measure dy is determined through the isomorphism $\mathbb{R}^n \simeq V$, which maps the orthonormal basis vectors $x_1, \ldots, x_n$ of $\mathbb{R}^n$ to the basis $e_1, \ldots, e_n$ of Λ. In this case $dy = V(\Lambda)dx$

Remark 4.14. In equation (4.49) there is a sign ambiguity, since at several steps in the derivation we had to take the square root of -1 and this can be set to either $+i$ or $-i$. We will set it to $+i$.

4.3.2 Theta functions and vector modular forms

For an integral lattice Λ in a vector space V, the lattice Λ^* has the following $Z(\Lambda)$ coset decomposition

$$\Lambda^* = \Lambda \cup (\lambda_1 + \Lambda) \cup \ldots \cup (\lambda_{|Z(\Lambda)|-1} + \Lambda), \tag{4.50}$$

where λ_α is a representative of the α-th coset and $\lambda_0 = 0$ is just the identity in the group structure of Λ^*. Recall that $V(\Lambda) = \sqrt{|Z(\Lambda)|}$. Let $\hat{\Omega}$ be a linear transformation on V satisfying the conditions in Proposition 4.13. Setting $x = \lambda_\alpha \in \Lambda^*$ in (4.49) gives us the relation

$$\sum_{v\in\lambda_\alpha+\Lambda} \exp(i\pi v \cdot \hat{\Omega} v) = \frac{(-i)^{b^-}}{\sqrt{|Z(\Lambda)|}\sqrt{\det(-i\tilde{\Omega})}} \sum_{\beta=0}^{|Z(\Lambda)|-1} \exp(2\pi i \lambda_\alpha \cdot \lambda_\beta) \sum_{l\in\lambda_\beta+\Lambda} \exp(-i\pi l \cdot \hat{\Omega}^{-1} l). \tag{4.51}$$

Now we assume in addition, that in a chosen basis $\{e_i\}$ for Λ, $\hat{\Omega} = \hat{\Omega}(\tau)$ is represented by the matrix

$$\tilde{\Omega} = \tilde{\Omega}(\tau) = \tau_1 I + i\tau_2 \tilde{G}, \tag{4.52}$$

where $\tau_2 > 0$ and $\tilde{G}$ satisfies $\tilde{G}^2 = I$. The matrix $\tilde{G}$ has eigenvalues ± 1. If we recall the discussion before (4.33), the matrix $(e_i \cdot e_j)$ has b^+ positive eigenvalues and b^- negative eigenvalues. The positive-definiteness of $\tilde{G}$ with respect to the scalar product implies that $\tilde{G}$ must have $b^\pm$ eigenvalues ± 1. Similarly as in the proof of Theorem 4.5 we now have

$$\det(-i\tilde{\Omega}) = (-i)^{(b^+ + b^-)} \tau^{b^+} \overline{\tau}^{b^-},$$

and

$$\tilde{\Omega}^{-1}(-1/\tau) = -\tilde{\Omega}(\tau).$$

Notice that the equation above implies that the corresponding linear transformation on V satisfies $\hat{\Omega}^{-1}(-1/\tau) = -\hat{\Omega}(\tau)$.

Now we define $|Z(\Lambda)|$ theta functions by

$$\Theta_\alpha(\tau) = \sum_{l\in\lambda_\alpha+\Lambda} \exp(i\pi l \cdot \hat{\Omega}(\tau) l). \tag{4.53}$$

Then the Poisson resummation formula (4.51) and the identities above give us the following relation between the theta functions

$$\Theta_\alpha(\tau) = \tau^{-\frac{b^+}{2}} \overline{\tau}^{-\frac{b^-}{2}} \frac{e^{2\pi i \eta/8}}{\sqrt{|Z(\Lambda)|}} \sum_{\beta=0}^{|Z(\Lambda)|-1} e^{2\pi i \lambda_\alpha \cdot \lambda_\beta} \Theta_\beta(-1/\tau), \tag{4.54}$$

where $\eta = \eta(\Lambda) = b^+ - b^-$. One can see furthermore that

$$\Theta_\alpha(\tau+1) = e^{\pi i \lambda_\alpha^2} \Theta_\alpha(\tau) \quad \text{for even } \Lambda, \tag{4.55}$$

$$\Theta_\alpha(\tau+2) = e^{2\pi i \lambda_\alpha^2} \Theta_\alpha(\tau) \quad \text{otherwise.} \tag{4.56}$$

The relations above fit properly into the framework of vector modular forms, if we recognize them as actions of the elements S, T and T^2 in $SL(2,\mathbb{Z})$. The setup is similar as in Section 4.2. Let B be the matrix

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}),$$

and $f : \mathbb{H} \to \mathbb{C}$ a function of τ and $\overline{\tau}$. Then we define the following action of B on f

$$\hat{A}_{k_1,k_2}(B)f(\tau) = (c\tau + d)^{-k_1}(c\overline{\tau} + d)^{-k_2} f\left(\frac{a\tau + b}{c\overline{\tau} + d}\right) \tag{4.57}$$

By extending this action linearly over f we get a group action of $SL(2,\mathbb{R})$, i.e. for two $B, B' \in SL(2,\mathbb{R})$ we have

$$\hat{A}_{k_1,k_2}(B)\hat{A}_{k_1,k_2}(B') = \hat{A}_{k_1,k_2}(BB'). \tag{4.58}$$

This again contains a sign ambiguity if k_1 and k_2 are not integers, but this is resolved by looking at the action of the metaplectic cover of $SL(2,\mathbb{Z})$.

Definition 4.15. Let $(f_1(\tau), \ldots, f_N(\tau))$ be a vector column of functions $f_i : \mathbb{H} \to \mathbb{C}$ and $\hat{A}_{k_1,k_2}(B)$ be defined as in (4.57) for $B \in SL(2,\mathbb{Z})$. Then $(f_1, \ldots, f_N)$ is said to be a *vector modular form* of weight (k_1, k_2) if for all $B \in SL(2,\mathbb{Z})$ (or a subgroup thereof) there exists matrices $D_{\beta\alpha}(B)$, such that

$$\hat{A}_{k_1,k_2}(B)f_\alpha(\tau) = \sum_{\beta=1}^{N} f_\beta(\tau)D_{\beta\alpha}(B). \tag{4.59}$$

Just as in Proposition 4.3 one only needs to check this condition for the generators S and T (or T^2 if one considers the Hecke subgroup Γ_θ). More details can be found in Chapter 13 of [Kac] and Chapter 1 of [Bru].

If the functions $\{f_i\}$ in the definition of a vector modular form are linear independent, the group structure in (4.58) gets reflected in the matrices:

$$D(B)D(B') = D(BB'). \tag{4.60}$$

Furthermore, we have the following (trivial) identities

$$D_{\alpha\beta}(I) = \delta_{\alpha\beta}, \quad D_{\alpha\beta}(-I) = (-1)^{k_1-k_2}\delta_{\alpha\beta}. \tag{4.61}$$

Now we look at the column vector with as entries the theta functions $\Theta_\alpha(\tau)$ as in (4.53). According to (4.55) and (4.56), we have

$$\begin{aligned} \hat{A}_{\frac{b^+}{2},\frac{b^-}{2}}(T)\Theta_\alpha(\tau) &= e^{\pi i \lambda_\alpha^2}\Theta_\alpha(\tau) \quad \text{for even } \Lambda, \\ \hat{A}_{\frac{b^+}{2},\frac{b^-}{2}}(T^2)\Theta_\alpha(\tau) &= e^{2\pi i \lambda_\alpha^2}\Theta_\alpha(\tau) \quad \text{otherwise.} \end{aligned}$$

Applying the identity ([Ser], Section VI.1)

$$\frac{1}{|Z(\Lambda)|}\sum_{\beta=0}^{|Z(\Lambda)|-1} e^{2\pi i(\lambda_\gamma - \lambda_\alpha)\cdot\lambda_\beta} = \delta_{\lambda_\gamma - \lambda_\alpha, 0} \tag{4.62}$$

to equation (4.54) one can deduce

$$A_{\frac{b^+}{2},\frac{b^-}{2}}(S)\Theta_\alpha(\tau) = \sum_{\beta=0}^{|Z(\Lambda)|-1} \frac{e^{-2\pi i\eta/8}}{\sqrt{|Z(\Lambda)|}} e^{-2\pi i \lambda_\alpha\cdot\lambda_\beta}\Theta_\beta(\tau).$$

Hence we conclude that the column vector $(\Theta_0, \ldots, \Theta_{|Z(\Lambda)|-1})$ is a vector modular form of weight $(b^+/2, b^-/2)$ with

$$D_{\alpha\beta}(S) = \frac{e^{-2\pi i\eta/8}}{\sqrt{|Z(\Lambda)|}} e^{-2\pi i\lambda_\alpha\cdot\lambda_\beta}, \tag{4.63}$$

$$D_{\alpha\beta}(T) = e^{\pi i\lambda_\alpha^2}\delta_{\alpha\beta} \quad \text{for even } \Lambda, \tag{4.64}$$

$$D_{\alpha\beta}(T^2) = e^{2\pi i\lambda_\alpha^2}\delta_{\alpha\beta} \quad \text{otherwise.} \tag{4.65}$$

It is immediately seen that $D(T)$ and $D(T^2)$ are unitary matrices. The identity in (4.62) shows that $D(S)$ is a unitary matrix as well and

$$D(S)^2 = D(-I)P,$$

where $P = \delta_{\lambda_\alpha+\lambda_\beta,0}$. We see that this conflicts (4.60), since one has $S^2 = -I$. This is due to the fact that if $\lambda_\alpha = -\lambda_\beta$ for some representatives $\lambda_\alpha, \lambda_\beta$, then $\Theta_\alpha(\tau) = \Theta_\beta(\tau)$, which means the condition on linear independence is not valid anymore. In any case, when acted upon through (4.59), the condition $S^2 = -I$ is still consistent with (4.58), since the matrix P acts as the identity on the theta functions. By using equation (4.58) and $(ST)^3 = -I$, it can be computed ([OA], pages 22-23) that for even lattices

$$\frac{1}{\sqrt{|Z(\Lambda)|}} \sum_{\gamma=0}^{|Z(\Lambda)|-1} e^{\pi i \lambda_\gamma^2} = e^{2\pi i \eta(\Lambda)/8}. \tag{4.66}$$

This result is known as *Milgram's formula*.

Remark 4.16. Since all λ_α lie in Λ^*, the matrix elements in the equations (4.63)-(4.65) are invariant under translations by elements in the lattice, i.e. if $\lambda_\alpha \mapsto \lambda_\alpha + l$, for some $l \in \Lambda$, the matrix elements remain the same. Therefore the matrix elements are independent of the choice of representatives in the coset decomposition (4.50). Since the coset decomposition was independent of the choice of the matrix $\tilde{G}$, the matrices in (4.63)-(4.65) do not depend on these parameters as well. Later on we will see that the matrix $Q^{-1}G$, with Q and G as in (4.9), has the same properties as we listed for the matrix $\tilde{G}$ in (4.52). The modular properties of the partition function that will be summarized in Section 4.3.4 are therefore only dependent on the *topology* of the underlying manifolds. As we will see in Section 4.4 the remaining $b^+ \times b^-$ parameters in G will define different partition functions (up to some discrete symmetries), but will have no effect on the modular properties.

Applications to unimodular lattices

For unimodular lattices the quotient group $Z(\Lambda)$ is trivial, hence only one theta function is associated to these lattices and the modular group action (4.58) will transform the theta function by multiplication with a phase factor. For even unimodular lattices, Milgram's formula implies $\exp(2\pi i\eta/8) = 1$ and by (4.63) and (4.64) the action of the modular group is trivial, i.e.

$$D(S) = D(T) = 1.$$

Odd lattices still can have any signature and the action of the Hecke subgroup is given by

$$D(S) = e^{-2\pi i\eta/8}, \quad D(T^2) = 1.$$

The object of our main interest will be $\Lambda_{total} = \Lambda^*_{even}$ as in (4.42). Using the decomposition (4.44), we can take as coset representatives in (4.50)

$$0, \quad \lambda_v, \quad \lambda_s = c/2, \quad \lambda_t = \lambda_v + c/2, \tag{4.67}$$

where λ_v is any element of Λ_{odd}. As a short intermezzo, we can deduce (4.39) by an application of Milgram's formula (4.66) and property (4.38) for c:

$$e^{2\pi i\eta/8} = \frac{1}{2} \sum_{\gamma=0,v,s,t} e^{\pi i \lambda_\gamma^2} = e^{2\pi i c^2/8}.$$

Associated to these coset representatives are four theta functions $\Theta_0(\tau), \Theta_v(\tau), \Theta_s(\tau)$ and $\Theta_t(\tau)$, which are defined through (4.53). Using (4.63), (4.64), (4.38), (4.39) and denoting $\omega = e^{-2\pi\eta/8}$, we compute $D(S)$ and $D(T)$:

$$D(S) = \frac{\omega}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \omega^2 & -\omega^2 \\ 1 & -1 & -\omega^2 & \omega^2 \end{pmatrix}, \tag{4.68}$$

$$D(T) = \mathrm{diag}(1, -1, \omega^{-1}, \omega^{-1}). \tag{4.69}$$

Let us fix the notation $\hat{A}_{\frac{b+}{2},\frac{b-}{2}}(B) = \hat{A}(B)$ for $B \in SL(2,\mathbb{Z})$. One can check that $\Theta_s - \Theta_t$ spans a subspace invariant under $SL(2,\mathbb{Z})$:

$$\hat{A}(S)(\Theta_s - \Theta_t) = \omega^3(\Theta_s - \Theta_t), \quad \hat{A}(T)(\Theta_s - \Theta_t) = \omega^{-1}(\Theta_s - \Theta_t).$$

This function vanishes if $\eta(\Lambda)$ is odd [OA].

