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Master thesis

Seiberg-Witten theory for symplectic manifolds

Author:
R.L. Klaasse

Supervisor:
Dr. G.R. Cavalcanti

Second examiner:
Prof. dr. M.N. Crainic

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Chapter 1

Introduction

This thesis is on the classification of four-dimensional smooth manifolds. More specifically, we give an introduction to the gauge theory used to study them called Seiberg-Witten theory. The aim of this chapter is to provide some background to the theory and give an outline of how it works, which will simultaneously provide an outline of the contents of this thesis. The guiding result which we want to prove is the following due to Taubes [31].

Theorem. *Let (X, ω) be a compact symplectic four-manifold with $b_2^+(X) > 1$. Then X does not admit a Riemannian metric of positive scalar curvature. Furthermore, it does not admit a connected sum decomposition $X \cong X_1 \# X_2$ with $b_2^+(X_i) > 0$ for $i = 1, 2$.*

This theorem will follow naturally from our development of Seiberg-Witten theory and is proven in this thesis as Theorem 5.6.7.

History

Before the introduction of gauge theory into the subject, the classification of four-dimensional manifolds up to diffeomorphism was proving troublesome. Indeed, the usual techniques for arbitrary n-dimensional manifolds could not capture their intricacies, due to the fact that these mostly dealt with invariants of the manifold preserved by homeomorphisms. This mostly suffices for other dimensions, but in dimension four there is a great distinction between the homeomorphism and diffeomorphism type of a given smooth four-manifold. A very nice introduction to the study of four-manifolds can be found in [12].

In 1983, Donaldson [3] wrote a paper using techniques from gauge theory to construct a new invariant for four-manifolds, later dubbed Donaldson invariants. These invariants were constructed by counting the amount of anti-self-dual (ASD) connections on a given SU(2)-bundle over the four-manifold. Solutions to these ASD equations are called instantons. They are invariant under bundle automorphisms called gauge transformations, and hence it is more natural to consider instead the quotient space called the moduli space of instantons. While the success of Donaldson theory can not be understated, the moduli space of instantons is generally not well-behaved. In particular, it is almost always not compact, and a great deal of work is required to find a suitable compactification necessary to define the Donaldson invariants. Standard references to learn more about Donaldson theory are [5, 8].

Then, in 1988, Witten [33] showed that the Donaldson invariants could be computed using what is called $N = 2$ supersymmetric Yang-Mills theory, a type of topological quantum field theory. Using this realization together with Seiberg in [28] he considered the dual to this theory, which turns out to give rise to much nicer spaces out of which one can construct invariants. Indeed, the associated Seiberg-Witten equations first written down in 1994 [34] have a moduli space which for example is always compact. After the introduction of Seiberg-Witten theory many outstanding problems on four-manifolds were quickly solved due to its similarity to Donaldson theory. In particular, Seiberg-Witten theory obtains the results from Donaldson theory in a much simpler manner. Mathematicians which were chiefly involved with the early development of Seiberg-Witten theory include Kronheimer, Mrowka, Morgan, Stern and Taubes, as can be read in [4].

Witten conjectured in [34] that the Donaldson and Seiberg-Witten invariants contain the same amount of information. More precisely, he proposed a formula expressing the Donaldson invariants in terms of the Seiberg-Witten invariants. More about this can also be found in [4]. Lastly we mention work by Leness and Feehan [6, 7] in this direction, following a program set out by Pidstrigach.

General strategy

The general strategy of Seiberg-Witten gauge theory is as follows. The reader familiar with Donaldson gauge theory should see many similarities, both in the strategy described here as in the techniques used to prove things. Throughout, X will be a compact oriented four-manifold. We wish to study the orientation preserving diffeomorphism type of X. In order to do this, we wish to find suitable vector bundles $E_1, E_2 \to X$ together with a differential operator $F : \Gamma(E_1) \to \Gamma(E_2)$. We then consider the equation

$$Fs = 0, \qquad s \in \Gamma(E_1),$$

called the Seiberg-Witten equations, and study its solutions. The differential operator will depend on some auxiliary data. More specifically, $F = F(g, \mathfrak{c})$, with g a Riemannian metric on X and $\mathfrak{c} \in \mathcal{S}^c(X)$ a so-called Spin^c-structure. Sometimes we will also need to consider perturbations of F, which we will then denote by F_η for a given perturbation parameter η.

Given a Spin^c-structure $\mathfrak{c}$, the Seiberg-Witten equations will be expressed in terms of variables taken from a configuration space $\mathcal{C}(\mathfrak{c})$. The solutions to $Fs = 0$ will be invariant under a group of automorphisms of the bundle associated to the configuration space, also called the gauge group, $\mathcal{G}$. Instead of considering the space $\mathcal{Z}(\mathfrak{c})$ of all solutions inside the configuration space, we will study the moduli space

$$M(\mathfrak{c}) = \mathcal{Z}(\mathfrak{c})/\mathcal{G},$$

which is a subset of the larger quotient $\mathcal{B}(\mathfrak{c}) = \mathcal{C}(\mathfrak{c})/\mathcal{G}$ of configurations modulo the action of the gauge group. If the gauge group acts freely, $M(\mathfrak{c})$ will be a smooth manifold. There will in general be points in the configuration space where the gauge group does not act freely. These are called reducible points; all other points are called irreducible. We must arrange our data such that none of these reducible points are solutions to $Fs = 0$. The operator F can be shown to be elliptic, so that away from reducible solutions we can linearize F to get a Fredholm map T. For a generic choice of η, $M(\mathfrak{c})$ will then be a smooth finite-dimensional manifold by the Sard-Smale theorem, as long as $b_2^+(X) > 0$. Its dimension is just the index of T, which we can compute using the Atiyah-Singer index theorem.

To arrive at the Seiberg-Witten invariants we wish to integrate certain canonical cohomology classes over the moduli space. To ensure this process does not depend on the metric g nor the perturbation η, we wish to show that any $M(g_0, \eta_0)$ and $M(g_1, \eta_1)$ are cobordant inside a larger space, and for that we use the topological condition $b_2^+(X) > 1$. When one only has $b_+^2(X) = 1$ one can still proceed, but things will depend on the class of Riemannian metric one picked. For this integration process to make sense, we need to establish the following wish list of properties of $M(\mathfrak{c})$.

Wish list. Let X be a compact oriented four-manifold satisfying $b_2^+(X) > 1$ equipped with a Riemannian metric g and a Spin^c-structure $\mathfrak{c}$. For the Seiberg-Witten moduli space $M(\mathfrak{c})$, establish the following.

