Universiteit Utrecht

How to juggle the proof for every theorem

An exploration of the mathematics behind juggling

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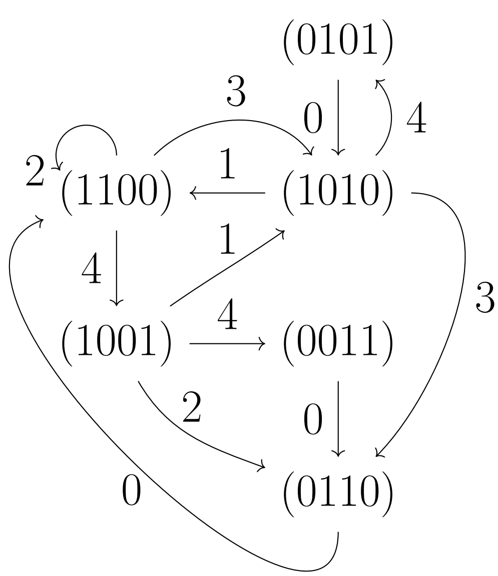

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1 Abstract

As the subtitle clarifies this thesis is on the mathematics behind juggling. Whenever we want to learn, talk about or describe something, we need a language which fits the subject well. Sometimes the language we speak suffices and all we need is to find some common ground by developing terminology. However, sometimes this does not suffice and we need to develop a language to fit the subject, taking music as an example. Imagine having to communicate a piece of music without having a dedicated language and notation, it's very hard! A similar thing is to say for communicating juggling so a language was developed. Given the precise nature of juggling, it is not much of a surprise that this became a mathematical language.

Having a mathematical language means having definitions and we need to check that these capture our phenomena we try to describe. We do so by making sure that, given our abstract definition of a juggling pattern, we can retrieve all the information necessary to actually juggle the pattern in the detail we are aiming to describe it. We answer practical questions like how many balls a pattern requires and where we need to start in the pattern.

Having a mathematical language also allows very well for seemingly nonsensical questions to be asked. Since we can be sure that we translated juggling into a piece of mathematics which makes sense for our topic, we can forget about the juggling for a moment and play around with just the mathematics. The theorems we then prove might not have any interesting physical interpretation at all, but sometimes we can relate these back to very nice properties of the juggling sequences.

This thesis tries to explore all of these facets. It should be viewed as a collection of theorems and proofs, which start rudimentary and build up to more technical results, which are hopefully interesting and useful for mathematicians and jugglers alike.

2 Preface

> "Jugglers and mathematicians share the joy of discovering new patterns" - Anneroos Everts

A few words on why I would write a mathematics thesis on juggling. The quote above captures the connection between the two fields very well. I am a juggler myself and already knew a fair bit on the notation and theory, but had a lot of gaps in this knowledge. It was a lot of fun to fill those in and get a better understanding and new found appreciation of my hobby. Personally, I learned a lot and I especially enjoyed writing on and learning about state diagrams, because it allowed me to think about a question I had in the back of my mind for a long time already, but never urged myself to think about. This would be "how can we find transitions between patterns in a more practical sense?" and I believe that I gained a lot of insight in this, which I hope you find as entertaining to read about, as I find it was to think about.

The way in which I wrote the proofs and structured the whole thesis is with an abundance of detail. I find myself often annoyed by how little detail is given in textbooks and how easily the words "clearly", "obviously", "trivially" etc. are used. This is personal of course, but as a thesis is quite personal as well, I took this opportunity to steer away from this and write in the way I would like to read it, which is with a lot of detail. I should admit that I also just struggle with being concise. The danger of this is writing too much detail, which I am sure many people will feel I have done if they would read it, and I can even imagine myself looking back on this in some years, being annoyed with how much detail I used. However, for now it feels right, and I do hope whoever might read this, finds it enlightening and the details helpful.

3 Preliminaries

3.1 Conventions and notation

We will use modular arithmetic a lot throughout this thesis and we would like to distinguish between evaluating some integer modulo p and saying that two integers are equivalent modulo p.

Definition 3.1. We define $r_p : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ to be the remainder function, which means that $r_p(a)$ is the remainder of a after division by p. So $a = r_p(a) + kp$ for some $k \in \mathbb{Z}$ and $0 \leq r_p(a) < p$.

Notation. *When we write $a \equiv b \pmod{p}$ we mean that $a = b + kp$ for some $k \in \mathbb{Z}$. Here we do really mean that a and b belong to the same class $\pmod{p}$.*

We will be working with the set $\{0, 1, .., p-1\}$ a lot so we want to have some notation for this.

Notation. *Let $\mathbb{Z}/p\mathbb{Z}$ denote the set $\{0, 1, .., p-1\}$. Even though $\mathbb{Z}/p\mathbb{Z}$ inherits a ring structure as an abstract quotient, and we will call on this from time to time by adding elements modulo p, we really want to see $\mathbb{Z}/p\mathbb{Z}$ as a set of these integers and not see the elements as representatives of their class.*

Notation. *Let $(a_i)_{i=0}^{p-1}$ and $(b_i)_{i=0}^{p-1}$ be sequences of integers. We will write $r_p((a_i)_{i=0}^{p-1})$ to denote the sequence $(r_p(a_i))_{i=0}^{p-1}$ where we evaluate every element modulo p. In this manner we also write $(b_i)_{i=0}^{p-1} \equiv (a_i)_{i=0}^{p-1} \pmod{p}$ if $b_i \equiv a_i \pmod{p}$ for all $i \in \mathbb{Z}/p\mathbb{Z}$.*

3.2 A mathematical description of juggling

We want to study juggling patterns and introduce mathematics into them and in order to do so we need to disregard some irrelevant aspects of a pattern and also make some assumptions. We will use this section to make clear what these assumptions are, give ourselves a feel for what we mean with juggling patterns and then abstractly define what a juggling pattern is.

First of all we ignore the objects a juggler uses and call them balls from now on. Furthermore we can safely assume that the number of balls stays constant during a juggling pattern and we disregard the path that a ball takes in the air, i.e. whether it goes in front of or behind the body or whether it goes underneath a leg or not etc. We can go on to disregard the number of jugglers involved in a pattern and even the number of hands involved. However, we do assume that, whatever the number of hands involved is, we decide on an order in which they throw and stick to this throwing order during the pattern. It will be convenient to say that the pattern has been going on forever and will continue to do so. We assume the throws in a pattern to happen at a constant rhythm and that a pattern as a whole is periodic. This constant rhythm means that throws happen at discrete equally spaced moments in time and we can use this to interpret a throwing moment t, from now on called a ***beat***, as an integer $t \in \mathbb{Z}$. We assume that, at every beat, either one or no balls land and once a ball is caught, it is thrown again immediately and if no balls land no throw is made at that beat. This last remark could also be seen as the requirement that no two or more balls thrown or caught at the same time, which, among jugglers, would be referred to as a "squeeze catch" or "multiplex throw" respectively. That these things have names indicates that it is indeed possible to perform them and there are lots of interesting patterns with this property, but we will not study these here.

With all of these assumptions we can study a given physical juggling pattern by studying the individual throws made at every beat. For every throw made we can see how many beats it takes for that ball to be thrown again or, in other words, after throwing this ball, how many other throws can we make before we throw this ball again. When we say that a ball spends h beats in the air we mean that, once the ball is thrown, it is thrown again h beats later.

Definition 3.2. An ***h throw*** is a throw which is thrown again for the first time h beats later.

Among jugglers it is also common to refer to the number of beats that a ball spends in the air as the height at which it is thrown or the time it spends in the air.

We can make an infinite list of all of the throws made at every beat and since we require a pattern to be periodic we know that this list will be periodic as well.

Definition 3.3. The ***period*** of a juggling pattern is the period of the corresponding infinite throw list.

It turns out that this list encodes all the information about a juggling pattern that we are interested in so we will define a juggling pattern accordingly. We stick to the definition as introduced in [2].

Definition 3.4. A ***juggling pattern*** of period p is a function $f : \mathbb{Z} \to \mathbb{Z}$ such that f is a permutation of $\mathbb{Z}$ and $f(t) \geq t$ for all $t \in \mathbb{Z}$. Furthermore, for the ***height function*** $h_f(t) := f(t) - t$ the following periodicity requirement should hold: $h_f(t + kp) = h_f(t)$ for any $k \in \mathbb{Z}$.

Intuitively, $f(t)$ is the landing beat of the throw made at beat t so this function assigns the landing spot for each throw. With this in mind the height of the throw at beat t would be $f(t) - t$ and this is exactly how we define it.

Requiring that $f(t) \geq t$ ensures us to make throws that don't go back in time which is a reasonable thing to want and asking f to permute $\mathbb{Z}$ translates to the fact that no two balls land at the same time. Note that we allow $f(t) = t$ which allows for throwing a 0 throw. We define it in this way, because we need to assign it some value when no balls land at this beat and by assigning it 0 we maintain the permutation property.

Indeed, when performing a juggling pattern it is straightforward to give the corresponding function, because we can list all of the throws we make. It is also clear that this function obeys all the rules we set it in the definition. A natural question would be to ask whether, given a juggling pattern as defined above, we have enough information to perform it (given we're skilled enough). How do you even throw a number? As this is a very practical question we dive more into this in Section 9.1. Now, does it matter where I start in the sequence? It turns out that, at what throw we start the pattern determines how we should hold the balls in our hand when we start. It is at first glance not obvious what these arrangements are and we need, later to be introduced, ***state diagrams*** to find out how to start patterns, which we will discuss in Section 7. Another rather important question is how many balls we need for a given pattern and even though this isn't clear from staring at the definition, we will see that this is easily computed by taking the average of a juggling pattern where we take the sum of all the heights and divide by the period; see Section 4.2. So even though, it is a bit harder to go from the definition to an actual physical pattern, we note at this point that all the information is there, but we need to do some work.

Some examples of these functions are in order at this point and we start with the most basic 3 ball juggling pattern which is known among jugglers as the cascade. In this pattern every ball is thrown every third beat and its function is

$$f(t) = 3 + t \quad \text{for all} \quad t \in \mathbb{Z}$$

and in general, the basic b ball pattern is

$$f(t) = b + t \quad \text{for all} \quad t \in \mathbb{Z}$$

The slightly more complicated 3 ball pattern 51 has function

$$f(t) = \begin{cases} 5 + t, & \text{if} \quad t \equiv 0 \pmod 2 \\ 1 + t, & \text{if} \quad t \equiv 1 \pmod 2 \end{cases}$$

and for the 3 ball pattern 441 we get the function

$$f(t) = \begin{cases} 4 + t, & \text{if} \quad t \equiv 0, 1 \pmod 3 \\ 1 + t, & \text{if} \quad t \equiv 2 \pmod 3 \end{cases}$$

Given a juggling pattern of period p, we can look at the sequence $(h_f(t))_{t=0}^{t=p-1}$ where we evaluate f at every throw of its period. This gives us all the information about the juggling pattern and it is in fact the juggling sequence which jugglers use to talk about juggling patterns.

Definition 3.5. A finite sequence of non-negative integers $(a_i)_{i=0}^{p-1}$ is a ***juggling sequence*** if there is some juggling pattern with associated function $f : \mathbb{Z} \to \mathbb{Z}$ such that $h_f(i) = a_i$ for all $i \in \mathbb{Z}/p\mathbb{Z}$. We also call a sequence ***jugglable*** if it is a juggling sequence.

At this point one might imagine that confusion arises whether the pattern 441 encodes 3 throws with corresponding heights $4, 4, 1$ or for example 2 throws with heights $44, 1$. In order to prevent this confusion the agreement is that throws 0 to 9 are noted with numbers and from 10 onwards we use letters a, b etc. So the pattern $db97531$ encodes a pattern of minimal period 7 where the first two throws are height 13 and 11 respectively. This finite alphabet might seem like a limiting factor, but in practice throws won't exceed a z due to physical limitations.

4 Practical problems with mathematical answers

Sometimes we ask questions because our mind ponders and wonders without any constraint and sometimes there are very practical questions in need of answers. It is up for debate how "practical" and "necessary" all of this really is, but for now we can safely say this chapter is focusing on the latter category. We will be looking into determining how many balls a pattern requires and checking whether or not a pattern is even jugglable.

4.1 When is a sequence jugglable?

The first thing we want to be able to do is check whether a given sequence is jugglable or not. There is a very clear and neat theorem for this, but we will need a few lemmas at our disposal to prove it. We provide the proof as stated in [2]. The first follows almost immediately from the definition of a period p pattern, but is worth mentioning.

Lemma 4.1. *If f is a period p juggling pattern then $f(t) + kp = f(t + kp)$ for any $t \in \mathbb{Z}/p\mathbb{Z}$ and $k \in \mathbb{Z}$.*

Proof. We use the fact that f is of period p which means that $h_f(t + kp) = h_f(t)$ for any $t \in \mathbb{Z}/p\mathbb{Z}$ and $k \in \mathbb{Z}$. Writing out both sides gives us $f(t + kp) - (t + kp) = f(t) - t$ which is equivalent to $f(t) + kp = f(t + kp)$. □

The following lemma shows that every juggling pattern of period p defines a permutation on $\mathbb{Z}/p\mathbb{Z}$.

Lemma 4.2. *Let f be a period p juggling pattern. Then the induced map*

$$\begin{aligned} f' : \quad & \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \\ & t \mapsto r_p(f(t)) \end{aligned}$$

is well defined and bijective and therefore a f' is a permutation of $\mathbb{Z}/p\mathbb{Z}$.

Proof. To show that f' is well defined we need to show that $s \equiv t \pmod p$ implies $f'(s) \equiv f'(t) \pmod p$. We know that $s = t + kp$ for some $\in \mathbb{Z}$ so $f'(s) = f'(t + kp)$ and with Lemma 4.1 we see $f'(s) = f'(t) + kp$ which is equivalent to saying that $f'(s) \equiv f'(t) \pmod p$.

For bijectivity it is enough to show either surjective or injective since f' is a map between a finite set and itself. We show f' is surjective, so let $t \in \mathbb{Z}/p\mathbb{Z}$, then we want to find some $s \in \mathbb{Z}/p\mathbb{Z}$ such that $r_p(f(s)) = t$ or equivalently, $f(s) \equiv t \pmod{p}$. Since f is a permutation of $\mathbb{Z}$ we know there is some $\tilde{s} \in \mathbb{Z}$ such that $f(\tilde{s}) = t$ and by Lemma 4.1 we know that the function values of f are equivalent modulo p for equivalent inputs. In other words, if we define $s := r_p(\bar{s})$ then, since $s \equiv \tilde{s} \pmod{p}$ we know that $f(s) \equiv f(\tilde{s}) \pmod{p}$ and since the latter equals t we find our desired equivalence of $f(s) \equiv t \pmod{p}$ and see that f' is indeed surjective, completing the proof.

$\square$

So, formally, we have now shown that for each period p juggling pattern f there is some permutation $\pi_f \in S_p$ such that

$$f(t) \equiv \pi_f(t) \pmod{p} \quad \text{for all} \quad t \in \mathbb{Z}/p\mathbb{Z}$$

We now have all the tools to prove a very useful theorem which allows to quickly check whether a sequence is jugglable or not.

Theorem 4.3 (Permutation Test)**.** *A finite sequence of non-negative integers $(a_i)_{i=0}^{p-1}$ is a juggling sequence if and only if the function*

$$\begin{aligned} \phi_{(a_i)} : \quad & \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \\ & i \mapsto r_p((a_i + i)) \end{aligned}$$

is a permutation of $\mathbb{Z}/p\mathbb{Z}$.

Before we give the proof it is useful to check why this should intuitively hold. The one really important thing that might go wrong with a sequence of non-negative integers is that, if we were to throw balls at the indicated heights of the sequence, more than would one ball land at the same beat. To see if this happens we simply compute when all the throws land and check whether all these numbers differ. It could very well happen that a throw takes more beats to land than the period of the pattern. For example, if we want to check that 901 is jugglable we compute the landing times by adding the beats at which a throw is made to that throw, i.e. $0 + 9 = 9$, $1 + 0 = 1$ and $2 + 1 = 3$ so the first three throws land at beats 9, 1 and 3 respectively.

Seems fine at first, but if we would continue this process until the highest throw has also landed, in this case the 9, we get the following.

beats	0	1	2	3	4	5	6	7	8
throws	9	0	1	9	0	1	9	0	1
landing times	**9**	1	3	12	4	6	15	7	**9**

We see that the first 9 and third 1 land at the same time. What this means is that we should repeat the pattern until all the throws have landed. Only then can we be sure that we did not miss any coinciding landings. The modulo condition in the theorem takes care of these coinciding landings which may happen outside the first p beats. After all, if we consider the throw a_i which is made in the k^{th} repetition of a sequence with period p, we know that its landing time is $a_i + (i + kp) \equiv a_i + i \pmod p$ which is the landing time of throw a_i made in the first set of p throws.

Proof of Theorem 4.3. Suppose that $(a_i)_{i=0}^{p-1}$ is a juggling sequence so there is some juggling pattern with associated function f such that $h_f(i) = a_i$ for all $i \in \mathbb{Z}/p\mathbb{Z}$. By Lemma 4.2 $f(i) \equiv \pi_f(i) \pmod p$ for some $\pi_f(i) \in S_p$, so there is some function $g : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}$ such that $f(i) = \pi_f(i) + g(i)p$ since $f(i)$ and $\pi_f(i)$ always differ a multiple of p, but that multiple depends on i. Now

$$\begin{aligned} h_f(i) &= f(i) - i \\ &= \pi_f(i) + g(i)p - i \end{aligned}$$

and at the same time $a_i = h_f(i)$ so $a_i + i = \pi_f(i) + g(i)p$ or equivalently $a_i + i \equiv \pi_f(i) \pmod p$. So we see that $\phi_{(a_i)}(i)$ is indeed a permutation.

Now suppose that $\phi_{(a_i)}$ is a permutation. To show that $(a_i)_{i=0}^{p-1}$ is a juggling sequence we need to find a pattern with associated function f such that $h_f(i) = a_i$ for all $i \in \mathbb{Z}/p\mathbb{Z}$. We first redefine (a_j) for all $j \in \mathbb{Z}$ by extending it periodically, i.e. $a_j := a_i$ if $j \equiv i \pmod p$. We then define

$$\begin{aligned} f &: \mathbb{Z} \to \mathbb{Z} \\ j &\mapsto a_j + j \quad \text{for all} \quad j \in \mathbb{Z} \end{aligned}$$

and the claim is that this f is the desired juggling pattern. To see this we note that the condition $h_f(i) = a_i$ holds by construction. We need to show that f is indeed a juggling pattern, so $h_f(j+kp) = h_f(j)$ needs to hold for any $k \in \mathbb{Z}$, $f(i) \geq i$ needs to hold for all $i \in \mathbb{Z}/p\mathbb{Z}$ and f needs to be a permutation. The periodicity requirement is taken care of by how we extended (a_i) since

$$\begin{aligned}
h_f(j+kp) &= f(j+kp) - (j+kp) \\
&= a_{j+kp} + (j+kp) - (j+kp) \\
&= a_{j+kp} \\
&= a_j \\
&= a_j + j - j \\
&= f(j) - j \\
&= h_f(j)
\end{aligned}$$

Now, a_j is non-negative for all $j \in \mathbb{Z}$ so clearly $f(j) = a_j + j \geq j$ holds.

We are left to show that f is a permutation. First we show injectivity so assume that $f(t) = f(s)$ for some $s, t \in \mathbb{Z}$. Then, by how we defined f, we see that $a_t + t = a_s + s$ or in other words, $\phi_{(a_i)}(t) = \phi_{(a_i)}(s)$. The function $\phi_{(a_i)}$ is a permutation of $\mathbb{Z}/p\mathbb{Z}$ by assumption, so, in particular, it is an injective function modulo p. This tells us that $s \equiv t \pmod{p}$ which implies that $a_s = a_t$. From $a_t + t = a_s + s$ it now follows that $s = t$ as desired for injectivity of f.

For surjectivity, let $t \in \mathbb{Z}$, then we need to find an $s \in \mathbb{Z}$ such that $f(s) = t$. Applying the remainder function gives us $r_p(t) \in \mathbb{Z}/p\mathbb{Z}$ and using the assumption that $\phi_{(a_i)}$ is a permutation of $\mathbb{Z}/p\mathbb{Z}$, we see there is some $s' \in \mathbb{Z}/p\mathbb{Z}$ such that $\phi_{(a_i)}(s') = r_p(t)$ or equivalently $\phi_{(a_i)}(s') \equiv t \pmod{p}$. This is the same as $a_{s'} + s' \equiv t \pmod{p}$ by the definition of $\phi_{(a_i)}$, so we see that $a_{s'} + s' + kp = t$ for some $k \in \mathbb{Z}$. Because of our periodic redefinition of a_i we see that $a_{s'} = a_{s'+kp}$ for any $k \in \mathbb{Z}$, which implies that

$$\begin{aligned}
f(s'+kp) &= a_{s'+kp} + s' + kp \\
&= a_{s'} + s' + kp \\
&= t
\end{aligned}$$

Indeed, for $s = s' + kp$ we have $f(s) = t$, proving surjectivity and finishing the proof as a whole. $\square$

Example 4.4. We will now give a bunch of examples of juggling sequences, which we encourage the reader to check are in fact jugglable using the Permutation Test from Theorem 4.3.

45141 663 801 bb86 88441 123456789 864 a8999 96636 78352 88536 9a551 77991133 858 744 771 66661 abc 633 534 7a1 85741 6631 9991 834 77781 6451 015 db97531 771177 55550 50505 b4444444 774 e01 13579 97531 75751 9631414 663 933 a6414 8448641 915 d797

We will learn in the next section how many balls each of these patterns requires and we will refer back to this example where the reader can then also check how many balls are required.

4.2 How many balls?

Once we know that a sequence is jugglable it would be nice to know what we are getting ourselves into and be able to compute the number of balls we will supposedly juggle. There is a very neat theorem stating that the number of balls is the average of the numbers in the sequence and we will use this section to introduce all the notions and tools required to prove it. These tools come in the form of operations we can apply to a juggling sequence to get a new one and they are useful for a lot more than just proving this theorem.

We start with a more important operation on juggling sequences which changes two numbers in such a way that these two numbers maintain their respective landing sites but take each others beats.

Definition 4.5. Let $s = (a_i)_{i=0}^{p-1}$ be a finite sequence of non-negative integers for $p \geq 2$ with $0 \leq m < n \leq p-1$ and $n - m \leq a_m$. Then we define the sequence $s_{m,n} := (a'_i)_{i=0}^{p-1}$ as $a'_i := a_i$ for $i \neq m, n$ and $a'_m := a_n + (n-m)$ and $a'_n := a_m - (n-m)$. The act of changing s to $s_{m,n}$ is called a ***site swap***.