For the subspace orthogonal to $\Theta_s - \Theta_t$, we choose the orthogonal basis $\Theta := \Theta_0 + \Theta_v$, $\Theta^T := \Theta_0 - \Theta_v$ and $\Theta^{ST} := \omega(\Theta_s + \Theta_t)$. For this basis one has

$$\Theta^{ST} = \hat{A}(S)(\Theta^T) = \hat{A}(S)\hat{A}(T)(\Theta) = \hat{A}(ST)(\Theta).$$

If one recognises Θ as the theta function corresponding to Λ, we see that the relation above corresponds to the decomposition of $SL(2,\mathbb{Z})$ as in (4.22), since Θ is invariant under Γ_θ. Under the action of S these functions get permuted into each other (up to phase factors):

$$\hat{A}(S)(\Theta) = \omega\Theta, \quad \hat{A}(S)(\Theta^T) = \Theta^{ST}, \quad \hat{A}(S)(\Theta^{ST}) = \omega^2\Theta^T. \tag{4.70}$$

The action of T results in

$$\hat{A}(T)(\Theta) = \Theta^T, \quad \hat{A}(T)(\Theta^T) = \Theta, \quad \hat{A}(T)(\Theta^{ST}) = \omega^{-1}\Theta^{ST}. \tag{4.71}$$

In fact this shows that the subspace spanned by these vectors is invariant under $SL(2,\mathbb{Z})$ and the rest of the actions of $SL(2,\mathbb{Z})$ can then be computed through combinations of (4.70) and (4.71).

Remark 4.17. There are some special cases depending on the values of the signature $\eta(\Lambda)$.

- $\eta(\Lambda)$ *is odd:* $\Theta_s - \Theta_t$ vanishes.
- $\eta(\Lambda)/4$ *is integral:* $c \cdot c/4$ is an integer and hence the two lattices in (4.46) are unimodular and odd or even depending on whether $\eta(\Lambda)/4$ is. Associated to these lattices are the theta functions $\Theta_0 + \Theta_s$ and $\Theta_0 + \Theta_t$, for which we have

  $$\hat{A}(S)(\Theta_0 + \Theta_s) = \omega(\Theta_0 + \Theta_s), \quad \hat{A}(T)(\Theta_0 + \Theta_s) = \Theta_0 + \omega\Theta_s,$$
  $$\hat{A}(S)(\Theta_0 + \Theta_t) = \omega(\Theta_0 + \Theta_t), \quad \hat{A}(T)(\Theta_0 + \Theta_s) = \Theta_0 + \omega\Theta_t,$$

  where $\omega = (-1)^{\eta(\Lambda)/4}$ now equals ± 1.
- $\eta(\Lambda)/8$ *is integral:* according to the relations above the linear combination

  $$2\Theta_0 + \Theta_s + \Theta_t = \Theta + \Theta^T + \Theta^{ST}$$

  is now a modular function for which $D(S) = D(T) = 1$.
- $\eta(\Lambda) \equiv 4 \mod 8$*:* according to (4.71) the linear combination

  $$\Theta - \Theta^T + \Theta^{ST}$$

  is now a modular function with $D(S) = D(T) = -1$.

4.3.3 Spin^c-structures and the adjusted Dirac quantization condition

An oriented, Riemannian n-manifold M has the property that the tangent bundle has structure group $SO(n)$ (see Section 3.1). It is known that if the second Stiefel-Whitney class $w_2(TM)$ of the tangent bundle vanishes there exists a lift of the structure group of the tangent bundle TM to its double cover $Spin(n)$ (Lemma 3.1.1, [Mor]). This means that we can find an open cover $\{U_\alpha\}$ of M and trivializations of TM over that cover such that we have transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(n)$ and $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(n)$ that satisfy $\rho \circ \tilde{g}_{\alpha\beta} = g_{\alpha\beta}$ for the double cover map $\rho : Spin(n) \to SO(n)$.

Definition 4.18. For an oriented, Riemannian n-manifold a lift of the structure group of the tangent bundle TM to its double cover $Spin(n)$ is called a *spin structure* on M. In this case the manifold M is said to be *spin*.

When such a lift of the structure group $SO(n)$ to $Spin(n)$ does not exist, there can still be another structure that is useful, called the *Spin^c-structure*. This Spin^c-structure consists of a bundle which can be constructed from the tangent bundle and has structure group $Spin^c(n)$, the Spin^c-group. The Spin^c-group is defined as

$$Spin^c(n) := SO(n) \times_{\mathbb{Z}_2} U(1) = SO(n) \times U(1)/\{(\pm I, \pm I)\}, \tag{4.72}$$

where the elements in the product get identified through the diagonal action of $\mathbb{Z}_2 \simeq \{(\pm I, \pm I)\}$.

Such a lift always exists when the oriented, Riemannian manifold M is four-dimensional (Lemma 3.1.2, [Mor]). This is equivalent to the fact that on these manifolds the second Stiefel-Whitney class $w_2(TM) \in H^2(M, \mathbb{Z}/2)$ can always be lifted to an element of $H^2(M, \mathbb{Z})$. Spin^c-structures therefore always exist for the manifolds we considered in the previous sections. A Spin^c-structure is necessary to construct spinor bundles. If a gauge field A couples to a fixed background spinor field, which is a section of this bundle, the Spin^c-structure will again induce consistency conditions for A, similar as the cocycle condition did for a connection on a line bundle. The result is the Dirac quantization condition as in (4.28). We will give a proof of this condition using ideas from [AO] and adapt the argument following Sections 3.1-3.3 of [Mor].

Associated to a spinor bundle P is the determinant line bundle $\det P$. The form $2A$ is then a connection on this line bundle, so we are back to the case in Section 4.1. Let $2F$ be the 2-form field strength of $2A$. For the cohomology class of $2F$ one has the relation

$$\frac{1}{2\pi}[2F] = c_1(\det P) \equiv w_2(TM) \mod 2,$$

where $w_2(TM)$ is the second Stiefel-Whitney class. In the case F is harmonic, we can apply (4.8) to $2F$:

$$\frac{2F}{2\pi} = \sum_I n^I \alpha_I, \quad n^I \in \mathbb{Z},$$

where the α_I are the unique harmonic representatives of the generators for $F^2(M, \mathbb{Z})$. Combining the two relations, we obtain

$$\frac{1}{2\pi} F = \sum_I m^I \alpha_I + w/2, \quad m^I \in \mathbb{Z}, \tag{4.73}$$

where w is the unique harmonic representative of the lift $w_2(TM)$ into $H^2(M, \mathbb{Z})$. This w should be torsion free, since it can be written as a linear combination of the α_I.

The connection $2A$ enters in the Dirac operator D_{2A} on the spinor bundle P ([Mor], Section 3.3). With the Atiyah-Singer index theorem we can compute its index. If we write $\sum_I m^I[\alpha_I] = m \in F^2(M,\mathbb{Z})$ and $c_1(\det P) = \frac{[2F]}{2\pi} = 2m + [w] = 2m + w$, we get

$$\begin{aligned}\operatorname{ind} D_{2A} &= \frac{1}{8}\int_M c_1(\det P)^2 - \frac{\sigma(M)}{8} \\ &= \frac{1}{8}Q(w+2m, 2m+w) - \frac{\sigma(M)}{8} \\ &= \frac{1}{8}(Q(w,w) - \sigma(M)) + \frac{1}{2}(Q(m,w) + Q(m,m)),\end{aligned}$$

where $\sigma(M)$ is given by (2.13). From the integrality of the index we deduce

$$Q(m,w) + Q(m,m) \in 2\mathbb{Z} \tag{4.74}$$

$$Q(w,w) \equiv \sigma(M) \mod 8. \tag{4.75}$$

In the context of Section 4.3.1, we can see $F^2(M,\mathbb{Z})$ as a unimodular lattice in $H^2(M,\mathbb{R})$, where the scalar product is given by (4.29). The signature of this lattice is $\sigma(M) = b^+ - b^-$ and (4.74) tells us w is a characteristic element of $F^2(M,\mathbb{Z})$. Equation (4.75) agrees with (4.39), but is now proved by using the Atiyah-Singer index theorem. We can determine w modulo 2 (which should be unique) by making use of (4.73). Writing $w = \sum_I w^I \alpha_I$ and choosing $m_I = -\sum \tilde{\alpha}^I$, where $\tilde{\alpha}^I$ is the Poincaré dual of Σ_I (see (4.7) and (4.12)), we get

$$w^I = Q(\tilde{\alpha}^I, \sum_J w^J \alpha_J) \equiv Q(\tilde{\alpha}^I, \tilde{\alpha}^I) \mod 2 \equiv Q^{II} \mod 2.$$

Therefore F as in (4.73) can be rewritten as

$$F = 2\pi \sum_I m^I \alpha_I + 2\pi w/2 = 2\pi \sum_I m^I \alpha_I + 2\pi \sum_I \frac{Q^{II}}{2}\alpha_I, \quad m^I \in \mathbb{Z} \tag{4.76}$$

This is the result we need in Section 4.3.4. Using this expansion for a harmonic F we can prove equation (4.28).

Recall that using the Poincaré duals $\tilde{\alpha}^I$ of Σ_I, we can write $Q^{II} = I(\Sigma_I, \Sigma_I) = Q(\tilde{\alpha}^I, \tilde{\alpha}^I)$. Furthermore, any $\Sigma \in F_2(M,\mathbb{Z})$ can be written as $\Sigma = \sum_I n^I \Sigma_I$ with $n^I \in \mathbb{Z}$. We can directly restrict to $F_2(M,\mathbb{Z})$, since torsion cycles give zero on the left- and right-hand side of equation (4.28). Using equation (4.74) we now compute

$$\begin{aligned}I(\Sigma,\Sigma) &= Q\left(\sum_I n^I \tilde{\alpha}^I, \sum_J n^J \tilde{\alpha}^J\right) \\ &= \sum_I n^I Q\left(\tilde{\alpha}^I, \sum_J Q^{JJ}\alpha_J\right) + 2k, \quad k \in \mathbb{Z} \\ &= \sum_I n^I Q\left(\tilde{\alpha}^I, \tilde{\alpha}^I\right) + 2k', \quad k' \in \mathbb{Z} \\ &= \sum_I n^I Q^{II} + 2k', \quad k' \in \mathbb{Z}.\end{aligned}$$

Integrating $F/2\pi$ over a 2-cycle Σ then results in equation (4.28). Notice that any other curvature F' is cohomologous to a harmonic curvature F, i.e. $F' = F + d\beta$ for some 1-form β. The integral of $d\beta$ over a two-cycle vanishes, hence (4.28) is also valid for F'.

Remark 4.19. If all $\Sigma \in H_2(M, \mathbb{Z})$ have even intersection number, which is equivalent to all Q^{II} being even, then F as in (4.76) satisfies $\frac{[2F]}{2\pi} \equiv 0 \mod 2$. From this we deduce that M has vanishing second Stiefel-Whitney class. Conversely, vanishing second Stiefel-Whitney class implies all self-intersection numbers are zero through (4.74). Hence we conclude that a closed, connected, oriented four-manifold M being spin is equivalent to all homology two-cycles having even self-intersection. The presence of a spin structure on M will leave the Dirac quantization condition as in (4.3) intact, and therefore the partition function as well. Notice that by equation (4.75) manifolds with an odd Hirzebruch signature cannot be spin.

In [AO] it is argued that a physical consequence of manifolds not being spin is that neutral complex spinor fields cannot exist. For manifolds that are not spin $F^2(M, \mathbb{Z})$ is an odd unimodular lattice.

4.3.4 Modular properties of the generalized Maxwell partition function

Let Λ be the lattice $F^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{Z})$ and the scalar product given by (4.29). Then the linear transformation $\hat{\Omega}(\tau)$ in (4.30) and (4.31) fits the description of the linear transformation corresponding to (4.52). We can namely pick a basis of $F^2(M, \mathbb{Z})$ such that $\hat{\Omega}(\tau)$ is represented by the matrix $\Omega(\tau) = \tau_1 + i\tau_2 Q^{-1}G$. This matrix $Q^{-1}G$ is positive-definite with respect to the given scalar product (represented by the matrix Q), since G is a positive-definite matrix. Furthermore, it satisfies

$$(Q^{-1}G)^2 = Q^{-1}(GQ^{-1}GQ^{-1})Q^{-1} = I.$$

The theory about theta functions in Section 4.3.1 can now be put to use to study the modular properties of the generalized Maxwell partition function, where $\Theta(\tau)$ in (4.30) is replaced by a linear combination of the other theta functions at hand.

Spin Manifolds

If M is a manifold with a spin structure, then the unimodular lattice $F^2(M, \mathbb{Z})$ is even and Section 4.3.2 tells us that the theta function $\Theta(\tau)$ in (4.30) is invariant under the action of $\hat{A}(S)$ and $\hat{A}(T)$, which is in agreement with Theorem 4.5. The value of $e^{-2\pi i(b^+ - b^-)/8}$ is now 1, since M is spin and therefore $\sigma(M) \in 8\mathbb{Z}$ by (4.75).

According to Remark 4.19, w can be set to zero in (4.31). Therefore the partition function in (4.31) is identical to that in (4.30).

Manifolds that are not spin

The case we are actually interested is the modular behaviour of (4.31) with $w \neq 0 \mod 2$, i.e. when M is not spin and the lattice $F^2(M, \mathbb{Z})$ is odd. We look at the lattice $F^2(M, \mathbb{Z})_{total}$ given by (4.42) and its decomposition in (4.44):

$$F^2(M,\mathbb{Z})_{total} = F^2(M,\mathbb{Z})_{even} \cup F^2(M,\mathbb{Z})_{odd} \cup (F^2(M,\mathbb{Z})_{even} + w/2) \cup (F^2(M,\mathbb{Z})_{odd} + w/2).$$

Associated to this lattice are the theta functions $\Theta(\tau)$ in (4.30), $\Theta_{w/2}(\tau) = e^{2\pi i(b^+ - b^-)/8}\Theta^{ST}(\tau)$ in (4.31) and

$$\Theta_0(\tau) = \sum_{m \in F^2(M,\mathbb{Z})_{even}} \exp(i\pi m \cdot \hat{\Omega}(\tau) m), \tag{4.77}$$

$$\Theta_v(\tau) = \sum_{m \in F^2(M,\mathbb{Z})_{odd}} \exp(i\pi m \cdot \hat{\Omega}(\tau) m). \tag{4.78}$$

The relations in (4.70) and (4.71) and the modular behaviour of the prefactor $(\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)}$ then give us the action of the full modular group $SL(2,\mathbb{Z})$ on the following partition functions: Z_{sc} as in (4.30) , Z_{sp} as in (4.31) and

$$Z_{sc^*}(\tau) \quad = \quad (\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)}(\Theta_0(\tau) - \Theta_v(\tau)) = (\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)}\Theta^T(\tau). \tag{4.79}$$

As a result of (4.70) we for example have

$$Z_{sp}(-1/\tau) = e^{-2\pi i(b^+-b^-)/8}\tau^{\frac{1}{2}(1-b^1+b^+)}\overline{\tau}^{\frac{1}{2}(1-b^1+b^-)}Z_{sc^*}(\tau). \tag{4.80}$$

As another example, $Z_{sp}(\tau)$ gets mulitplied with a factor $\exp(2\pi i(b^+ - b^-)/8)$ under the action of T.