- Have a formula for its formal dimension $d(M(\mathfrak{c}))$. This was mentioned earlier, and is done by an index calculation;
- Show compactness of M, which is automatic in Seiberg-Witten theory coming from an a priori estimate.
- Show orientability of M, which is done by considering the determinant index bundle $\det \operatorname{ind}(T)$;
- Ensure that M does not contain reducible points using transversality. Essentially, one can perturb F to F_η so that there are no reducible solutions, for generic choices of η;
- Find suitable cohomology classes to integrate. This will turn out to be rather straightforward. In fact, in most cases the moduli space M will be zero-dimensional, so that the integration process boils down to a count of points.

Once we have all the ingredients on the wish list, we will obtain a map called the Seiberg-Witten invariant SW_X, i.e.

$$\mathrm{SW}_X : \mathcal{S}^c(X) \to \mathbb{Z}.$$

For a given manifold X, there are only finitely many Spin^c-structures giving a nonzero Seiberg-Witten invariant. It turns out that manifolds with an almost complex structure have a canonical choice of Spin^c-structure. We show that given a compact symplectic manifold (X, ω) with a compatible almost complex structure, this Spin^c-structure will have Seiberg-Witten invariant equal to 1 mod 2. On the other hand, it follows from the Weitzenböck formula that manifolds admitting a Riemannian metric with positive scalar curvature have vanishing Seiberg-Witten invariants, as all solutions must be both reducible and irreducible. Similarly, a manifold admitting a connected sum decomposition as in the theorem above will have vanishing Seiberg-Witten invariants as well. From this it is clear that the theorem follows.

This thesis is organized as follows. In Chapter 2 we discuss some of the background required to formulate the Seiberg-Witten equations. The equations will take place on the space of sections of a vector bundle which is most readily obtained through the associated vector bundle construction out of a principal G-bundle for some Lie group G. We recall some of the theory on principal G-bundles and describe the notion of a connection. We then continue by recalling some theory on almost complex and symplectic manifolds. We finish off with some generalities from functional analysis which will be important for us, namely the theory of elliptic operators, Sobolev spaces and Hilbert manifolds.

In Chapter 3 we discuss some notions from spin geometry. We describe the Lie group G called $\mathrm{Spin}^c(4)$ and make explicit how to then obtain the aforementioned vector bundle on X called the spinor bundle $S(X)$. The spinor bundle admits an action of T^*X called Clifford multiplication, and through it we can define an important class of differential operators on spinor bundles called Dirac operators $D_A : \Gamma(S) \to \Gamma(S)$. Afterwards we discuss these concepts in the presence of an almost complex or symplectic structure and relate the Dirac operator to the Cauchy-Riemann operator in this setting.

In Chapter 4 we are then able to state the Seiberg-Witten equations and describe the group of automorphisms leaving it invariant. In order to study the moduli space M of solutions to the Seiberg-Witten equations, we equip the spaces involved with Sobolev norms making them into Hilbert manifolds. After we have done this, we follow the wish list of properties of M and proceed to establish that the moduli space indeed satisfies them.

Finally, in Chapter 5 we define the Seiberg-Witten invariant SW_X and proceed to study its properties. In particular, we show that it is independent of the choice of Riemannian metric g and perturbation η we made to construct it. Furthermore, we show the aforementioned vanishing theorem in the case of admitting a metric of positive scalar curvature and outline the proof in the case of a connected sum decomposition. Lastly, we describe the proof by Taubes of the non-vanishing of the Seiberg-Witten invariants in the case of symplectic manifolds.

In describing Seiberg-Witten theory as outlined above we have of course been forced to leave out many interesting aspects and results due to the space constraints of this thesis. We would like to stress that none of the results in this thesis are new, yet hope that it provides a readable introduction to the subject. Since its full inception in 1994 there have appeared several texts on the subject, each differing either in approach or scope. These include [1, 4, 15, 19, 22, 23, 26, 27]. Our main references are [23] and [27]. The former gives a practical introduction to Seiberg-Witten theory, while the latter is more thorough in its explanations. Lastly we mention [26], which goes to great length to discuss the analytical aspects the theory.

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Chapter 2

Preliminaries

This chapter is devoted to covering most of the background necessary for Seiberg-Witten theory. As was mentioned in the introduction, we develop in this chapter some theory on principal G-bundles and connections thereon, some on almost complex and symplectic manifolds and some tools from functional analysis. We will be brief with regard to proofs and will instead give a general reference for each topic.

2.1 Principal G-bundles

In this section we discuss theory of principal bundles insofar as it is used in this thesis. A general reference for this section is [16]. We will see that the Seiberg-Witten equations take place in the space of sections of a certain vector bundle over the manifold. The construction of this vector bundle is most easily described using the language of principal bundles. We go over the basic definitions of principal bundles and show how to create vector bundles out of them using the associated vector bundle construction. Throughout, let G be a Lie group.

Definition 2.1.1. A *smooth free right action* of G on a manifold P is a smooth map $P \times G \to P$ written as $(p, g) \mapsto pg$ such that

- $(pg_1)g_2 = p(g_1g_2)$ for all $p \in P$, $g_1, g_2 \in G$ (*associativity*);
- For all $p \in P$, $pg = p$ if and only if $g = e$, the identity element of G (*freeness*).

It follows readily from this that for any fixed $g \in G$, right multiplication by g, i.e. the map $(p, g) \mapsto pg$, $P \times \{g\} \to P$ is a diffeomorphism.

Definition 2.1.2. A *principal G-bundle* P over B is a smooth submersion $\pi : P \to B$ onto a manifold B along with a smooth free right action $P \times G \to P$ such that

- As a set we have $P/G = B$;
- For any $b \in B$ there exists a neighbourhood $U \subseteq B$ of b and a diffeomorphism $\Phi_U : \pi^{-1}(U) \to U \times G$ such that $\mathrm{pr}_1 \circ \Phi_U = \pi|_{\pi^{-1}(U)}$, or in other words that the following

diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\Phi_U\quad} & U \times G \\ & {\scriptstyle \pi}\searrow \qquad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array} \quad , \tag{2.1.1}$$

and Φ_U commutes with the action of G, i.e. for each $h \in G$ we have $\Phi_U(ph) = (b, gh)$, where $(b, g) = \Phi_U(p)$ and $\pi(p) = b \in U$.

Remark 2.1.3. We see that a principal G-bundle is a fiber bundle along with a right G-action compatible with its local trivializations. In particular, the group action preserves the fibers and acts transitively. The space B is also called the *base space*, with P being called the *total space*.