Let's dive into what this definition is really saying. Say our sequence is

beats	0	1	...	m	...	n	...	$p-1$
throws	a_0	a_1	...	a_m	...	a_n	...	a_{p-1}

When we perform a site swap we want to throw a_m at beat n and a_n at beat m to arrive at

beats	0	1	...	m	...	n	...	$p-1$
throws	a_0	a_1	...	a_n	...	a_m	...	a_{p-1}

We want to make the shifted throws land at their original landing times, these being $a_m + m$ and $a_n + n$, so since we shifted a_m forward by $n - m$, we need to subtract this from the original height, resulting in $a_m - (n - m)$. Similarly, we shifted a_n backwards by $n - m$ to throw it at beat m. In order to make this shifted throw land at the original landing time of a_n, we need to raise the throwing height by how much we shifted backwards, resulting in $a_n + (n - m)$. So the new sequence is given by

beats	0	1	...	m	...	n	...	$p-1$
throws	a_0	a_1	...	$a_n + (n-m)$	...	$a_m - (n-m)$	...	a_{p-1}

Indeed, if we now compute the landing times of the throws at beats m and n we find that $a'_m + m = a_n + (n-m) + m = a_n + n$ and $a'_n + n = a_m - (n-m) + n = a_m + m$ as desired. Note that we ensured that the term $a_m - (n - m)$ stays non-negative by only taking m, n such that $n - m \leq a_m$.

We defined a site swap for a sequence that does not have to be a juggling sequence and Theorem 4.7 elaborates on this. A proof using diagrams is given in [4] and we provide an algebraic proof.

Definition 4.6. Let $s = (a_i)_{i=0}^{p-1}$ be a juggling sequence. Then we define its ***average*** as $\frac{1}{p}\sum_{i=0}^{p-1} a_i$

The following theorem essentially states that applying a site swap does not change whether a sequence is jugglable or not and that the average of the sequence remains the same.

Theorem 4.7 (Site swap)**.** *Let $s = (a_i)_{i=0}^{p-1}$ be a finite sequence of non-negative integers for $p \geq 2$ with $0 \leq m < n \leq p-1$ and $n - m \leq a_m$. Then s is a juggling sequence if and only if $s_{m,n} := (a'_i)_{i=0}^{p-1}$ is. Furthermore the average of s is the same as the average of $s_{m,n}$.*

Proof. Suppose that s is a juggling sequence. Using the Permutation Test (Theorem 4.3) we see that $(b_i)_{i=0}^{p-1} := r_p((a_i + i)_{i=0}^{p-1})$ is some permutation of

$\mathbb{Z}/p\mathbb{Z}$ where we evaluate every component modulo p. For $s_{m,n} := (a'_i)_{i=0}^{p-1}$ we know that $a'_i := a_i$ for $i \neq m, n$ so we know that $a'_i + i \equiv a_i + i \pmod{p}$ for $i \neq m, n$. Furthermore, we defined $a'_m := a_n + (n-m)$ and $a'_n := a_m - (n-m)$ exactly such that their landing sites did not change so indeed $a'_m + m \equiv a_n + n \pmod{p}$ and $a'_n + n \equiv a_m + m \pmod{p}$. If we then consider $r_p((a'_i + i)_{i=0}^{p-1}) =: (b'_i)_{i=0}^{p-1}$ we see that $(b'_i)_{i=0}^{p-1}$ is still a permutation of $\mathbb{Z}/p\mathbb{Z}$ and conclude that $s_{m,n}$ must also be a juggling sequence. If we now assume that $s_{m,n}$ is a juggling sequence, the same argument stands so we see that s is a juggling sequence if and only if $s_{m,n}$ is.

To show they share the same average it would be enough to note that all we changed to the sequence is adding some number to one element of the sequence and subtract the same from some other number, leaving the average untouched. However, for completion can compute the average of $s_{m,n}$. We note that $\frac{1}{p}\sum_{\substack{i=0\\i\neq m,n}}^{p-1} a_i = \frac{1}{p}\sum_{\substack{i=0\\i\neq m,n}}^{p-1} a'_i$, because the elements a_i and a'_i only differ for $i = m, n$.

$$\begin{aligned}
\frac{1}{p}\sum_{i=0}^{p-1} a'_i &= \frac{1}{p}\sum_{\substack{i=0\\i\neq m,n}}^{p-1} a'_i + \frac{1}{p}(a'_m + a'_n) \\
&= \frac{1}{p}\sum_{\substack{i=0\\i\neq m,n}}^{p-1} a'_i + \frac{1}{p}(a_n + (n-m) + a_m - (n-m)) \\
&= \frac{1}{p}\sum_{\substack{i=0\\i\neq m,n}}^{p-1} a'_i + \frac{1}{p}(a_n + a_m) \\
&= \frac{1}{p}\sum_{\substack{i=0\\i\neq m,n}}^{p-1} a_i + \frac{1}{p}(a_n + a_m) \\
&= \frac{1}{p}\sum_{i=0}^{p-1} a_i
\end{aligned}$$

and we see that they indeed share the same average. □

A somewhat obvious, but important remark needs to be made here.

Remark. A site swap preserves the number of balls being juggled, since all we do is take two balls in the pattern and change the times at which they are thrown.

Example 4.8. A popular 5 ball juggling sequence is 97531 and we will show a few site swaps of this pattern to gain some intuition. We start with the following situation

beats	0	1	2	3	4
throws	9	7	5	3	1

and let's say we perform a site swap of beats 1 and 2 resulting in

beats	0	1	2	3	4
throws	9	6	6	3	1

Another lovely 5 ball trick, but we can keep going. Nothing restricts us to perform a site swap on non-adjacent beats so let's make beats 0 and 2 swap sites.

beats	0	1	2	3	4
throws	8	6	7	3	1

and to wrap it up, let's do beats 3 and 4 to get

beats	0	1	2	3	4
throws	8	6	7	2	2

We now introduce another tool in our sequence changing arsenal, which does not generate new patterns out of old ones, but helps us to decide whether or not two patterns are different.

Definition 4.9. A ***cyclic shift*** of a sequence $s = (a_i)_{i=0}^{p-1}$ is the sequence $s_{\rightarrow} := (a_{r_p(i-1)})_{i=0}^{p-1} = a_{p-1}a_0a_1...a_{p-2}$. We define a ***k-shift*** to be $s_{\xrightarrow{k}}$ which is the resulting sequence after applying k cyclic shifts to s for some $k \in \mathbb{Z}/p\mathbb{Z}$.

We will call upon the following fact a few times so it's good to have stated and proven as a Lemma.

Lemma 4.10. *The function*

$$\begin{aligned} f :&\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \\ &i \mapsto r_p(i+c) \end{aligned}$$

is a permutation for any constant $c \in \mathbb{Z}$.

Proof. We see that it has an inverse

$$\begin{aligned} f^{-1} : & \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \\ & j \mapsto r_p(j-c) \end{aligned}$$

which shows it is a bijection and therefore permutation of $\mathbb{Z}/p\mathbb{Z}$. □

Theorem 4.11 (Cyclic shift)**.** *Let $s = (a_i)_{i=0}^{p-1}$ be a finite sequence of non-negative integers. Then s is a juggling sequence if and only if $s_{\rightarrow} := (a_{r_p(i-1)})_{i=1}^{p-1}$ is. Furthermore the average of s is the same as the average of $s_{\rightarrow}$.*

Proof. The proof is straightforward and similar to the proof of the site swap theorem. If s is a juggling sequence we know with the Permutation Test (Theorem 4.3) that $(b_i)_{i=0}^{p-1} := r_p((a_i + i)_{i=0}^{p-1})$ is some permutation of $\mathbb{Z}/p\mathbb{Z}$ where we evaluate every component modulo p. Now, $s_{\rightarrow}$ contains the exact same elements as s, but cyclically permuted by shifting every element one spot to the right. When computing $a_i + i$, this shifting translates to adding 1 to every $a_i + i$ modulo p and we know that adding a constant modulo p is a permutation of $\mathbb{Z}/p\mathbb{Z}$ by Lemma 4.10. So $r_p((a_{r_p(i-1)} + i)_{i=1}^{p-1})$ will again give a permutation of $\mathbb{Z}/p\mathbb{Z}$ proving that $s_{\rightarrow}$ is a juggling sequence. The other implication follows the same reasoning. Furthermore, it is immediate that s and $s_{\rightarrow}$ have the same average as they have the same elements but with different indices. □

Contrary to site swaps, cyclic shifts do not generate new patterns since every cyclic shift of a pattern is the same sequence in practice. For example, we do not distinguish 414 from 441 as they generate the same juggling sequence. So in general it is interesting to consider juggling patterns up to cyclic shifts.

The remark we made for site swaps holds here as well.

Remark. A cyclic shift preserves the number of balls being juggled.

Lemma 4.12. *Site swaps and cyclic shifts are invertible.*

Proof. We see readily that applying a site swap of beats m, n to $s_{m,n}$ results in s so indeed, site swaps are invertible and in particular its inverse is applying another siteswap to the same beats.

For a cyclic k-shift of s, say $s_{\xrightarrow{k}}$ for some $k \in \mathbb{Z}/p\mathbb{Z}$, we see that applying a $p-k$ shift to $s_{\xrightarrow{k}}$ results in s. □

With site swaps and cyclic shifts at our disposal we can prove the following theorem as given in [4] on page 21:

Theorem 4.13. *Every b ball juggling sequence of period p can be constructed from $\underbrace{bb...b}_{p\ times}$ using only site swaps and cyclic shifts.*

Proof. The proof is given by the following algorithm, known as the flattening algorithm, which is devised by Allen Knutson as mentioned in [4].

We start by inputting any sequence $s = (a_i)_{i=0}^{p-1}$ for $p \geq 1$ and we perform the following steps in which we redefine s using site swaps and cyclic shifts:

1. We check if s is constant. If so we terminate and output s. If not we proceed.

2. Apply cyclic shifts as often as needed to rearrange s in such a way that a throw of maximal height within s, say a_i, is thrown at beat 0 and the next throw, a_{i+1} is of non maximal height. We check if $a_i - a_{i+1} = 1$. If so we terminate and output this new sequence. If not we proceed.

3. We perform a site swap of beats 0 and 1 and return to step 1.

There are some things to note here. The first being that this procedure will terminate, since, whenever we perform a site swap as described, we lower a throw of maximal height and heighten a throw of non-maximal height by only 1. Since this non-maximal height throw differs more than 1 from maximal height we eliminated one maximal height throw from the pattern which implies this will stop after finitely many steps.

Secondly, the jugglability and average of the initial sequence is maintained throughout as we have proven with the site swap and cyclic shift theorems. So whenever we input a juggling sequence it will stay jugglable throughout the process so in particular this procedure will never stop at step 2. If it would, we found that a_i and a_{i+1} differ by 1, but this is not possible for a juggling sequence, since these two throws would then land at the same beat. This implies that, since the program will terminate and it will not do so on step 2, it must terminate at step 1 resulting in a constant sequence. Now, let's say the average of $(a_i)_{i=0}^{p-1}$ is b. This constant sequence will be $\underbrace{bb...b}_{\text{p times}}$, since site swaps and cyclic shifts preserve the average of the sequence and this is the only period p sequence with average b.

In a similar fashion, upon input of a non-juggling sequence this procedure will terminate at step 2, since the only thing that could make it non-jugglable is the property that at least two throws land at the same time. These throws will reveal themselves by enough cyclic shifts and site swaps in step 2.

To finish the proof we note that we can start with $\underbrace{bb...b}_{\text{p times}}$ and apply all the steps of the algorithm in reverse to end up with $(a_i)_{i=0}^{p-1}$, since we have seen in Lemma 4.12 that site swaps and cyclic shifts are invertible. $\square$

Example 4.14. As an example we flatten the 4 ball sequence 6451.

$$6451 \xrightarrow{swap} 5551 \xrightarrow{shift} 1555 \xrightarrow{shift} 5155 \xrightarrow{swap} 2455 \xrightarrow{shift} 5245 \xrightarrow{swap} 3445 \xrightarrow{shift} 5344 \xrightarrow{swap} 4444$$

and similarly we apply the algorithm for the non-juggling sequence 713. We can see it is not jugglable since $(713) + (012) = (725)$ and $(122) = r_3((725))$ which is not a permutation. In particular, the 1 and 3 land at the same time.

$$713 \xrightarrow{swap} 263 \xrightarrow{shift} 326 \xrightarrow{shift} 632 \xrightarrow{swap} 452 \xrightarrow{shift} 245 \xrightarrow{shift} 524 \xrightarrow{swap} 344 \xrightarrow{shift} 434$$

At this point the procedure stops because we have found that the first 4 and the 3 land at the same time. Note that we already see sets of throws a_i and a_{i+1} with a difference of 1 well before this point (for example, the 32 in 326), but as we set up the algorithm it does not detect these.

The average theorem is now essentially a corollary of Theorem 4.13, but because of its importance we state it as a theorem nonetheless. The proof as we give it was suggested in [4] on page 22 although there, as well as in [2], a different proof is given based on the existence of a certain limit. We chose this approach as it feels more intuitive.

Theorem 4.15 (Average theorem)**.** *Let $(a_i)_{i=0}^{p-1}$ be a juggling sequence. Then*

$$\frac{1}{p}\sum_{i=0}^{p-1} a_i = b$$

for some non-negative integer b and b will be the number of balls juggled.

Proof. First of all we can be sure that the average of $(a_i)_{i=0}^{p-1}$ will be some non-negative integer b. This is because, by the Permutation Test (Theorem 4.3) we know that the point wise sum of the beats i with the heights a_i is some permutation of $\mathbb{Z}/p\mathbb{Z}$ if we evaluate modulo p. Since the beats themselves are also a permutation of $\mathbb{Z}/p\mathbb{Z}$ we see that the sum of the heights a_i must be 0 modulo p. Or stated in terms of equations

$$\sum_{i=0}^{p-1} a_i + i \equiv \sum_{i=0}^{p-1} i \pmod{p}$$

which implies that

$$\sum_{i=0}^{p-1} a_i \equiv 0 \pmod{p}$$

or equivalently, the average is an integer. It being non-negative is assured since we only make non-negative throws.

It is clear that the sequence $\underbrace{bb...b}_{\text{p times}}$ is juggled with b balls and that the average is b. Furthermore, we already noted that site swaps and cyclic shifts preserve the number of balls being juggled so we see that, by constructing $(a_i)_{i=0}^{p-1}$ from $\underbrace{bb...b}_{\text{p times}}$, it must also be juggled with b balls. $\square$

Just for fun, we encourage the reader to go back to Example 4.4 and workout how many balls some of these patterns need. Furthermore, we introduced the notion of a site swap and it is also nice to have a look and see which of the patterns in this example are site swaps of each other (where potentially more than one were applied). One way the reader might go about this is by classing the patterns in groups of how many balls are required, as site swaps do not change this, and then looking at the period, as site swaps do nothing with this either. At this point it is up to the reader to have a play and try and make some lovely new patterns, by happily applying site swaps and cyclic shifts to the patterns in the example in order to get a feel for what happens.

5 Answers only generate more questions

Indeed, whenever we find answers to questions, these generally instigate all sorts of new questions, which is only a good thing! It keeps us off the streets. In this particular instance we learned in Theorem 4.3 about the Permutation Test, which essentially states that every period p juggling pattern induces a permutation of $\mathbb{Z}/p\mathbb{Z}$. This raises the question: given a permutation, can we construct a juggling pattern of period p and how do we go about it?

We also learned that the average of a sequence is the number of balls and that this should be an integer by how we defined a juggling sequence. Having said this we know that a sequence is not jugglable if the average is not an integer. However, what do we know when the average is an integer? It certainly has potential, but is by no means directly jugglable. As an example the sequence 513 has an integer average of 3, but is not jugglable. We can see this with the Permutation Test, or we inspect closely and see that the 5 and 3 will land at the same time, which is always a violation of our definition of a juggling pattern. However, 531 is a juggling sequence, which the reader is encouraged to check with the Permutation Test. This begs the question whether we can always find a jugglable permutation if the average is an integer.

This chapter is devoted to answering these two questions as well as introducing the necessary notions and lemmas to do so.

5.1 Changing juggling sequences

We have now seen site swaps and cyclic shifts which are operations on sequences which do not change the jugglability of them and in this section we will introduce a few more of these as described in [4]. Contrary to site swaps and cyclic shifts, these operations do change the number of balls in the pattern. The first states that we can add any number, given that it is large enough, to every number in the sequence and the other one states that we can add any multiple of the period to any number in the sequence. As well as being necessary to discuss for the upcoming theorems, these constructions of juggling sequences are just fun to know. Both are easy to prove with the Permutation Test (Theorem 4.3) and Lemma 4.10

The following concept is introduced in [4] as Vertical shifts on page 23.

Theorem 5.1. *Let $(a_i)_{i=0}^{p-1}$ be a juggling sequence with b balls and $d \in \mathbb{Z}$ such that $d \geq -\min_{i \in \mathbb{Z}/p\mathbb{Z}}(a_i)$. Then $(a_i + d)_{i=0}^{p-1}$ is a juggling sequence with $b+d$ balls.*

Proof. Firstly, note that all elements of the new sequence are non-negative integers by how we chose d. Since $(a_i)_{i=0}^{p-1}$ is jugglable we see with the Permutation Test (Theorem 4.3) that $(b_i)_{i=0}^{p-1} := r_p((a_i + i)_{i=0}^{p-1})$ is some permutation of $\mathbb{Z}/p\mathbb{Z}$ where we evaluate every component modulo p. If we evaluate $(a_i + d + i)_{i=0}^{p-1}$ for our new sequence we see that $(a_i + d + i)_{i=0}^{p-1} \equiv (b_i + d)_{i=0}^{p-1} \pmod{p}$, so all we need to do is show that the latter, once evaluated modulo p, is again a permutation of $\mathbb{Z}/p\mathbb{Z}$. Applying Lemma 4.10 to $\phi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ with $\phi(b_i) = b_i + d$ gives the required result since (b_i) is a permutation.

To see how many balls we need we compute the average

$$\begin{aligned}
\frac{1}{p}\sum_{i=0}^{p-1}(a_i + d) &= \frac{1}{p}\sum_{i=0}^{p-1} a_i + \frac{1}{p}\sum_{i=0}^{p-1} d \\
&= b + \frac{1}{p}\sum_{i=0}^{p-1} d \\
&= b + \frac{1}{p}pd \\
&= b + d
\end{aligned}$$

□

Theorem 5.2. *Let $(a_i)_{i=0}^{p-1}$ be a juggling sequence with b balls and $f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}$ any map such that $a_i + f(i)p \geq 0$ for all $i \in \mathbb{Z}/p\mathbb{Z}$. Then $(a_i + f(i)p)_{i=0}^{p-1}$ is again a juggling sequence with $b + \sum_{i=1}^{p-1} f(i)$ balls.*

Proof. Due to the restriction on f our new sequence still has non-negative values. With the Permutation Test (Theorem 4.3) the result is almost immediate, since $(a_i + f(i)p + i)_{i=0}^{p-1} \equiv (a_i + i)_{i=0}^{p-1} \pmod{p}$ and we know that $(b_i)_{i=0}^{p-1} := r_p((a_i + i)_{i=0}^{p-1})$ is some permutation of $\mathbb{Z}/p\mathbb{Z}$ where we evaluate every component modulo p. So our new sequence still passes the Permutation Test since its landing beats are equivalent to those of the sequence we started with.

To see how many balls we need we compute the average

$$\begin{aligned}\frac{1}{p}\sum_{i=0}^{p-1}(a_i + f(i)p) &= \frac{1}{p}\sum_{i=0}^{p-1} a_i + \frac{1}{p}\sum_{i=0}^{p-1} f(i)p \\ &= b + \frac{1}{p}\sum_{i=0}^{p-1} f(i)p \\ &= b + \sum_{i=0}^{p-1} f(i)\end{aligned}$$

$\square$

5.2 Constructing all sequences with the Permutation Test

Since we know that every period p juggling sequence induces a permutation of $\mathbb{Z}/p\mathbb{Z}$ in a very concrete way, we can try to reverse this process. Given a permutation of $\mathbb{Z}/p\mathbb{Z}$, how can we find the corresponding b-ball juggling sequences of period p. By corresponding we of course mean that get the permutation we started with if we apply the Permutation Test (Theorem 4.3) on any of the generated juggling sequences. The following approach was given in [4] on page 24.

We start with a permutation π of $\mathbb{Z}/p\mathbb{Z}$ and let $(a_i)_{i=0}^{p-1}$ be the ordered list with $a_i := \pi(i)$. Since the Permutation Test will add the index to every element of our juggling sequence and evaluate modulo p, we define $(a'_i)_{i=0}^{p-1}$ to be the sequence with elements $a'_i := r_p((a_i - i))$ to reverse this process. The average of $(a'_i)_{i=0}^{p-1}$ is an integer value a, since we know that

$$\sum_{i=0}^{p-1} a_i = \sum_{i=0}^{p-1} i \tag{1}$$

which implies that

$$\begin{aligned}\sum_{i=0}^{p-1} a'_i &= \sum_{i=0}^{p-1} r_p((a_i - i)) \\ &\equiv \sum_{i=0}^{p-1} a_i - \sum_{i=0}^{p-1} i \pmod p \\ &= 0\end{aligned}$$

so in particular

$$\sum_{i=0}^{p-1} a'_i \equiv 0 \pmod p$$

which is equivalent to $(a'_i)_{i=0}^{p-1}$ having an integer average a. Because we evaluate the elements of $(a'_i)_{i=0}^{p-1}$ modulo p we can also be sure that $0 \leq a \leq p-1$. So thus far we have a juggling sequence $(a'_i)_{i=0}^{p-1}$ with a balls, which corresponds to the permutation $(a_i)_{i=0}^{p-1}$, but we would like to find all sequences with b balls. To do so, we see how much we differ from b by defining $d := b - a$ and then generating all sequences of non-negative integers $(d_i)_{i=0}^{p-1}$ of length p whose sum is d. Given such a sequence $(d_i)_{i=0}^{p-1}$, the sequence $(pd_i)_{i=0}^{p-1}$ has average d and therefore $(pd_i + a'_i)_{i=0}^{p-1}$ has average b. It is also jugglable as $(a'_i)_{i=0}^{p-1}$ is jugglable and all we did is add non-negative multiples of the period to every element. We have seen in Theorem 5.2 that this results in a juggling sequence with, in our case, b balls and computing these sequences for all such $(d_i)_{i=0}^{p-1}$ will indeed give all possible b-ball sequences with period p which correspond to the permutation $(a_i)_{i=0}^{p-1}$. To see this we take a b-ball sequence $(b_i)_{i=0}^{p-1}$ corresponding to a permutation $(a_i)_{i=0}^{p-1}$. We see that evaluating every element modulo p results in $(a'_i)_{i=0}^{p-1}$ and therefore $(b_i)_{i=0}^{p-1}$ is uniquely determined by how many multiples of p each b_i differs from a'_i. We note that there are no b ball juggling sequences of period p corresponding to π if $a > b$, or $d < 0$.

This approach lends itself very well to be programmed so it seemed like a waste not do so. A python code can be found through a link in the Appendix (12). As suggested in [4], there is a check, which is built into the program to speed things up. After generating $(a'_i)_{i=0}^{p-1}$ for every permutation of $\mathbb{Z}/p\mathbb{Z}$ there might be some of these which are the same up to cyclic shift. We will see that continuing with all of these will result in a lot of redundancy.