The partition functions Z_{sc} and Z_{sc^*} correspond to theories in which the Abelian gauge fields couple to scalar fields, but with different theta parameters: $Z_{sc^*}(\tau)$ is the same as $Z_{sc}(\tau)$, but with $2\pi\mathrm{Re}(\tau) = \theta + 2\pi$. Recall that Z_{sp} corresponds to a theory in which the Abelian gauge field couple to spinor fields.

Remark 4.20. Whenever the underlying manifold becomes important in the partition function, we will distinguish this in the notation by a subscript.

Depending on the signature of $F^2(M,\mathbb{Z})$, which is the Hirzebruch signature $\sigma(M)$, certain linear combinations of these partition functions become modular forms of weight $\frac{1}{2}(1 - b^1 + b^+, 1 - b^1 + b^-)$ according to Remark 4.17. The partition functions on del Pezzo surfaces cover all the cases, since $\sigma(dP_l)$ runs from 1 to -7.

Remark 4.21. Formulas (2.11)-(2.13) in [Ver] contain a similar result as in Theorem 4.5 for the scalar partition function (4.30) and as equation (4.80) for the spinor partition function (4.31). Verlinde computes that the scalar partition function remains invariant up to a phase factor under T (or T^2 when M is not spin) and transforms under S as

$$Z_{sc}(-1/\tau) = \epsilon\tau^{\frac{1}{2}(1-b^1+b^+-b^-)}\overline{\tau}^{\frac{1}{2}(1-b^1)}Z(\tau),$$

where ϵ is some eight root of unity. The spinor partition function still remains invariant under T^2 and under S transforms as

$$Z_{sp}(-1/\tau) = \epsilon\tau^{\frac{1}{2}(1-b^1+b^+-b^-)}\overline{\tau}^{\frac{1}{2}(1-b^1)}(\mathrm{Im}\tau)^{\frac{1}{2}(b^1-1)}\sum_{m\in F^2(M,\mathbb{Z})}\exp(i\pi m\cdot\hat{\Omega}(\tau)\cdot m + 2\pi i m\cdot w/2).$$

Using (4.74), we see $m \cdot w$ is even when $m \in F^2(M,\mathbb{Z})_{even}$ and odd if $m \in F^2(M,\mathbb{Z})_{odd}$. This shows

$$Z_{sp}(-1/\tau) \quad = \quad \epsilon\tau^{\frac{1}{2}(1-b^1+b^+-b^-)}\overline{\tau}^{\frac{1}{2}(1-b^1)}Z_{sc^*}(\tau).$$

Comparing this with Theorem 4.5 and with (4.80), we see that a factor $\overline{\tau}^{b^-/2}$ is replaced with $\tau^{b^-/2}$. The derivation of the results above uses properties for electric and magnetic flux operators associated to the gauge field A.

The computations done in Section 2.2 in [Wit3] give the same result as Theorem 4.5 as we mentioned in Remark 4.6. However, Witten uses a field-theoretic approach, instead of analyzing theta functions.

4.4 Moduli space $\mathcal{M}_{b^+,b^-}$ of partition functions

The computation of the Maxwell partition function on M is greatly simplified, since the saddle points form a lattice isomorphic to $F^2(M,\mathbb{Z})$. The scalar product induced by the unimodular intersection form Q then determines the modular properties, which are only dependent on the topology of M. However, the matrix G in the partition function is still dependent on the detailed geometry of M. As equation (4.10) shows, this is determined by the action of $\star$ on the basis of the lattice. According to equations (4.16) and (4.17), this is equivalent to how the space $\mathcal{H}^2(M)$ splits as in (2.12). What we would like to establish in this section is that the moduli space of parameters in the partition function [Ver] is given by

$$\mathcal{M}_{b^+,b^-} \simeq SO(b^+) \times SO(b^-) \setminus SO(b^+,b^-)/O(b^+,b^-,\mathbb{Z}). \tag{4.81}$$

First we note that the group $SO(b^+,b^-)$ can take various forms depending on the rank, type and signature of the intersection form Q. After (2.12) we commented that $\mathcal{H}^+$ and $\mathcal{H}^-$ form maximal positive and negative subspaces of Q when applied to $\mathcal{H}^2(M)$. The splitting of $\mathcal{H}^2(M)$ is orthogonal and we can look at both the projection operators $\pi_+ = \frac{1}{2}(1+\star)$ and $\pi_- = \frac{1}{2}(1-\star)$ onto $\mathcal{H}^+$ and $\mathcal{H}^-$, respectively. From these projection operators it is possible to reconstruct $\star$, namely $\star = \pi_+ - \pi_-$. Now all possible maximal and negative subspaces of Q can be obtained by a linear transformation $\tilde{A}$ of $\mathcal{H}^2(M)$ which leave Q invariant, i.e. $Q(\tilde{A}\alpha,\tilde{A}\beta) = Q(\alpha,\beta)$ for all $\alpha,\beta \in \mathcal{H}^2(M)$. This means that in the basis $\{\alpha_I\}$ this transformation can be represented by a matrix A, such that $A^tQA = Q$, where $Q = Q_{IJ}$ is the matrix in (4.10). This A hence defines an element in $O(b^+,b^-)$ and we can even take $A \in SO(b^+,b^-)$ to obtain all possible maximal and negative subspaces. Under such a linear transformation we obtain the projection operators onto the new subspaces:

$$\pi'_\pm = \tilde{A}\pi_\pm\tilde{A}^{-1}. \tag{4.82}$$

Equation (4.13) shows the Hodge star operator on the basis $\{\alpha_I\}$ is determined by the matrix $G = G_{IJ}$, so let us denote $\star = \star_G$ accordingly. We now have $m \cdot Gm = Q(\alpha,\star_G\alpha)$, where $\alpha = \sum_I m^I\alpha_I$ and $m = (m^1,\ldots,m^{b^2})$. If $\pi'_\pm$ satisfies (4.82) with $A \in SO(b^+,b^-)$, then for $\star'_G = \star' = \pi'_+ - \pi'_-$ and $\star_G = \pi_+ - \pi_-$ we have

$$m \cdot G'm = Q(\alpha,\tilde{A}(\pi_+ - \pi_-)\tilde{A}^{-1}\alpha) = Q(\tilde{A}^{-1}\alpha,\star_G\tilde{A}^{-1}\alpha) = m \cdot (A^{-1})^tGA^{-1}m.$$

So by letting $A \in SO(b^+,b^-)$ act on one suited choice of G as A^tGA, we obtain all the possible matrices. This establishes the main group in (4.81).

After picking one G to act on, the subgroup $SO(b^+)\times SO(b^-)$ corresponds to transformations $\tilde{A}$ leaving invariant the spaces $\mathcal{H}^+_G$ and $\mathcal{H}^-_G$ associated to $\star = \star_G$. These transformations then commute with the projection operators $\pi_\pm$, so that $\star'$ in (4.82) satisfies $\star' = \pi'_+ - \pi'_- = \star$. If two matrices $A', A \in SO(b^+,b^-)$ satisfy $A' = OA$, where $O \in SO(b^+) \times SO(b^-)$, then O corresponds to a transformation that leaves invariant $\mathcal{H}^+_G$ and $\mathcal{H}^-_G$, so that $(A')^tGA' = A^tGA$. Furthermore if $A' = AO'$, where $O' \in SO(b^+) \times SO(b^-)$, then O' leaves invariant $\mathcal{H}^+_{A^tGA}$ and $\mathcal{H}^-_{A^tGA}$, so that $(A')^tGA' = A^tGA$. This establishes the quotient by $SO(b^+)\times SO(b^-)$ in (4.81).

Any automorphism of the lattice $\Lambda = \bigoplus_I \mathbb{Z}\alpha_I$ is represented by an element in $GL(b^+ + b^-,\mathbb{Z})$, i.e. matrices that have integer-entries and determinant ± 1. The subgroup $O(b^+,b^-,\mathbb{Z})$ corresponds to the elements in the intersection $GL(b^+ + b^-,\mathbb{Z}) \cap O(b^+,b^-.\mathbb{R})$. Suppose two $A, A' \in SO(b^+,b^-,\mathbb{R})$ satisfy $A' = AO$, where $O \in O(b^+,b^-,\mathbb{Z})$. Then the matrix $\Omega(\tau) = \tau_1 Q + \tau_2 A^tGA$ in (4.25) can be replaced by $O^t\Omega(\tau)O = \tau_1 Q + \tau_2 (A')^tGA'$, since O is an automorphism of $\mathbb{Z}^{b^2}$ (which leaves invariant the sum) and $O^tQO = Q$.

This confirms that the moduli space of partition functions is indeed given by (4.81), which is an orbifold with the discrete subgroup $O(b^+,b^-,\mathbb{Z})$ acting on it. The dimension of this space is

$$\begin{aligned}\dim SO(b^+,b^-) - \dim(SO(b^+)\times SO(b^-)) &= \frac{1}{2}(b^+ + b^-)(b^+ + b^- - 1) \\ &\quad -\frac{1}{2}(b^-)(b^- - 1) - \frac{1}{2}(b^+)(b^+ - 1) \\ &= b^+ \times b^-.\end{aligned}$$

The isomorphism for $\mathcal{M}_{b^+,b^-}$ is affine, since we had to start from one matrix G and corresponding $\star$ operator in order to define the action of $SO(b^+,b^-)$. The elements in $\mathcal{M}_{b^+,b^-}$ then act on the matrix G in the adjoint representation.

Remark 4.22. Notice that varying G does not affect our constructions in Section 4.3.4, since these only involve the properties of the intersection form Q.

The group $O(b^+,b^-,\mathbb{Z})$ depends on the signature, rank and type of Q. In general a large part of $O(b^+,b^-,\mathbb{Z})$ will correspond to permutations and reflections of vectors in $\bigoplus_I \mathbb{Z}\alpha_I$ that leave invariant Q. For the intersection form equivalent to

$$Q = n\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the group $O(n,n,\mathbb{Z})$ has an interesting structure, which has been worked out in detail in [GMR, GPR].

The action of the mapping class group of M is reflected in $O(b^+,b^-.\mathbb{Z})$. An action of this group on M should leave invariant the partition function and this invariance gets reflected in the action of $O(b^+,b^-.\mathbb{Z})$. Fixed points in $\mathcal{M}_{b^+,b^-}$ of the group $O(b^+,b^-,\mathbb{Z})$ correspond to metrics of the manifold for which there are extra symmetries. We will see an example of this in Section 4.5.

4.4.1 Computation of the moduli parameters

Given the lattice $\Lambda = \bigoplus_I \mathbb{Z}\alpha_I \simeq F^2(M,\mathbb{Z})$ and matrix $Q = Q_{IJ} = Q([\alpha_I],[\alpha_J])$, we can compute the matrix G_{IJ} by using the properties of $\star$ and the fact that it should be a positive-definite symmetric matrix. For $\star$ we have the identity $\star^2|_{\Omega^2(X)} = \mathrm{id}$. We already saw in equation (4.14) that this means $(GQ^{-1})^2 = I$ or equivalently $GQ^{-1}G = Q$. We can solve for these conditions directly, but we can also start from one G that satisfies the conditions and let $SO(b^+,b^-)$ act on it in the adjoint representation. Given such a G, then the matrix $G' = A^tGA$ is symmetric and positive-definite as well for $A \in SO(b^+,b^-)$ and it satisfies $G'Q^{-1}G' = Q$, since A leaves Q and Q^{-1} invariant.

If we have $Q^{-1} = Q$, the condition $(GQ^{-1})^2 = I$ just means that G leaves Q invariant. Such a G must be a symmetric positive-definite matrix in $SO(b^+,b^-)$. Since the indentity matrix I satisfies the conditions, all the other matrices have the form $G = A^tA$ with $A \in SO(b^+,b^-)$. If $Q^{-1} = Q$, these constructions will sometimes make it easier to find the possible G.

The matrix G can also be determined though a direct calculation of $\star$ and the harmonic 2-forms $\{\alpha_I\}$. The operator $\star|_{\Omega^2(M)}$ depends on the conformal class of the metric (see Proposition 2.5) and in order to compute the harmonic forms, we need to solve the differential equations $d\alpha = d^*\alpha = 0$. Such a direct calculation needs specific details of the metric and will generally not be easy.

However, we only have to know that these harmonic 2-forms exist and then the possible G can be computed using only algebraic relations as described above. The coefficients that return in the matrix G, will generally be a function of $b^+ \times b^-$ parameters that are dependent on the detailed geometry of the manifold. Only in some cases the problem is tractable enough to find the explicit geometric origin of these parameters. In the next section we will calculate the partition function for $d\tilde{P}_1 = S^2 \times S^2$ endowed with the product metric on two round spheres. According to (4.81) we would just have 1 moduli parameter and we will see that this parameter corresponds to the ratio between the radii of the two spheres. For the other del Pezzo surfaces dP_l the number of parameters is $b^+ \times b^- = l$. For $l > 0$, the partition functions will be calculated in Section 4.6. By looking at some qualitative behaviour of the partition function, we will heuristically describe how these parameters are related to the geometry of dP_l.