Given two principal G-bundles $\pi : P \to B$ and $\pi' : P' \to B'$, a *bundle map* is an equivariant map $\sigma : P \to P'$, i.e. a map which commutes with the actions of G on P and P'. If $B = B'$, we call such a map σ a *morphism* of principal G-bundles. If further $P = P'$, we call such a map σ an *automorphism* of P. We denote the collection of all principal G-bundles over B by $\mathcal{P}_G B$, and the set of bundle automorphisms of P by $\mathrm{Aut}(P)$, which is a group under composition. A principal G-bundle is said to be *trivial* if it is isomorphic to the product principal bundle $B \times G \to B$. Of course, by definition every principal bundle is locally trivial. It is in fact quite a strong condition for a local product (P, π) over B with fiber G to be a principal G-bundle. In fact, we have

Lemma 2.1.4. *Any morphism of principal G-bundles is an isomorphism.*

Proof. Let $\sigma : P \to Q$ be such a morphism between principal bundles P, Q over B. Assume that $P = Q = B \times G$, the product principal bundle. Then $\sigma(b, g) = (b, f(b)g)$ for some continuous function $f : B \to G$. But then σ is an isomorphism with inverse $\sigma^{-1}(b, g) = (b, f(b)^{-1}g)$. As every principal bundle is locally trivial, the general case immediately follows from this. $\square$

Definition 2.1.5. A *local section* of a principal bundle P is a smooth map $s : U \to P$ defined on a neighbourhood $U \subset B$ such that $\pi \circ s = \mathrm{Id}_U$. A *section* is a local section defined on $U = B$.

To see the distinction with vector bundles more clearly, note that we have

Corollary 2.1.6. *A principal G-bundle $\pi : P \to B$ is trivial if and only if it admits a section.*

Proof. If P is trivial then it has a section. Indeed, any local trivialization $\Phi_U : \pi^{-1}(U) \to U \times G$ gives rise to a local section $s(x) = \Phi^{-1}(x, e)$. Conversely, if $s : B \to P$ is a section, then the map $\phi : B \times G \to P$ given by $\phi(b, g) = s(b)g$ is a morphism of principal bundles, so that by Lemma 2.1.4 it is an isomorphism. $\square$

As with vector bundles, we can consider a purely local description of a principal G-bundle in terms of transition functions. Consider a pair of overlapping trivializing neighbourhoods $U_\alpha, U_\beta \subseteq B$. Then we have

$$\Phi_\beta \circ \Phi_\alpha^{-1}(b, g) = (b, \psi_{\beta\alpha}(b, g)), \tag{2.1.2}$$

for some map $\psi_{\beta\alpha} : B \times G \to G$. For each fixed $b \in U_\alpha \cap U_\beta$, the map $g \mapsto \psi_{\beta\alpha}(b, g)$ is just a map $G \to G$. By associativity of the action we then see that for each such b we must have

$$\psi_{\beta\alpha}(b, g)h = \psi_{\beta\alpha}(b, gh) \qquad \text{for all } g, h \in G. \tag{2.1.3}$$

Hence if we now take $g = e$ to be the unit element, we see that the map $\psi_{\beta\alpha}(b, \cdot)$ is just left multiplication by $\psi_{\beta\alpha}(b) = \psi_{\beta\alpha}(b, e) \in G$.

Definition 2.1.7. Given two local trivializations $U_\alpha, U_\beta \subseteq B$, the map $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ is called the *transition function* associated to U_α, U_β.

We see that principal G-bundles have transition functions just like vector bundles do. In particular, these transition functions satisfy the same cocycle conditions.

Lemma 2.1.8. *Given local trivializations $U_\alpha, U_\beta, U_\gamma$, the associated transition functions satisfy*

$$\psi_{\alpha\alpha} = \mathrm{Id}_G, \qquad \psi_{\alpha\beta}\psi_{\beta\alpha} = \mathrm{Id}_G, \qquad \psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha} = \mathrm{Id}_G, \tag{2.1.4}$$

whenever the above equations make sense.

In fact, as with vector bundles, all of the information of a principal bundle is encoded in these transition functions, as can be seen through the following Steenrod construction. Let $\{U_\alpha\}$ be an open cover of B by trivializing neighbourhoods together with a system of maps $\{\psi_{\beta\alpha}\}$ satisfying the cocycle conditions. Consider now for each $U_\alpha \subset B$ the product $U_\alpha \times G$, and define a relation $\sim$ between elements $(b, g) \in U_\alpha \times G$ and $(b', g') \in U_\beta \times G$ by setting

$$(b, g) \sim (b', g') \qquad \text{if and only if} \qquad b = b' \text{ and } g' = \psi_{\beta\alpha}(b)g. \tag{2.1.5}$$

It is then easy to check using the cocycle conditions that this is an equivalence relation. Now consider the quotient

$$P = \bigsqcup_\alpha (U_\alpha \times G) \,/ \sim, \tag{2.1.6}$$

consisting of the disjoint union of all $U_\alpha \times G$ glued together by the equivalence relation. We then have

Theorem 2.1.9. *The quotient P as defined by equation (2.1.6) is a principal G-bundle.*

In particular, we see that given a vector bundle E over B, we can take its transition functions and construct a principal G-bundle P over B. A special case of this is when one considers a manifold's tangent bundle and out of it creates the principal $\mathrm{GL}(n, \mathbb{R})$-bundle called the *frame bundle*. The construction can in fact be reversed to obtain from a principal G-bundle P over B a vector bundle E over B.

Definition 2.1.10. Let $\pi : P \to B$ be a principal G-bundle and let F be a space with a left G-action, i.e. for each $g \in G$ we have a map $T_g : F \to F$ satisfying $T_g \circ T_{g'} = T_{gg'}$ for all $g, g'G$. Then the *associated bundle* is the bundle

$$P \times_G F \to B, \tag{2.1.7}$$

constructed as follows: the total space is $P \times_G F = (P \times F)/\sim$, with the equivalence relation $\sim$ being given by the action of G on both factors, i.e.

$$(p, f) \sim (p', f') \qquad \text{if and only if } p' = pg \text{ and } f' = gf \text{ for some } g \in G. \tag{2.1.8}$$

The projection map for this bundle sends the class $[(p, f)] \in P \times_G F$ of $(p, f) \in P \times F$ to $\pi(p)$.

Remark 2.1.11. In general, the associated bundle is not a principal bundle. Moreover, the vector bundle associated to the frame bundle of B via the fundamental representation of $\mathrm{GL}(n; \mathbb{R})$ is isomorphic to the tangent bundle TB of B.