These cyclic shifts will have the same average, say d, and therefore have the same corresponding lists of sequences $(d_i)_{i=0}^{p-1}$ with this average. Now given a sequence $(d_i)_{i=0}^{p-1}$, any cyclic shift of this sequence will be in the list as well, since the sum will be the same. As a result of this, we get the exact same list of patterns (up to cyclic shifts), if we compute $(pd_i + a'_i)_{i=0}^{p-1}$ for two $(a'_i)_{i=0}^{p-1}$ which are cyclic shifts of each other. As an example to see that this will give the same list, say that, for permutations $(a_i)_{i=0}^{p-1}$ and $(b_i)_{i=0}^{p-1}$, the sequence $(a'_i)_{i=0}^{p-1}$ is the same as $(b'_i)_{i=0}^{p-1}$, but shifted forward once. So $(a'_i)_{i=0}^{p-1} = b'_{p-1}b'_0...b'_{p-2}$. After generating all the sequences with the needed average, we fix one $(d_i)_{i=0}^{p-1}$ and know that any cyclic shift, so in particular $(\bar{d}_{r_p(i-1)})_{i=0}^{p-1} = \bar{d}_{p-1}\bar{d}_0...\bar{d}_{p-2}$, will also be in this list. We see that $(p\bar{d}_i + a'_i)_{i=0}^{p-1}$ is just $(pd_i + b'_i)_{i=0}^{p-1}$ but shifted one forward as well. It is now clear why the generated lists of patterns for $(a'_i)_{i=0}^{p-1}$ and $(b'_i)_{i=0}^{p-1}$ will be the same.

It should be noted that the program, as it is described above, will return all b ball sequences of period p which implies that there will be some which are not of a minimal period. For example, the sequence $\underbrace{bb...b}_{\text{p times}}$ will always be generated and, as a more specific example, it will generate 4242 as a 3 ball sequence of period 4, even though we would really see it as a sequence of period 2. Because of this, we find another way to avoid some very similar redundancies. Say we find repeatable sub sequences of length n in $(a'_i)_{i=0}^{p-1}$ for some divisor n of p. Then if $(d_i)_{i=0}^{p-1}$ also has repeatable sub sequences of length n we only have to compute $(pd_i + a'_i)_{i=0}^{p-1}$ for the first n shifts of $(d_i)_{i=0}^{p-1}$, because doing so for any more shifts will create shifted versions of what we have already computed.

5.3 The converse to the average theorem

We will now dive into proving the converse of the average theorem as mentioned in [4], or otherwise known by the very catchy name of "Converse Average Theorem". To reiterate the goal, we will show that a sequence with an integer average is always permutable to a juggling sequence. We apply the same approach as in [4].

Definition 5.3. A ***qualifying sequence*** is a finite sequence of non-negative integers $(a_i)_{i=0}^{p-1}$ where $a_i \geq 0$ for $i \in \{0, .., p-1\}$ such that it has an integer average.

Note that a finite sequence of non-negative integers is qualifying if and only if the sum of its elements is 0 modulo the period. We can now state the theorem.

Theorem 5.4 (Converse Average)**.** *Given a qualifying sequence $(a_i)_{i=0}^{p-1}$ there is a permutation $\pi \in S_p$ such that $(a_{\pi(i)})_{i=0}^{p-1}$ is a juggling sequence.*

Here S_p denotes the finite symmetric group which contains the permutations of p elements. In order to prove this we will rely heavily on the following lemma.

Lemma 5.5. *Let $(a_i)_{i=0}^{p-1}$ be a qualifying sequence for which there exists a permutation $\pi \in S_p$ such that $(a_{\pi(i)})_{i=0}^{p-1}$ is a juggling sequence. If we replace any two elements of $(a_i)_{i=0}^{p-1}$ such that the resulting sequence still qualifies, there exists a permutation $\pi' \in S_p$ such that $(a_{\pi'(i)})_{i=0}^{p-1}$ is a juggling sequence.*

Assuming Lemma 5.5 holds we can now give a proof of the converse average theorem.

Proof. Let $(a_i)_{i=0}^{p-1}$ be our qualifying sequence for which we will show there exists a permutation which turns it into a juggling sequence. Observe that the sequence $(b_i)_{i=0}^{p-1} := \underbrace{00...0}_{\textbf{p times}}$ is clearly a qualifying sequence, since the sum of the elements is 0 modulo p. It is also a juggling sequence, since it is the trivial sequence with 0 balls of period p, so in particular it can be permuted to a juggling sequence. Now we describe a procedure which turns (b_i) into (a_i) by switching two numbers each step. Broadly speaking we work through (b_i) from left to right, check if the elements match up with those of (a_i) and we adjust them if needed. When we change a b_i we also change $\sum_{i=0}^{p-1} b_i$ and we want this to stay 0 modulo p. In order to achieve this, we take b_{i+1} and change this in order to get $\sum_{i=0}^{p-1} b_i \equiv 0 \pmod{p}$. In the next step we move to b_{i+1}, change this number to a_{i+1} and then use b_{i+2} to again fix the sum. This process continues and eventually we will have our desired (a_i).

We will define this process more formally. To this end, we use b_i^n and b_i^o to denote the new and old b_i's respectively. Furthermore, $d_j = a_j - b_j^o$ denotes the difference between our desired sequence and our current sequence in step j. Starting at $j = 0$, in step $j < p - 2$ of the process we redefine (b_i)

as follows:

$$b_i^n := \begin{cases} b_i^o & \text{for} \quad i < j \\ b_i^o + d_j = a_j & \text{for} \quad i = j \\ (kp - d_j) & \text{for} \quad i = j+1 \\ b_i^o = 0 & \text{for} \quad i > j+1 \end{cases}$$

Here kp is the smallest multiple of p such that $kp \geq d_j$. In particular, $k = \left\lceil \frac{d_j}{p} \right\rceil$. Indeed, $\sum_{i=0}^{p-1} b_i^n \equiv 0 \pmod p$ for these choices of b_i^n. In step $p-2$ we do something different than for the other steps. We still set $b_{p-2}^n = a_{p-2}$ and $b_{p-1}^n = kp - d_{p-2}$, but now we note that $a_i = b_i$ for $i < p-1$ and both (a_i) and (b_i) are qualifying so $\sum_{i=0}^{p-1} a_i \equiv \sum_{i=0}^{p-1} b_i^n \equiv 0 \pmod p$. Looking at their difference we see that this is still 0 modulo p and, since the first $p-1$ terms coincide, that is terms 0 up to including $p-2$, we have that $r_p(a_{p-1} - b_{p-1}) = 0$. This means that there is some $n \in \mathbb{N}$ such that $a_{p-1} = b_{p-1} + np$. Since $kp - d_{p-1} \equiv (k+n)p - d_{p-1} \pmod p$, we could set $b_{p-1} = (k+n)p - d_{p-1}$ since we would still have $\sum_{i=0}^{p-1} b_i^n \equiv 0 \pmod p$. We see that, by doing so, we have now fixed the last two numbers in one step, which ensures that process terminates after $p-1$ steps.

After each step we find ourselves with a new intermediate sequence which has $\sum_{i=0}^{p-1} b_i^n \equiv 0 \pmod p$ and thus, according to Lemma 5.5, will again be permutable, allowing us to keep applying the lemma after every step. $\square$

We should note that we could have chosen any multiple of p in every step for setting b_i^n, but we chose for kp, since this seems like the cleanest choice. We also see that, after every step we find a permutable sequence, so we end with more permutable sequences than just our target sequences, which is a nice bonus. Furthermore, the number of balls is not constant in this process so the intermediate sequences, once permuted, will be with different numbers of balls.

To clarify this somewhat abstract procedure we will demonstrate it in an example.

Example 5.6. Let $(a_i) = 54024$ with average 3 but not a juggling sequence yet since by the Permutation Test (Theorem 4.3) we see that $(54024) + (01234) = (55258)$ and $r_5((55258)) = (00203)$, which is not a permutation of (01234) so multiple balls collide. We apply the described process, starting at

00000 moving left to right and having the numbers we changed in each step underlined, with the number that we matched up to our desired sequence also in bold.

$$\begin{array}{lll} 00000 & & \\ \underline{\mathbf{5}0}000 & \textit{step} & 0 \\ 5\underline{\mathbf{4}1}00 & \textit{step} & 1 \\ 54\underline{\mathbf{0}1}0 & \textit{step} & 2 \\ 540\underline{\mathbf{2}4} & \textit{step} & 3 \end{array}$$

It remains to prove Lemma 5.5.

Proof of Lemma 5.5. Firstly, we use the fact that we can permute our given qualifying sequence into a juggling sequence by starting with a juggling sequence in the proof. Like the previous proof, there is a certain algorithm we use and it is very useful in this case to compute an example within the proof in order to explain this algorithm. The juggling sequence we will use is the 4 ball sequence 5961511 of period 7. To prove the lemma, we will change two numbers such that the sum of the numbers is still 0 modulo 7 and then find a permutation of the resulting numbers which is a juggling sequence. Before swapping we introduce the following matrix which we use to keep track of things.

Beats	0	1	2	3	4	5	6
Throwing heights	**5**	9	6	1	**5**	1	1
Landing moments	5	3	1	4	2	6	0

The first row indicates at what moment a throw of the second column is made and the third column is the sum of the first and the second evaluated modulo 7 which, by the Permutation Test (Theorem 4.3), gives a permutation of 0123456.

We now choose to change two numbers at times 0 and 4, say $b_0 = 5$ and $b_1 = 5$, as assigned in the matrix above and change them to get the matrix below.

Beats	0	1	2	3	4	5	6
Throwing heights	**3**	9	6	1	**0**	1	1
Landing moments	5	3	1	4	2	6	0

Indeed, we preserve the sum of the throws being 0 modulo 7 as we subtracted 7 in total. We reorganise this matrix a bit;

Beats	0	1	2	3	4	5	6		
Throwing heights		9	6	1		1	1	3	0
Landing moments		3	1	4		6	0	5	2

To clarify, we singled out the new throwing heights and their old landing times and put them aside. In this new matrix we are interested in the numbers $a_0, a_1, \tilde{b}_0, \tilde{b}_1, c_0$ and c_1 as marked below in a matrix which we call M. Here a_0 and a_1 are the times at which we were supposed to throw the throws of which we changed the height. Then $\tilde{b_0}$ and $\tilde{b_1}$ are these changed heights and c_0 and c_1 are the times at which the old throws b_0 and b_1 were supposed to land.

Beats	a_0	1	2	3	a_1	5	6		
Throwing heights		9	6	1		1	1	$\tilde{b}_0$	$\tilde{b}_1$
Landing moments		3	1	4		6	0	c_0	c_1

Observe that

$$a_0 + a_1 + \tilde{b}_0 + \tilde{b}_1 \equiv c_0 + c_1 \pmod 7 \tag{2}$$

since $b_0 + b_1 \equiv \tilde{b_0} + \tilde{b_1} \pmod 7$ and $c_0 = r_7(a_0 + b_0)$, $c_1 = r_7(a_1 + b_1)$. We will now start searching for a permutation and we want to start by checking if perhaps $a_0 + \tilde{b}_0$ or $a_0 + \tilde{b}_1$ might equal c_0 or c_1 once evaluated modulo 7. Due to (2) we get the following four cases where all the sums are modulo 7:

$$\begin{cases} a_0 + \tilde{b}_0 = c_0, & \text{if} \quad a_1 + \tilde{b}_1 = c_1 \\ a_0 + \tilde{b}_0 = c_1, & \text{if} \quad a_1 + \tilde{b}_1 = c_0 \\ a_0 + \tilde{b}_1 = c_0, & \text{if} \quad a_1 + \tilde{b}_0 = c_1 \\ a_0 + \tilde{b}_1 = c_1, & \text{if} \quad a_1 + \tilde{b}_0 = c_0 \end{cases} \tag{3}$$

In any of these cases we have already found ourselves a permutation that works out to be a jugging sequence. To illustrate, say that $c_0 = r_7(a_1 + \tilde{b}_0)$ and therefore also $c_1 = r_7(a_0 + \tilde{b}_1)$. We place the numbers back in the matrix in the following manner

Beats	a_0	1	2	3	a_1	5	6
Throwing heights	$\tilde{b}_1$	9	6	1	$\tilde{b}_0$	1	1
Landing moments	c_1	3	1	4	c_0	6	0

and we readily see that the second row is indeed a juggling sequence, since the third row is still a permutation of 0123456. The same argument works for the other cases.

Now let's assume that we don't have any of these cases. Set $a_2 := r_7(c_0 - \tilde{b}_0)$ and note that we are making a choice here and could just as well set a_2 to be $r_7(c_1 - \tilde{b}_1)$. By assumption a_2 is not a_0 or a_1 so the column of a_2 in matrix M is non empty and has entries, say b_2 and c_2. Since $c_0 - \tilde{b}_0 = a_2$, changing the entries below a_2 with $\tilde{b}_0$ and c_0 doesn't change the jugglability of this sequence. In our specific example $a_2 = 5 - 3 = 2$ so $b_2 = 6$ and $c_2 = 1$. In general, we place b_2 and c_2 in our redefined matrix M as follows;

Beats	a_0	1	2	3	a_1	5	6		
Throwing heights		9	$\tilde{b}_0$	1		1	1	b_2	$\tilde{b}_1$
Landing moments		3	c_0	4		6	0	c_1	c_2

So in our example we get;

Beats	0	1	2	3	4	5	6		
Throwing heights		9	3	1		1	1	6	0
Landing moments		3	5	4		6	0	2	1

It will become apparent soon why we did not place c_2 underneath b_2 as might be expected. First we show that equation 2 still holds for this matrix. We know that

$$a_0 + a_1 + \tilde{b}_0 + \tilde{b}_1 = c_0 + c_1$$

or equivalently

$$a_0 + a_1 + \tilde{b}_1 = c_0 - \tilde{b}_0 + c_1$$

and since $c_0 - \tilde{b}_0 = c_2 - b_2$

$$a_0 + a_1 + \tilde{b}_1 = c_2 - b_2 + c_1$$

or equivalently

$$a_0 + a_1 + \tilde{b}_1 + b_2 = c_2 + c_1$$

as desired.

Now, we can check again if we can end this process by checking whether $a_0 + b_2$ or $a_0 + \tilde{b}_1$ might equal c_2 or c_1. If so, because equation 2 still holds, we can put the numbers outside of the matrix back in to find the desired permutation. If this isn't the case we continue our search and set $a_3 := c_1 - b_2$ and here we see why we didn't set c_2 underneath b_2. If we would have done this, we would now be stuck with $a_3 := c_2 - b_2 = a_2$ and we have already explored this.

It is clear that, if this process terminates in a finite number of steps, we are now done as we will find a permutation this way. We will prove this by showing that every beat a_i for $i \in \{0, 1, .., 6\}$ will only be found once when computing which columns we need to swap out. In order to refer to this computation during the rest of this proof we will call it the ***beat computation*** from now on. So in our example our first beat computation was $5 - 3$ to find that we need to swap out column 2. What the statement above is saying is that this column will stay untouched from this point onwards.

To check none of these a_i appear twice we pick two arbitrary swaps in this algorithm, which are depicted in matrix A and C in Figures 1 and 3 respectively. We find $c_0 - b_0 = a_i$ in our first beat computation and change the matrix A to B. We find $c_0' - b_0' = a_j$ for the second beat computation and change matrix C to D, as depicted in Figures 2 and 4 respectively.

Beats	...	a_i	...		
Throwing heights	...	b_i	...	b_0	$\tilde{b}_1$
Landing moments	...	c_i	...	c_0	c_1

Figure 1: Matrix A

Beats	...	a_i	...		
Throwing heights	...	b_0	...	b_i	$\tilde{b}_1$
Landing moments	...	c_0	...	c_1	c_i

Figure 2: Matrix B

Beats	...	a_j	...		
Throwing heights	...	b_j	...	b'_0	$\tilde{b}_1$
Landing moments	...	c_j	...	c'_0	c'_1

Figure 3: Matrix C

Beats	...	a_i	...		
Throwing heights	...	b'_0	...	b_j	$\tilde{b}_1$
Landing moments	...	c'_0	...	c'_1	c_j

Figure 4: Matrix D

In general, for any two such moments in our algorithm, we have shown that (2) holds so for matrices A and C this gives us

$$a_0 + a_1 + b_0 + \tilde{b}_1 = c_0 + c_1$$

and

$$a_0 + a_1 + b'_0 + \tilde{b}_1 = c'_0 + c'_1$$

We note that, a_0 and a_1 are always there and by construction of this algorithm we also fix $\tilde{b}_1$. Because of this, subtracting the second from the first gives us

$$b_0 - b'_0 = c_0 + c_1 - c'_0 - c'_1$$

with some rewriting

$$c'_0 - b'_0 = c_0 - b_0 + c_1 - c'_1$$

and since $a_i = c_0 - b_0$ and $a_j = c'_0 - b'_0$ we are left with

$$a_i = a_j + c_1 - c'_1 \tag{4}$$

Now, if we assume that entries can be visited twice, there has to be an entry which is the first to be visited twice. Say this entry is a_i. Since the discussion above holds in general, this in particular holds when we come across a_i again. So, applying equation 4 in this instance we see that $a_i = a_i + c_1 - c_1'$ or equivalently $c_1 = c_1'$. However, we will show that this is not possible.

Note that, after making matrix B we would perform the beat computation and find some $a_i' := c_1 - b_i$ and then move b_i and c_1 to the column of a_i'. We can be sure that $a_i \neq a_i'$ since, if $a_i = a_i'$ then $c_i - b_i = c_1 - b_i$ implying that $c_1 = c_i$, but the c entries are all unique elements modulo p, so this cannot be possible.

Furthermore, c_1 stays at the bottom of column a_i' unless a subsequent value of the beat computation also equals a_i'. This will not happen before we reach a_i again in matrix C, since it would contradict our assumption that a_i is the first to be repeated. So, once we find ourselves at C, we have c_1 still sitting at the bottom op column a_i' and since every c_j is a unique element from $\{0, 1, .., 6\}$ we conclude that $c_1 \neq c_1'$. This proves that no a_i appears more than once in the beat computation and therefore concluding the proof as a whole. □

To clarify the algorithm introduced in the proof we finish the example which we started.

Example 5.7. To recall, we take the period 7 juggling sequence 5961511 and change the first and second 5 to a 3 and 0 respectively. The matrix we get is

Beats	0	1	2	3	4	5	6		
Throwing heights		9	6	1		1	1	3	0
Landing moments		3	1	4		6	0	5	2

With $a_0 = 0$, $a_1 = 4$, $b_0 = 3$, $\tilde{b}_1 = 0$, $c_0 = 5$ and $c_1 = 2$. We see that none of the desired sums $a_0 + b_0$ or $a_0 + \tilde{b}_1$ match up with c_0 or c_1 so evaluate $5 - 3 = 2$ and look at the column of 2 to find $b_2 = 6$ and $c_2 = 1$. We find the new matrix

Beats	0	1	2	3	4	5	6		
Throwing heights		9	3	1		1	1	6	0
Landing moments		3	5	4		6	0	2	1

Again the sums don't match up so we evaluate $2-6=3$ (remember that we evaluate modulo 7) and look at the column of 3 to find $b_3=1$ and $c_3=4$. The new matrix is

Beats	0	1	2	3	4	5	6		
Throwing heights		9	3	6		1	1	1	0
Landing moments		3	5	2		6	0	1	4

Now we see that $0+1=1$ and $4+0=4$ so $a_0+b_0=c_0$ and $a_1+\tilde{b}_1=c_1$. Putting the numbers back into the matrix we get

Beats	0	1	2	3	4	5	6
Throwing heights	1	9	3	6	0	1	1
Landing moments	1	3	5	2	4	6	0

and see that the permutation 1936011 of our starting sequence 3961011 is indeed a juggling sequence.

As mentioned in [4], this constructive proof lends itself very well to be programmed, but apparently no one seems to have done so yet, which seemed like a waste. A fully working program in Python, although be it probably not the most efficient, can be found in the Appendix (12).

6 Mathematical problems with mathematical answers

At this point we start to think about questions which are more interesting and fun in a mathematical sense than a practical sense. We should note that mathematical fun is a very specific type of fun, which is not understood by everyone. We wonder a bit about special types of juggling sequences, which relate strongly to the previous section, but also about properties of juggling sequences, which can again be related to the very practical juggler. The main theorem discussed in this chapter is on a very reasonable and natural question which is about how many patterns there are given a few constraints.

6.1 Scramblable and magic sequences

A natural followup question of Theorem 5.4 is to ask whether there are sequences such that all of its permutations are jugglable and it turns out there are.

Definition 6.1. A ***scramblable juggling sequence*** is a juggling sequence such that every permutation of it is jugglable.

Some examples include the constant sequence $\underbrace{bb...b}_{\text{p times}}$, 714, 963 and 1915 and there is a very nice way to categorise all of such sequences as given in [4] on page 35.

Theorem 6.2 (Scramblable sequences)**.** *A finite sequence of non-negative integers is scramblable if and only if it is of the form $(c+f(i)p)_{i+0}^{p-1}$ for some non-negative integer c and $f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}$ a map such that $c + f(i)p \geq 0$ for all $i \in \mathbb{Z}/p\mathbb{Z}$.*

Proof. Firstly, we note that indeed $(a_i)_{i=0}^{p-1} = \underbrace{cc...c}_{\text{p times}}$ is trivially scramblable for any non-negative integer c. By Theorem 5.2 we see that $(c + f(i)p)_{i=0}^{p-1}$ is still a juggling sequence and it will even still be scramblable. The latter is due to the fact that, before adding multiples of p, every permutation is jugglable and therefore every permutation passes the Permutation Test (Theorem 4.3). Since adding multiples of p will not change the outcome of the Permutation Test we see that any permutation of the resulting sequence will still be jugglable. We see that sequences of the form $(c + f(i)p)_{i+0}^{p-1}$ are indeed scramblable.

Conversely, if we have a scramblable sequence $(a_i)_{i=0}^{p-1}$ we can evaluate all of its elements modulo p and note that, with a similar reasoning as above, this resulting sequence is still scramblable. Suppose it is not constant so there are elements a_i and a_j such that $a_i > a_j$. We now permute the sequence to get a_i on beat 0 and a_j on beat $a_i - a_j$. Since We evaluated both a_i, a_j modulo p, their difference will be less then p, so this shifting is always possible. This should give us a juggling sequence since it is scramblable, but the throw on beat 0 lands at beat a_i and the throw on beat $a_i - a_j$ lands at beat $a_j + (a_i - a_j) = a_i$ as well which is not possible for juggling sequences. So we found a contradiction and conclude that the sequence we found after reducing modulo p is indeed constant, say $(a_i)_{i=0}^{p-1} = \underbrace{cc...c}_{\text{p times}}$. This implies that the original sequence $(a_i)_{i=0}^{p-1}$ is of the form $(c + f(i)p)_{i+0}^{p-1}$ for $f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}$ a map such that $c \geq f(i)p$ for all $i \in \mathbb{Z}/p\mathbb{Z}$. $\square$

We see that modulo p there is one juggling scramblable sequence for every non-negative integer c, which is not a very big set of sequences, but good to have explored.

Sticking to theme of jugglable permutations we might wonder the following. Are there jugglable sequences of period p which contain every integer in $\mathbb{Z}/p\mathbb{Z}$ exactly once?

Definition 6.3. A ***magic juggling sequence*** of period p is a juggling sequence which contains every integer in $\mathbb{Z}/p\mathbb{Z}$ exactly once.