Notice that if $b^+ = 0$ or $b^- = 0$, $\mathcal{M}_{b^+,b^-}$ should be trivial, i.e. the partition functions are not dependent on any parameters. The partition functions for $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$ are an example of this.

Example 4.23. One can show $G = Q$ for $\mathbb{CP}^2$ and $G = -Q$ for $\overline{\mathbb{CP}}^2$. The form $\alpha = \frac{\omega}{2\pi}$, with ω the Fubini–Study metric on $\mathbb{CP}^2$ as in (3.45),[3] is a self-dual harmonic form. Its cohomology class equals minus the first Chern class of the tautological line bundle. Furthermore, (3.46) shows $\int_{\mathbb{CP}^2} \alpha^2 = 1$, so its cohomology class forms the basis for $H^2(M, \mathbb{Z})$ and the intersection form is equivalent to $Q = (1)$. Since ω is self-dual, we have $G = Q = (1)$ and equation (4.18) then gives us

$$Z_{sc}(\tau)_{\mathbb{CP}^2} = \frac{1}{\sqrt{\mathrm{Im}\tau}} \sum_m \exp(i\pi\tau m^2). \tag{4.83}$$

On $\overline{\mathbb{CP}}^2$, ω is an anti-self-dual harmonic form. Since the orientation is reversed, the form $\alpha = \frac{\omega}{2\pi}$ now satisfies $\int_{\overline{\mathbb{CP}}^2} \alpha^2 = -1$ and its cohomology class lies in $H^2(\overline{\mathbb{CP}}^2, \mathbb{Z})$. We conclude that $G = -Q = (1)$ and therefore

$$Z_{sc}(\tau)_{\overline{\mathbb{CP}^2}} = \frac{1}{\sqrt{\mathrm{Im}\tau}} \sum_m \exp(-i\pi\overline{\tau} m^2). \tag{4.84}$$

According to Theorem 4.5 these partition functions are invariant under S and T^2.

For the other cases where $b^+ = 0$ or $b^- = 0$, the partition function is still not dependent on any parameters, since in these cases $\star|_{\Omega^2(M)} = \pm\mathrm{id}$ and hence $Q = G$ and $Q = -G$, respectively. Donaldson's diagonalisability theorem (see Remark 4.11) shows that if in addition the manifold M is *simply-connected* the partition function is given by

$$Z_{sc}(\tau) = \frac{1}{\sqrt{\mathrm{Im}\tau}} \left(\sum_m \exp(i\pi\tau m^2) \right)^{b^2},$$

if $b^- = 0$ or

$$Z_{sc}(\tau) = \frac{1}{\sqrt{\mathrm{Im}\tau}} \left(\sum_m \exp(-i\pi\overline{\tau} m^2) \right)^{b^2},$$

if $b^+ = 0$.

We make the final remark that for $b^+, b^- \neq 0$ we can pick Q just to be one of the matrices in Theorem 4.9 depending on the rank, signature and type of the intersection form Q. We are namely still free to pick the generators of $F^2(M, \mathbb{Z})$ and their unique harmonic representatives. In general we will determine Q^{-1} using generators of $F_2(M, \mathbb{Z})$ and then look at harmonic forms dual to these generators (see (4.7)). The cohomology classes of these forms then generate $F^2(M, \mathbb{Z})$ and these will form the basis in which we express Q.

[3] Recall that we have absorbed the factor i in all the connections.

4.5 The Maxwell partition function for $S^2 \times S^2$

Let $S^2 \times S^2 = S \times S'$, where S and S' are round spheres with radius R and R', respectively. The product $S \times S'$ is simply-connected and has $b^2(M) = 2$. We compute the intersection form with the two-cycles $\Sigma_1 = S \times \{(R', 0)\}$ and $\Sigma_2 = \{(R, 0)\} \times S'$. These two-cycles have zero self-intersection numbers, since a small perturbation of the two-cycles in $S \times S'$ is disjoint. For example, Σ_1 and $S \times \{(\sqrt{(R')^2 - \epsilon^2}, \epsilon)\}$ are disjoint for small ϵ. Furthermore, we have $I(\Sigma_1, \Sigma_2) = I(\Sigma_2, \Sigma_1) = 1$, since these cycles meet transversly in $(R, 0) \times (R', 0)$ and $T\Sigma_1 \oplus T\Sigma_2$ obviously matches the orientation of $T(S \times S')$ at that point. The intersection matrix is hence given by:

$$Q^{IJ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = Q_{IJ}. \tag{4.85}$$

Remark 4.24. Notice that Σ_1 and Σ_2 correspond to an A- and B-line (see Section 2.2.3) when we realize $S \times S'$ as a quadric in $\mathbb{CP}^3$. In this realization, the form of the intersection matrix is just implied by the rules for the A- and B-lines. The two-cycles Σ_1 and Σ_2 are sometimes also called an A- and B-cycle, respectively.

If we pick double spherical coordinates $(\theta, \varphi, \theta', \varphi')$, the product metric on $S \times S'$ takes the following form:

$$g_{ij} = \begin{pmatrix} R^2 & 0 & 0 & 0 \\ 0 & R^2 \sin^2\theta & 0 & 0 \\ 0 & 0 & (R')^2 & 0 \\ 0 & 0 & 0 & (R')^2 \sin^2\theta' \end{pmatrix} \tag{4.86}$$

The forms dual to Σ_1 and Σ_2 are given by:

$$\alpha_1 = \frac{1}{4\pi} \sin\theta d\theta \wedge d\varphi, \quad \alpha_2 = \frac{1}{4\pi} \sin\theta' d\theta' \wedge d\varphi'. \tag{4.87}$$

Let $\pi_1 : S \times S' \to S$ and $\pi_2 : S \times S' \to S'$ be the projection maps, then under pullbacks one has $\alpha_1 = \frac{1}{4\pi R^2}\pi_1^*(\mathrm{vol}_S)$ and $\alpha_2 = \frac{1}{4\pi (R')^2}\pi_2^*(\mathrm{vol}_{S'})$, where vol_S and $\mathrm{vol}_{S'}$ are the volume forms on S and S', respectively. Since the volume forms are closed and for pullbacks we have $d \circ f^* = f^* \circ d$, α_1 and α_2 are closed forms. Now by the rules for the Hodge star operator on Cartesian products [4] and its action on volume forms:

$$\begin{aligned} \star\,\alpha_1 = \frac{1}{4\pi R^2} \star (\pi_1^*(\mathrm{vol}_S)) &= \frac{1}{4\pi R^2} \star (\pi_1^*(\mathrm{vol}_S) \wedge \pi_2^*(1)) \\ &= \frac{1}{4\pi R^2} (\pi_1^*(\star \mathrm{vol}_S) \wedge \pi_2^*(\star 1)) \\ &= \frac{1}{4\pi R^2} (\pi_1^*(1) \wedge \pi_2^*(\mathrm{vol}_{S'})) \\ &= \frac{(R')^2}{R^2} \alpha_2. \end{aligned} \tag{4.88}$$

Similarly we have $\star\alpha_2 = \frac{R^2}{(R')^2}\alpha_1$. Since $d^* = -\star d \star$ we conclude that α_1 and α_2 are unique harmonic representatives of the generators of $F^2(M, \mathbb{Z}) = H^2(M, \mathbb{Z})$.

[4] The following identity holds for forms on a Cartesian product $X \times Y$:

$$\star(\pi_1^*(\alpha) \wedge \pi_2^*(\beta)) = (-1)^{(\dim X - k)l} \pi_1^*(\star\alpha) \wedge \pi_2^*(\star\beta),$$

where $\alpha \in \Omega^k(X)$, $\beta \in \Omega^l(Y)$ and π_1, π_2 are the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$.

Remark 4.25. We could have also obtained (4.88) by a calculation in local coordinates. The basis vectors for the 1-forms in spherical coordinates are given by $d\theta, d\varphi, d\theta'$ and $d\varphi'$. The Hodge dual of $d\theta \wedge d\varphi$ in local coordinates can now be computed through the metric tensor (4.86) and its inverse g^{ij}:

$$\begin{aligned}\star(d\theta \wedge d\varphi) &= \sum_{i,j,k,l} \frac{\sqrt{\det(g_{ij})}}{(4-2)!} g^{1i} g^{2j} \epsilon_{ijkl} dx^k \wedge dx^l \\ &= \sum_{k,l} \frac{R^2 \sin\theta (R')^2 \sin\theta'}{2} (R^2)^{-1} (R^2 \sin^2\theta)^{-1} \epsilon_{12kl} dx^k \wedge dx^l \\ &= \frac{(R')^2 \sin\theta'}{R^2 \sin\theta} d\theta' \wedge d\varphi'.\end{aligned}$$

By linearity of $\star$, this will give (4.88). Notice that the first identity is just the Euclidean version of (1.1).

Setting $r = \frac{R'}{R}$, the matrix G_{IJ} in (4.10) is now seen to be given by

$$G_{IJ} = \begin{pmatrix} r^2 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}. \tag{4.89}$$

On $S \times S'$ we have the following self-dual and anti-self-dual harmonic forms

$$\alpha^+ = \frac{1}{\sqrt{2}} \left(\frac{1}{r}\alpha_1 + r\alpha_2 \right), \quad \alpha^- = \frac{1}{\sqrt{2}} \left(\frac{1}{r}\alpha_1 - r\alpha_2 \right), \tag{4.90}$$

satisfying $\langle \alpha^\pm, \alpha^\pm \rangle = 1$ and $\langle \alpha^\pm, \alpha^\mp \rangle = 0$. The forms α_1 and α_2 can now be written as:

$$\alpha_1 = \frac{r}{\sqrt{2}}(\alpha^+ + \alpha^-), \quad \alpha_2 = \frac{1}{r\sqrt{2}}(\alpha^+ - \alpha^-)$$

According to equation (4.16), the Maxwell partition function is equal to

$$\begin{aligned}Z_{sc}(\tau, \overline{\tau}, r)_{S^2 \times S^2} &= \frac{1}{\sqrt{\mathrm{Im}\tau}} \sum_{(p^+, p^-) \in \Gamma_{b^+, b^-}} \exp(i\pi\tau(p^+)^2 - i\pi\overline{\tau}(p^-)^2)) \\ &= \frac{1}{\sqrt{\mathrm{Im}\tau}} \sum_{n,m} \exp\left(\frac{i\pi\tau}{2}((nr + m/r)\alpha_+)^2 - \frac{i\pi\overline{\tau}}{2}((nr - m/r)\alpha_+)^2 \right) \\ &= \frac{1}{\sqrt{\mathrm{Im}\tau}} \sum_{n,m} \exp\left(\frac{i\pi\tau}{2}(nr + m/r)^2 - \frac{i\pi\overline{\tau}}{2}(nr - m/r)^2 \right).\end{aligned} \tag{4.91}$$

Furthermore, equation (4.19) should hold for $e_I^\pm$ in the decomposition $\alpha_I = e_I^+ + e_I^-$, since the $\alpha^\pm$ are already an orthonormal basis for $\mathcal{H}^2(M)$. This is indeed the case:

$$\begin{aligned}\frac{1}{2}(G_{IJ} \pm Q_{IJ}) &= \frac{1}{2} \begin{pmatrix} r^2 & \pm 1 \\ \pm 1 & \frac{1}{r^2} \end{pmatrix} \\ &= \begin{pmatrix} (r/\sqrt{2})(r/\sqrt{2}) & (r/\sqrt{2})(\pm 1/r\sqrt{2}) \\ (\pm 1/r\sqrt{2})(r/\sqrt{2}) & (\pm 1/r\sqrt{2})(\pm 1/r\sqrt{2}) \end{pmatrix} \\ &= (e_I^\pm)(e_J^\pm).\end{aligned}$$

In fact we could have plugged this matrix in equation (4.18) to get the same result as (4.91).

Modular transformation properties

According to Theorem 4.5, the scalar partition function we computed is a modular form of weight $(1,1)$ under the action of the full group $SL(2,\mathbb{Z})$, since the matrix Q_{IJ} is even or equivalently $S^2 \times S^2$ is spin. We know that in this case the spinor partition function Z_{sp} in (4.31) is also given by (4.91).