There are two important examples of associated bundles.

Example 2.1.12. Consider $F = G$ with the adjoint action of G on itself by conjugation, i.e. for given fixed $g \in G$ we have $T_g f = gfg^{-1}$. From this we get the adjoint bundle $\mathrm{Ad}(P) = P \times_G G$.

Example 2.1.13. Consider $F = V$ a vector space with a linear representation of G, i.e. a homomorphism $\rho : G \to \mathrm{End}(V)$. From this we get an associated vector bundle $P \times_\rho V$. In particular, taking ρ to be the adjoint representation ad of G on its Lie algebra $\mathfrak{g}$, we get the associated vector bundle $\mathrm{ad}(P) = P \times_{\mathrm{ad}} \mathfrak{g}$.

We see from the above example that the notion of a principal bundle makes it simple to describe all the vector bundles one can create out of a given one. Examples of this are creating the cotangent bundle T^*B and its exterior powers $\bigwedge^\bullet T^*B$ out of the tangent bundle, or taking connected sums $E \oplus F$ or tensor products $E \otimes F$. Indeed, the reason these constructions worked for vector bundles is essentially encapsulated in the slogan "everything that works for vector spaces works for vector bundles". We now see that this can be described in terms of representations and the associated bundle construction.

There is one more construction that bears a special name. Note firstly that given two Lie groups G and H and a homomorphism $f : H \to G$, there is a natural left action of H on G given by $(h, g) \mapsto f(h)g$ for $g \in G, h \in H$.

Definition 2.1.14. Let P be a principal G-bundle and let a homomorphism $H \to G$ from another Lie group H be given. We say P admits a *reduction of the structure group* from G to H if there exists a principal H-bundle P_H such that through the associated bundle construction we have

$$P_H \times_H G \cong P. \tag{2.1.9}$$

This construction makes sense for any map $H \to G$, which in particular need not necessarily be an inclusion, despite the terminology used. In terms of the transition functions of the bundles, a G-bundle can be reduced if and only if its transition functions can be taken to have values in H. This is not always the case, and in general there are topological obstructions for this. Furthermore, even when it can be reduced, this reduction need not be unique.
Many structures on the tangent bundle of a manifold can be expressed in terms of reductions of the structure group. Recall that the tangent bundle is a priori a $\mathrm{GL}(n)$-bundle.

- The reduction to $O(n) < \mathrm{GL}(n)$ is the same as equipping it with a Riemannian metric. Note that because $O(n)$ is the maximal compact subgroup of $\mathrm{GL}(n)$, the inclusion $O(n) \hookrightarrow \mathrm{GL}(n)$ is a deformation retract, so that this reduction is always possible;
- The reduction $\mathrm{GL}^+(n) < \mathrm{GL}(n)$ is the same as choosing an orientation. We know that a manifold X is orientable if and only if its first Stiefel-Whitney class $w_1(X) = w_1(TX)$ vanishes, so that this reduction is not always possible. Furthermore, note that orientable manifolds admit exactly two orientations, or two different reductions of the structure group;
- The reduction $\mathrm{GL}(n, \mathbb{C}) < \mathrm{GL}(2n, \mathbb{R})$ is the same as choosing an almost complex structure. For this to exist certainly the manifold must then be orientable, and for four-manifolds X we in fact have the following criterion due to Wu [35].

  Theorem 2.1.15. *A four-manifold X admits an almost complex structure J if and only if there exists a $h \in H^2(X; \mathbb{Z})$ such that $h^2 = 3\sigma(X) + 2\chi(X)$ and $h \equiv w_2(X) \mod 2$, in which case $h = c_1(TX, J)$.*

Other examples include the existence of a volume form through the reduction $\mathrm{SL} < \mathrm{GL}$, which because $\mathrm{SL} \to \mathrm{GL}^+$ is again a deformation retract is possible if and only if the bundle is orientable. Lastly, the reduction $\mathrm{GL}(k) \times \mathrm{GL}(n-k) < \mathrm{GL}(n)$ is the same as expressing a vector bundle as a Whitney sum of two sub-bundles of rank k and $n-k$. We will see two other important examples of reduction of the structure group in the next section, when we discuss the $\mathrm{Spin}(n)$ and $\mathrm{Spin}^{\mathrm{c}}(n)$ groups.

Let B be a manifold and consider again the set $\mathcal{P}_G(B)$ of principal G-bundles over B. Something we have not yet mentioned is that given principal G-bundle $P \to B$ and a map $f : X \to B$ from some other space X we can consider the pullback bundle

$$f^*P \to X, \tag{2.1.10}$$

which is now a principal G-bundle over X. One can prove that homotopic maps give rise to isomorphic pullback bundles. In particular, any principal G-bundle over a contractible space is trivial. Now recall the notion of a *weakly contractible space*, i.e. a space with all homotopy groups vanishing. Given a space X, let $[X, B]$ denote the space of homotopy classes of maps $f : X \to B$. From the pullback construction there is the natural question if there is a space B such that every principal G-bundle can be obtained as the pullback bundle in this manner.

Definition 2.1.16. Let G be a Lie group. A *classifying space* for G is a space B together with a principal G-bundle $P \to B$ such that P is weakly contractible.

It is common practice to write BG for the classifying space of G, and using the notation EG for its universal principal G-bundle P. Note however that BG and EG are only well-defined up to homotopy equivalence. The following theorem answers the previous question in the affirmative.

Theorem 2.1.17. *Let G be a Lie group. Then there exists a classifying space for G. Moreover, for every CW-complex B, the map*

$$\varphi : [B, BG] \to \mathcal{P}_G(B), \qquad f \mapsto f^* EG \tag{2.1.11}$$

is a bijection.

The previous theorem explains why BG is called the classifying space.

Example 2.1.18. The classifying space for $G = \mathrm{U}(1)$ is given by $BG = \mathbb{C}P^\infty$, the infinite-dimensional projective plane.

Let $\pi : P \to B$ be a principal G-bundle. We defined earlier the group $\mathrm{Aut}(P)$ of bundle automorphisms of P. We can now prove the following.