We note that the average of such a sequence $(a_i)_{i=0}^{p-1}$ is

$$\begin{aligned}\frac{1}{p}\sum_{i=0}^{p-1} a_i &= \frac{1}{p}\sum_{i=0}^{p-1} i \\ &= \frac{1}{p}\frac{(p-1)(p-1+1)}{2} \\ &= \frac{p-1}{2}\end{aligned}$$

If a sequence is a magic juggling sequence it is in particular jugglable, so by the average theorem we see that the average has to be an integer. This implies that the period has to be odd. Furthermore, given some odd p, any sequence containing all elements of $\mathbb{Z}/p\mathbb{Z}$ exactly once has an integer average by the above formula. With the converse average theorem we see that there is a jugglable permutation which means there exists a magic sequence for this odd p. So clearly these sequences exist if and only if we have an odd period p. The following theorem shows one way of generating magic sequences as given in [4] on page 36.

Theorem 6.4 (Magic sequences)**.** *Let p, q be positive integers such that $gcd(p, q) = 1$, $gcd(p, q-1) = 1$, p odd and $q > 1$. Then*

$$r_p(((q-1)i)_{i=0}^{p-1})$$

is a magic juggling sequence.

We will prove this after proving the following two number theoretic results.

Lemma 6.5. *Let p, q be positive integers with $gcd(p, q) = 1$. If $p|qr$ then $p|r$.*

Proof. By Bézout's identity there exist integers a, b such that $ap + bq = 1$. Multiplying by r results in $apr + bqr = r$ and since $p|apr$ and $p|bqr$ we see that $p|(apr + bqr)$ or $p|r$. $\square$

Lemma 6.6. *Given positive integers p, q with $gcd(p, q) = 1$ the following map is a permutation of $\mathbb{Z}/p\mathbb{Z}$*

$$\begin{aligned} f :& \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \\ & i \mapsto r_p(iq) \end{aligned}$$

Proof. Since f is a map from a finite set to itself it is enough to show either surjectivity or injectivity to show bijectivity. We show injectivity and assume that $f(i) = f(j)$ for some $i, j \in \mathbb{Z}/p\mathbb{Z}$. So $iq \equiv jq \pmod{p}$ or equivalently $q(i-j) = kp$ for some $k \in \mathbb{Z}$. We see that $p|q(i-j)$ and since $gcd(p, q) = 1$ we know with Lemma 6.5 that $p|(i-j)$. The latter statement is equivalent to $i \equiv j \pmod{p}$ as desired for injectivity. We see that f is indeed a permutation. $\square$

We can now prove Theorem 6.4.

Proof of Theorem 6.4. Since, by assumption $gcd(p, q-1) = 1$, we see by Lemma 6.6 that the sequence $(q-1)i$ is a permutation of $\mathbb{Z}/p\mathbb{Z}$ and is therefore magic, but we need to check that it is jugglable as well. To do so we apply the Permutation Test (Theorem 4.3) once again by checking that

$$r_p((i + (q-1)i)_{i=0}^{p-1})$$

is a permutation as well. We see that $i + (q-1)i = iq$ and now we use that $gcd(p, q) = 1$ as well to see that

$$r_p((qi)_{i=0}^{p-1})$$

is a permutation as well, proving jugglability. $\square$

Remark. Since we construct these sequences modulo p, choosing equivalent q and q' will result in the same sequence, so we can restrict ourselves to choosing $q \in \{2, .., p-1\}$ to find different sequences with a given period p. In fact, all different valid choices for q in this set will also result in different sequences. Since, say we take q, q' different valid choices in this set we get $0q-1...$ and $0q'-1...$ as corresponding sequences. Clearly $q-1$ and $q'-1$ are also different and since every element from $\mathbb{Z}/p\mathbb{Z}$ only appears once we see the generated sequences are also different.

We see that this algorithm misses quite a few sequences, as it generates $p-2$ sequences in the best case when p is prime, but we see that for $p = 7$ there are already 19 magic sequences in total. We see this is also the first p where the two numbers differ so it is interesting to see which patterns our algorithm is finding in comparison to the ones we are missing. We list them in Figure 5 where the entries in bold are the ones our algorithm finds.

Algorithm (finds 5)	All sequences (total 19)				
0123456	**0123456**	0236415	0315246	0413562	0512463
0246135	0135264	0241536	0346152	**0415263**	0526134
0362514	0142635	0245163	**0362514**	0416235	**0531642**
0415263		**0246135**		0425613	
0531642		0263145		0461253	

Figure 5: Magic sequences of Theorem 6.4 of period 7 compared with all of the magic sequences

6.2 Orbits

Given a juggling pattern we might be interested in the behaviour of each ball within the pattern which means that we wonder what kind of throws it traverses.

Definition 6.7. Given a juggling pattern and a ball in this pattern, we define the ***orbit*** of a ball to be the sequence of throws a ball traverses within this pattern.

As the pattern itself is periodic, the orbit of a ball should be periodic as well. Questions arise such as how we find these orbits of a pattern and how many orbits does a given pattern have? To find orbits we simply need to find out where every ball lands and then have a look at what throw is made at that landing beat. Since we only need to know on what number each ball lands, and not necessarily on what beat, we compute every number modulo the period which tells us how many beats later it lands within the first cycle of the pattern. We then read of what this number is, see where this number lands and keep going until we are back at the number we started.

Example 6.8. As an example we consider the 5 ball pattern 645. For clarity, we take the somewhat ignorant approach first by writing out where each ball lands in Figure 19, without using the modulo condition.

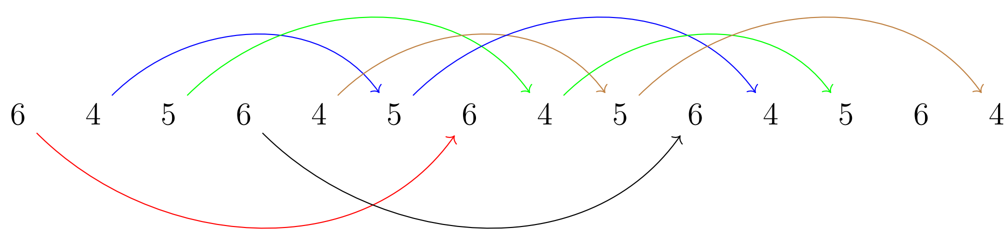

Figure 6: Ignorant approach to finding orbits of 645

We see that the first ball thrown as a 6 lands on what will be the third 6 (in red) so we have already found 1 orbit of a single throw. We see that the fourth ball thrown is also a 6 and lands on what will be the fourth 6 (in black) so we have another ball doing the same thing. We see that blue, green and brown are also all orbits with the throws 4 and 5 in it. So we have 2 balls with the same orbit, which is 6, and 3 other balls also with the same orbit, which is 45. We denote these orbits as O_6 and O_{45}.

Using the modulo approach we quickly see that $r_3(6) = 0$ so it will land on itself and it is its own orbit whereas $r_3(4) = 1$ so it lands 1 beat further on the 5 which in turn lands $r_3(5) = 2$ beats further which is on the 4. One way of writing this is depicted in Figure 7

$$\begin{tikzcd} 6 \arrow[loop left] & 4 \arrow[r, bend right] & 5 \arrow[l, bend right] \end{tikzcd}$$

Figure 7: Orbits of 645

So we can find the orbits, but it might not be clear how many balls are in each orbit, using only the modulo approach. One way of going about this is by noting that every ball is in one of these orbits and no more than one so the pattern decomposes in a disjunct union of orbits. In fact, every orbit is a pattern in itself and with this we will see how this decomposition of the whole pattern into orbits comes about. As an example, we can say that, because the orbits are all disjoint, it would not make difference to the balls in O_6 if the balls in O_{45} were not there at all. This implies that the pattern 600 is perfectly jugglable, since 645 is jugglable and taking away the throws 4 and 5 does not do anything to the throw 6 since the balls making the throws 4 and 5 have nothing to do with any of the other throws. We can now compute the number of balls in O_6 by computing the average of 600. For the same reason we see that 045 is jugglable by taking away all the balls in O_6. We see that O_6 has 2 balls and O_{45} has three because 600 is a 2 ball pattern and 045 is a 3 ball pattern. We might denote this decomposition as a formal sum, so $645 = 600 + 045$.

We can see that any pattern decomposes as a sum of the patterns corresponding to the orbits. Reversing this process, we can take the sum of two period p juggling patterns $(a_i)_{i=0}^{p-1}$ and $(b_i)_{i=0}^{p-1}$ if and only if for all $i \in \mathbb{Z}/p\mathbb{Z}$ we have the condition that if $a_i \neq 0$ then $b_i = 0$ and if $b_i \neq 0$ then $a_i = 0$, or equivalently, if for all $i \in \mathbb{Z}/p\mathbb{Z}$ we have that $a_i b_i = 0$.

Scramblable patterns can play an interesting role in this regard, which, we recall, is a pattern for which every permutation is jugglable. We recall from

Theorem 6.2 that a pattern is scramblable if and only if every number in the sequence is some non-negative integer c plus a multiple of the period, so when computing the orbits we see that each throw lands the same amount of beats further, namely c. Using this, we can say something about the number of orbits we expect to have. If $c = 0$ we know that every ball is thrown again from the same position as it was thrown before, so we have an orbit for every number in the sequence, hence p orbits. If $c = 1$ we see that every ball is thrown again from the next position in the sequence, which means that every ball traverses every throw, hence we have 1 orbit. A more general statement is given below.

Proposition 6.9. *Let $(a_i)_{i=0}^{p-1}$ be a scramblable juggling sequence of period p where $a_i = c + f(i)p$ for some function $f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}$ such that $c + f(i)p \geq 0$ for all $i \in \mathbb{Z}/p\mathbb{Z}$ and $c \geq 0$. The number of orbits is $g := gcd(c, p)$ and the balls thrown on the beats $\{0, 1, .., g-1\}$ are generators of the orbits.*

Proof. We will show that two beats are in the same orbit if and only if they are equivalent modulo g. First we assume for beats i, j that $i \equiv j \pmod g$ so $i + kg = j$ for some non-negative integer k. Generally we find orbits by starting at some beat, add c and then take away p by evaluating modulo p. We then repeat this until we found the beat we started, as this would assign that we found an orbit, so we would like to find j by starting at i in this manner. We apply Bézout's identity on integers c, p so we know that there are integers x, y such that $xc + yp = g$ or equivalently $xc \equiv g \pmod p$. Now, starting at beat i we add kxc as a multiple of c and then evaluate modulo p to get $i + kxc \equiv i + kg \pmod p$ of which we know the right hand side is j.

Conversely, if i, j are in the same orbit we will show they are equivalent modulo g. Saying they are in the same orbit is the same as saying that $i + ac + bp = j$ for some integers a, b and since both c and p are divisible by g we see that $i \equiv j \pmod g$ □

In practice orbits can be very useful as well as aesthetically pleasing to the spectators eye of both jugglers and non-jugglers. What makes them useful is the decomposition property, which can be applied when learning a new pattern. As with anything that is hard, it is good to build op to it and make gradual steps, rather than going for it straight away. The orbits of a pattern provide these steps, as they each contribute a small piece of the pattern, which can be learned individually, as the orbits are patterns themselves with fewer balls. Once all of these are mastered, it will be easier to combine them in the pattern we want to learn and in the case of having more than two orbits it is possible to try combinations of some, but not all of them, to have even smaller steps in our learning process. Practicing an orbit is not very helpful when an orbit only contains 1 ball, as this does not shed much light on what the pattern as whole feels and does. Having 2 or more is actually useful, but what we can do is compose orbits containing 1 ball and still have useful practice patterns.

Example 6.10. We discuss a few examples to cover a few different situations in which orbit decomposition can be useful. The reader is encouraged to check that the orbits we give are in fact the corresponding ones of the pattern at hand.

Starting small we have the 3 ball pattern 531 which has orbits 501 and 030. The 501 is very useful, as it has 2 balls, and the 030 not so much as there is only 1 ball and no orbits to combine it with to better practice patterns, apart from going straight to the real pattern 531.

An example where we can combine our 1 ball orbit is the 6 ball pattern 963. The orbits are 900 with 3 balls, 060 with 2 balls and 003 with 1 ball. Again, we have a boring 1 ball orbit, but even the 2 and 3 ball orbits are not very useful as they turn out to be higher versions of the basic 2 and 3 ball pattern. There is some use to them as we learn the needed accuracy and rhythm at these heights, but not as useful as 501 is for 531 as this combines throws of different heights which is one of the trickiest aspects when learning a new pattern. In general, when a number in the sequence is divisible by the period we see that the corresponding orbit contains only throws of the same height, making it not particularly useful. However, we can make them more useful by composing them. We can make a 3 ball practice pattern by composing 003 and 060 to get 063 or a 4 ball exercise by composing 003 with 900 to get 903. If we feel like doing a hard practice pattern we can compose 060 with 900 to try the 5 ball pattern 960.

We also mentioned that orbits have a certain aesthetics to them. We can showcase the orbits very nicely by using the same colour for balls which are in the same orbit and then use different colours for the different orbits.

6.3 How many patterns?

6.3.1 Preliminaries and a strategy

When some object is introduced in a mathematical context it makes sense to ask how many such objects we have. To plainly ask how many juggling patterns there are does not result in an interesting answer as there is at least one pattern for any non-negative integer b, so infinitely many. However we have some parameters we can play with such as the number of balls, the period and the maximum throw height. We also need to be clear in what sense we want to differentiate patterns when counting them. We have already mentioned that cyclic shifts are really no different from each other and that a period is only of interest to a juggler if it is the minimal period. So 423 and 324 are no different to us and similarly, we would like to view 4242 and 42 as the same. An interesting question turns out to be how many juggling patterns there are for some given number of balls b and period p. We will give the proof as it was introduced in [2] with a large amount of detail here, apart from a combinatorial identity which we will not delve into. We will also stick to the notation of [2], so let $N(b,p)$ be the number of juggling patterns with b balls and period p. Note that we will start by determining a larger number than we are actually interested in, as we will only later filter out cyclic shifts of patterns and patterns which do not have minimal period p. The way we want to determine $N(b,p)$ is by using the fact that every pattern corresponds to a permutation. We can then count the patterns associated to a fixed permutation and we can find $N(b,p)$ if we do this for each permutation of $\mathbb{Z}/p\mathbb{Z}$.

Before we start our journey towards finding $N(b,p)$, we deal with a few lemmas here, as their proofs would only clutter the main story line.

Lemma 6.11. *For positive integers b,p and k the following equality holds*

$$\sum_{a=0}^{b-1} \binom{a-k+p-1}{p-1} = \binom{b-k+p-1}{p}$$

Proof. We will show that the final equality follows by induction on b and *Pascal's identity.* We recall Pascal's identity for positive integers n,k

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{5}$$

We want to perform induction on b in Equation 5 by first checking the base case $b = 1$. Filling in $b = 1$ on the right hand side gives us something we can expand using Pascal's identity.

$$\begin{aligned}\binom{b-k+p-1}{p} &= \binom{p-k}{p} \\ &= \binom{p-k-1}{p-1} + \binom{p-k-1}{p} \\ &= \binom{p-k-1}{p-1} + 0\end{aligned}$$

The second term is defined to be 0, since $p - k - 1 < p$ so we have our base case. Now for the induction step we assume that

$$\sum_{a=0}^{m-1} \binom{a-k+p-1}{p-1} = \binom{m-k+p-1}{p}$$

for some non-negative integer m and show that the statement also holds for $m + 1$ which means we want to show

$$\sum_{a=0}^{m} \binom{a-k+p-1}{p-1} = \binom{m-k+p}{p}$$

Expanding the left hand side as our previous sum plus one more term and using Pascal's identity once more provides the answer.

$$\begin{aligned}\sum_{a=0}^{m} \binom{a-k+p-1}{p-1} &= \sum_{a=0}^{m-1} \binom{a-k+p-1}{p-1} + \binom{m-k+p-1}{p-1} \\ &= \binom{m-k+p-1}{p} + \binom{m-k+p-1}{p-1} \\ &= \binom{m-k+p}{p}\end{aligned}$$

□

As another preliminary, we define drops and descents for permutations and show that the number of permutations with k drops is the same as the number of permutations with k descents. We do this, because we will naturally encounter drops in our search for $N(b, p)$ and want to apply a combinatorial identity which involves descents

Definition 6.12. Given a permutation π of $\mathbb{Z}/p\mathbb{Z}$, we call $t \in \mathbb{Z}/p\mathbb{Z}$ a ***drop*** of π if $\pi(t) < t$.

Definition 6.13. If $\pi \in S_p$ then $t \in \mathbb{Z}/p\mathbb{Z}$ is a ***descent*** of π if $\pi(t+1) < \pi(t)$.

We denote the number of permutations of $\mathbb{Z}/p\mathbb{Z}$ with k descents as $\left\langle {p \atop k} \right\rangle$ and the number of permutations with k drops as $\delta_p(k)$.

Lemma 6.14. *The number of permutations with k descents is the same as the number of permutations with k drops or equivalently*

$$\left\langle {p \atop k} \right\rangle = \delta_p(k)$$

In order to prove Lemma 6.14, we need to take a look at a somewhat weird construction of permutations. Given a permutation $\pi \in S_p$ we will represent this as the sequence $(a_i)_{i=1}^p$ where $a_i = \pi(i)$. A descent can then be recognised as an index in this list for which the next element is smaller. An example could be 563214. Another way of writing out permutations is in cycle form, where we decompose the permutation in cycles and write $(a_i)_{i=0}^m$ for some $0 \le m \le p-1$ and mean that $\pi(a_i) = a_{i+1}$ for $i < m$ and $\pi(a_m) = a_0$. If we include singletons, the example from above would be $(15)(264)(3) \in S_6$ where $\pi(1) = 5$, $\pi(5) = 1$ and $\pi(2) = 6$, $\pi(6) = 4$, $\pi(4) = 2$ and $\pi(3) = 3$. We can choose to have a preference in how we arrange these cycles and also shift them as we please of course. One such arrangement could be to shift each cycle until the largest element in the cycle is in the first position and then arrange all the cycles from low to high based on the first element. For our example we get $(3)(51)(642)$. We now define $\hat{\pi}$ as the sequence we get by going through the whole process as described above plus forgetting about the parentheses. So in our example for $\pi = 563214$, as we started, the corresponding $\hat{\pi}$ is 351642.

We can uniquely reconstruct π from $\hat{\pi}$ by starting at the first entry and inserting left parentheses. Then we set this as our current maximum and read left to right until we find a new maximum, before which we insert right and left parentheses in this order. This will give us π back by how $\hat{\pi}$ is constructed so this construction is bijective. Furthermore, $\hat{\pi}$ is clearly an element of S_p again so we have a bijection of S_p to itself.

Why do we consider this seemingly weird construction? Our goal is to prove that there are as many permutations with k drops as there are with k descents and this construction will allow us to speak about how descents of $\hat{\pi}$ relate nicely to drops of π. Without further ado we will now give the proof.

Proof of Lemma 6.14. We will show that a descent of $\hat{\pi}$ corresponds to a drop of π and that a drop of π corresponds to a descent of $\hat{\pi}$. If we can show this, we know that the number of drops of π is the same as the number of descents of $\hat{\pi}$ and since these are in one to one correspondence we now know that, for every permutation with k drops, there is a permutation with k descents and vice versa. This would prove the claim that the number of permutation with k drops is the same as the number of permutations with k descents.

Let $t \in \mathbb{Z}/p\mathbb{Z}$ be a descent of $\hat{\pi}$, so $\hat{\pi}(t+1) < \hat{\pi}(t)$. Now we can view $\hat{\pi}(t)$ as an element of the cycle decomposition of π and by how we constructed $\hat{\pi}$, this $\hat{\pi}(t)$ cannot be the last element of a cycle of π, because we arranged the cycles such that the last element is followed by a larger element, that being the first element of the next cycle. So $\hat{\pi}(t)$ lies within a cycle of π which means that applying π to $\hat{\pi}(t)$ results in the next element in $\hat{\pi}$, i.e. $\pi(\hat{\pi}(t)) = \hat{\pi}(t+1)$. Since $\hat{\pi}(t+1) < \hat{\pi}(t)$ we now see that, if t is a descent of $\hat{\pi}$, then $\hat{\pi}(t)$ is a drop of π because $\pi(\hat{\pi}(t)) < \hat{\pi}(t)$.

Now let t be a drop of π, so $\pi(t) < t$. If we look at the cycles of π arranged as we wish for constructing $\hat{\pi}$, we see that, when viewing t as an element and not as an index, t cannot be the last element of such a cycle. This is because the last element is sent to the first, which, by our construction, should be larger than any of the others in the cycle. However, as argued above, if t would be the last element, the first element would be smaller than the last element, i.e. $\pi(t) < t$. Now viewing t as an element of $\hat{\pi}$ assigns it an index within this list, say k, so $\hat{\pi}(k) = t$. Because t is not at the end of a cycle in π, we can be sure that it does not go back to the beginning of the cycle, which implies that $\pi(t)$ is the element after t as we view it in the list of $\hat{\pi}$, i.e. $\hat{\pi}(k+1) = \pi(t)$. Because of this we see that $\hat{\pi}(k+1) < \hat{\pi}(k)$, which means that k is a descent of $\hat{\pi}$. We have now shown both correspondences and, by the argument in the beginning, finished the proof. $\square$

6.3.2 Computing $N(b, p)$

We have already seen as a direct consequence of Lemma 4.2 that for every juggling pattern with associated function f of period p there is a permutation $\pi_f \in S_p$ such that $f(t) \equiv \pi_f(t) \pmod p$ for all $t \in \mathbb{Z}/p\mathbb{Z}$. Equivalently we can write every such f as $f(t) = \pi_f(t) + g(t)p$ for some function $g : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}$ for all $t \in \mathbb{Z}/p\mathbb{Z}$. So given a permutation $\pi \in S_p$, the number of juggling patterns associated to π is fully determined by how many "good" functions $g(t)$ we can find. Here "good" means that the resulting f as defined by $f(t) = \pi(t) + g(t)p$ becomes a juggling pattern with b balls. Recall that a sequence of non-negative integers is jugglable if and only if it passes the Permutation Test (Theorem 4.3). The Permutation Test is defined for the throwing heights of a pattern and we interpret the values $f(t)$ as the landing moments of throws made at beat t, whereas we defined $h_f(t) := f(t) - t$ to be the height function of a juggling pattern. Therefore, we would apply the Permutation Test to the sequence $(h_f(t))_{t=0}^{p-1}$ and we see that, because of how we construct f, it passes the test immediately. We do need to make sure that the function f is admissible as a juggling function, i.e. the height function has to be non-negative, and we want to be juggling b balls. We have seen with the Average Theorem (Theorem 4.15) that the number of balls of a juggling sequence is the average of the heights. When given a juggling function of period p, we can find the corresponding juggling sequence by looking at p consecutive values of $h_f(t)$. So we want $h_f(t) \geq 0$ for all $t \in \mathbb{Z}$ and we want the average of $h_f(t)$ over $0 \leq t \leq p-1$ to be b.