The partition function being a modular function of weight $(1,1)$ is equivalent (see Proposition 4.3) to invariance under $\tau \mapsto \tau + 1$ and the transformation rule

$$Z_{sc}(-1/\tau) = \tau\overline{\tau} Z_{sc}(\tau).$$

As a check of the general theory we will compute these two results directly. First we look at the action of the transformation $\tau \mapsto \tau + 1$ on the partition function:

$$\begin{aligned}
Z_{sc}(\tau+1) &= Z_{sc}(\tau+1, \overline{\tau}+1, r) \\
&= \frac{1}{\sqrt{\mathrm{Im}(\tau+1)}} \sum_{n,m} \exp\left(\frac{i\pi(\tau+1)}{2}(nr+m/r)^2 - \frac{i\pi(\overline{\tau}+1)}{2}(nr-m/r)^2 \right) \\
&= \frac{1}{\sqrt{\mathrm{Im}(\tau)}} \sum_{n,m} \exp\left(\frac{i\pi\tau}{2}(nr+m/r)^2 - \frac{i\pi\overline{\tau}}{2}(nr-m/r)^2 \right) \\
&\quad \times \exp\left(\frac{i\pi}{2}((nr+m/r)^2 - (nr-m/r)^2 \right) \\
&= \frac{1}{\sqrt{\mathrm{Im}(\tau)}} \sum_{n,m} \exp\left(\frac{i\pi\tau}{2}(nr+m/r)^2 - \frac{i\pi\overline{\tau}}{2}((nr-m/r))^2 \right) \exp\left(\frac{i\pi}{2}(4nm) \right) \\
&= \frac{1}{\sqrt{\mathrm{Im}(\tau)}} \sum_{n,m} \exp\left(\frac{i\pi\tau}{2}(nr+m/r)^2 - \frac{i\pi\overline{\tau}}{2}(nr-m/r)^2 \right) \\
&= Z_{sc}(\tau)
\end{aligned} \tag{4.92}$$

Now we directly apply the Poisson resummation formula (4.23) to compute the action of S on the partition function: [5]

$$
\begin{aligned}
& Z_{sc}(-1/\tau) = Z_{sc}(-1/\tau, -1/\overline{\tau}, r) \\
= \; & \frac{1}{\sqrt{\mathrm{Im}(-1/\tau)}} \sum_{n,m} \exp\left(-\frac{i\pi}{2}\frac{1}{\tau}(nr + m/r)^2 + \frac{i\pi}{2}\frac{1}{\overline{\tau}}(nr - m/r)^2 \right) \\
= \; & \sqrt{\tau\overline{\tau}}\mathrm{Im}(\tau) \sum_{k,l} \int dx dy \exp(2\pi i k x + 2\pi i l y) \exp\left(-\frac{i\pi}{2\tau}(x/r + yr)^2 + \frac{i\pi}{2\overline{\tau}}(x/r - yr)^2 \right) \\
= \; & \sqrt{\frac{\tau\overline{\tau}}{\mathrm{Im}(\tau)}} \sum_{k,l} \int dx dy \exp\left(2\pi i k x + 2\pi i l y - \frac{i\pi}{2}\left(\frac{\overline{\tau} - \tau}{\overline{\tau}\tau}\right)\frac{x^2}{r^2} - \frac{i\pi}{2}\left(\frac{\overline{\tau} - \tau}{\overline{\tau}\tau}\right) r^2 y^2 \right. \\
& \left. - i\pi \left(\frac{\overline{\tau} - \tau}{\overline{\tau}\tau}\right) xy \right) \\
= \; & \sqrt{\frac{\tau\overline{\tau}}{\mathrm{Im}(\tau)}} \sum_{k,l} \int dx dy \exp\left(-\frac{i\pi}{2r^2}\underline{\tau}\left(x - \left(\frac{2k}{\underline{\tau}} - \beta y\right) r^2 \right)^2 + \frac{i\pi\underline{\tau} r^2}{2}\left(\frac{2k}{\underline{\tau}} - \beta y\right)^2 \right. \\
& \left. + 2\pi l y - \frac{i\pi\underline{\tau}}{2} r^2 y^2 \right) \\
= \; & \sqrt{\frac{\tau\overline{\tau}}{\mathrm{Im}(\tau)}} \sqrt{\frac{2r^2}{i\underline{\tau}}} \sum_{k,l} \int dy \exp\left(\frac{i\pi\underline{\tau} r^2}{2}\left(\frac{2k}{\underline{\tau}} - \beta y\right)^2 \right) \exp\left(2\pi l y - \frac{i\pi\underline{\tau}}{2} r^2 y^2 \right) \\
= \; & \sqrt{\frac{\tau\overline{\tau}}{\mathrm{Im}(\tau)}} \sqrt{\frac{2r^2}{i\underline{\tau}}} \sum_{k,l} \int dy \exp\left(\frac{i\pi r^2}{\underline{\tau}} 2k^2 - i\pi 2\beta k y r^2 + 2\pi l y - \frac{i\pi}{2} r^2 y^2 \frac{4}{\overline{\tau} - \tau} \right) \\
= \; & \sqrt{\frac{\tau\overline{\tau}}{\mathrm{Im}(\tau)}} \sqrt{\frac{2r^2}{i\underline{\tau}}} \sum_{k,l} \int dy \exp\left(-\frac{i 2\pi r^2}{\tau - \overline{\tau}}\left(y - \frac{1}{2}\left(\frac{l}{r^2} - \beta k\right)(\tau - \overline{\tau}) \right)^2 + \frac{i\pi 2 k^2 r^2}{\underline{\tau}} \right. \\
& \left. + \frac{i\pi r^2}{2}\left(\frac{l}{r^2} - \beta k\right)^2 (\tau - \overline{\tau}) \right) \\
= \; & \sqrt{\frac{\tau\overline{\tau}}{\mathrm{Im}(\tau)}} \sqrt{\frac{2r^2}{i\underline{\tau}}} \sqrt{\frac{\tau - \overline{\tau}}{i 2 r^2}} \sum_{k,l} \exp\left(\frac{i\pi}{2}\left(-4k^2 r^2 \frac{\tau\overline{\tau}}{\tau - \overline{\tau}} + r^2(\tau - \overline{\tau})\left(\frac{l^2}{R^4} + \beta^2 k^2 - \frac{2\beta k l}{r^2} \right) \right) \right) \\
= \; & \frac{\tau\overline{\tau}}{\sqrt{\mathrm{Im}(\tau)}} \sum_{k,l} \exp\left(\frac{i\pi\tau}{2}\left(k^2 r^2 + 2kl + l^2/r^2\right) - \frac{i\pi\overline{\tau}}{2}\left(k^2 r^2 - 2kl + l^2/r^2\right) \right) \\
= \; & \frac{\tau\overline{\tau}}{\sqrt{\mathrm{Im}(\tau)}} \sum_{k,l} \exp\left(\frac{i\pi\tau}{2}(kr + l/r)^2 - \frac{i\pi\overline{\tau}}{2}(kr - l/r)^2 \right) \\
= \; & \tau\overline{\tau} Z_{sc}(\tau).
\end{aligned}
\tag{4.93}
$$

Moduli parameters in $\mathcal{M}_{1,1}$

According to equation (4.81) the space of moduli parameters is equal to

$$
SO(1,1)/O(1,1,\mathbb{Z}),
$$

[5] In this calculation we set $\beta = \frac{\overline{\tau}+\tau}{\overline{\tau}-\tau}$, $\underline{\tau} = \frac{\overline{\tau}-\tau}{\tau\overline{\tau}}$ and use the relations $1 - \beta^2 = \frac{-4\tau\overline{\tau}}{(\overline{\tau}-\tau)^2} = -\frac{4}{\underline{\tau}(\overline{\tau}-\tau)}$ and $\beta^2(\tau - \overline{\tau}) + \frac{4\tau\overline{\tau}}{\overline{\tau}-\tau} = \tau - \overline{\tau}$.

where $SO(1,1)$ should leave invariant Q and have determinant 1. This group of matrices is given by

$$A = \pm A_r = \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}.$$

The matrix $G = I$ is positive-definite and leaves Q invariant, hence according to our description in Section 4.4.1 all possible matrices are now of the form

$$G_r = (A_r)^t A_r = \begin{pmatrix} r^2 & 0 \\ 0 & 1/r^2 \end{pmatrix}.$$

This matrix matches (4.89), which was the result we computed directly.

Notice that $Z_{sc}(\tau, \overline{\tau}, r) = Z_{sc}(\tau, \overline{\tau}, 1/r)$. This is due to the fact that we could have switched the roles of S and S', which corresponds to an action of the mapping class group of $S^2 \times S^2$. This symmetry corresponds to an action of $O(1,1,\mathbb{Z})$ on $SO(1,1)$, where $O(1,1,\mathbb{Z})$ is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We see that the adjoint action of this matrix on G_r transforms it into $G_{1/r}$. A special point in $SO(1,1)$ is $G = 1$, which is a fixed point of $O(1,1,\mathbb{Z})$. Geometrically this corresponds to S and S' having the same radius and $S^2 \times S^2$ then has an extra $\mathbb{Z}_2$ symmetry.

4.6 The Maxwell partition function for $dP_l = \mathbb{CP}^2 \# l\overline{\mathbb{CP}}^2$ $(l > 0)$

For $8 \geq l \geq 1$ these surfaces dP_l correspond with the del Pezzo surfaces we listed at the beginning of Section 2.2. When realized as blown-up complex surfaces, Theorem 2.18 tells us that the second homology group $H_2(dP_l, \mathbb{Z})$ has basis $\{H, E_1, \ldots, E_l\}$, where H is the Poincaré dual of the Fubiny-study metric on $\mathbb{CP}^2$ and the E_i are the homology cycles corresponding to the exceptional divisors of the l blown-up points. The intersection form can then be calculated through Proposition 2.20 and our result for $\mathbb{CP}^2$ in Example 4.23:

$$Q = Q_{IJ} = Q^{IJ} = (1) \oplus l(-1) = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & -1 & 0 & \cdots \\ 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For more generic metrics on dP_l, we could have computed this using the more general statement in Proposition 2.8 and the calculations in Example 4.23. From Q we read off that $b^+(dP_l) = 1$ and $b^-(dP_l) = l$.

Recall that if $l > 1$, we can realize dP_l as well by blowing up $S^2 \times S^2$ in $l-1$ points, i.e. as $S^2 \times S^2 \#(l-1)\overline{\mathbb{CP}}^2$. Using Proposition 2.8 and (4.85), the associated intersection form is represented by the matrix Q' given by:

$$Q' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus (l-1)(-1). \tag{4.94}$$

In Example 4.10, we already mentioned that this form is equivalent to the matrix Q considered above, which reflects the fact $\mathbb{CP}^2 \# l\overline{\mathbb{CP}^2}$ is diffeomorphic to $S^2 \times S^2 \#(l-1)\overline{\mathbb{CP}^2}$.

As seen from (4.10), G_{IJ} can be calculated if we know how the Hodge star operator acts on the unique harmonic representatives of $H^2(dP_l, \mathbb{Z})$, which are dual to the generators of

$H_2(dP_l, \mathbb{Z})$. As we mentioned in Section 4.4, we can circumvent the (difficult) explicit calculation of $\star|_{\Omega^2(dP_l)}$ and the harmonic 2-forms and use algebraic relations to determine G_{IJ}.

Recall that the matrix $G = G_{IJ}$ is a positive-definite symmetric matrix satisfying $GQ^{-1}G = Q = Q^{-1}$. According to Section 4.4.1 we can write all G as $G = A^tA$ with $A \in SO(1,l)$, since $G = I$ is a positive-definite matrix satisfying $GQG = Q$. In $SO(1,l)$, we are now allowed to quotient out the group $SO(1) \times SO(l)$ according to (4.81). The matrix G then corresponds to a proper orthochronous Lorentz transformation of a $1+l$ flat Minskowski space time, which takes the following form:

$$G = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & -\frac{\beta_i}{\sqrt{1-\beta^2}} \\ -\frac{\beta_i}{\sqrt{1-\beta^2}} & I + \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)\frac{\beta_i\beta_j}{\beta^2} \end{pmatrix}, \quad \beta^2 = \sum_{i=1}^{n} \beta_i^2 < 1. \tag{4.95}$$

Insertion of Q and G in equation (4.18) gives us the partition function for $dP_l = \mathbb{CP}^2\#l\overline{\mathbb{CP}}^2$:

$$\begin{aligned} Z_{sc}(\tau, \overline{\tau}, \beta_1, \ldots, \beta_l)_{dP_l} \;=\; & \frac{1}{\sqrt{\mathrm{Im}\tau}} \sum_{n, m_1, \ldots, m_l} \exp\left(\frac{i\pi\tau}{2}\left[n^2\left(1 + \frac{1}{\sqrt{1-\beta^2}}\right)\right.\right. \\ & \left.\left. -\frac{2}{\sqrt{1-\beta^2}}\sum_i n\beta_i m_i + \frac{1}{\beta^2}\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)\sum_{i,j} m_i m_j \beta_i\beta_j\right]\right) \\ & \times \exp\left(-\frac{i\pi\overline{\tau}}{2}\left[n^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) - \frac{2}{\sqrt{1-\beta^2}}\sum_i n\beta_i m_i\right.\right. \\ & \left.\left. +2\sum_i m_i^2 + \frac{1}{\beta^2}\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)\sum_{i,j} m_i m_j \beta_i\beta_j\right]\right). \end{aligned} \tag{4.96}$$

In the limit $\beta_i \to 0$ the partition function factorizes as:

$$Z_{sc}(\tau, \overline{\tau}, \beta_1, \ldots, \beta_l)_{dP_l}|_{\beta_i = 0} = \left(\sum_m \exp(-i\pi\overline{\tau}m^2)\right) Z_{sc}(\tau, \overline{\tau}, \beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_l)_{dP_{l-1}}, \tag{4.97}$$

which would be the partition function if we consider the disjoint union $(\mathbb{CP}^2\#(l-1)\overline{\mathbb{CP}}^2) \coprod \overline{\mathbb{CP}}^2$ (recall equation (4.83)). This manifold can be obtained by collapsing the 'neck' in the smooth connected sum $(\mathbb{CP}^2\#(l-1)\overline{\mathbb{CP}}^2)\#\overline{\mathbb{CP}}^2$ (see Section 2.1.4), i.e. letting the annuli diffeomorphic to $S^3 \times [0,1]$ collapse to a point.

Doing this $l-1$ times, we get to the case $(\mathbb{CP}^2\#\overline{\mathbb{CP}^2}) \coprod (\coprod_{i=1}^{l-1} \overline{\mathbb{CP}^2})$. The partition function is then given by

$$Z_{sc}(\tau, \overline{\tau}, \beta, 0, \ldots, 0)_{dP_l} = \tilde{Z}_{sc}(\tau, \overline{\tau}, \beta)_{dP_l} = \left(\sum_m \exp(-i\pi\overline{\tau}m^2)\right)^{l-1} Z_{sc}(\tau, \overline{\tau}, \beta)_{dP_1}, \tag{4.98}$$

where

$$\begin{aligned}
Z_{sc}(\tau,\overline{\tau},\beta)_{dP_1} &= \frac{1}{\sqrt{\mathrm{Im}\tau}}\sum_{n,m}\exp\left(\frac{i\pi\tau}{2}\left[n^2\left(1+\frac{1}{\sqrt{1-\beta^2}}\right)-\frac{2}{\sqrt{1-\beta^2}}n\beta m\right.\right.\\
&\quad\left.\left.+\frac{1}{\beta^2}\left(\frac{1}{\sqrt{1-\beta^2}}-1\right)m^2\beta^2\right]\right)\exp\left(-\frac{i\pi\overline{\tau}}{2}\left[n^2\left(\frac{1}{\sqrt{1-\beta^2}}-1\right)\right.\right.\\
&\quad\left.\left.-\frac{2}{\sqrt{1-\beta^2}}n\beta m+\frac{1}{\beta^2}\left(\frac{1}{\sqrt{1-\beta^2}}+1\right)m^2\beta^2\right]\right)\\
&= \frac{1}{\sqrt{\mathrm{Im}\tau}}\sum_{n,m}\exp\left(\frac{i\pi\tau}{2\sqrt{1-\beta^2}}\left(n\sqrt{1+\sqrt{1-\beta^2}}-m\sqrt{1-\sqrt{1-\beta^2}}\right)^2\right)\\
&\quad\times\exp\left(-\frac{i\pi\overline{\tau}}{2\sqrt{1-\beta^2}}\left(n\sqrt{1-\sqrt{1-\beta^2}}-m\sqrt{1+\sqrt{1-\beta^2}}\right)^2\right). \qquad (4.99)
\end{aligned}$$

Letting $\beta\to 0$ this partition function as well factorizes as

$$Z_{sc}(\tau,\overline{\tau},\beta=0)_{dP_1}=\frac{1}{\sqrt{\mathrm{Im}\tau}}\left(\sum_n\exp(i\pi\tau n^2)\right)\left(\sum_m\exp(-i\pi\overline{\tau}m^2)\right). \qquad (4.100)$$

When we compare (4.99) with (4.91), we see that the parameters r and $1/r$ in $Z_{sc}(\tau,r)_{S^2\times S^2}$ get replaced with

$$\frac{1}{1-\beta^2}\sqrt{1+\sqrt{1-\beta^2}} \quad \text{and} \quad \frac{1}{1-\beta^2}\sqrt{1-\sqrt{1-\beta^2}}\ \text{, respectively.}$$

Notice that in this case a symmetry like $r\mapsto 1/r$ for (4.91) is absent. This is not in contradiction with our expectation, since the intersection forms of $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ and $S^2\times S^2$ are not isomorphic.