Proposition 2.1.19. *The group $\mathrm{Aut}(P)$ can be identified with the space of smooth sections $\Gamma(\mathrm{Ad}(P))$ of the associated bundle $\mathrm{Ad}(P)$.*

Proof. Let $f : P \to P$ be an element of $\mathrm{Aut}(P)$. Then from this we can define a map $\psi : P \to G$ by setting $f(p) = p\psi(p)$ for all $p \in P$. As we have by equivariance of f that $f(pg) = f(p)g$ for any $g \in G$ we get equivariance of ψ, i.e. $\psi(pg) = g^{-1}\psi(p)g$. In other words, ψ defines a section s of $\mathrm{Ad}(P)$ by $s : B \to P \times_G G$ given by $b \mapsto [p, \psi(p)]$, with $p \in \pi^{-1}(b)$ chosen arbitrarily. Conversely, a section s of $\mathrm{Ad}(P)$ defines an equivariant function ψ and thus a bundle automorphism f by the above formulas. $\square$

From the above proposition we see that we can view $\mathrm{Aut}(P)$ as an infinite-dimensional Lie group. Its Lie algebra can then be identified with the space of sections of $\mathrm{ad}(P)$ through a fiberwise exponential map. Namely, given a section σ of $\mathrm{ad}(P)$, i.e. a map $\sigma : B \to P \times_G \mathfrak{g}$ given by $b \mapsto [p, X]$, we can define a section $s = \exp \sigma$ of $\mathrm{Ad}(P)$, $s : B \to P \times_G G$ given by $b \mapsto [p, \exp X]$. This is well-defined as for any $g \in G$ and $X \in \mathfrak{g}$ we have $\exp(g^{-1}Xg) = g^{-1}(\exp X)g$, i.e. the regular exponential map is equivariant with respect to the conjugation action.

Definition 2.1.20. Given a principal G-bundle P, its *gauge group* is the space $\mathrm{Aut}(P) = \Gamma(\mathrm{Ad}(P))$.

An important case for us will be when $G = \mathrm{U}(1)$. Hence consider a principal $\mathrm{U}(1)$-bundle $\mathcal{L}$. Any automorphism of $\mathcal{L}$ is given fiberwise by multiplication by an element of $\mathrm{U}(1)$. This immediately implies the following result.

Lemma 2.1.21. *For $G = \mathrm{U}(1)$, the gauge group of any principal $\mathrm{U}(1)$-bundle over B can be identified with the space $C^\infty(B; \mathrm{U}(1))$ of smooth maps from B to $\mathrm{U}(1)$.*

2.2 Connections

In this section we discuss connections on principal G-bundles and describe how they are related to connections on the associated vector bundles. Again, all of this can be found in [16, 27]. There are several different ways of looking at connections. Each has its merits depending on the situation. Let $\pi : P \to B$ be a principal G-bundle. Taking differentials we get a map $d\pi : TP \to TB$. We can then consider the subbundle $\ker(d\pi) \subset TP$, whose fiber over a point $p \in P$ is just $\ker(d\pi_p)$. Of course, this fits pointwise into the short exact sequence

$$0 \to \ker(d\pi_p) \hookrightarrow T_pP \overset{d\pi_p}{\to} T_{\pi(p)}B \to 0. \tag{2.2.1}$$

A connection on P is then a way of choosing a complementary subspace to each $\ker(d\pi_p)$ which behaves nicely under the G-action. More precisely, we have the following.

Definition 2.2.1. A *connection* on a principal G-bundle $\pi : P \to B$ is a distribution $\{H_p\}_{p\in P}$, i.e. a smooth family of linear subspaces of the tangent bundle TP, such that for all $p \in P$

$$\ker(d\pi_p) \oplus H_p = T_pP, \tag{2.2.2}$$

and such that it is invariant under the G-action, i.e. $g_*(H_p) = H_{g(p)}$ for all $p \in P$ and $g \in G$.

Remark 2.2.2. One also calls $\ker(d\pi)$ the vertical distribution, where then a choice of H_p satisfying equation (2.2.2) is called a horizontal distribution. The condition of equation (2.2.2) can also be stated by saying that $(d\pi_p)(H_p) = T_{\pi(p)}B$ for all $p \in P$.

A second way of looking at connections is in terms of a $\mathfrak{g}$-valued one-form. Given an element $X \in \mathfrak{g}$, recall that for $t \in \mathbb{R}$, $\exp(tX) \in G$ is such that

$$\frac{\partial}{\partial t}\exp(tX)\bigg|_{t=0} = X. \tag{2.2.3}$$

Now define $\sigma : \mathfrak{g} \to \Gamma(TP)$ by setting for $p \in P$

$$\sigma(X)(p) = \frac{\partial}{\partial t} p \cdot \exp(tX)\bigg|_{t=0}. \tag{2.2.4}$$

The vector field $\sigma(X)$ is called the *fundamental vector field* associated to X.

Definition 2.2.3. A *connection one-form* on a principal G-bundle $\pi : P \to B$ is an element $\omega \in \Omega^1(P, \mathfrak{g})$ such that

$$\omega_p(\sigma(X)(p)) = X_p, \qquad \forall X \in \mathfrak{g}, \tag{2.2.5}$$

and such that for any $g \in G$, $p \in P$ and $v \in T_pP$ we have

$$\omega_{g(p)}(v \cdot g) = g^{-1}\omega_p(v)h. \tag{2.2.6}$$

In other words, ω is equivariant with respect to the adjoint action.

The proof of the equivalence of these descriptions will not be given here. Essentially, the one-form ω is defined out of a connection by letting ω_p be the projection from T_pP onto $\ker(d\pi_p)$ with kernel H_p, and then noting that $\ker(d\pi_p)$ can be identified with $\mathfrak{g}$ using the action of G. Hence applying ω to an element $v \in T_pP$ gives back an element of $\mathfrak{g}$.

Lemma 2.2.4. *Given a principal G-bundle P, every connection one-form uniquely determines a connection and vice-versa.*

Given a connection on a principal G-bundle, we get induced connections on associated vector bundles. First let us recall the definition of such an object. Let X be a manifold.

Definition 2.2.5. A *connection* ∇_A on a vector bundle $E \to X$ is a linear map

$$\nabla_A : \Omega^0(X; E) \to \Omega^1(X; E) \tag{2.2.7}$$

such that for all $f \in C^\infty(X)$ and $s \in \Omega^0(X; E)$ we have

$$\nabla_A(fs) = df \otimes s + f\nabla_A(s). \tag{2.2.8}$$

Out of a connection ∇_A and a vector field $Y \in \mathcal{X}(X)$ we can form the *covariant derivative in direction of* Y $\nabla_A(Y) : \Gamma(E) \to \Gamma(E)$. It is $C^\infty(X)$-linear with respect to Y and linear with respect to the sections it acts on. If we denote $\nabla_A(Y)$ by ∇_Y, it furthermore satisfies $\nabla_Y(fs) = f\nabla_Y s + Y(f)s$ for $f \in C^\infty(X)$ and $s \in \Gamma(E)$. Connections on vector bundles always exist through a partition of unity argument. We denote the space of all connections on a given vector bundle E by $\mathcal{A}(E)$.