For the second criterion we can simply compute the average

$$\begin{aligned}
\frac{1}{p}\sum_{t=0}^{p-1} h_f(t) &= \frac{1}{p}\sum_{t=0}^{p-1} f(t) - t \\
&= \frac{1}{p}\sum_{t=0}^{p-1} \pi(t) + g(t)p - t \\
&= \sum_{t=0}^{p-1} g(t)
\end{aligned}$$

Since π is a permutation the terms cancel with the t terms. So we see that in order for f to be a b ball sequence, we need to have functions g whose terms add up to b. To ensure that $\pi(t) - t + g(t)p \geq 0$, i.e. $h_f(t) \geq 0$, we need to do a bit more work. We note that, since $0 \leq \pi(t), t \leq p-1$, the term $g(t)p$ will always dominate $\pi(t)-t$, because of the factor p, i.e. $|g(t)p| \geq |\pi(t)-t|$. For our period we always have that $p \geq 1$ so as soon as $g(t) < 0$ we see that $\pi(t) - t + g(t)p < 0$. So $g(t) \geq 0$ has to hold for all $t \in \mathbb{Z}/p\mathbb{Z}$. Furthermore, we would also be in trouble if $\pi(t) < t$ and $g(t) = 0$ so we also want $g(t) > 0$ when $\pi(t) < t$ to avoid this.

Thus far, for a given permutation π, we need functions $g(t)$ whose elements sum to b, is always non-negative and strictly positive when $\pi(t) < t$. The last criterion is exactly what we defined to be a drop of a permutation π; see Definition 6.12. To count the drops of a permutation and at the same time invoke that $g(t)$ is always non-negative and strictly positive for those t which are drops, we define the binary function

$$d_\pi(t) := \begin{cases} 1 & \text{if } t \text{ is a drop of } \pi \\ 0 & \text{if } t \text{ is not a drop of } \pi \end{cases}$$

If we then define $G(t) := g(t) - d_\pi(t)$, requiring that $G(t)$ is non-negative for all $t \in \mathbb{Z}/p\mathbb{Z}$ is equivalent to requiring that $g(t)$ is always non-negative and strictly positive for those t which are drops of π. Since, if $G(t)$ is non-negative, certainly $g(t)$ is as well, because we subtract a non-negative value from $g(t)$. Furthermore, if t is a drop of π we see that $g(t)$ has to be strictly positive in order for $G(t)$ to be non-negative. Conversely, if $g(t)$ is always non-negative we see that the only option for $G(t)$ to be negative is when we subtract a non-zero value from $g(t)$, i.e. when t is a drop of π. In this case we assume $g(t)$ to be strictly positive so indeed we find that $G(t)$ is always non-negative.

We can also use G to invoke that the sum of the values of g is b, because, if the number of drops of π is k, we know that the sum of the values of G has to be $b-k$. We have now squeezed the conditions on g into one function G and in this manner turned the problem into finding all such satisfying G. So for a given permutation π, finding all satisfying g is now equivalent to finding all sequences of length p with non-negative values such that their sum is $b-k$ if k is the number of drops of π.

Lemma 6.15. *The number of sequences of length p with non-negative integer values and a given sum x is*

$$\binom{x+p-1}{p-1}$$

Proof. We give a combinatorial argument. One way of interpreting such a list is by thinking of every integer in the list as a sum of 1's. So we have a list of p slots which are each filled with a certain number of 1's, where the total number of 1's is x. We can also think of this list of p slots filled with some number of 1's, as a list with $x + p - 1$ elements where $p - 1$ are separating elements and the remaining x are 1's. Choosing all the different ways of dividing these $p - 1$ separating elements among all of the $x + p - 1$ elements gives all the different lists of p non-negative elements with a sum of x, if we interpret a non-separated sequence of 1's as the integer corresponding to the number of 1's in this sequence. We see that the given identity gives the answer to the latter question, which we argued is the same as the initial question. A similar argument can be found in [5] on page 119. □

Using Lemma 6.15 we see that the number of satisfying functions G for a given permutation π with k drops is

$$\binom{b-k+p-1}{p-1}$$

Multiplying this with the number of permutations with k drops and then summing over all possible values of drops gives us our answer in a rough form. Let $\delta_p(k)$ be the number of permutations of $\mathbb{Z}/p\mathbb{Z}$ with k drops. Then

$$N(b,p) = \sum_{k=0}^{p-1} \delta_p(k) \binom{b-k+p-1}{p-1}$$

Now we would like to do a few things now that we have an expression for $N(b,p)$. Firstly, it will be convenient at a later stage in the proof if we start considering the number of patterns with period p and ***up to*** b balls, so anything from 0 to $b - 1$ balls which we will refer to as $N_<(b,p)$. We find this by the following expression

$$
\begin{aligned}
N_<(b,p) &= \sum_{a=0}^{b-1} N(a,p) \\
&= \sum_{a=0}^{b-1}\sum_{k=0}^{p-1} \delta_p(k)\binom{a-k+p-1}{p-1} \\
&= \sum_{k=0}^{p-1} \delta_p(k)\sum_{a=0}^{b-1} \binom{a-k+p-1}{p-1} \\
&= \sum_{k=0}^{p-1} \delta_p(k)\binom{b-k+p-1}{p}
\end{aligned}
$$

The final equality holds by Lemma 6.11. We see that

$$N_<(b,p) = \sum_{k=0}^{p-1} \delta_p(k)\binom{b-k+p-1}{p} \tag{6}$$

Our next step would be to work out what $\delta_p(k)$ is. If we are on top of our combinatorial identities, the following might spring to mind, which we will not prove here.

Identity (Worpitzky).

$$x^p = \sum_{k=0}^{p-1} \left\langle {p \atop k} \right\rangle \binom{x+k}{p} \tag{7}$$

By Lemma 6.14 we can now write Equation 6 as

$$N_<(b,p) = \sum_{k=0}^{p-1} \left\langle {p \atop k} \right\rangle \binom{b-k+p-1}{p} \tag{8}$$

which is not quite good enough if we want to apply Worpitzky's Identity (6.3.2). However, we note the following property of $\left\langle {p \atop k} \right\rangle$.

$$\left\langle {p \atop k} \right\rangle = \left\langle {p \atop p-1-k} \right\rangle \tag{9}$$

We quickly see why Equation 9 holds. For any permutation π we can construct $\tilde{\pi}$ which we would define as the sequence $\pi(p-1)\pi(p-2)..\pi(0)$, if we denote π as the sequence $\pi(0)\pi(1)..\pi(p-1)$. We see that this construction is again bijective and the resulting $\tilde{\pi}$ is again an element of S_p. Furthermore, because a descent is defined to be an index t such that $\pi(t+1) < \pi(t)$, it is clear that if t is not a descent then $\pi(t+1) > \pi(t)$, since $\pi(t+1) = \pi(t)$ would contradict that π is a bijection. Since a permutation of $\mathbb{Z}/p\mathbb{Z}$ can have at most $p-1$ descents we now know that, if π has k descents, then there are $p-1-k$ indices t for which $\pi(t+1) > \pi(t)$ holds. This implies that the constructed $\tilde{\pi}$ has exactly $p-1-k$ descents. So for every permutation with k descents there is one with $p-1-k$ descents and vice versa of course. So we see that the number of permutations with k descents is equal to the number of permutations with $p-1-k$ descents as desired.

Applying Equation 9 to our formula for $N_<(b,p)$ we get

$$N_<(b,p) = \sum_{k=0}^{p-1} \left\langle {p \atop p-1-k} \right\rangle \binom{b-k+p-1}{p} \tag{10}$$

We know that summing k from 0 to $p-1$ is the same as summing k from $p-1$ down to 0. Since we can start at $k = p-1$ or $p-1-k = 0$ and $p-1-k$ appears in both elements of the sum we get the idea to change variables in the sum so we redefine the variable we sum over by defining $s := p-1-k$ and summing s from 0 to $p-1$, since this is where $k=0$. Filling in gives us

$$N_<(b,p) = \sum_{s=0}^{p-1} \left\langle {p \atop s} \right\rangle \binom{b+s}{p} \tag{11}$$

Now we are in a good place where we can apply Worpitzky's Identity to obtain Theorem 6.16

Proposition 6.16. *The number of patterns with up to b balls and period p is*

$$N_<(b,p) = b^p$$

Such a clean and neat result for something that started as a pretty hairy equation. A direct corollary of this is computing $N(b,p)$

Corollary 6.17.

$$N(b,p) = N_<(b+1,p) - N_<(b,p) = (b+1)^p - b^p$$

6.3.3 Filtering out redundancies

As pleased as we should be with Corollary 6.17, we are not done yet! We recall that $N(b,p)$ counts cyclic shifts as distinct as well as counting pattern which do not have minimal period p. We handle both problems at the same time by defining $M(b,p)$ as the number of patterns with minimal period p where we do not count cyclic shift as distinct patterns. For every divisor d of p we can have patterns of minimal period d being counted as patterns of period p in $N(b,p)$ by repeating them $\frac{p}{d}$ times. Since we can shift these minimal patterns d times, we see that each of these repetitions occur d times in $N(b,p)$. We see that

$$N(b,p) = \sum_{d|p} dM(b,d) = (b+1)^p - b^p \tag{12}$$

The readers who are familiar with Möbius inversion should recognise this kind of expression. See [5] for more information as well as the proof on page 93.

Theorem 6.18 (Möbius Inversion)**.** *We call a positive integer n square free if every prime number in its prime factorisation appears no more than once. Let μ be the Möbius function, which is defined for all positive integers n as*

$$\mu(n) := \begin{cases} 1 & \textit{if } n \textit{ is square-free with an even number of prime factors} \\ -1 & \textit{if } n \textit{ is square-free with an odd number of prime factors} \\ 0 & \textit{if } n \textit{ is not square-free} \end{cases}$$

Then given functions $f, g : \mathbb{N} \to A \subset \mathbb{C}$ such that

$$f(p) = \sum_{d|p} g(d)$$

the following holds

$$g(p) = \sum_{d|p} \mu(d) f(\frac{p}{d})$$

Applying Theorem 6.18 with $f(p) = N(b,p) = (b+1)^p - b^p$ and $g(d) = dM(b,d)$ gives us

$$pM(b,p) = \sum_{d|p} \mu(d)((d+1)^p - d^p)$$

So we have found an expression for $M(b,p)$ as given in Theorem 6.19, which is really the number we were interested in all along.

Theorem 6.19. *The number of patterns with b balls and minimal period p where cyclic shift are not counted as distinct is*

$$M(b,p) = \frac{1}{p}\sum_{d|p} \mu(d)((d+1)^p - d^p)$$

7 State diagrams

7.1 What are they?

Say that we are juggling some pattern, which means that we know at every beat how high we should throw a ball. After a while you might want to mix it up and start throwing a different pattern. As per usual in life you want to consider your options and to do so we stop throwing balls momentarily before making our next throw.

Example 7.1. As an example we are juggling the 3 ball pattern 441 and we stop throwing after throwing the 4 which is underlined in the table below.

beats	0	1	2	3	4	5	6	7	8
throws	4	4	1	$\underline{4}$	4	1	4	4	1
landing times	4	5	3	7	8	6	10	11	9

Even though we stopped throwing momentarily after beat 3, we are still expecting all the balls to come down from all the previous throws we made. The following table gives a clearer view at this point since we are momentarily breaking the pattern at beat 3 and it is not sure what we will do after this beat

beats	0	1	2	3	4	5	6	7	8
throws	4	4	1	$\underline{4}$					
landing times	4	5	3	7					

Looking at the table we see that, from beat 3 onwards, there will be balls landing on beats 4, 5 and 7. Now let's actually consider our options for our next throw on beat 4. Since a ball landed we need to throw it somewhere, but we can't throw it one beat further since, as mentioned, there is already a ball landing at beat 5 at this point. We can however throw it 2 beats further to make it land at beat 6, since nothing is landing here yet, but throwing it 3 beats further would make it land at beat 7 which is also already occupied. Then from beat 8 onwards there are no balls landing yet, so we could also throw it 4 or more beats further. Indeed, throwing a 4 is what is scheduled if we would continue throwing the pattern 441, but we see that we can do any of the above mentioned throws to change the pattern.

To clearly record what our options are at a given moment t_0 in a pattern we introduce the idea of a juggling state. It is sensible to stick to a maximum throwing height as we only want finitely many options for our next throw, so say this maximum height is h. We denote any of the h beats after t_0 with a 1 if a ball is landing and a 0 if no ball is landing.

Definition 7.2. Given a juggling pattern, a ***juggling state*** of a throw is a string of 1's and 0's which denote respectively whether or not balls are coming down at the consecutive beats after this throw.

If we consider a maximum height of 5 for Example 7.1 we get the binary string 11010 for the first 5 beats after we stopped throwing at beat 3. The first 1 here represents the ball at beat 4 which will be thrown somewhere. Choosing where to throw this ball is now a case of choosing a 0 in this string. We can approach the choice of where to throw this ball by removing the first 1, since it will be thrown, adding a 0 to the list at the end, since we can throw as high as a 5, and scanning where the zeros are. To illustrate this procedure we choose the second 0 and note that this really means that we make a 4-throw, i.e. a throw which spends 4 beats in the air, since the second 0 is 4 beats further than the 1 we removed.

$$11010 \xrightarrow{\text{remove first 1}} 1010 \xrightarrow{\text{add a 0}} 10100 \xrightarrow{\text{replace second 0 with 1}} 10110$$

We see that throwing a 4 results in a different state and we have a 1 upfront so we can repeat and have a look where we will throw this ball.

$$10110 \xrightarrow{\text{remove first 1}} 0110 \xrightarrow{\text{add a 0}} 01100 \xrightarrow{\text{replace third 0 with 1}} 01101$$

We have now chosen the third 0 which means we throw a 5 this time. Note that we have a 0 zero upfront now which means that at this beat no ball is landing. To meet the requirements of a juggling pattern we have to make a 0 throw which is equivalent to removing the first 0, but still adding a 0 to the end to maintain this maximum height of 5.

$$01101 \xrightarrow{\text{remove first 0}} 1101 \xrightarrow{\text{add a 0}} 11010$$

We are making sure in every choice that we make a throw which is allowed, in the sense that no other ball will land wherever we throw this one. In this manner we see that we can happily keep making throws, but a juggling pattern needs to be periodic so at what point will this become a pattern? One can imagine that periodicity is met as soon as we find a way to get back to our original state. To further describe how juggling patterns connect to these states we first introduce the notion of a ***state diagram***.

Given some number of balls b and maximum height $h \geq b$ we can list all of the possible states. We can then turn these states into a graph which we call the ***state diagram*** of a pattern and denote with $S_{b,h}$. It is constructed by connecting two states with a pointed edge/arrow if there is a throw to go from one state to the other. An example for 2 balls with max height 4 is shown in Figure 8. Now let's start at a state and follow some arrows through the diagram such that we eventually make it back to our original state, where we close the loop. If we denote all the labels from the arrows we followed it is clear that these numbers form a juggling sequence since all the throws we made were allowed, in the sense that they will not land at the same time as any other throw, and it is periodic by returning to our beginning state.

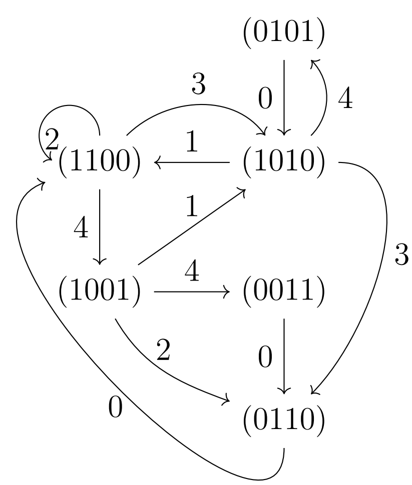

Figure 8: State diagram for 2 balls with max height 4

Example 7.3. We use the state diagram for 2 balls with max height 4 as depicted in Figure 8. We choose the state (1100) as a starting state and follow the arrow which has the label 4. We end up in the state (1001) and follow the arrow labeled 1 to end up in (1010) and the the arrow with a 3 to get to (0110). Finally we follow the arrow with a 0 and get back to our beginning state. Recording the labels we followed we see that 4130 is a juggling sequence. For peace of mind we can perform the permutation test to see that this is indeed a juggling sequence. We see that $(0123) + (4130) = (4253)$ and $r_4((4253)) = (0213)$ which is a permutation.

The following theorem, as stated in [4], shows that all juggling patterns with some number of balls and maximum height correspond to such a loop in the state diagram.

Theorem 7.4. *Given b and h positive integers such that $h \geq b$. Every b-ball juggling sequence with max height h corresponds to a closed path in $S_{b,h}$ which start and end at the same state and contain at least one vertex and one edge.*

Proof. As mentioned, it is clear that such a loop will give us a juggling sequence. This is a direct consequence of how we constructed our state diagram where we assure that following an edge out of a state is equivalent to making an allowed throw.

We are left to show that, given a juggling pattern $(a_i)_{i=0}^{p-1}$ it corresponds to exactly one such loop in the state diagram. In order to find the corresponding loop we reconstruct the states which this pattern visits after every throw. To do so we pick a starting throw in the pattern, say a_0, and construct the table for the first h beats after a_0 as we did in Example 7.1 for 441. All the information required to make this table is given by the permutation. This gives us the state s_0 where we end up after making the throw a_0 and we choose this state as our starting state in the path we are constructing. We can continue to do this for every throw in the period of the sequence, drawing an arrow labelled with a_i towards the state s_i in every step. Once we computed the state s_{p-1} corresponding to the throw a_{p-1}, we compute a_0 once more, as it is the next throw in the sequence, and can be sure that we will once again find the state s_0 so we link up the state s_{p-1} with s_0 through a_0. This is due to the fact that our juggling sequence is periodic so the information in the table will be exactly the same as when we computed a_0 previously, apart from the values being shifted by the period p. This does not change the output state though. We conclude that we find exactly one closed path corresponding to our sequence the with same beginning and ending state.

We can always be sure to have at least one vertex and edge since we make throws in our pattern and every throw results in a state and edge. $\square$

7.2 Grounded or Excited?

Note that we would distinguish between 441 and 414 in Theorem 7.4, since they start at different states, even though they result in exactly the same juggling pattern. We will also say that a pattern is ***based*** at a particular state if it starts, and therefore also ends, there. In practice this is a bit weird to do as we don't want to distinct them as juggling patterns. This distinction only makes sense if we are considering going back and forth between patterns. To discuss this further we will introduce the notion of ***ground state*** patterns and ***excited state*** patterns (which are both equally exciting). We can see in Figure 8 that there is only one edge which starts and ends in the same state. This is the edge containing the throw of height b, or 2 in this case, and it corresponds to the basic pattern with b balls which is always the only pattern of minimal period 1. The state around which this edge loops is $(\underbrace{11..1}_{\text{b times}}\underbrace{00..0}_{\text{h-b times}})$ which corresponds to the base pattern and this state we define to be the ***ground state***. Its name is tied to the correspondence to the basic pattern which, for most jugglers, is the pattern they learn first and whenever a juggler performs a trick he or she will usually start juggling the basic pattern and then goes back and forth between other patterns and the basic pattern. This is just because it is the easiest pattern to continuously juggle and therefore a good base from which jugglers can perform other patterns and go back to. Some of these other patterns can be shifted to start and end at the ground state, making the transition between the base pattern and the other pattern direct. Though there are so many interesting patterns which do not visit the ground state. We will call patterns which visit the ground state also ***ground state patterns*** and similarly we call patterns which do not visit the ground state an ***excited state patterns***. Note that, for a ground state pattern we only require it to visit the ground state, which means that we have the option to start it at ground state by taking a possibly shifted version of the pattern. For example, we have seen that 414 has a shifted version which starts at ground state, which is 441, and because of this we would call 414 also ground state even though we cannot start the pattern from ground state as it is written now.

Example 7.5. We consider our 2 ball max height 4 state diagram again in Figure 8. An example of a ground state pattern was already given in Example 7.3 and another pattern which is ground state would be 3401. The reader can check this by starting at ground state, following the arrows with the corresponding labels of the pattern and ensuring that we do end up at the ground state again.

An example of an excited state pattern would be 40 which is based at (1010). Again, the reader is encouraged to go through the diagram and follow the arrows and more so, the reader is encouraged to find patterns themselves. Pick a state, follow some arrows and make sure you eventually make it back to the state you started.

7.3 Transitions

7.3.1 The superior approach

We can finally answer a very practical question which was raised in the very beginning of this thesis. Given a juggling pattern as we defined it in Definition 3.4, do we have all the information to actually juggle the pattern and how do we extract this information? We have since learned that the average of the corresponding juggling sequence is the number of balls we will need and it was clear from the beginning how high to throw each ball at each moment in time as all this information is contained in the permutation. What remained unclear is how to start juggling the pattern. Given a throw in the sequence which we want to be our starting throw, how are the balls divided in our hands? Note that this is the first time in our discussion that we will employ the fact that we are using two hands to juggle.

We will dive into our 441 example and for notating states we limit ourselves to a maximum height of 4. We start by making the subdiagram with the corresponding states to each throw in the pattern as depicted in Figure 9. We discussed in the proof of Theorem 7.4 how to find these states.

$$\begin{tikzcd} (1110) \arrow[r, "4"] & (1101) \arrow[d, "4"] \\ & (1011) \arrow[ul, "1"] \end{tikzcd}$$

Figure 9: Subdiagram for 441

Say we want to start juggling on the first 4. We go to the corresponding state right before that throw in the diagram. This would be 1110 which the reader should recognise as the special ground state. Let's say that we want to start juggling with our right hand and keep throwing balls in the order right left right etc. This tells us that balls are coming down on the first three beats so if we want to start here, we need to have one ball in our right hand, one in our left hand and again one in our right hand (or two in the right and one in the left for short). Throwing the sequence 441 will now work flawless. However, if we want to start with the second 4, we start one state later in the diagram on (1101). This means we need one in the right, one in the left, none in the right and one in the left (so one in the right and two in the left). Starting like this makes it feel like we are throwing 414, but we are throwing the exact same pattern as 441 of course. For completion, starting on state (1011) requires one in the right, none in the left, one in the right and one in the left (or two in the right and one in the left). Note that the number of balls in each hand is the same as for the ground state (1110), but the order in which they are thrown is different.

Remark. We see that cyclic shifts of a juggling pattern relate to picking different starting throws.

So we see that we can start a period p pattern at any of the p states it visits and it is clear how to derive from the state diagram how the balls should be divided in our hands. Excellent! We have all the information to start juggling. Not all of these starts are ideal though, in the sense that some are awkward in practice. Let's examine our options in a bit more detail. Starting at (1011) requires you to start by passing 1 ball over into a hand which is already holding a ball, which can be annoyingly hard, and then immediately throw this ball again as a 4. There is no reason to make this harder than it needs to be, so we want to pick the "easiest" and most "natural" start. Another option was starting at (1101) where our first throw is the second 4 in 441. This might still feel somewhat weird since we need to start throwing with the hand with the least amount of balls in it. Besides that it is an okay start and a lot better than the previous one. However, we see that one of the options is to start at ground state and this is always a good place to start. This is because the balls are divided equally among our hands and, if we are dealing with an odd number of balls the hand with the most balls will start. Besides this every ball is thrown once before the first gets thrown again, contrary to the start of (1011) where we the first ball gets thrown twice before any of the others. This is what we mean when we say that a start is "easy" and "natural". Another way of saying this is that the start feels a lot like the basic pattern apart from having to throw balls at different heights. It might, understandably, be hard to imagine why any of the described starts are awkward if the reader has no juggling experience. However, if the reader feels at home with juggling a sequence of numbers, it is recommended to try these starts of 441 and experience this for themselves. We we will come back to certain combinations of throws feeling nicer than others in Section 7.3.2.