According to Theorem 4.5 the partition function (4.96) is a modular form of weight $\frac{1}{2}(2,1+l)$ under the Hecke subgroup Γ_θ. Under the transformation S there is an additional phase factor $e^{i\pi\sigma(dP_l)/4}=e^{i\pi(1-l)/4}$. For an action of the full modular group one also has to look at the partition functions $Z_{sc^*}(\tau)_{dP_l}$ given by (4.79) and $Z_{sp}(\tau)_{dP_l}$ in (4.31), which equals (4.96) but with $(n,m)\in\mathbb{Z}\times\mathbb{Z}^l+w/2$, where $w=(1,-1,\ldots,-1)$. The theory of Section 4.3 then shows how these partition functions then transform into each other under $SL(2,\mathbb{Z})$: one applies the relations (4.70) and (4.71) to (4.30), (4.31) and (4.79). For dP_1 the sum of these partition functions is modular covariant. For dP_5 the sum $Z_{sc}-Z_{sc^*}+Z_{sp}$ is modular covariant. One can see directly that the factorizations as in (4.97), (4.98) and (4.100) still apply to Z_{sc^*} and Z_{sp}.

A part of the symmetry group $O(1,l,\mathbb{Z})$ (as in (4.81)) for (4.96) is formed by relabeling the parameters β_i, which corresponds to an exchange of the blown-up points. This exchange of blown-up points can indeed be realized by an action of the mapping class group of dP_l [INV].

The parameters β_i in the partition function seem to be put there by hand. The parametrization was found using algebraic relations and besides topological data no specific geometric data of the del Pezzo surface was used. The limit $\beta_i\to 0$ seems to correspond to exclusion of a neighbourhood of the exceptional divisor E_i (diffeomorphic to $\overline{\mathbb{CP}}^2$ minus a point) which results in the disjoint union $dP_{l-1}\coprod\overline{\mathbb{CP}}^2$. In combination with our comment regarding $O(1,l,\mathbb{Z})$ this suggests that β_i is associated to the geometry of E_i. A parameter in which β_i might be expressed is the Kähler parameter of the exceptional divisor, which corresponds to its volume relative to the blown-up $\mathbb{CP}^2$ [INV]. The following two subsections further try to clarify the geometric origin of the β_i in (4.96), although an *exact* geometric correspondence is still unclear.

4.6.1 Correspondence with $S^2 \times S^2$

Just as we obtained the partition function $Z_{cl}(\tau, \overline{\tau}, \beta)_{dP_1}$ in (4.98) by letting all but one β_i go to zero, it is also possible to obtain $Z_{cl}(\tau, \overline{\tau}, r)_{S^2\times S^2}$ in (4.91) as a factor by taking specific values for the parameters β_i. The starting point in obtaining (4.96) was to look for G such that $GQG = Q = (1) \oplus l(-1)$, which resulted in (4.95). We could have also looked for G' such that $G'Q'G' = Q'$, with Q' as in (4.94). The matrix B as in (4.41) with $m = l-1$ is an automorphism of the lattice $\mathbb{Z}^{b^2}$ satisfying $B^tQ'B = Q$. The desired matrix G' is then given by $G' = (B^t)^{-1}GB^{-1}$ with G as in (4.95), which is also symmetric and positive-definite. The inverse of B has the form

$$B^{-1} = \begin{pmatrix} \begin{matrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{matrix} & O \\ O & I_{l-2} \end{pmatrix} = \begin{pmatrix} A & O \\ O & I_{l-2} \end{pmatrix}.$$

Denoting

$$G = \begin{pmatrix} D & E \\ E^t & \tilde{G} \end{pmatrix},$$

where D is the upper-left 3×3-submatrix and $\tilde{G}$ the lower-right $(l-2)\times(l-2)$-submatrix, we calculate G':

$$G' = \begin{pmatrix} A^tDA & A^tE \\ E^tA & \tilde{G} \end{pmatrix}.$$

Setting $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, the components A^tE and A^tDA are given by

$$A^tE = \begin{pmatrix} -\gamma\beta_3 - (\gamma-1)\beta_1\beta_3/\beta^2 & \dots & -\gamma\beta_l - (\gamma-1)\beta_1\beta_l/\beta^2 \\ -\gamma\beta_3 - (\gamma-1)\beta_2\beta_3/\beta^2 & \dots & -\gamma\beta_l - (\gamma-1)\beta_2\beta_l/\beta^2 \\ -\gamma\beta_3 - (\gamma-1)\beta_1\beta_3/\beta^2 & \dots & -\gamma\beta_l - (\gamma-1)(\beta_1+\beta_2)\beta_l/\beta^2 \end{pmatrix} \tag{4.101}$$

$$A^tDA = \begin{pmatrix} \begin{matrix}\gamma(1+2\beta_1)+1 \\ +(\gamma-1)\beta_1^2/\beta^2\end{matrix} & \begin{matrix}\gamma(1+\beta_1+\beta_2) \\ +(\gamma-1)\beta_1\beta_2/\beta^2\end{matrix} & \begin{matrix}\gamma(1+2\beta_1+\beta_2)+1 \\ +(\gamma-1)(\beta_1^2+\beta_1\beta_2)/\beta^2\end{matrix} \\ \begin{matrix}\gamma(1+\beta_1+\beta_2) \\ +(\gamma-1)\beta_1\beta_2/\beta^2\end{matrix} & \begin{matrix}\gamma(1+2\beta_2)+1 \\ +(\gamma-1)\beta_2^2/\beta^2\end{matrix} & \begin{matrix}\gamma(1+\beta_1+2\beta_2)+1 \\ +(\gamma-1)(\beta_2^2+\beta_1\beta_2)/\beta^2\end{matrix} \\ \begin{matrix}\gamma(1+2\beta_1+\beta_2)+1 \\ +(\gamma-1)(\beta_1^2+\beta_1\beta_2)/\beta^2\end{matrix} & \begin{matrix}\gamma(1+\beta_1+2\beta_2)+1 \\ +(\gamma-1)(\beta_2^2+\beta_1\beta_2)/\beta^2\end{matrix} & \begin{matrix}\gamma(1+2\beta_1+2\beta_2)+2 \\ +(\gamma-1)(\beta_1+\beta_2)^2/\beta^2\end{matrix} \end{pmatrix} \tag{4.102}$$

If we let all β_i go to zero for $i \neq 1, 2$, we have

$$G' \to G' = \begin{pmatrix} A^tDA & O \\ O & I_{l-2} \end{pmatrix},$$

where now $\gamma = \frac{1}{\sqrt{1-\beta_1^2-\beta_2^2}}$. We consider the set

$$\begin{aligned} \mathcal{B} &= \left\{(\beta_1, \beta_2) \in [-1, 0]^2 \,|\, \beta_1^2 + \beta_1 + \beta_2 + \beta_1\beta_2 + \beta_2^2 = 0\right\} \\ &= \left\{(\beta_1, \beta_2) \in [-1, 0]^2 \,|\, \beta_1 = -\frac{1+\beta_2}{2} - \frac{1}{2}\sqrt{(1+\beta_2)(1-3\beta_2)}\right\}, \end{aligned} \tag{4.103}$$

which are the solutions to $\gamma^{-1} = \sqrt{(1+\beta_1)(1+\beta_2)}$. These solutions simultaneously solve

$$\begin{aligned}\gamma(1+\beta_1+\beta_2) = -1 &= -(\gamma-1)\beta_1\beta_2/\beta^2 \\ \gamma\beta_1 + (\gamma-1)\beta_1^2/\beta^2 = -1 &= \gamma\beta_2 + (\gamma-1)\beta_2^2/\beta^2,\end{aligned}$$

as one can directly check. For these values of β_1 and β_2 the matrix A^tDA becomes

$$G' = \begin{pmatrix} \begin{matrix} r^2 & 0 \\ 0 & \frac{1}{r^2} \end{matrix} & O \\ O & I_{l-1} \end{pmatrix}, \tag{4.104}$$

where $r^2 = \sqrt{\frac{1+\beta_1}{1+\beta_2}}$. We recognize the upper-left 2×2-matrix as the matrix (4.89).

Finally, if we take the partition function as in (4.18) but plug in G' and Q' instead of G and Q, the resulting partition function factorizes as

$$Z_{sc}(\tau,\overline{\tau},\beta_1,\beta_2,0,\ldots,0)_{dP_l}|_{(\beta_1,\beta_2)\in\mathcal{B}} = \left(\sum_m e^{-i\pi\overline{\tau}m^2}\right)^{l-1} Z_{sc}\left(\tau,\overline{\tau},r=\left(\frac{1+\beta_1}{1+\beta_2}\right)^{\frac{1}{4}}\right)_{S^2\times S^2}, \tag{4.105}$$

where $Z_{sc}(\tau,\overline{\tau},r)_{S^2\times S^2}$ is the partition function (4.91). Notice that we would get this result as well (but with a little more work) if we would have set $\beta_i = 0$ in (4.96) for $i > 2$ and let $(\beta_1,\beta_2)\in\mathcal{B}$.

What we would like is that similar to (4.105) Z_{sp} factors as

$$Z_{sp}(\tau,\overline{\tau},\beta_1,\beta_2,0,\ldots,0)_{dP_l}|_{(\beta_1,\beta_2)\in\mathcal{B}} = \left(\sum_m e^{-i\pi\overline{\tau}(m+1/2)^2}\right)^{l-1} Z_{sc}\left(\tau,\overline{\tau},r=\left(\frac{1+\beta_1}{1+\beta_2}\right)^{\frac{1}{4}}\right)_{S^2\times S^2},$$

since the upper-left 2×2-submatrix of Q' that corresponds to the intersection form on $S^2\times S^2$ is even. If the sequence of integers in the sum in (4.96) is translated by $\frac{1}{2}(1,-1,\ldots,-1)$, we see that under B this vector gets mapped to $\frac{1}{2}(0,0,-1,\ldots,-1)$. After we translate the sequence of integers in (4.105) by these numbers, we indeed get the desired factorization.

Starting from $S^2\times S^2\#\overline{\mathbb{CP}}^2$

In the calculations above we actually circumvented the computation of G' by directly solving $G'Q'G' = Q'$, with Q' given by (4.94). For $l = 2$ the direct computation is not very long and will give us a parametrization for G' that is not as extensive as $G' = A^tDA$ in (4.102). Plugging Q' and G in (4.18) will then result in a different paramatrization of $Z_{sc/sp}(\tau)_{dP_2}$ than as we had before.

The ansatz for G' is

$$G' = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}.$$

The equation we have to solve is $G'Q'G' = Q'$, where

$$Q' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This gives us the following set of equations:

$$\begin{aligned} 2ab - c^2 &= 0 & 2bd - c^2 &= 0 \\ 2ce - f^2 &= -1 & ad + b^2 - ce &= 1 \\ ae + bc - cf &= 0 & eb + dv - ef &= 0. \end{aligned}$$

One can check that a, d and b all should have the same sign. The parameters c, e could have the same or opposite sign. In the first case, we must have

$$b = \sqrt{ad} \pm 1, \quad c = \pm\sqrt{2a(\sqrt{ad} - 1)}, \quad e = \pm\sqrt{2d(\sqrt{ad} - 1)}, \quad f = b + \sqrt{ad} = 2\sqrt{ad} \pm 1,$$

and in the other case

$$b = -(\sqrt{ad} \pm 1), \quad c = \pm\sqrt{2a(\sqrt{ad} - 1)}, \quad e = \mp\sqrt{2d(\sqrt{ad} - 1)}, \quad f = b - \sqrt{ad} = -(2\sqrt{ad} \pm 1).$$

Now the signs are being fixed by imposing positive-definiteness for G'. We know that the possible matrices should lie in the same connected component of $GL(3, \mathbb{R})$, so under a continuous limit we should be able to obtain G' as in (4.104). The only solution that is consistent with this limit, where we must have $ad = 1$, is given by

$$G' = \begin{pmatrix} a & \sqrt{ad} - 1 & \pm\sqrt{2a(\sqrt{ad} - 1)} \\ \sqrt{ad} - 1 & d & \pm\sqrt{2d(\sqrt{ad} - 1)} \\ \pm\sqrt{2a(\sqrt{ad} - 1)} & \pm\sqrt{2d(\sqrt{ad} - 1)} & 2\sqrt{ad} - 1 \end{pmatrix}. \tag{4.106}$$

In the solution above, a and d must be positive parameters such that $ad \geq 1$. One can check that this solution is consistent with (4.102) if we write

$$a = \gamma(1 + 2\beta_1) + 1 + (\gamma - 1)\beta_1^2/\beta^2 \tag{4.107}$$
$$d = \gamma(1 + 2\beta_2) + 1 + (\gamma - 1)\beta_2^2/\beta^2. \tag{4.108}$$

In the parametrization above we indeed have $a, d > 0$ and

$$\begin{aligned} \sqrt{ad} - 1 &= \gamma(1 + \beta_1 + \beta_2) + (\gamma - 1)\beta_1\beta_2/\beta^2 \geq 0 \\ 2\sqrt{ad} - 1 &= \gamma(1 + 2\beta_1 + 2\beta_2) + 2 + (\gamma - 1)(\beta_1 + \beta_2)^2/\beta^2 \\ \pm\sqrt{2a(\sqrt{ad} - 1)} &= \gamma(1 + 2\beta_1 + \beta_2) + 1 + (\gamma - 1)(\beta_1^2 + \beta_1\beta_2)/\beta^2 \\ \pm\sqrt{2d(\sqrt{ad} - 1)} &= \gamma(1 + \beta_1 + 2\beta_2) + 1 + (\gamma - 1)(\beta_2^2 + \beta_1\beta_2)/\beta^2, \end{aligned}$$

where the sign in the last two equations is negative when (β_1, β_2) lie in the set

$$\left\{(\beta_1, \beta_2) \in [-1, 0]^2 \,|\, \beta_1^2 + \beta_2^2 < 1, \quad \beta_1^2 + \beta_1 + \beta_2 + \beta_1\beta_2 + \beta_2^2 \geq 0\right\}.$$

When $ad = 1$, we obtain (4.105) with $r^2 = a$.