Lemma 2.2.6. *Given a principal G-bundle $\pi : P \to X$ with a connection and an associated vector bundle $E = P \times_G V$ associated to a linear representation $\rho : G \to \mathrm{GL}(V)$, E is equipped with an induced connection.*

Proof. The definition of this connection ∇_A on E can be described as follows. Let s be a local section of E defined near some $x \in X$, and let $v \in T_xX$ be given. By the definition of the tangent space we can choose a curve γ in X such that $\gamma(0) = x$ and $dg(0) = v$. Pulling back this curve through π we get a curve $\widetilde{\gamma}$ such that $\pi \circ \widetilde{\gamma} = \gamma$ which is horizontal, i.e. for all t we will have $d\widetilde{\gamma}(t) \in H_{\widetilde{\gamma}(t)}$. Now consider $s(\gamma(t)) \in E|_{\gamma(t)}$. By the definition of an associated vector bundle we see that s will be locally given as $[\widetilde{\gamma}(t), V(t)]$ where $V(t) \in V$ is some smooth V-valued function. Given this, we set

$$\nabla_A(s)(v) = \left[p, \left.\frac{\partial V}{\partial t}\right|_{t=0}\right] \tag{2.2.9}$$

The linearity of this object is clear, and one can readily check that it is well-defined and satisfies the required Leibniz identity. Note that we have here used the induced representation $d\rho : \mathfrak{g} \to \mathrm{End}(V)$. □

If G is a matrix subgroup, i.e. is embedded in $\mathrm{GL}(V)$ as a Lie group, then one can recover the connection on the principal G-bundle from this associated connection ∇_A on E. Furhermore, one can check that the difference of two connections $\nabla_A, \nabla_{A'}$ on E is C^∞-linear. Indeed, we have by the Leibniz identity that

$$(\nabla_A - \nabla_{A'})(fs) = df \otimes s + f\nabla_A(s) - df \otimes s - f\nabla_{A'}(s) = f(\nabla_A - \nabla_{A'})(s). \tag{2.2.10}$$

This immediately implies the following result.

Lemma 2.2.7. *Given any two connections $\nabla_A, \nabla_{A'}$ on E, their difference $a = \nabla_A - \nabla_{A'} \in \Omega^1(X; \mathrm{End}(E))$ is an $\mathrm{End}(E)$-valued one-form.*

Moreover, we get that given a connection ∇_A we can construct any other connection on E as $\nabla_{A'} = \nabla_A + a$ for some one-form a. We say the space $\mathcal{A}(E)$ of connections on E forms an *affine space* over $\Omega^1(X; \mathrm{End}(E))$: there is a free transitive action of $\Omega^1(X; \mathrm{End}(E))$ through addition, but there is no canonical choice of origin. Hence only after a choice of origin do we get a (non-canonical) isomorphism $\mathcal{A}(E) \cong \Omega^1(X; \mathrm{End}(E))$. Nevertheless, we now have a local description of a connection. Let E have a trivializing cover $\{U_\alpha\}$ with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to U_\alpha \cap U_\beta$. Then a connection ∇_A is nothing more than a collection of matrices $\{A_\alpha\}$ of E-valued one-forms on U_α: given a section $s \in \Gamma(E)$ with local representative s_α on U_α, we have

$$(\nabla_A s)_\alpha = ds_\alpha + A_\alpha s_\alpha. \tag{2.2.11}$$

In this local description the exterior derivative is the trivial connection on the trivial bundle over U_α, and we have described ∇_A locally as being constructed out of d through the addition of an E-valued one-form. The matrices $\{A_\alpha\}$ transform according to

$$A_\alpha = g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} - g_{\alpha\beta}^{-1} dg_{\alpha\beta}. \tag{2.2.12}$$

Hence we might as well identify ∇_A with its *connection matrix* $A = \{A_\alpha\}$, as we will do in later chapters. Now, given a connection ∇_A on a vector bundle E we can extend it to a map $\nabla_A : \Omega^p(X; E) \to \Omega^{p+1}(X; E)$ between E-valued forms by setting

$$\nabla_A(\alpha \otimes s) = d\alpha \otimes s + (-1)^p \alpha \wedge \nabla_A(s) \qquad \text{for } \alpha \in \Omega^p(X), s \in \Gamma(E). \tag{2.2.13}$$

Sometimes this extension is denoted by d_A instead. We can further consider the *curvature* F_A of a connection, namely

$$F_A = \nabla_A \circ \nabla_A : \Omega^0(X; E) \to \Omega^2(X; E). \tag{2.2.14}$$

One can check that it is C^∞-linear and hence defines an element of $\Omega^2(X; \mathrm{End}(E))$. A similar thing can be done for connections on principal bundles. Now suppose the connection ∇_A on E comes from a connection on a principal G-bundle through a representation ρ. Then F_A will be the image of the curvature of the connection on the principal G-bundle through the map $d\rho : \mathfrak{g} \to \mathrm{End}(V)$. We can also give a local description of the curvature. Using the same trivializing cover $\{U_\alpha\}$ as before, the curvature applied to a section $s \in \Gamma(E)$ is described by a *curvature matrix* which we will also denote by F_A, given locally by

$$(F_A s)_\alpha = (dA_\alpha + A_\alpha \wedge A_\alpha) s_\alpha. \tag{2.2.15}$$

It will be useful to relate the curvature of two different connections to each other. This is answered by the following lemma.

Lemma 2.2.8. *Given a connection ∇_A and a one-form $a \in \Omega^1(X; \mathrm{End}(E))$, we have*

$$F_{A+a} = F_A + da + a \wedge a. \tag{2.2.16}$$

Now consider the action of the gauge group on connections, which we describe in the principal G-bundle situation. Given a connection ∇ on a principal G-bundle P with connection one-form ω and a gauge transformation $u \in \mathcal{G}(P)$, we can form the pull-back connection $u^*(\nabla) = u\nabla u^{-1}$. This pull-back connection will have connection one-form $u^*(\omega)$, more explicitly given by

$$u^*(\omega) = u^{-1}\omega u + u^{-1}du. \tag{2.2.17}$$

This defines an action $u \cdot \nabla = u^*(\nabla)$. We can also easily identify what happens to the curvature: if $\Omega = \nabla \circ \nabla$ is the curvature of ω, then the curvature Ω' of $u^*(\omega)$ will simply be its conjugate by u, i.e.

$$\Omega' = u^{-1}\Omega u. \tag{2.2.18}$$

The same is true for the associated gauge transformations on any associated vector bundle E coming from P. Lastly we mention one result coming from Chern-Weil theory. Let E be a $\mathrm{U}(m)$-bundle. One can check that given a connection A on E, its curvature satisfies

$$d_A F_A = 0 \tag{2.2.19}$$

This is called the Bianchi identity. Now specify to the case $m = 1$, as this will be the only case we care about. Then equation (2.2.19) merely reads $dF_A = 0$, as the Lie algebra of $\mathrm{U}(1)$ is $i\mathbb{R}$, which is abelian. In other words, F_A is closed. Because of this it defines a cohomology class $[F_A] \in H^2(X; \mathbb{C})$. Up to a scalar multiple it can be identified with the first Chern class of E.