Excited state patterns are generally very interesting and fun to juggle, but we recall that they do not visit the ground state so a sub optimal start, i.e. not ground state, cannot be avoided. At this point it is good to note that, even though this discussion has been about how to start juggling a pattern when all the balls are still in your hands, we also have the option to juggle something we already know and go into the desired pattern from there. At any state you visit during your current pattern, you can find a sequence of throws in the state diagram which will get you to any of the states of the excited pattern. These throws are commonly known as ***transition throws***. If a state appears both in the pattern you are juggling and in the pattern you want to start juggling, there are no transition throws required and you can swap between the two patterns at the appropriate state. We will extensively cover this in Example 7.6

Example 7.6. Say that we are juggling the 3 ball pattern 531 (which the reader can check is ground state) and want to start juggling 450, which is based at (11010). We can see in Figure 10 that these patterns both include the state (10110). This implies that we can stop throwing the continuous pattern 531 at some point to flawlessly start juggling 450. A glance at the diagram tells us that we should stop 531 after the throws 53 to find ourselves in the state (10110) and start throwing 504. To reiterate, this is the same pattern as 450 of course, but it is based at a different state and therefore starts at a different point in the cycle. So we can disregard the numbers in the patterns when looking for a transition from one pattern to another and purely focus on the states these patterns visit.

However, if we wanted to go from 531 to the pattern 51, based at (11010) we have to make transition throws as these patterns have no states in common. Now we can have some fun. The pattern 531 visits three states and 51 visits

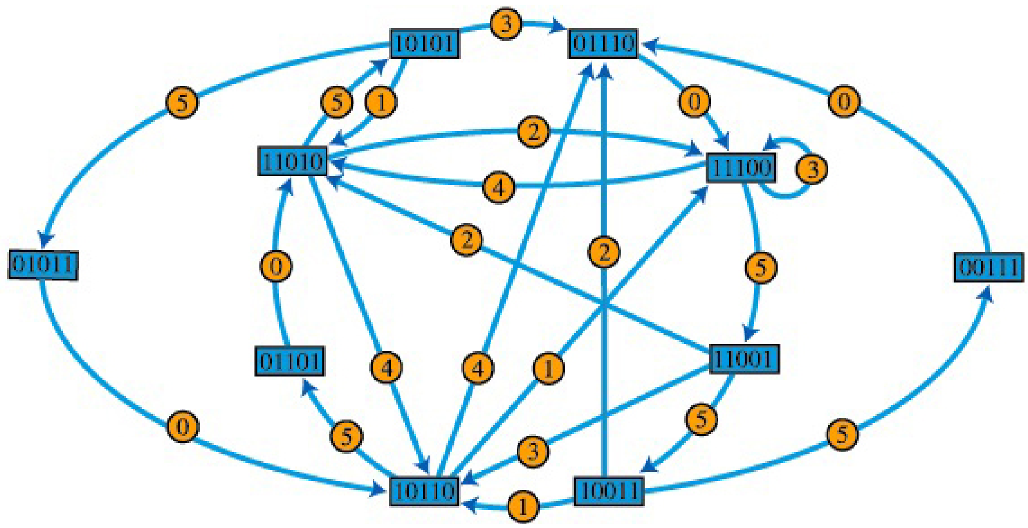

Figure 10: State diagram for 3 balls with maximum height 5 taken from [4]

two. A priori we can pick any combination of these and just trace whatever path between them to create a set of transition throws between the patterns. We have already seen that we can get to (11010) since we could transition to 504 which visits this state. However, we might want a transition which involves less throws. We see that we can get to (11010) in one throw by throwing a 4 from ground state. It turns out that entering at the other state which 51 visits is not an option. This is (10101), which we see has only one incoming arrow and this happens to be from the other state 51 visits. This means that any way of getting from ground state to (10101) will imply visiting (11010) as well, so we don't actually enter the pattern at (10101) this way.

7.3.2 There is a preference

When looking for transitions there is definitely a bias towards certain transitions and we have seen the reason for this already. Some transitions are unnecessarily hard or awkward as described before and others just feel very natural. We should note that a lot of this discussion is a generalisation of jugglers opinions. For one, these can differ as jugglers are human and besides this, there is no rock solid mathematical argument here. Regardless, it is fun and instructive to make the link to the jugglers interpretation.

One particular instance of this is that many jugglers tend to try and not break up the pattern they are juggling when they want to transition. What we mean by this is that, when starting to juggle many rounds of 531 for example, we pick a starting point and this choice determines how we view the pattern. We could start with a 5 which makes us view the pattern as blocks of 531, but starting at the 3 makes it feel like we are throwing blocks of 315. Whenever we wish to leave the pattern, it feels most natural to do so at the point we entered, so to not break a full round of the pattern. It might seem like this goes against our argumentation that cyclic shifts of a pattern are the same and it is still fair to say that they are when viewing the pattern as a whole. Though, if we allow our feelings about a pattern to be involved when looking for transitions, the completionist will take over and have a strong preference. And even in choosing the starting point there is a bias as in our example, most jugglers would pick 531 over 315. The reason for this has to do with our assumptions. We assumed that throwing moments are equally spaced in time and that, as soon as a ball is caught, it is thrown again immediately. However, in real life we hold on to a ball for a brief moment and depending on what throw we make with this ball influences heavily how long we hold on to it. This is negligible for our system we use to describe juggling, but it does play a role here. When making a high throw, we need to lower our arms more in order to generate the momentum to get the ball at the desired height. When throwing a 3, we do not need to lower our arms as much as when throwing a 5 and therefore we hold on to a ball which becomes a 5 a bit longer than a ball which becomes a 3. So, when we need to make a choice where to start, choosing 531 gives us descending holding times, which means we can hold on to a ball a little shorter every next throw. This means our holding times speed up which gives the pattern as a whole a very nice feel, contrary to choosing 315 or 153 where the holding times slow down, which is a lot harder to do in practice. Of course it is important to mention that this "feeling" only exists for a few throws depending on where we start, as we will be doing cycles of the pattern as whole after this, which feels all the same.

Now imagine we have been doing our 531 for a while and want to do many cycles of 51 which we wish to enter at the state (11010). If we allow ourselves to be driven by the human urge to want to complete the things we started, we look for transitions which allow us to leave at the state we started. We started at ground state with throwing a 5 and can for example transition by throwing a 4. However, nostalgic as we are, we miss our beloved 531 and want to return to this pattern. We need to return the way we came, which means we want to leave 51 at (11010) and enter 531 at ground state. One option would be to throw 41 and we would also end up in ground state. We notice that, concatenating the transition throws we used to get in (4) and out (41) of 51 gives us an old friend, 441! Indeed, If we want to transition from 531 to 51, but at the same time respect the order of throws within a pattern, the transition throws used for this will always concatenate into another pattern. With the knowledge of state diagrams this is a trivial fact, because we are forcing this concatenation of throws to be a loop, which we learned corresponds to a juggling pattern.

7.3.3 Finding transitions using the flattening algorithm

It is found to be notoriously "difficult" among jugglers to work out these transition throws. Difficult is not quite the right word though, as we have seen that it is quite straight forward given that you have access to the state diagram, but making one takes a lot of time. A juggler is usually interested in a transition, because there is an interesting pattern they thought of and they want to try. The excitement and enthusiasm to try this pattern makes it all the more tedious to work the whole thing out. You just want to try it! It is therefore also common in practice to juggle the basic pattern and just try throwing some balls a bit higher until it works as a transition and later work out what it is you did. After many years of playing around with these excited patterns, one definitely gets a feel for whether a given pattern is excited or not or what might be a sensible transition to try to get into the pattern. Sometimes it just feels right.

But what if there is no app, pen and paper or clever friend who knows all the transitions and trying out throws does not seem to work. Making a state diagram is tedious work, event though it is a lot of fun to play with once we have it. We want a more direct approach. Given a pattern $(a_i)_{i=0}^{p-1}$ we want to juggle, a good first question is whether it is ground state or not, because if it is, we can shift it until our pattern starts there, meaning we won't need any transition throws. This involves going through all the cyclic shifts to see if there is one which starts at ground state, but how do we check this?

Say the highest throw in our pattern is of height h. We use the following observation.

Observation 1. A pattern is based at ground state if and only if $a_i \geq b - i$ for all $i \in \{0, 1, .., b-1\}$.

If a cyclic shift starts at ground state we can make the first throw from ground state, i.e. $(\underbrace{11..1}_{\text{b times}}\ \underbrace{00..0}_{\text{h-b times}})$. This implies the condition, because the first b spots are taken by incoming balls. We quickly see that ground state is the only state with this property. Assume to the contrary that there is an excited state in our b-ball pattern for which this does hold. Because it is excited, there must be a 0 entry within the first b entries of the state. But this directly contradicts the assumption that $a_i \geq b - i$ for all $i \in \{0, 1, .., b-1\}$. Even though this criterion is a good start, it is not a general approach.

We will discuss an approach to finding transitions from the basic pattern to the desired pattern. In other words, we look for a transition from ground state to the state at which our desired pattern starts. If we want to find a transition between arbitrary states, we could use the following approach in an inefficient way by first going from one state to ground state and then from ground state to the other state.

We need two important remarks about how site swaps and cyclic shifts interact with a state. Given a pattern $(a_i)_{i=0}^{p-1}$, we will use s_i to denote the state in which we find us before throw i. So here s_0 is the state where we find ourselves at the beginning of the pattern.

Remark. Given a pattern $(a_i)_{i=0}^{p-1}$, applying a site swap will not change the state we find ourselves in.

This is because a site swap changes when balls are thrown, but preserves when they land, which is exactly what a state captures.

Remark. We recall that a cyclic shift of 1 step of $(a_i)_{i=0}^{p-1}$ is $a_{p-1}a_0a_1...a_{p-2}$. Given a pattern $(a_i)_{i=0}^{p-1}$, with corresponding states s_i, applying a cyclic shift of k steps, changes the state we find ourselves in from s_0 to $s_{r_p(p-k)}$. More importantly, it is equivalent to making $p-k$ throws of the pattern, which means that we know exactly what throws we made to get from the state in which we were before the cyclic shift to the state we find ourselves in after the shift.

We are now going to use Theorem 4.13, also known as the flattening algorithm, which states that we can construct any b ball juggling sequence using only site swaps and cyclic shifts. Let $(a_i)_{i=0}^{p-1}$ be the b ball pattern we want to transition to from the basic pattern. As described in Theorem 4.13 we can apply the flattening algorithm in reverse to find the site swaps and cyclic shifts needed to go from the basic pattern to $(a_i)_{i=0}^{p-1}$. We just learned what these two operations do for the state we find ourselves in, which is ground state when we begin, because we begin in the basic pattern. In the case of making a site swap, nothing happens and when we shift the pattern, we know exactly what throws we make. Keeping track of all these throws which are induced by the shifts we find our desired transition.

Example 7.7. As an example we compute the transition for the beautiful 6 ball pattern 963. We apply the flattening algorithm below.

$$963 \xrightarrow{swap} 783 \xrightarrow{shift} 378 \xrightarrow{shift} 837 \xrightarrow{swap} 477 \xrightarrow{shift} 747 \xrightarrow{swap} 567 \xrightarrow{shift} 756 \xrightarrow{swap} 666$$

Doing this in reverse to go from 666 to 963, we only consider the shifts we made and we apply these in reverse, because only the shifts influence the state we are in. So we shift from 756 to 567 by throwing a 7, which we note. the next shift happens from 747 to 477 by again throwing a 7. So far our transition is 77. Then we shift from 837 to 378 by throwing an 8 and again from 378 to 783 by throwing a 3. Our total transition is 7783.

$$963 \leftarrow 783 \xleftarrow[throw \quad 3]{} 378 \xleftarrow[throw \quad 8]{} 837 \leftarrow 477 \xleftarrow[throw \quad 7]{} 747 \leftarrow 567 \xleftarrow[throw \quad 7]{} 756 \leftarrow 666$$

Although it gets the job done, it is a very sloppy algorithm. As we see in Example 7.7 we find the transition 7783 to go from the basic 6 ball pattern to 963. By close inspection, we see that 783 is a pattern in itself, which is based in the same state as 963. Therefore it is redundant in the transition and throwing just a 7 would suffice to go from the basic pattern to 963. It is generally desired to make transitions of a minimal length and we discuss another approach for this in the next section. It would be interesting to see if we can filter this redundancy out of the algorithm. We could look at whether some shifts might cancel each other in some way. We could also consider making different swaps than just the first two. For example, swapping not the 9 and 6 in the first step, but rather swapping the 9 and 3 results in 567.

This takes us straight to the final few steps where we shift from 567 to 756 and then swap once more to get 666. Reversing this would give us only the 7 throw from 756 to 567 as an entry, which is indeed minimal. We will assign a site swap of two entries with a $\sim$ if the entries we swap are not the first two.

$$963 \xrightarrow{swap \quad 9\sim 3} 567 \xrightarrow{shift} 756 \xrightarrow{swap} 666$$

$$963 \leftarrow 567 \xleftarrow[throw \quad 7]{} 756 \leftarrow 666$$

This poses the idea of altering the algorithm slightly, when using it to find transitions. We still shift until a throw of maximal height, say a_m, has index 0, but now we need to reconsider with which throw we swap a_m. Let's call this throw a_l and the throws they become after the site swap a'_m and a'_l. We want to make as much progress towards reaching the base pattern in one step, so taking a low a_l is good, but the position of a_l also matters for what the site swap does to the heights. Choosing a_l further to the back will increase how much a_m decreases and a_l increases so we have two factors which we need to balance out. By making these bigger steps, the whole pattern gets closer to the base pattern in a quicker manner and with that, skipping more redundancies. However, we do want to actually get to the base pattern and the fact that we always reduced a throw of maximal height is what got us there in the flattening algorithm. So we need to select an l such that swapping a_m with a_l decreases a_m the most, but still ensure that a_l stays smaller then a_m, such that we do not accidentally create an a'_l, which becomes larger or equally large as a_m.

We can also make some progress in minimising the number of throws in the transition if we are more careful when shifting. If we want to get a throw of maximum height upfront we have been shifting in one direction up to this point, but we can always choose to shift the other way if this seems better. The "better way" here is choosing the direction which shifts the most entries, because this implies we shift the least entries when going back and those are the once which will end up in our transition. However, we have just seen that it matters what position the throw is in which we choose to perform a site swap with. So maximising the number of shifts we do, in order to minimise the number of transition throws later on, may not be optimal if we need to perform another site swap in the next step, as the positions of the throws might be more optimal for a site swap after a different number of shifts. It would be nice to have some real time measure of what happens with the length of our transition, because it is hard at this point to compare the two options of optimisation. They clearly influence each other, but we would like to know how. As we don't manage to clarify this exactly, we say that, when using this algorithm in practice, a good rule of thumb is to site swap high numbers in the beginning with low numbers in the back in order to end up with transitions of minimal length.

So we proceed by using this algorithm by making site swaps which seem best, which is swapping large numbers in the front with small numbers in the back. We see that we can still do something to get to a transition of minimal length, once the algorithm is done. What we mean by this is, once we the algorithm spits out a transition, we can no longer make cyclic shifts, as these would influence the transition, but we can still make site swaps. These can reveal the redundancies we created. We illustrate what we mean by this with the example where we generated the redundant 7783 to get into 963 from the basic 6 ball pattern.

Example 7.8. We can, swap the first 7 with the 3 in 7783 to get 6784. We see that, since we are aiming to go from the basic 6 ball pattern to 963, the first 6 is redundant so we cancel it. We are now left with the transition 784. Again we apply a site swap, now on the 7 and the 4 which gives us 685. Again, we find a redundant 6 in the front which we eliminate, which leaves us with the transition 85. Now, site swapping the 8 and 5 gives us 67 where, again, the first 6 is redundant. We see that the minimal transition turns out to be just a 7 as we saw before.

Applying site swaps at the end also reduces the maximum height of the throws, which is also a desired property of a transition, which we will discuss in more detail in Section 7.3.4. We see that we can still get very far using this algorithm and it would be interesting to try and make this always work systematically, using the comments made above about optimal shifting and swapping. When do we need to swap which numbers and how far do we need to shift in order to be sure that we will always end with a transition?

We should note that, even though it may not look like it, this becomes very practical when used in the real world and it becomes more efficient when used by a more experienced juggler. The reason for this is that, even though our goal is to find the basic pattern, we can stop the flattening algorithm as soon as we find a ground state pattern. This would be 477 which is a shifted version of 774 which starts at ground state. Knowing this allows us to stop and realise that we can go straight to 774 from the basic pattern and then throw 77 to get from 774 to 477 (as it is a shifted version). We can now work our way back from there. Of course, this comment really says that we can stop whenever we recognise a pattern of which we know a shifted version is ground state **or** of which we know the transition throws from ground state. This final comment is very powerful as it implies that the more transitions you know, the quicker you will be done. Hence in practice, when using this for the first time, a more experienced juggler will find more use in this then someone who does not have a vast database of transitions and juggling patterns in their repertoire. However, the less experienced juggler can definitely benefit from this approach as well, by putting in a bit more work in the beginning and as the juggler progresses in their knowledge and recognition of patterns, benefit more from this approach.

7.3.4 Transitions of minimal length

The approach discussed in the previous section is very practical for the experienced juggler as it generates an answer quickly, but it lacks in terms of satisfying minimality of both the number of transition throws as well the maximal throwing height required in our transition. We commented on how we could improve on this, but did not figure it out. Having minimal length is nice, because we make transition throws to get somewhere and we would generally like to get there as soon as possible. Not focusing on this would be like telling the taxi driver the address you need to go and that he can take all day to get you there. Our previous approach sometimes drove straight past the front door in order to make one more lap of the neighborhood. Besides being inefficient this is also expensive in the sense that making these throws costs energy which we would like to invest in the pattern itself. Besides minimal length, we mentioned that minimal height is also ideal and this comes from a certain experience that is associated with transitions. When going from ground state, making transition throws really feels like building up to a pattern, and exceeding the throw height of the pattern would break this build up feeling. This poses the following conjecture.

Conjecture 7.9. *Let $(a_i)_{i=0}^{p-1}$ be a juggling sequence for which we want to find a transition from the base pattern. It is possible to find a set of transition throws from ground state such that the maximal height of these transition throws does not exceed the maximal height of the throws in the pattern itself.*

We will show that this conjecture holds and more so, give an algorithm to go from the base pattern to a desired pattern with a set of transition throws, which has minimal necessary length and with a maximum throw height which is no more than the maximum height of the pattern itself.

The following algorithm was devised by Wouter van Doorn as a follow up on the previous section and it has been fine tuned in conversations between the author and Wouter. The proof of Conjecture 7.9 was done by Anneroos Everts.

The set up is a b ball juggling sequence $(a_i)_{i=0}^{p-1}$ of period p, which we will refer to as the ***target pattern***, and we wish to start juggling from the base pattern, so we need to find a transition from ground state to any state in this target pattern. We can shift the target pattern to pick a desired starting throw and by doing so we fix a state upon which we wish to enter, but we do not know what the states are at this point and we will see that we do not need to know them either. Say our preferred starting throw is a_0. The essence of this algorithm is that it attempts to start from the base pattern and finds all the things that will go wrong, i.e. finds the base pattern throws which would land at the same time as throws of the target pattern. By focusing on only the necessary problems and fixing those with transition throws, we will end up with an entry of minimal length.

We start by making throws of the target pattern and are interested in whether any of the base pattern throws we made before making these throws, will land at the same time as the target pattern throws. We compute where these throws of our target pattern land by $a_i + i$, where we start making these throws at beat 0. If a base pattern throw lands on $a_i + i$ we can check when it was thrown which is at beat $a_i + i - b$. If any of these computations gives a negative number we know that a base pattern throw we made in the past lands at the same time as one of the throws of our target pattern we want to throw, which means trouble. Note that any non-negative beat we get by this computation implies that, if a base pattern throw lands at $a_i + i$, it was thrown instead of one of the throws of our target pattern at this non-negative beat, which is fine. In fact, if we would just continue making, for example p, throws of height b the list of computations $a_i + i - b$ would be 012..p-1. Note that we can make any other pattern of this period which starts at ground state by applying site swaps and that applying site swaps changes when balls were thrown within this period so it permutes this list of 012..p-1. What this means is that we find a permutation of the set $\mathbb{Z}/p\mathbb{Z}$ if we list all the outputs of $a_i + i - b$ for a pattern which starts at ground state. This is if and only if, since the only way for a pattern to not start at ground is that there is some base pattern throw from the past which lands at a planned landing spot of the pattern at hand, which means a negative output in the computation. So having a permutation as output of this computation, but not starting at ground state would contradict each other directly.

Our goal at this point is to fix the negative entries we get when computing $a_i + i - b$ and we see that, as this pattern does not start at ground state, the list of computed values $a_i + i - b$ is not a a permutation so it misses some elements of $\mathbb{Z}/p\mathbb{Z}$, and it misses exactly one for each non-negative entry we have. We want to throw the balls at the computed negative beats differently in order to make them land at landing spots which are not taken by any of the throws in our pattern yet. We know exactly what these landing spots are as they correspond to the missing elements of $\mathbb{Z}/p\mathbb{Z}$ in our computation list. Since, say i is such a missing element. Then no ball of our target pattern will land at $i + b$, since i would have otherwise been in our computation list. Furthermore, $i + b$ will also not be the landing time of any of the base pattern throws from the past as these throws are of height b implying that this throw was made at beat i, but that would contradict this throw being made in the past. We conclude that $i + b$ is simply not hit yet, while it does need to be hit in order for this to be a permutation, which we see if we go back to our definition of a juggling function $f(t)$ (Definition 3.4), which gives the landing times for each beat t. Intuitively, and quite importantly, we have two problems here, which is that there are balls landing at the same time and that there are landing beats which are not hit by our juggling function and the solution to each problem lies in the existence of the other.

We can now match up the negative values with the missed landing times by not throwing a b at these negative beats, but throwing whatever it takes to make that ball land at a beat $i + b$ for some i which is not in the computation list. To be exact, say this negative beat is j, then we throw a height of $i + b - j$. The first basic pattern throw we need to change is at the most negative beat, say beat n, so we find as many transition throws as $|n|$, as there might be throws after beat n which do not need altering, but they are still included in the transition.

A natural question is how far we need to write out our pattern and compute $a_i + i - b$ to find all of the negative values. For any beat $j > b$ we see that $j - b$ cannot be negative, so any negative values of our computation appear in the first b throws of our pattern. However, we also want to know which elements of $\mathbb{Z}/p\mathbb{Z}$ we are missing and it is less clear how far we need to write out the pattern and compute to make sure we know which elements are missing. Let's clarify with an example.

Example 7.10. It's good to consider extreme cases, which are patterns including very high numbers. We consider the 5 ball pattern $e01$ and recall that $e = 14$ as a shorthand.

beats i	0	1	2	3	4
heights a_i	14	0	1	14	0
landing times $a_i + i$	14	1	3	17	4
$a_i + i - b$	9	-4	-2	12	-1

As we wrote out the first 5 beats we can be sure that we have found all the negative entries, so we know there are 3 problems to fix, however, we haven't found a single entry of $\mathbb{Z}/p\mathbb{Z}$ in the list $a_i + i - b$ so we have no idea yet what we are missing.

beats i	0	1	2	3	4	5	6	7	8	9	10	11
heights a_i	14	0	1	14	0	1	14	0	1	14	0	1
landing times $a_i + i$	14	1	3	17	4	6	20	7	9	23	10	12
$a_i + i - b$	9	-4	-2	12	-1	1	15	2	4	18	5	7

Writing out more entries will definitely get us there as we can just move through the entries of $\mathbb{Z}/p\mathbb{Z}$ and see how far in the list $a_i + i - b$ it can maximally appear. So for example, we are missing a 3 and the furthest away in the list it might appear is at beat 8 if would throw a 0 there as then 8+0-5 would give us 3. After beat 8 it is impossible to still have a zero and depending on the lowest throw in your pattern you can work out what number can still appear where.