We can invert equation (4.107) and (4.108) by comparing $B^t G' B$ with equation (4.95), where B is as in (4.41) with $m = 1$. This gives us the following three equations:

$$\begin{aligned} \gamma &= 4\sqrt{ad} - 3 + a + d - 2(c + e) \\ \gamma\beta_1 &= 3\sqrt{ad} - 2 + d - (c + 2e) \\ \gamma\beta_2 &= 3\sqrt{ad} - 2 + a - (e + 2c). \end{aligned}$$

After filling in c and e, we get the following expressions for β_1 and β_2 in terms of a and d:

$$\beta_1 = \frac{3\sqrt{ad} - 2 + d \mp \sqrt{2(\sqrt{ad}-1)(\sqrt{a}+2\sqrt{d})}}{4\sqrt{ad} - 3 + a + d \mp 2\sqrt{2(\sqrt{ad}-1)(\sqrt{a}+\sqrt{d})}} \tag{4.109}$$

$$\beta_2 = \frac{3\sqrt{ad} - 2 + a \mp \sqrt{2(\sqrt{ad}-1)(2\sqrt{a}+\sqrt{d})}}{4\sqrt{ad} - 3 + a + d \mp 2\sqrt{2(\sqrt{ad}-1)(\sqrt{a}+\sqrt{d})}}. \tag{4.110}$$

4.6.2 Comparison with $n(S^2 \times S^2)$

According to Theorem 2.7 the homology for the connected sum $n(S^2 \times S^2) = S^2 \times S^2 \# \ldots \# S^2 \times S^2$ is generated by the A- and B-cycles (see Remark 4.24) in the n connected sum components. Let Σ_i and $\tilde{\Sigma}^i$ be the A-cycle and B-cycle on the i-th copy of $S^2 \times S^2$. Let α_i, β^i be the corresponding harmonic representatives of the duals, i.e. $\int_{\Sigma_i} \alpha_j = \int_{\tilde{\Sigma}^i} \beta^j = \delta_{ij}$. Proposition 2.8 and (4.85) then show the intersection form is given by

$$n\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If we reorganize the basis $H^2(M, \mathbb{Z})$ to $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$, this intersection form is seen to be equivalent to the $2n \times 2n$-matrix

$$Q_{IJ} = Q^{IJ} = \int_M \alpha_i \wedge \beta^j = \begin{pmatrix} O & I \\ I & O \end{pmatrix}.$$

According to our discussion in Section 4.4.1, the matrix G_{IJ} expressed in this basis should be a positive-definite symmetric matrix satisfying

$$G^{-1} = Q^{-1} G Q^{-1}.$$

By writing

$$\int_M \beta^i \wedge \star\beta^j = G^{ij}, \quad \int_M \alpha_i \wedge \star\beta^j = B_i^j, \quad \int_M \alpha_i \wedge \star\alpha_j = C_{ij},$$

the matrix $G = G_{IJ}$ can be expressed as

$$G_{IJ} = \begin{pmatrix} C_{ij} & B_i^j \\ B_j^i & G^{ij} \end{pmatrix}.$$

This is a $2n \times 2n$-matrix, composed of 4 $n \times n$ -submatrices, where the indices $i, j = 1, \ldots, n$ denote the row- and column-vector, respectively. The condition $G^{-1} = Q^{-1} G Q^{-1}$ induces the following relations

$$\sum_k B_k^i G^{kj} + \sum_k G^{ik} B_k^j = 0, \quad C_{ij} = G_{ij} - \sum_{k,l} B_i^l B_l^k G_{kj}, \tag{4.111}$$

where G_{ij} is the inverse of G^{ij}. If we define $B_{ij} := \sum_k B_i^k G_{kj}$, these relations turn into $B_{ij} = -B_{ji}$ and $C_{ij} = G_{ij} - \sum_{k,l} B_{ik} G^{kl} B_{lj}$. The $b^+ \times b^- = n \times n$ moduli parameters are given by the entries in the symmetric matrix G_{ij} and the anti-symmetric matrix B_{ij}. We still have to impose positive-definiteness on G_{IJ}, but this doesn't change the number of parameters.

Remark 4.26. The ansatz we use to compute G for $n(S^2 \times S^2)$ is different than the one we used to compute G for dP_l as in (4.95). We start with a symmetric $2n \times 2n$-matrix, which consist of $n(2n+1)$ parameters. The relation $Q^{-1}GQ^{-1} = G^{-1}$ then gives us the relations in (4.111). This kills the $\frac{1}{2}n(n+1)$ parameters in C_{ij} and relates G^{ij} and B_i^j to each other through $\frac{1}{2}n(n-1)$ plus n equations. The number of independent parameters then reduces to $n \times n$. A part of the $O(n, n, \mathbb{Z})$ symmetry in (4.81) is formed by relabeling the A- and B-cycles and this corresponds to an action of the mapping class group of $n(S^2 \times S^2)$. Another part of the symmetry group is the permutation of an A-cycle and its corresponding B-cycle, which for $S^2 \times S^2$ was seen to give the $O(1, 1, \mathbb{Z})$ symmetry $r \mapsto 1/r$.

After filling the paramaters B_{ij}, G_{ij} and G^{ij} into equation (4.9), the classical action of the saddle point configurations becomes

$$S[m^I] = S[m,n] = \frac{4\pi}{g^2}\left(\sum_{i,j} m^i G_{ij} m^j + \sum_{i,j}(n_i - \sum_k B_{ik}m^k)G^{ij}(n_j - \sum_l B_{jl}m^l)\right) - i\theta m^i n_i, \tag{4.112}$$

where n_i and m^i are the flux over the B-cycle $\tilde{\Sigma}^i$ and A-cycle Σ_i, respectively. By performing a Poincaré summation over n_i one obtains [Ver]

$$Z_{sc}(G,B)_{n(S^2\times S^2)} = \sum_{m,n} \exp(-S[m,n]) \rightarrow Z_{sc}(G,B,\tau,\overline{\tau})_{n(S^2\times S^2)} = \sum_{m,n} \exp(-2\pi E[m,n]), \tag{4.113}$$

with

$$E[m,n] = \frac{1}{\mathrm{Im}\tau} \sum_{i,j} (n^i + \tau m^i)(G_{ij} + B_{ij})(n^j + \overline{\tau} m^j).$$

According to [Ver], pinching the A-cycle Σ_i corresponds to G_{ii} blowing up and going to infinity. In this limit factorization takes place as

$$Z[m,n] = \exp(-2\pi E[m,n]) \rightarrow \prod_i \exp(-2\pi G_{ii}|n_i + m_i\tau|^2/\mathrm{Im}\tau) W_{m_i n_i},$$

where the numbers $W_{m_i n_i}$ are quantities that can be computed using Abelian versions of the *Wilson–'t Hooft line operators.*

The case $n = 1$ was covered in Section 4.5, where pinching the A-cycle just means letting $r = R'/R$ go to infinity. One indeed sees from (4.89) that $G_{ii} = G_{11}$ goes to infinity and $G^{ii} = G_{22}$ to zero. The dominant factor in the partition function (4.91) becomes

$$Z_{sc}(\tau,\overline{\tau},r)_{S^2\times S^2} \approx \frac{1}{\sqrt{\mathrm{Im}\tau}} \sum_{n,m} \exp\left(\frac{i\pi(\tau-\overline{\tau})}{2} r^2 n^2\right),$$

where n is the quantized flux over the A-cycle.

The same sort of blowing-up of G in (4.95) happens when $\beta^2 \rightarrow 1$. A special case is letting one β_i go to ± 1 and setting the rest of the β_j to zero. In this limit the leading contribution is given by the asymptotic behaviour of the factor (4.99):

$$Z_{sc}(\tau,\overline{\tau})_{dP_1} \approx \frac{1}{\sqrt{\mathrm{Im}\tau}} \sum_{n,m} \exp\left(\frac{i\pi(\tau-\overline{\tau})}{2}\frac{(n-m)^2}{\sqrt{1-\beta^2}}\right).$$

Equation (4.98) then turns into

$$\tilde{Z}_{sc}(\tau,\overline{\tau},\beta)_{dP_l} \approx \left(\sum_m \exp(-i\pi\overline{\tau}m^2)\right)^{l-1} \sum_{n,m} \exp\left(\frac{i\pi(\tau-\overline{\tau})}{2}\frac{(n-m)^2}{\sqrt{1-\beta^2}}\right).$$

Another case is the special limit (4.105), where $\beta_i = 0$ for $i > 2$ and $(\beta_1, \beta_2) \in \mathcal{B}$ as in (4.103). There, the analogy with pinching a cycle is more direct: $(\beta_1, \beta_2) \to (-1, 0)$ corresponds to r going to zero and $(\beta_1, \beta_2) \to (0, -1)$ corresponds to r going to infinity.

By the similarity in the limiting behaviour of the partition functions, one would guess that letting β_i go to 1 in (4.96) corresponds to pinching some surface in dP_l that is diffeomorphic to S^2. This might be the exceptional divisor E_i. Pinching this divisor means letting the associated Kähler parameter go to zero, i.e. the volume of E_i is shrinked to zero.

Chapter 5

Outlook

In Chapter 2 we have discussed some basic facts about four-manifolds and studied del Pezzo surfaces in more detail using blow-ups. Both in the algebraic and smooth sense these surfaces had some interesting properties. In Chapter 3 we covered vector bundles endowed with a structure group. Four-dimensional Yang–Mills theory and instantons were described in this setting and we discussed the class of Yang–Mills instantons formed by anti-self-dual connections. We covered some central results regarding the moduli space of these ASD connections, with a focus on $SU(2)$ bundles. For Kähler surfaces we could describe the $SU(2)$ ASD connections in terms of stable holomorphic vector bundles. This correspondence could be applied to del Pezzo surfaces and we used methods from algebraic geometry to model the ASD moduli space for $S^2 \times S^2$. The blow-up construction that related $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ was central in this model.

Abelian gauge theories have been studied in Chapter 4 in the form of Euclidean Maxwell theory. The associated partition function could be calculated using a semi-classical approximation. Electric-magnetic duality could be studied by looking at how this function transformed under modular transformations of the complex coupling constant. Gauge fields coupling to (background) scalar fields fitted in the framework of connections and line bundles. The underlying four-manifold was taken to be closed, connected, oriented and Riemannian, so that the resulting partition function became a modular form. Crucial tools in the calculations were Poincaré duality, the intersection form, Hodge theory and the Dirac quantization condition. When gauge fields coupled to scalar fields, the Dirac quantization was shifted and the altered partition function had to be studied in a more generalized framework using theta functions and lattices. The scalar and spinor partition function together with a third function allowed for an action of $SL(2, \mathbb{Z})$ and the modular behaviour of this set of functions was determined. If the manifold was spin, both the scalar and spinor partition functions were equal to each other and modular covariant under the full group $SL(2, \mathbb{Z})$. The dependence of the partition functions on the metric was reduced to algebraic relations involving the intersection form. The price we paid was that the geometric origin of the parametrization was unclear and could be computed only in some cases. We applied our results to del Pezzo surfaces and were able to find an exact expression for the partition functions. These functions had interesting factorization behaviour under certain limits and these reflected the geometrical properties of the del Pezzo surfaces.

Electric-magnetic duality and Yang–Mills instantons both appear in recent examples of exact models in gauge theory. Although a great deal of extra symmetry and simplification of the physics is needed, the hope is that the study of these models will shed light on the mathematical structures that run through physics. In this setting we will first comment on the duality of Euclidean Maxwell theory with toroidal models in string theory. Next we mention how similar approaches using saddle points as in Chapter 4 are nowadays applied to $SU(2)$

gauge theories. The ASD moduli spaces naturally appear in some models. In Seiberg–Witten gauge theory electric-magnetic duality is encountered as well.

Duality with toroidal compactifications in string theory

The partition functions we obtained for Abelian gauge theories on four-manifolds are very similar to those one encounters for toroidal models in string theory. The geometric origin of the moduli parameters that are present in the partition functions is still unclear. These parameters also appear in the toroidal models, so studying this duality might give new information.