Theorem 2.2.9. *Given a connection A on a* $\mathrm{U}(1)$*-bundle E, we have*

$$[F_A] = -2\pi i c_1(E). \tag{2.2.20}$$

Proof. Essentially, this can be taken to be true by definition of the Chern classes. For more information see [21, Appendix C]. □

2.3 Almost complex and symplectic manifolds

In this section we recall some of the theory on almost complex and symplectic manifolds. In particular we recall the notion of compatibility between these structures giving almost Kähler manifolds, and pay special attention to the Cauchy-Riemann operator $\overline{\partial}$ in this setting. A general reference for symplectic manifolds is [20], with the theory of almost Kähler manifolds following [26, 27].

Throughout, let X be a four-manifold unless otherwise specified. Most of the results below of course hold true in arbitrary dimension, but we will only focus on the four-dimensional case. First, let us recall the basic definitions.

Definition 2.3.1. An *almost complex structure* on X is an endomorphism $J : TX \to TX$ such that $J^2 = -1$. It is called *integrable* if it arises from a complex structure on X. A manifold X together with an almost complex structure will be referred to as an *almost complex manifold.*

Here by an almost complex structure "arising from a complex structure" we mean the endomorphism J locally given by $J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$ in complex coordinates $z_i = x_i + iy_i$. Furthermore, an almost complex manifold is orientable with an orientation specified by the ordered basis of pairs $\{\frac{\partial}{\partial x_i}, J\frac{\partial}{\partial x_i}\}$.

Definition 2.3.2. A *symplectic form* on X is a closed non-degenerate two-form $\omega \in \Omega^2(X;\mathbb{R})$, i.e. for all $x \in X$ and nonzero $X \in T_xX$ there exists a $Y \in T_xX$ such that $\omega_x(X,Y) = 1$. A manifold X together with a symplectic form will be referred to as a *symplectic manifold.*

Now let (X, J, g) be an almost complex manifold equipped with a Riemannian metric g. There is a notion of these two structures being compatible.

Definition 2.3.3. A Riemannian metric g is called J-invariant or *almost Hermitian*, if

$$g_x(X,Y) = g_x(JX, JX) \qquad \text{for all} \qquad x \in X \text{ and } X, Y \in T_xX. \tag{2.3.1}$$

We will see shortly how these notions relate to symplectic forms, but first we introduce the notion of self-duality.

Definition 2.3.4. Let X be compact oriented Riemannian n-dimensional manifold with volume form dvol_g. The *Hodge star operator* $* : \Omega^k(X;\mathbb{R}) \to \Omega^{n-k}(X;\mathbb{R})$ for $0 \le k \le n$ is defined by the equation

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \mathrm{dvol}_g, \tag{2.3.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product induced by g.

It is readily verified that $*^2 = (-1)^{k(n-k)}$ on k-forms. In particular when X is four-dimensional, $n = 4$, so that

$$* : \Omega^2(X;\mathbb{R}) \to \Omega^2(X;\mathbb{R}), \qquad *^2 = 1. \tag{2.3.3}$$

Hence because $*$ is an involution on $\Omega^2(X)$, we can split Ω^2 up into its ± 1-eigenspaces denoted by $\Omega^2_+(X)$ and $\Omega^2_-(X)$ respectively. In other words, we have

$$\Omega^2(X;\mathbb{R}) = \Omega^2_+(X;\mathbb{R}) \oplus \Omega^2_-(X;\mathbb{R}). \tag{2.3.4}$$

We write $\alpha = \alpha_+ + \alpha_-$ for the decomposition of $\alpha \in \Omega^2(X)$ into $\alpha_\pm \in \Omega^2_\pm(X)$.

Definition 2.3.5. Given $\alpha \in \Omega^2(X)$, we say α is *self-dual* (or SD) if $\alpha_- = 0$ and *anti-self-dual* (or ASD) if $\alpha_+ = 0$.

Remark 2.3.6. The forms α_+ and α_- are also called the self-dual and anti-self-dual parts of α, respectively. We see that α is ASD if and only if $*\alpha = -\alpha$. Note furthermore that the notion of (anti)-self-duality of a given form depends on the Riemannian metric, and that the projection $\mathrm{pr}_\pm\alpha$ of a given form onto its (anti)-self-dual part is explicitly given by $\mathrm{pr}_\pm\alpha = \frac{1}{2}(\alpha \pm *\alpha)$.

Using the Hodge star operator, we can define an inner product $(\cdot,\cdot)$ on the space of forms by

$$(\alpha,\beta)=\int\limits_X \alpha\wedge *\beta, \qquad \alpha,\beta\in\Omega^k(X). \tag{2.3.5}$$

Using this inner product we can define the codifferential d^*, which is the formal adjoint of d with respect to this inner product. We can express it in terms of the Hodge star operator and d, and viewed as an operator on k-forms it then satisfies

$$d^*=(-1)^{k(n-k)} * d*, \tag{2.3.6}$$

as one can verify using Stokes' theorem. We can then define the Hodge Laplacian $\Delta = dd^*+d^*d$. A form $\omega\in\Omega^2$ is said to be *harmonic* if $\Delta\omega=0$, which is seen to be identical to requiring that $d\omega=d^*\omega=0$. Note that $\Delta=(d+d^*)^2$. We denote the space of harmonic k-forms on X by $\mathcal{H}^k(X)$. Now recall that when X is compact, Hodge theory tells us that each cohomology class has a unique harmonic representative, i.e. $H^k(X;\mathbb{R})\cong\mathcal{H}^k(X;\mathbb{R})$. Furthermore, in this case there is a Hodge decomposition

$$\Omega^k(X;\mathbb{R})=\mathcal{H}^k(X;\mathbb{R})\oplus\operatorname{im}(d)\oplus\operatorname{im}(d^*). \tag{2.3.7}$$

Moreover, when X is compact the de Rham cohomology groups $H^k(X;\mathbb{R})$ are finite dimensional.