Even though we highlighted in the example how to make it work anyway, it would nice to know upfront how many beats we need to find all the missing elements of $\mathbb{Z}/p\mathbb{Z}$, without adjusting the number of beats we write out as we go. The key to this is Conjecture 7.9 which we mentioned in the beginning. Assuming that we make transition throws up to beat 0, where we start making throws of the target pattern at beat 0, the furthest beat a transition throw a_j can reach is $a_j - 1$, if this throw is made at beat -1. In particular, if a_m is the highest throw in the transition, the whole transition can reach as far as beat $a_m - 1$. This is of importance to us, because, as we already noted, the problems of balls landing at the same time and certain landing beats not being hit, need each other to be solved. In order to make a pattern work by adding transition throws, a landing beat which is not hit, has to be filled by a throw made before the pattern, which would otherwise collide with a throw of the pattern. In other words, whatever the furthest beat is that is not hit yet, it has to be within reach of the transition, so it can appear in the absolute worst case on beat $a_m - 1$ if we make our throw a_m on beat -1. If we can assume that the maximal throw height a_m of the transition is of at most the maximal throw height of the pattern, say a_h, we can be sure that all of the elements we miss will appear if we write out a_h beats.

Proof of Conjecture 7.9. We first fine tune the statement we would like to prove. First of all we note that the highest throw in our transition is the length of the state of the juggling pattern we wish to enter, if we disregard any potential 0's on the end of the state when we talk about the length. So for this statement we mean to say that the state 100110100 has length 7, even though there are still two 0's after the seventh 1. If our pattern is not based at ground state, there will be some 0's in between the 1's in our state and we can be sure that all the 1's after the first 0 we encounter, correspond to transition throws. If there are no zeros separating consecutive 1's, we can be sure these two 1's were throws of the same height and conversely, the more 0's separate two 1's, the higher the corresponding throw to the second 1 is. With all this said it is now clear that the highest throw of the transition indeed corresponds to the length of the state as we define it.

We proceed by showing that the length of a state corresponding to the pattern $(a_i)_{i=0}^{p-1}$ is $max(a_i - (p - 1 - i))$ where we take the maximum over all $i \in \mathbb{Z}/p\mathbb{Z}$. Intuitively the length of the state is the position of the 1 corresponding to the last throw we make before being in this state. Since $(a_i)_{i=0}^{p-1}$ has period p we can make all the throws once and be sure that we are back in the state we started. Whenever we then make a throw, a 1 which is in front of the state is moved to a zero whose index corresponds to the height of the throw a_i. However, this 1 does not stay here as it shifts one place to the left in the state for every throw we make after this, which are still $p-1-i$, so the position of this 1 after making all of the throws is $a_i-(p-1-i)$. Since we want the position of the 1 which managed to stay all the way at the end after all of the throws we take the maximum over all $i \in \mathbb{Z}/p\mathbb{Z}$ to find the position of this 1 and therefore also the length of the state. Since i is at most $p-1$, we see that $p-1-i$ is always non-negative so $a_i-(p-1-i)$ is at most a_i, from which we conclude that $max(a_i-(p-1-i))$ is at most $max(a_i)$, i.e. the highest throw in the pattern.

To conclude this discussion, we have learned that the highest throw in the transition is equal to the length of the state of the pattern and that this length is at most the maximum height of the pattern we wish to enter, which is exactly what we wanted to prove. □

So we can conclude that we are sure to find all the missing entries i of $\mathbb{Z}/p\mathbb{Z}$ if we write out the first $max(a_i)$ throws. We needed to write out the first b throws to be sure to find all of the possibly negative entries and since $max(a_i) \geq b$ this is covered.

We would like to make a final few remarks regarding this algorithm as we have some freedom to how we combine the negative entries we find after computing a_i+i-b and the entries $i \in \mathbb{Z}/p\mathbb{Z}$ we miss in this computation. It is favourable to also have minimal height in our transition and the choices we make at this point will influence this and this is best shown with an example so we continue our Example 7.10 here. We have found that we have negative entries -4,-2 and -1, which implies our transition will have length 4 and that we are looking for 3 non-negative entries of $\mathbb{Z}/p\mathbb{Z}$ which are missing. We see that we are missing entries 0, 3 and 6 and we now have the freedom to match any of the negative entries to the non-negative ones. One obvious choice would be to match them in order of appearance, so, for example, we match the -4 to the 0, which means that, at beat -4 we will not throw a base pattern 5, but we will throw whatever it takes to make this ball land at beat $0+5$, so we throw a 9. We match -2 to 3 which means we throw a 10 at beat -2, in order to make it land at beat $3+5=8$. For completion, we throw a 12 at beat -1, which then lands at beat $6+5=11$. Note that beat -3 is fine as it is so we can just throw a 5 here. Our transition would be 95ac and remember that $a=10$ and $c=12$ to avoid confusion. We can do better in terms of minimal height though. We see that we have 4 negative throws (beat -4,-3,-2 and -1) of which 3 were problematic (beat -4,-2,-1) and 1 was okay (beat -3). We would like to make all of the "problematic" throws as far to the front as possible, so at beats -3,-2 and -1, and then match them in order of appearance with the missing entries of $\mathbb{Z}/p\mathbb{Z}$. So -3 with 0, -2 with 3 and -1 with 6. This means we need to shift our originally scheduled b throw from beat -3 back to beat -4 by throwing a b+1 there. This would give nearly the same entry as before, because we now get 68ac, which is not much better because we could not shift anything in the first few negative beats where the highest throws are, but we will give an example in which this approach is indeed fruitful. We should note that the same result can be obtained by applying a site swap.

Example 7.11. We perform this algorithm in its full glory on the 6 ball pattern 7a1. We write out the first 10 throws of the pattern as 10 is the highest throw in the pattern.

beats i	0	1	2	3	4	5	6	7	8	9
heights a_i	7	10	1	7	10	1	7	10	1	7
landing times $a_i + i$	7	11	3	10	14	6	13	17	9	16
$a_i + i - b$	1	5	-3	4	8	0	7	11	3	10

We see that we have one problematic beat which is -3, so our transition will have 3 throws, and that we are missing beat 2 in the lowest row. Without applying our most recent addition to the algorithm, it might seem logical to match -3 with 2, so we would throw something which lands on beat $2+6=8$ at beat -3, i.e. an 11. Beats -2 and -1 are okay so we would throw 6's which makes the total transition b66, since $b=11$ to avoid confusion. However, if we now apply the the shifting idea, we want to make the throw at beat -3 at beat -1, and we get it there by shifting the 6's planned at beats -2 and -1, both 1 beat backwards to beats -3 and -2 respectively by throwing 7's instead. This allows us to match beat -1 to beat 2 by throwing something which would land at beat $2+5$ which would be a 9. Our transition is now 779 and we should note that we could have just as well gotten here by applying site swaps to b66, as we do in the flattening algorithm approach.

7.4 Counting states, arrows and patterns

Having played around with $S_{2,4}$ and seen some examples, it is instructive to build these state diagrams from the ground up and have a look at a few small cases as we increase the number of balls and the maximum height. Starting with some small cases, we take the minimal value for h which is $h = b$. This is because we would not be able to juggle a b-ball pattern with a maximum height being less than the number of balls, since, for example, this would contradict the average theorem as the average could never be b. The most basic case is depicted in Figure 11 for $b = h = 1$ and we quickly see that in general for b balls the minimal value for the max height $h = b$ only gives us the basic pattern as seen in Figure 12. If we fix the number of balls at 1 and let the height vary as in Figures 13 and 14 we also see a general pattern arise for the state graphs as depicted in Figure 15. When looking at state diagrams for 2 balls these already get quite complicated going from max height 3 in Figure 16 to max height 4 as we have already seen in Figure 8.

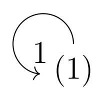

Figure 11: State diagram for 1 ball with max height 1

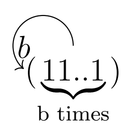

Figure 12: In general: State diagram for b balls with max height b

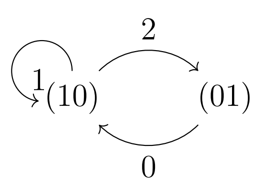

Figure 13: State diagram for 1 ball with max height 2

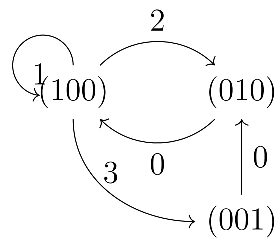

Figure 14: State diagram for 1 ball with max height 3

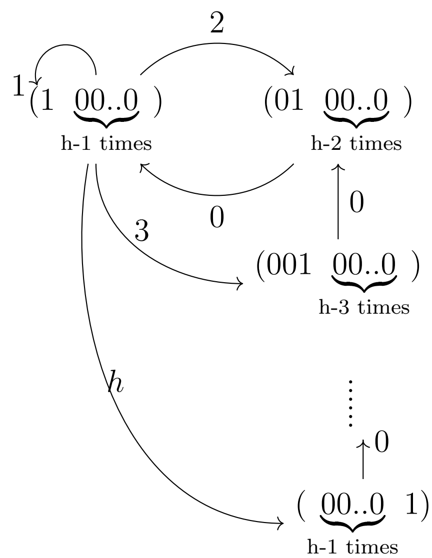

Figure 15: State diagram for 1 ball with max height h

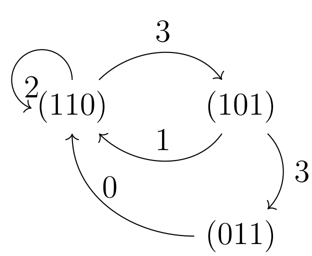

Figure 16: State diagram for 2 balls with max height 3

In general, given some number of balls b and max height h the number of possible different states is easily found by thinking of a state as a list of h zeros where we choose b of these to turn into 1's, since there are b balls in the pattern so also always b balls coming down. This would be

$$\binom{h}{b} = \frac{h!}{(h-b)!b!}$$

Furthermore we can think about how many incoming and outgoing arrows we expect at a given state s. Looking at outgoing arrows first we consider the scenarios of the state starting with a 1 or a 0. If it starts with a 0 we only have one option for an outgoing arrow which is making a 0 throw. If we have a 1 at the beginning we have as many options for outgoing arrows as there are zeros in the string after taking away this 1 at the beginning and adding a 0 at the end as these are all the places a ball can be thrown, without coinciding with other balls, given our max height. So the state itself has $h-b$ zeros, but we add a zero at the end so this gives $h-b+1$ outgoing arrows.

For the incoming arrows we distinguish between the cases where our state ends in a 0 or a 1. If a state ends in a 1, we know that a ball was thrown at maximum height to get there and there is only one state from which this ball could have come, that is the unique state s' which starts with a 1and of which the last $h-1$ entries match the first $h-1$ entries of s. So in this case we are expecting 1 incoming arrow. However, if a state s ends in a 0 there are more options for incoming arrows. It could be that the state it originates from started with a 0 which means we were forced to make a 0 throw. If the originating state started with a 1 there were several options as to where this throw could have come from. Though, we know that it must have been thrown to a 1 in our current state, which we call $\bar{1}$. Again, there is a unique state s' corresponding to throwing to $\bar{1}$ and that is the state which starts with a 1, has a 0 in the position of $\bar{1}$ and matches up with s for the other positions. Since we have b 1's we know there are b incoming arrows plus the option of a 0 throw which makes for a total of $b+1$ incoming arrows.

Given a state graph, and therefore a number of balls b and period p, we would like to know how many patterns of a particular state we might expect. A number for the somewhat special ground state is given in [4] on page 48 and we give the proof here.

Theorem 7.12. *Let G be the number of ground state patterns with b balls and period p. Then*

$$G = \begin{cases} p! & \textit{for} \quad b \geq p \\ b!(b+1)^{p-b} & \textit{for} \quad b < p \end{cases}$$

Proof. To find the number of ground state patterns for a given period p and number of balls b we imagine to be juggling the basic b-ball pattern, where we at some point switch to make p throws of any other ground state pattern of period p and then go back to the basic pattern. We will now consider all of the options for these p throws. A diagram to sketch the situation is helpful at this point.

$$\text{beats:} \quad \underbrace{\underbrace{1 \quad .... \quad b}_{b\text{ balls landing}} \quad \underbrace{b+1 \quad .... \quad p}_{\text{no balls landing yet}}}_{p\text{ throws to be made}} \quad \underbrace{p+1 \quad .... \quad p+b}_{\text{throws made again}} \quad \underbrace{p+b+1}_{\text{first ball lands}}$$

Figure 17: Number of ground state patterns

In order for it to be helpful it will need some explanation though. What we see depicted are the beats 1 up to $p+b+1$ of which the first p are of the new pattern we choose to do. Note that, since we are juggling a ground state pattern to begin with, there will be balls landing on the first b beats of the period p pattern. Then there are $p-b$ more beats where no balls are scheduled to land yet. After this we start making throws again for our basic pattern. From this we observe that, from the p throws made at beats 1 to p, b of them have to land at beats $p+1$ to $p+b$, since it is scheduled that throws will be made here. More importantly, none of the p throws can land further than at beat $p+b$, since we have incoming balls from the basic pattern we started throwing again. Furthermore, there are balls landing on the first b beats of our period p pattern, so we cannot make 0-throws here. This implies that any 0-throws we might want to make have to happen between beats $b+1$ and p.

With all of this said, let's consider all of our options for this period p pattern in the middle and to do so, we consider the two cases $b < p$ and $b \geq p$.

$$\text{beats: } \underbrace{1 \quad \text{....} \quad p}_{p \text{ throws to be made}} \quad \underbrace{\underbrace{p+1 \quad \text{....} \quad b}_{\text{balls landing and thrown}} \quad \underbrace{b+1 \quad \text{....} \quad b+p}_{\text{available places}}}_{b \text{ throws made for basic pattern}}$$

Figure 18: Number of ground state patterns if $b < p$

This distinction comes from wanting to distinct between having the option to make a 0-throw or not. In either case we are going to consider our options starting at throw p and work our way back to throw 1. We describe the situation for $b \geq p$ in Figure 18.

If $b \geq p$ there is no space for 0-throws, since there are throws landing on the first b beats, so in particular on all of the p beats. This implies that, for throw p, we have as many options as there are non-taken spots in front of us. We know that, from the first b beats, the remaining $b-p$ are all taken by incoming throws from the basic pattern. Furthermore, we know that the balls landing on these beats are thrown again as basic pattern b-throws, which implies that they will land from beat $b+p+1$ onwards. We see that, for throw p, we have beats $b+1$ up to $b+p$, which are p options in total. Once we picked a beat to throw it, we have $p-1$ options remaining for throw $p-1$ and so on up to throw 1. This means we have $p!$ options for different patterns if $b \geq p$.

If $b < p$ the situation is very similar, but we are allowed to make 0-throws. The more general diagram in Figure 17 depicts this well. Starting again at throw p we see that we can now decide to make a 0-throw or throw it to any of the beats $p+1$ up to $p+b$. This gives us $b+1$ options for this throw and any of the throws happening at beats larger than b have $b+1$ options. Assume that $p-1 > b$. There is again the option to make a 0-throw and we can have all of the available spots in front of this throw. From the perspective of $p-1$ there are $b+1$ spots, but one is taken by throw p so this leaves b. Together with the 0-throw option this makes a total of $b+1$ options. There are $p-b$ beats which have the options of a 0-throw and, analogous to the case $b \geq p$, there are $b!$ options for the remaining b throws. This gives a total of $b!(b+1)^{p-b}$ patterns if $b < p$. □

In Theorem 6.19 we found an expression for $M(b,p)$, which is the total number of patterns with b balls and minimal period p where cyclic shifts are not counted as distinct. Combining this with what we just learned, we see that the total number of patterns which are excited is $M(b,p) - G$. However, it would be nice to find the number of patterns corresponding to a particular excited state. This turns out to be a bit harder to do, but the following lemma, as given in [3], is a good start as it gives a requirement for period p patterns to exist for a particular state. See [3] for a proof on computing the number patterns based at a particular state.

Lemma 7.13. *Let s be an arbitrary state for b balls. If a period p pattern is based at s, then for all $k > p$ we know that if $s_k = 1$ then $s_{k-p} = 1$ as well. Furthermore, for all k we have that if $s_k = 0$ then $s_{k+p} = 0$. Here $s_k = i$ denotes that entry k of s has the value i.*

Proof. We assume that $s_k = 1$ and $s_{k-p} = 0$ to find a contradiction. Since there exists a period p pattern by assumption, say $(a_i)_{i=0}^{p-1}$, we choose to make all p throws once. This means that the value of s_k shifts by p so s_{k-p} now has the value of whatever was the value of s_k. Note that we started in s and should also end in s after the p throws since this is what it means for the pattern to be based in s. However, we see that $s_{k-p} = 0$ before and $s_{k-p} = 1$ after the p throws so we cannot be in the same state.

By a very similar reasoning as above, if $s_k = 0$ for some k and $s_{k+p} = 1$, we could make p throws of our pattern based at s to see that we would not end up at s, since $s_k = 0$ before the throws, but $s_k = 1$ after since it takes the value of what used to be s_{k+p}. So we see that if $s_k = 0$ for any k, then $s_{k+p} = 0$ as well.

□

Remark. The converse is not true. If, for some $k > p$, $s_{k-p} = 1$ then it is not necessary for $s_k = 1$ to hold, since if $s_k = 0$, there are p throws which could possibly land at this spot to turn the 0 into a 1.

7.5 Prime patterns

Learning about state diagrams and that a pattern corresponds to a loop in them, gives us a lot of new insight into a juggling sequence. Not only did we gain more understanding for what a juggling sequence is, but we could develop a lot of theory on transitions between juggling patterns which become very trivial with the knowledge of the corresponding state diagram. We learned that a pattern is based at a particular state, which means we come back to this state after a full cycle of the pattern, and that this is determined by the first throw we make in the pattern. Since loops in the diagram correspond to real life patterns, we can concatenate loops which start at the same state to concatenate juggling sequences in practice which generates a longer pattern, starting at the same state. Doing so allows us to repeat any of the two patterns within the longer one as often as we want and we still get a valid juggling sequence. In general, whenever a loop traverses a state twice we have a sub loop which can be repeated which translates to a repeatable sub sequence of throws in the juggling sequence. Given this knowledge, a natural question is how long we can make a loop in a state diagram before it returns to its starting state, or interpreting this as a pattern, how long we can make a pattern with a given maximal height without having repeatable sub sequences.

Definition 7.14. A ***prime pattern*** with b balls and maximal throw height h is a juggling sequence with no repeatable sub sequences, or equivalently a loop in the corresponding state diagram $S_{b,h}$ which does not visit any state more than once.

Their name is fitting as we can see that any juggling pattern with b balls and max height h in a way decomposes into these prime patterns, similarly to how the natural numbers decompose into prime numbers. We can consider the corresponding loop of a pattern and whenever we visit a state twice, we shift the pattern until this becomes our starting state. We can then separate the sub loop from the pattern as whole and search both loops for more repeatable sub loops to repeat the process which in the end will return only prime patterns. So one thing we might be interested in is the longest prime pattern and we will introduce several facts and theorems to help us further along to find the answer. One thing we can already note is that, since a prime pattern does not visit any state more than once, a clear upper bound for the length of prime patterns is the number of states in $S_{b,h}$ which is $\binom{h}{b} = \frac{h!}{(h-b)!b!}$. The upcoming discussion is based around and uses theorems from [1].

To get a feel for the topic, a few examples of prime patterns are in Figure 10 are 5520, 42, 4530, 4440, 55500 and 55150530 based at (11100). Others include 40551, 504, 50550, 502551 based at (10110) and we encourage the reader to play around and find more. In particular, we try to find patterns which are long and looking at this we notice that longer examples tend to includes lots of 0's and 5's, so throws of minimal and maximal height. Johannes Waldmann was the first to notice this fact; see [1]. When many of these throws appear we get consecutive sequences with only h and 0 throws and these sequence are separated by one or more throws which are not of minimum or maximum height. The following theorem, as given in [1], is a nice constraint for the length of these sequences in prime patterns of a certain minimal length, which is of interest to us as we are looking for prime patterns of maximal length.

Theorem 7.15. *Given a prime pattern $(a_i)_{i=0}^{p-1}$ with $p > h$, a consecutive sequence of minimal and maximal $0, h$ throws will not be longer than $h - 2$.*

Proof. First we note that making an h or 0 throw effectively shifts our state, as the respective 1 and 0 which are at the front get placed at the back of the state, when making said throws. Furthermore, it is always possible to make exactly h of these throws consecutively and get back to our original state. Since, if our state starts with a 1, it is always an option to make an h throw and if the state starts with a 0 we are forced to make a 0 throw. This effectively shifts our beginning state h times. It is clear that a prime pattern can therefore not have more than h of these throws in a row.

We now prove the claim by showing that, if a consecutive sequence has more than $h-2$ throws within a prime pattern, it necessarily has exactly h throws. This would imply that, if the pattern has more than h throws in total, we cannot have a consecutive sequence containing more than $h - 2$ of these $0, h$ throws.

So if the total pattern has more than h throws, we clearly can't have a consecutive sequence of h of these $0, h$ throws as we would already return to our beginning state before finishing the pattern, making it not prime. If we make $h - 1$ consecutive $0, h$ throws, we know that one more of these throws should take us back to the beginning state. There are two options now, which is that the state we find ourselves starts with a 1 or a 0. If it starts with a 0, we are forced to throw a 0, allowing us to use the previous argument as we would have completed a consecutive sequence of h of these throws. However, if we have a 1 in the front it might feel like we are not forced to throw a 5 and return to the base state, as a 1 can become any height we wish. This is not quite true in this case, because of the fact that we already know the next state ends in a 1, because it is 1 shift away from the state we are in now. The problem with this is that there is only one way to get there, namely by throwing a 5 from the state we are in right now. Choosing to not do this, implies we have to eventually cycle back to this state in order to make the final 5 throw, which means we cannot be a prime pattern. We can conclude that $h-1$ of these consecutive $0, h$ is not an option either. So we can be sure that, if our pattern contains more than h throws in total, any consecutive sequence of $0, h$ throws will not be longer than $h-2$. □

So, as remarked in [1], long prime patterns contain long sequences of these $0, h$ throws, but we just learned it is never more $h-2$. To understand why long prime patterns contain these long sequences we introduce the following definitions as has been done in [1].

Definition 7.16. We refer to throws of heights $0, h$ as ***shift throws*** as these throws shift the state we are in. Given a juggling pattern, a sequence of no more than $h-2$ shift throws is called a ***block*** and a throw of non minimal or maximal height is called a ***link throw*** (the reason for this will become clear). The number of throws in a block is called the ***block length***. Finally, a pattern with b balls and max height h is in ***block form*** if it is composed of one or more of these blocks.