The space $\mathcal{M}_{b^+,b^-}$ as in (4.81) was first studied in the context of string theory. In fact, the results we used to describe this space in Section 4.4 are actually leant from papers in this field. Verlinde argued that the appearance of the space $\mathcal{M}_{b^+,b^-}$ is not a coincidence. In Section 4 of [Ver] a six-dimensional theory on $M^4 \times T^2$ is studied. It is computed that the compactification on the torus results in a Euclidean Maxwell theory on M^4. The modular parameter of the torus then gets mapped to the complex coupling constant τ in Euclidean Maxwell theory. It is observed that the partition function of $n(S^2 \times S^2)$ (or a manifold with the same intersection form) in (4.113) is equal to the partition function of a specific toroidal model, where both the left- and right moving sectors of the bosonic string are compactified on a n-dimensional torus T^n. Subsequently it is showed how one obtains this model after compactification on the four-manifold. The parameter τ in the Euclidean Maxwell action then gets mapped to the modular parameter of the torus, which returns in the genus one partition sum of the toroidal compactification. These toroidal models have been extensively studied in [GMR, GPR]. The actions of $SL(2,\mathbb{Z})$ on τ in both theories are dual to each other as well. The moduli space of inequivalent toroidal models then exactly corresponds with the space $\mathcal{M}_{n,n}$ given by (4.81).

Verlinde extended this idea to four-manifolds with $b^+ \neq b^-$, of which the del Pezzo surfaces are an example. He argues that then a toroidal model is obtained with unequal left- and right movers. For del Pezzo surfaces this duality might be related to the 'mysterious duality' as covered in [INV]. This last duality relates the moduli space of the extended Kähler metric on dP_l to the $l+1$ moduli parameters of M-theory on T^l. The extended Kähler metric on dP_l is determined by $l+1$ Kähler parameters: the volumes of the l exceptional divisors and the volume of the blown-up $\mathbb{CP}^2$. M-theory on T^l is described through the l compactification radii and the Planck scale. Just before Section 4.6.1 we suggested that the parameters β_i in (4.96) could be related to l of these Kähler parameters.

Modern $SU(2)$ gauge theories

One of the things we needed for the factorization of the partition function in a classical saddle point contribution and determinants, was a Faddeev–Popov procedure. For $U(1)$ gauge theories, this procedure would not encounter any major problems, since the theories are Gaussian. For $SU(2)$ gauge theories however (and other non-Abelian gauge theories), the regularized determinants are usually not exactly computable or could contain anomalies, since the action is not quadratic anymore. It is still an outstanding problem in mathematics to resolve the mathematical issues related to this [JW]. In some supersymmetric $SU(2)$ gauge theories, it is possible to compute an exact answer for the partition function and other correlation functions using a saddle point approximation. Famous examples are the twisted $N = 2$ version Witten used to compute the Donaldson invariants [Wit1] and Seiberg–Witten $SU(2)$ gauge theory [SW1, SW2]. Since the ASD connections form saddle points of the Yang–Mills functional and gauge transformations are usually modded out in the calculations, the ASD moduli spaces $\mathcal{M}_k$ from Chapter

3 naturally appear in these theories. Other examples where these moduli spaces are used, can be found in [VW, LNS].

The saddle points of the Yang–Mills functional not necessarily have to be self-dual or anti-self-dual, so that there still exist classes of Yang-mills instantons that we have not described yet. The question whether such instantons exist is not a simple one. In section 2.1 of [VN] these *non-self-dual* instantons are discussed and further references regarding this problem are given.

In general it is difficult to give an explicit model of $\mathcal{M}_k$ and its compactification. As we saw in Section 3.4, there was a correspondence between the moduli space of ASD connnections and stable rank 2 holomorphic vector bundles in the presence of a Kähler structure. This enabled us to give a description of the ASD moduli space using algebro-geometric methods. We explicitly described the moduli space $\mathcal{M}_2$ and its compactification in terms of quadrics that intersect the embedding of $S^2 \times S^2$ as a quadric in $\mathbb{CP}^3$. Such models for the ASD moduli spaces might make it possible to exactly perform certain computations in this field of gauge theory, where integrals over these spaces are encountered. The ADHM construction has for example been used in Nekrasov instanton calculus [Nek, NO, NS].

Sections 3 and 4 of [Wit3] give a description of how the Abelian gauge theories we considered in Chapter 4 are embedded in Seiberg–Witten $SU(2)$ gauge theory. The $U(1)$ connections in Euclidean Maxwell theory appear as a connection on specific line bundles in this theory. These bundles appear in the splitting $E = L \oplus L^{-1}$ for reducible $SU(2)$ connections, which we covered in Section 3.3.1. In Euclidean Maxwell theory we had two gauge fields A and A_D and under the transformation $S \in SL(2,\mathbb{Z})$ their field strengths transformed into each other as

$$F_D \mapsto -F, \quad F \mapsto F_D.$$

In Seiberg–Witten gauge theory these $U(1)$ connections A and A_D are defined similarly and the S-duality present in this theory is similar to the electric-magnetic duality we covered. When the underlying manifold is not spin, there exist non-trivial Spin^c-structures, which causes the scalar and spinor partition function for Euclidean Maxwell theory to be different. Such non-trivial Spin^c-structure also play a role in Seiberg–Witten gauge theory. The results we covered in Chapter 4 might be useful when studying the embedding of these gauge fields as described by Witten. For the conjecture relating the Donaldson invariants and Seiberg–Witten invariants (as stated in [Wit2]), many deep connections still have to be established.

Appendix A

Complex projective space

In this appendix we cover some basic properties of complex projective space and the Segre embedding .

A.1 Complex projective n-space

Let $\mathbb{C}^n$ be the standard n-dimensional Euclidean vector space over $\mathbb{C}$. We say $z \sim z'$ if and only if there exists a $\lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$ such that $z' = \lambda z$. Complex projective n-space, $\mathbb{CP}^n$, is defined as the quotient $\mathbb{C}^{n+1} - \{0\}/\sim$.

If $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$, then its equivalence class in $\mathbb{CP}^n$ is denoted by $(z_0 : \ldots : z_n)$. These are the so-called *homogeneous coordinates* on $\mathbb{CP}^n$ and the colon ":" indicates we are dealing with ratios, i.e.

$$(z_0 : \ldots : z_n) = (\lambda z_0 : \ldots : \lambda z_n).$$

The space $\mathbb{CP}^n$ is also called the set of *complex lines* in $\mathbb{C}^{n+1}$.

If $Z_i \neq 0$, $z_j = Z_j/Z_i$ defines a local coordinate chart on $\mathbb{CP}^n$. We can then write $(Z_0 : \ldots : Z_n) = (z_0 : \ldots z_{i-1} : 1 : z_{i+1} : \ldots : z_n)$. These charts are holomorphic, so that $\mathbb{CP}^n$ is a complex manifold of dimension n. For $n = 1$ these maps coincide with the stereographic projections for the two-dimensional sphere S^2 when it is realized as a complex manifold. This shows $\mathbb{CP}^1 \simeq S^2$ and S^2 realized in this way is also called the *Riemann sphere*.

To get a better geometric intuition of $\mathbb{CP}^n$, one can look at the following embedding of $\mathbb{CP}^{n-1}$ in $\mathbb{CP}^n$:

$$\mathbb{CP}^n \cap \{Z_n = 0\} = \{(Z_0 : \ldots : Z_{n-1} : 0) \,|\, 0 \neq (Z_0, \ldots, Z_{n-1}, 0) \in \mathbb{C}^{n+1})\} \simeq \mathbb{CP}^{n-1}. \qquad \text{(A.1)}$$

When $Z_n \neq 0$, we have just seen that $\mathbb{CP}^n \cap \{Z_n \neq 0\} \simeq \mathbb{C}^n$ under the correspondence

$$(Z_0 : \ldots : Z_n) = (z_0 : \ldots z_{i-1} : 1 : z_{i+1} : \ldots : z_n).$$

This altogether gives us the following decomposition for $\mathbb{CP}^n$:

$$\mathbb{CP}^n \simeq \mathbb{C}^n \sqcup \mathbb{CP}^{n-1} = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \ldots \sqcup \mathbb{C} \sqcup \{(1 : 0 : \ldots : 0)\}.$$

For $n = 1$ this decomposition is called adding 'the point at infinity' to $\mathbb{C}$, for $n = 2$ adding the 'projective line at infinity', et cetera. Notice that the choices of our coordinates are arbitrary, so that in general we can embed any $\mathbb{CP}^m$ in $\mathbb{CP}^n$ (for $m < n$) using the description above.

Dividing by the norm of $(Z_0, \ldots, Z_n)$, we see $(Z_0 : \ldots : Z_n) = (z_0 : \ldots : z_n)$, where $\sum_i |z_i|^2 = 1$. Under the isomorphism $\mathbb{C}^{n+1} \simeq \mathbb{R}^{2(n+1)}$, the $(2n+1)$-sphere can be realized as

$S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^n \mid |\sum_i |z_i|^2 = 1\}$. For $z, z' \in S^{2n+1}$, we have $z \sim z'$ if $z = \lambda z'$ with $|\lambda| = 1$, meaning $\lambda \in S^1 \subset \mathbb{C} - \{0\}$. This shows $\mathbb{CP}^n \simeq S^{2n+1}/S^1$.

Modeling $\mathbb{CP}^n$ in this way, one can compute the homology of $\mathbb{CP}^n$. Let B^n be the closed unit n-ball. One can check that the map

$$z = (z_0, \ldots, z_{n-1}) \in B^{2n} \mapsto (z_0 : \ldots : z_{n-1} : 1 - \|z\|) \in \mathbb{CP}^n \simeq S^{2n+1}/S^1,$$

is a characteristic map $(B^{2n}, S^{2n-1}) \to (\mathbb{CP}^n, \mathbb{CP}^{n-1})$, i.e. it is surjective, injective on $B^{2n} \backslash S^{2n-1}$ and the image of S^{2n-1} is $\mathbb{CP}^{n-1}$. Using tools from algebraic topology, we get $H_k(\mathbb{CP}^n) = 0$ for k odd and $H_k(\mathbb{CP}^n) = H_k(B^k, S^{k-1}) = \mathbb{Z}$ for k even.

A.2 The Segre embedding

The Segre embedding ([Har], page 13) is given by:

$$\Psi : \mathbb{CP}^m \times \mathbb{CP}^n \to \mathbb{CP}^{mn+n+m}; ((s_0 : \ldots : s_m), (t_0 : \ldots t_n)) \mapsto (s_0 t_0 : s_0 t_1 : \ldots : s_m t_n). \quad \text{(A.2)}$$

Let us denote Z_{ij} for the homogeneous coordinates of $\mathbb{CP}^{mn+m+n}$ where $0 \leq i \leq m$ and $0 \leq j \leq n$ and the coordinates are ordered lexicographically. We have the following proposition for the Segre embedding.

Proposition A.1. *The Segre embedding is well-defined, injective and its image in $\mathbb{CP}^{mn+m+n}$ is given by*

$$\bigcap_{i \neq k, j \neq l} \{Z_{ij} Z_{kl} - Z_{il} Z_{kj} = 0\} = \bigcap_{i \leq k, j \leq l} \{Z_{ij} Z_{kl} - Z_{il} Z_{kj} = 0\}.$$

Proof. The map is well-defined, since $\Psi(\lambda s, \lambda' t) = \lambda \lambda' \Psi(s, t) = \Psi(s, t)$. To see injectivity, we first notice $\Psi(s, t) = \Psi(s', t')$ implies that for some $\lambda \in \mathbb{C}^*$ we have $s_i t_j = \lambda s'_i t'_j \ \forall i, j$. If $s_i = 0$, then there is at least one $t'_k \neq 0$, so that $0 = \lambda t'_k s'_i$ implies $s'_i = 0$. Similarly $t_j = 0$ implies $t'_j = 0$. Since not all s_i can be equal to zero, $s_k, s'_k \neq 0$ for at least one k. For all j we now have $t_j = (\lambda s'_k / s_k) t'_j$, hence $t = (t_0 : \ldots : t_n) = (t'_0 : \ldots : t'_n) = t'$. Similarly $s = s'$.

The inclusion $\Psi(\mathbb{CP}^m \times \mathbb{CP}^n) \subset \bigcap_{i \neq k, j \neq l} \{Z_{ij} Z_{kl} - Z_{il} Z_{kj} = 0\}$ is clear, since for arbitrary i, j, k, l with $i \neq k, j \neq l$, we have $s_i t_j s_k t_l - s_i t_l s_k t_j = 0$. The reverse inclusion can be proved by studying the sets $\{Z_{ij} Z_{kl} - Z_{il} Z_{kj} = 0\}$ with $i \neq k, j \neq l$ in the local charts given by (2.16) and (2.17). The sets U_i are more properly indexed as $U_{ij} = \{Z_{ij} \neq 0\}$, with $0 \leq i \leq m, 0 \leq j \leq n$. Let us fix i, j. Now we have

$$\{Z_{ij} Z_{kl} - Z_{il} Z_{kj} = 0\} \cap U_{ij} \quad = \quad \{Z_{kl}/Z_{ij} - (Z_{il}/Z_{ij})(Z_{kj}/Z_{ij}) = 0, Z_{ij} \neq 0\}$$

For every $Z_{ij} \neq 0$ there exists $s_i, t_j \neq 0$ such that $Z_{ij} = s_i t_j$. For every $l \neq j$ and $k \neq i$ we can now find s_k and t_l such that $Z_{il} = s_i t_l$ and $Z_{kj} = s_k t_j$. On U_{ij} we now see that $\{Z_{ij} Z_{kl} - Z_{il} Z_{kj} = 0\}$ is a subset of

$$\{Z_{kl} - s_k t_l = 0\} \cap \{Z_{ij} - s_i t_j = 0\} \cap \{Z_{il} - s_i t_l = 0\} \cap \{Z_{kj} - s_k t_j = 0\}, \text{ where } s_i, t_j \neq 0.$$

The intersection of these sets over all $k \neq i, l \neq j$ obviously is the image of Ψ intersected with U_{ij}. Furthermore, if $Z_{ij} \neq 0$ and $Z_{ij} Z_{kl} - Z_{il} Z_{kj} = 0$ for all $k \neq i, l \neq j$, then $Z_{k'l'} Z_{k''l''} - Z_{k'l''} Z_{k''l'} = 0$ for all combinations $k' \neq k'', l' \neq l''$.

Letting i, j vary again, we see that on all charts the set $\bigcap_{i \neq k, j \neq l} \{Z_{ij} Z_{kl} - Z_{il} Z_{kj} = 0\}$ lies in the image of Ψ. □

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