Because of what we learned in Theorem 7.15, every prime pattern with more than h throws is automatically in block form.

It became clear in the proof of Theorem 7.15 that we can always make h shift throws, by simply throwing an h if we have a 1 at the beginning of our state and throwing a 0 if the state starts with a 0. These h throws take us back to the state we started and we see that these throws always form a prime pattern, which we call a ***shift cycle***. When $gcd(b, h) = 1$ the shift cycles always have length h and it is possible to have shift cycles with a shorter length if b, h have common divisors. It is clear that each state is in exactly 1 such shift cycle, so the state diagram $S_{b,h}$ decomposes into shift cycles. All the other throws, which we know as link throws, serve the purpose of moving between the cycles, hence their appropriate name.

These shift cycles are very helpful in studying maximal patterns, as it is clear we are a prime pattern as long as we are in one cycle. If we can find a way to traverse all of the shift cycles fully using link throws we would be of maximal length. However, it is not possible to traverse each shift cycle completely of course, as we would eventually get back to where we started. So the goal is to stay in each shift cycle as long as we can and leave just before we would be forced to return to the state we first visited in this shift cycle. We therefore want to think about how these link throws behave and at what point in a shift cycle we can enter and leave.

Let $s = (s_i)_{i=0}^{h-1}$ denote a state where s_i can take the values 1 and 0 and let's say we make a link throw to go from one shift cycle to another. The state s of the shift cycle in which we enter has to end in a 0, since, states which end in a 1, say $\bar{s}$ with $\bar{s}_{h-1} = 1$, can only be reached from the unique shifted version of $\bar{s}$, say $\bar{s}' := (\bar{s}_{h-1}\bar{s}_0\bar{s}_1..\bar{s}_{h-2})$, by making an h throw. This would imply that the state we came from is in the same shift cycle, but we are entering from a different one. By this we know that the just before s, say s', starts with a 0, which means that, once we reach this state we are forced to make a 0 throw and get back to s. This implies that s' has not been visited yet by our prime pattern, since, if it would have been visited, so would s, making our pattern not prime anymore. For the same reason we are sure to not visit the state when traversing the shift cycle we find ourselves in so in total it will not appear in the pattern. If we wish to leave this shift cycle on some state t we know that $t_0 = 1$ since, we need to make some non-zero throw in order to leave. In this manner we need to also miss the state $t' = (t_1t_2..t_{h-1}t_0)$, which comes after t, since the only way to get here is by throwing an h from t, which we choose to not do in order to get to the next shift cycle. We can luckily combine these two states we have to miss by entering on a state s with $s_{h-2} = 1$ and $s_{h-1} = 0$, such that we can leave after $h-2$ shifts and only have to miss the state $s' = (s_{h-1}s_0s_1..s_{h-2})$ enjoying a total of $h-1$ states of this shift cycle. We conclude that, by trying to stay on these shift cycles as long as possible in order to have maximal length, we will at least miss 1 state for each shift cycle in $S_{b,h}$. This refines our previous upper bound to

$$MPL = \binom{h}{b} - SC$$

where MPL is the maximal length of a prime pattern in $S_{b,h}$ and SC is the number of shift cycles. This bound was also found by Johannes Waldmann; see [1].

8 Sometimes we do not find the answers

During this exploration we have seen a variety of questions being answered and some of those only raised more questions and others did not get an answer at all, but merely an explanation of what we like to ask. We would like to use this section to go through some of these questions chronologically and highlight what might be possible in further research.

8.1 The converse average theorem

Starting with the proof of Theorem 5.4, where we explicitly construct a permutation of a given qualifying sequence which will be jugglable. We have already learned that it is very well possible for a sequence to have more than one jugglable permutation as we even have scramblable sequences for which each permutation is jugglable. This makes us wonder if there is a neat way to find all of them by tweaking the algorithm used in the proof of 5.4 as suggested in [4]. In particular, we are looking for making changes in Lemma 5.5 which is really where all the action is in this proof. In this proof, after checking whether any of sums in Equation 3 might hold, we need to start searching for a permutation which is jugglable and when doing so we move the column with entries $\bar{b}_0, c_0$ in matrix M and we do so by computing $c_0 - \bar{b}_0$ to see what beat would work with this corresponding column. However, this is a choice! We could just as well move the column with entries $\bar{b}_1, c_1$ or even any move any of the combinations $\bar{b}_0, c_1$ and $\bar{b}_1, c_0$ by making a similar computation where we subtract the b value from the c value and evaluating modulo p. This approach would be fine since Equation 2 would still hold. In fact, every time we need to change a column (or set of two entries b, c) we could choose a different one than the previous time and the algorithm would still work fine. This is up to the point that the algorithm does need to stop and it is very hard to ensure this when choosing a random column every time. We know that sticking to the same choice every time gets us to the finish line, as this is the algorithm we proved, so one thing we can say is that we can choose alternating and if we find ourselves in a loop, we can start choosing the same column to make it stop. We could also vary in what way we alternate the columns of course. All of this is with the end goal to find as many different permutations as we can with this algorithm and requires further playing around with examples and research.

Within the same proof of Lemma 5.5 we find ourselves with a problem. This is not a problem which jeopardizes the correctness of the proof, but more so how nice we find it. A very important point in this proof is that we find a convincing answer as to why the algorithm we introduce actually terminates. We base this around the fact that no throwing moment is visited or computed more than once, implying we are done in finitely many steps. The proof of this fact is algebraically correct, but lacks intuition as to what is happening both in terms of the numbers as well as in real life, which is a bit of a shame, because the subject as a whole is so tangible. It has to do with our consistent choice of what column we swap out, and certain entries being fixed in the matrix as we keep swapping these columns (we are talking about $a_0, a_1, \bar{b}_1$). This also contributes to the difficulty of changing the algorithm to find more permutations as changing different columns every step messes with these fixed inputs. One way of proving finiteness is having some measure which outputs a number representing how far a pattern is from being jugglable and outputs 0 when it is jugglable. The hope would be that we find the output of this measure is monotonically decreasing while keeping track of this measure as we swap out columns, meaning we are getting closer to being jugglable every step. This would prove finiteness as we know our algorithm terminates when we find something jugglable. The point here is that we would like to get better intuition for why this process terminates and it has proven to be hard to gain this.

8.2 Magic sequence construction

In Theorem 6.4 we introduced a way to generate magic sequences, which is neat to have, but also noted that it does not perform very well as the period gets larger. As mentioned, in the best case when p is prime we find $p-2$ sequences with our algorithm, but we already have 19 magic sequences in total for $p = 7$ as listed in Figure 6.1. The algorithm we use is based around a number theoretical fact that the function

$$\begin{aligned} f :& \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \\ & i \mapsto iq \end{aligned}$$

with q such that $gcd(p, q) = 1$ is a permutation of $\mathbb{Z}/p\mathbb{Z}$. We applied this twice, since we require q to be coprime with p to find a permutation as a sequence, because it had to be magic, and then we also require $q-1$ to be coprime with p to find a permutation when applying the Permutation Test (Theorem 4.3). It would be nice to look for other functions with a similar permutation property, which could be applied in a similar fashion. We could start looking be exploring what magic sequences we do generate and which we are missing. Maybe we can find a pattern in a subset of the missing ones, which points to a similar function as the one we already used or some other method of constructing them.

8.3 finding transitions with flattening algorithm

We spoke about using the flattening algorithm to find transitions in Section 7.3.3 and this approach works very well in practice, but we found that it is hard to make explicit how to go about using it. We found that site swapping high numbers in the front with low numbers in the back is generally a good approach and enough to make this algorithm very usable in real life. We explicitly stated what the best choice for making site swaps is, but noted that we also need to take into account how we shift the pattern as this influences not only how many transition throws we make, but also what our options are for the next site swap. Therefore, optimising for just one of the two, site swaps or shifts, may not be ideal for the total outcome as the two influence each other and it is hard to see how they exactly influence each other. It would be very interesting to study how these two interact some more, in order to make this algorithm more detailed and be able to exactly say how to go about using it, rather than having the approach be a touch vague.

8.4 A lot more to be said about prime patterns

We only touched the tip of the iceberg on prime patterns and introduced a few concepts which can be useful to try and find maximal prime patterns. Even though a lot can be done in this area, this thesis did not swing in that direction. What we want to mention here is a question, which came up in the very beginning of this exploration, but never really got studied properly. Since state diagrams are essentially graphs we can endow them with a CW-structure and start thinking about computing fundamental groups. This becomes particularly interesting when we think about prime patterns, as every pattern is composed of prime patterns. Because of this, the question is raised whether prime patterns in a state diagram, correspond to generators of the corresponding fundamental group. We let go of this idea early on, mostly because there is enough interesting mathematics in the "basic" concepts, but also because in the normal study of graphs as CW-complexes, we are dealing with undirected graphs and that is not the case. In particular, inverses become a problem as there is not always a direct arrow back to the node we came from, so we would need a weaker notion of a groupoid to begin with.

9 Of interest to the juggler

This section is intended for the juggler who might wonder if there is anything useful he or she might be able to take away from this to apply to their juggling, but, understandably, does not want to read the whole thing.

9.1 The basics

We start with the basics which are finding out if an arbitrary sequence is jugglable and how many balls you might need. The first can be done with the Permutation Test (Theorem 4.3) and the second by computing the average as stated in Theorem 4.15. However, knowing these two might not clarify how we should actually juggle this, because we never really went into the practicality of throwing a number. We explain this using the pattern 423 as an example.

Example 9.1. So we know this is jugglable by applying the Permutation test.

$$(423) + (012) = (435) \quad \text{and} \quad r_3((435)) = (102)$$

and we know that it is a 3 ball sequence by computing the average. To understand how we should actually throw these balls, we assume we are juggling with two hands and that the order we use them in is alternating. Because of this we see that making a throw of an even height will land in the same hand. This is because the height of a throw indicates how many beats pass before this ball gets thrown again and since we assign our hands to every beat in an alternating fashion, after an even number of beats pass we throw with the same hand. Similarly, making an odd height throw implies this ball gets thrown next by our other hand. The most straight forward way to make these two things happen is by throwing even numbers straight up and odd numbers crossing.

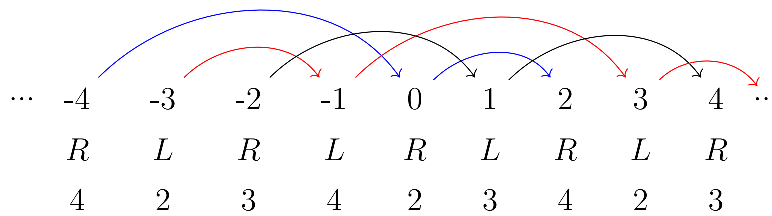

Figure 19: Schematic representation of 423

Now, how high this should be is not set in stone, as the only thing that matters is that the relative heights are correct. What we mean by this is that, after throwing a ball at a height of, for example, 4, there are 3 more beats at which we should make throws before we throw this ball again. One way of figuring out for yourself how high this 4 should then be, is by taking one ball in your right hand, throwing it straight up and then snapping your fingers in the order of first your left hand, then your right hand and finally your left hand again in a constant rhythm, where the ball should land back in your right hand at the moment you would expect to make the next finger snap. You can then vary the height of your 4 throw by varying how fast you snap your fingers.

9.2 Changing, constructing and decomposing sequences

Once we have a juggling sequence, we can adapt it in various ways to create new sequences, some of which with the same number of balls and some with more or less balls. The first operation we introduced were site swaps in Theorem 4.7 where we pick two throw in the sequence and maintain where these balls are landing, but swap the moments at which they are thrown, which changes their heights. Applying this does not change the number of balls we juggle. Two operations which do change the number of balls we juggle are presented in Theorems 5.1 and 5.2 where we add either a carefully chosen value to every number in the sequence or some multiple of the period to any number in the sequence.

We have seen in Section 6.2 that a juggling sequence decomposes in its orbits, where each orbit is a pattern in itself, but with fewer balls than the original pattern. Since orbits appear within the pattern they generally give a good feel for what the final pattern will be, so a good practice strategy when learning a pattern could be to find the orbits of a pattern and practice these first. See Example 6.10.

9.3 Transitions

Finding transitions between patterns is found notoriously difficult among jugglers and for good reason. Say we are juggling patterns with b balls and of maximal height h, we can find all of the possible transitions by constructing the corresponding state diagram as discussed in Section 7. Then we find the loops corresponding to the tricks we want to move between and pick our favourite way to go from the one to the other. This is far superior to any other approach since it gives us every transition we might want, but this overload of information comes with a price, which is that it takes a lot of time to make a state diagram. As this is a chapter on a practical take away for a juggler, we would not advice this. Favourably, we would focus on just two patterns and find a transition between these. We have shown two ways of doing this, one of which uses the flattening algorithm, as described in Section 7.3.3, and the other finding transitions of minimal length, as described in Section 7.3.4. Both of these approaches give us a transition from the basic pattern to some other given pattern, which seems like a bit of a restriction as we might want to transition between two non-basic patterns. However, say we have two of these non-basic patterns, we can then use the flattening approach to find transitions from the basic pattern to the respective patterns and then reverse one of these quite easily. If we proceed to concatenate this with the other transition, we find a total transition between the two patterns. We illustrate this with an example.

Example 9.2. Say we want to go from the 5 ball pattern $a14$ to the pattern 771. We apply our flattening algorithm approach to both patterns below.

$$a14 \xrightarrow{swap \quad a\sim 4} 618 \xrightarrow{shift} 861 \xrightarrow{swap \quad 8\sim 1} 366 \xrightarrow{shift} 663 \xrightarrow{swap \quad 6\sim 3} 564 \xrightarrow{swap \quad 6\sim 4} 555$$

Working backwards and looking at only the shifts we can work out what throws we need to make in order to go from the basic pattern to $a14$.

$$a14 \leftarrow 618 \xleftarrow[throw \quad 8]{} 861 \leftarrow 366 \xleftarrow[throw \quad 66]{} 663 \leftarrow 564 \leftarrow 555$$

So the transition from the basic pattern to $a14$ is 668. However, if we work in the other direction and keep track of just the shifts as well we get exactly the throws we need to go to the basic pattern.

$$a14 \to 618 \xrightarrow[\textit{throw} \quad 61]{} 861 \to 366 \xrightarrow[\textit{throw} \quad 3]{} 663 \to 564 \to 555$$

So the transition from $a14$ to the basic pattern is 613. Doing the same for 771 gives the following diagrams.

$$771 \xrightarrow{\textit{swap} \quad 7{\sim}1} 375 \xrightarrow{\textit{shift}} 753 \xrightarrow{\textit{swap} \quad 7{\sim}3} 555$$

Working back and keeping track of shifts only shows that the transition from the basic pattern to 771 is 75, which we can site swap to get 66 as a transition of minimal height.

$$771 \leftarrow 375 \xleftarrow[\textit{throw} \quad 75]{} 753 \leftarrow 555$$

Working in the other direction shows that the transition from 771 to the basic pattern is a 3.

$$771 \to 375 \xrightarrow[\textit{throw} \quad 3]{} 753 \to 555$$

Having all this information we can now make the following diagram

$$\begin{tikzcd}
 & a14 \arrow[d, "613"'] \\
771 \arrow[r, "3"] & 555 \arrow[u, bend right, "668"'] \arrow[l, bend left, "66"]
\end{tikzcd}$$

Figure 20: Transitioning from $a14$ to 771

Reading Figure 20 works straightforward, where the numbers at an arrow assign what throws we need to make to get from where the arrow starts to where it ends. We see that a way to go from $a14$ to 771 is through the basic pattern 555 by throwing 61366. When going from 771 to $a14$ we see that there might be some redundancies in this as depicted in Figure 21.

$$\begin{tikzcd}
& a14 \arrow[d, "613"'] \\
771 \arrow[ur, bend left, "8"] \arrow[r, "3"] & 555 \arrow[l, bend left, "66"] \arrow[u, bend right, "668"']
\end{tikzcd}$$

Figure 21: Transitioning from 771 to $a14$

Going through the basic pattern we would make the throws 3668 to get from 771 to $a14$. However, we note that, from 555 it takes the throws 668 to get to $a14$ and since we are already doing 771 at this point, we have done the throws 66 already from the perspective of 555. A more sophisticated way of stating the same is that we find ourselves in the same state when doing cycles of 771, as we would after throwing 66 from the basic pattern. This implies that we can "finish" the transition we would make from 555, by simply throwing one more 8 from the point of doing cycles of 771.

We note that it is not clear what the best way is to make site swaps and shifts, as we get redundant sub sequences in our transition if we simply follow the flattening algorithm, as described in Example 7.7. As mentioned, it is best to site swap large numbers in the front with small numbers in the back as a general approach. Then, when we have a transition we can apply more site swaps to filter out potential redundancies we created, as shown in Example 9.2.

The flattening approach feels more practical, but also more fiddly, than the minimality approach and becomes a lot faster over time as the juggler gets more experienced with transition throws. The minimality approach, being a bit less practical, is superior to the flattening approach as it always finds us a transition of minimal length and even of minimal throw height. The first is a favourable condition for a transition as we don't want to waist time on making lots of transition throws if we already be in the pattern we want to transition to. This minimal throw height turns out to be less then or equal to the maximal throw height of the pattern we wish to transition to.

At some point we might want to go back to the base pattern, or whatever pattern we came from. As mentioned above and in Example 9.2, we can use the flattening algorithm for this. A fun remark is that, if we transition from the pattern $(a_i)_{i=0}^{p-1}$ to $(b_i)_{i=0}^{p-1}$ and then back into $(a_i)_{i=0}^{p-1}$, we can concatenate these transitions to find a new pattern, as noted in Section 7.3.2. We can see this in Example 9.2 where we transition from 555 into 771 by throwing 66 and 3 to get out of it, and indeed, 663 is a 5 ball pattern in itself. Similarly, we throw 668 to get to $a14$ and 613 to get back to 555, and indeed, 668613 is a 5 ball pattern. Besides this begin a cool feature, it can also be useful if we know how to transition into a pattern from, say, the basic pattern, but not yet how to go back. If we happen to recognise the transition as the start of some pattern $(c_i)_{i=0}^{p-1}$ which is based at, in this case, ground state, we know that whatever we did not throw yet of $(c_i)_{i=0}^{p-1}$ can be used to get back to the basic pattern. The pattern 663 is quite common, so knowing this and knowing that 66 is the way to get to 771 from 555, allows us to already know that a 3 will do to get back to 555. If we prefer another 5 ball ground state pattern which starts with 66, like for example 66661 or 66751, we can just as well go back to 555 from 771 by throwing either 661 or 751.

10 Some final remarks

10.1 The impact on juggling in general and myself

Firstly, this theory has boosted the overall technical juggling level immensely. People started to realise and discover these crazy patterns on paper, which were never done before, which must be very different from how new patterns were discovered before this era of mathematical juggling. Besides people discovering these new things, there was now also a very concise and practical way of sharing the patterns. No more vague descriptions of how high to throw each ball and in what order. Sure, there are still plenty of patterns we cannot describe with this theory, which involve throwing balls behind the back or from some other on orthodox place around the body, but we are a lot better at it now then we were before!

In a way, the introduction of this theory made juggling patterns more open source. Where there was no way to describe a complicated pattern before, a juggler had to go into conversation with someone, ask for explanation, or study the pattern in great detail and translate their findings into something they could throw themselves. However, nowadays people get so comfortable with these numbers and recognising patterns based on these numbers, that it becomes possible to watch someone juggling a pattern and learn it based of this alone. If you can work out the sequence from looking at it, it is much easier to learn. You could decompose it into orbits are think of other practice patterns. Some jugglers even take this notion as far as seeing a pattern, working out what the numbers are and not needing any practice at all to juggle the pattern. They just throw the numbers and do it! This comes from years of experience with juggling these sequences and working out how high balls should be thrown and long they should be held in your hands when throwing certain combinations of numbers in a row. As discussed in Section 7.3.2, these things vary and are exactly what makes throwing balls at different heights very difficult.

From personal experience, I can say that, once I got more acquainted with this notation and mathematics behind juggling, I started to appreciate very different things. Things which may seem dull for someone who does not know this theory, became very interesting and pretty in their own way. Certain sequences of numbers have interesting properties or just look pretty on paper, where admittedly, it takes a mathematician to actually find a sequence of numbers pretty. An example would be a particular case of a magic sequence, namely 123456789, which is a 5 ball sequence. Everyone sees the ascending pattern in here and it turns out that this pattern is delightfully beautiful when thrown in the air as well. However, the sequence 13579 of ascending odd numbers, which is also a 5 ball juggling sequence, is hard to distinguish as anything when thrown and many would say, is not very pretty. So these aspects of when a sequence is beautiful, seems to not always translate back and forward between the sequences on paper and in the air, although it a very personal subject of course. You can imagine that this changes very much what kind of a performance a juggler might give, because it now depends greatly on his or her audience. Do they know this theory or not and can they appreciate this beautiful sequence I am about to do or will they not tell the difference between this and the basic pattern? Besides this appreciation of what is pretty, I also got a lot better at juggling in general. Because holding times and throwing times are so important here, you get to try and do all kinds of different combinations of these, which is just great for your general juggling.

10.2 About the title

If you were unfamiliar with the theory we discussed, the title cannot really make sense, and if you were familiar with the theory, it still probably does not make much sense. I quickly want explain it here. Recall that we assign letters to double digit throwing heights to avoid confusion, i.e. is 201 a 3 period pattern with throws 2, 0 and 1 or maybe a 2 period pattern with throws 20 and 1? To this end, we always use single digit throwing heights and define $a = 10$, $b = 11$ etc. In this sense, certain words become juggling sequences with a ridiculous amount of balls. We encourage the reader to check that the sequences "the", "proof", "of", "every" and "theorem" and indeed jugglable.

11 Acknowledgements

There is a handful of people without whom this thesis wouldn't be what it is now or simply without whom this thesis wouldn't be. I very much appreciate and enjoyed being able to write on and think about my hobby, so I would like to thank Gil Cavalcanti for taking a leap of faith in supervising me on this unusual topic. There is a lot to thank Wouter van Doorn for. I started taking an interest in this side of juggling because of him and he pushed me to try mathematics, which turned out to be a wonderful choice. His continued support through writing this in terms of proofreading pieces, thinking about proofs with me and helping me many times when I could not figure it out myself has been irreplaceable. Most of all, I want to thank Wouter for being such a good friend and his continued support over the past few years. In a similar fashion, I want to thank Anneroos Everts for her proof of Conjecture 7.9. This thesis was written during the Corona pandemic, making for a less then ideal situation. I want to thank Hanna van der Stap and Piet van Steen for housing me during this period, as being locked up in a tiny room would not have been beneficial for writing. Thank you for enduring my, often present, grumpy moods when the mathematics didn't quite work out and giving me such a "gezellige" atmosphere to live and work in. Finally, as this thesis marks the end of my bachelor, I want to thank Timo Franssen for making my bachelor so much fun and all the great working sessions we had together. I learned a lot in terms of work ethic and mathematics from you and I honestly don't think I would have made it to the end without such a good friend to work with.

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12 Appendix

Here are links to an online repository where the two codes can be found and used as mentioned in Sections 5.3 and 5.2.

Code of the converse average theorem

Code for generating all patterns with b balls and period p