Definition 2.3.7. Let X be compact. The *kth Betti number* b_k is defined by

$$b_k=\dim H^k(X;\mathbb{R}) \qquad \text{for} \qquad 0\le k\le n. \tag{2.3.8}$$

Now note that the Hodge star operator swaps the kernels of d and d^*. Hence if ω is harmonic then $*\omega$ is again harmonic. In particular, the (anti-)self-dual part of a harmonic two-form is again harmonic. Because of this, we can split $\mathcal{H}^2(X;\mathbb{R})$ into two subspaces

$$\mathcal{H}^2(X;\mathbb{R})=\mathcal{H}^2_+(X;\mathbb{R})\oplus\mathcal{H}^2_-(X;\mathbb{R}). \tag{2.3.9}$$

We denote the dimensions of these subspaces by $b_2^+(X)$ and $b_2^-(X)$ respectively. At this point it is not clear that these numbers are independent of the chosen Riemannian metric g. To see that this is true, consider the *intersection form* Q_X of X. This is a symmetric bilinear form, defined for compact oriented four-manifolds X by

$$Q_X:\, H^2(X;\mathbb{Z})\times H^2(X;\mathbb{Z})\to\mathbb{Z}, \qquad (\alpha,\beta)\mapsto\int_X\alpha\wedge\beta. \tag{2.3.10}$$

It can be shown that it vanishes only for torsion classes, so that it descends to a non-degenerate pairing on $H^2(X;\mathbb{Z})$ modulo torsion. After picking a basis for this space, we can represent it by a matrix, also denoted by Q_X. From Poincaré duality we can interpret the integral in equation (2.3.10) as an signed count of points of intersection of embedded surfaces of X, i.e. representatives of homology classes in $H_2(X;\mathbb{Z})$. This shows that Q_X is indeed integer valued. Moreover, if we tensor with the reals and extend Q_X to $H^2(X;\mathbb{R})$, we can diagonalize it as it is a symmetric matrix. By non-degeneracy of Q_X it then contains only the entries $+1$ and -1. The numbers $b_2^\pm(X)$ are then the amount of entries equal to ±1 respectively. In particular, this argument shows that they are both indeed independent of g, and in fact depend only on topological information of X.

Remark 2.3.8. The intersection form Q_X we just described captures most of the topological information of X. More precisely, Freedman [9] and Donaldson [3] showed that the homeomorphism type of a simply-connected compact smooth four-manifold X is determined by Q_X. The study of Q_X is indeed central in understanding four-manifolds, as can be read in [12].

Now note that we have

$$b_2(X) = b_2^+(X) + b_2^-(X). \tag{2.3.11}$$

Their difference is called the *signature* of X, denoted by $\tau(X) = b_2^+(X) - b_2^-(X)$. Furthermore, the expresssion

$$\chi(X) = b_0(X) - b_1(X) + b_2(X) - b_3(X) + b_4(X) \tag{2.3.12}$$

is called the *Euler characteristic* of X. Of course by Poincaré duality and the fact that $H_0(X;\mathbb{R}) \cong \mathbb{R}$ we see that

$$\chi(X) = 2 - 2b_1(X) + b_2(X). \tag{2.3.13}$$

For later use we also introduce the notation $d^+ = \mathrm{pr}_+ \circ d$ for the self-dual part of the exterior derivative. With it we can form the following complex.

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d^+} \Omega^2_+(X) \to 0. \tag{2.3.14}$$

Indeed, this is a complex because $d^+ \circ d = \mathrm{pr}_+ d^2 = 0$. Let us determine its cohomology groups.

Lemma 2.3.9. *The cohomology groups of the above complex are isomorphic to $\mathcal{H}^0(X;\mathbb{R})$, $\mathcal{H}^1(X;\mathbb{R})$ and $\mathcal{H}^2_+(X;\mathbb{R})$ respectively.*

Proof. The first cohomology group consists of all 0-forms ω in $\ker d \subset \Omega^0(X;\mathbb{R})$, which because for a 0-form we automatically have $d^*\omega = 0$ immediately implies that ω is harmonic. Similarly, the third cohomology group is equal to $\mathrm{coker} d^+ \subset \Omega^2_+(X;\mathbb{R})$. Hence let $\omega \in \Omega^2_+(X;\mathbb{R})$ be orthogonal to the image of d^+. Then for all $\alpha \in \Omega^1(X;\mathbb{R})$ we have

$$0 = (\omega, d^+\alpha) = (\omega, d\alpha) = (d^*\omega, \alpha). \tag{2.3.15}$$

This implies that $d^*\omega = 0$, but ω is self-dual so that $*\omega = \omega$, and hence $0 = d^*\omega = - * d * \omega = - * d\omega$. In other words, $d\omega = 0$ as well, so that ω is harmonic. Finally, consider a one-form $\alpha \in \Omega^1(X;\mathbb{R})$ which lies in the kernel of d^+ but is orthogonal to the image of d. Then for all $\omega \in \Omega^0(X)$ we have

$$0 = (\alpha, d\omega) = (d^*\alpha, \omega), \tag{2.3.16}$$

so that $d^*\alpha = 0$. Now note that we have

$$2d^*d^+ = 2d^*\frac{1}{2}(d + *d) = d^*d + d^* * d = d^*d. \tag{2.3.17}$$

Hence consider

$$(d\alpha, d\alpha) = (d^*d\alpha, \alpha) = (2d^*d^+\alpha, \alpha) = 0, \tag{2.3.18}$$

so that $d\alpha = 0$. We conclude that α is harmonic. □

Now let J be an almost complex structure on a compact Riemannian four-manifold X. Given an almost Hermitian metric g, define a two-form ω on X by

$$\omega(X, Y) = g(JX, Y). \tag{2.3.19}$$

By construction ω is J-invariant and g-self-dual. Furthermore, it is non-degenerate because g is. However, there is no guarantee that ω is closed, so that ω is not necessarily a symplectic form. However, when it is, we know that any two of the triple (J, g, ω) determine the third one.

Definition 2.3.10. Let ω be a symplectic form on X. Then an almost complex structure J is said to be *ω-compatible* if g defined by $g(X, Y) = \omega(X, JY)$ is a Riemannian metric. A triple (X, ω, J) consisting of a symplectic form ω with an ω-compatible almost complex structure J is called an *almost Kähler manifold.*

By a result of Gromov the space of such J is non-empty and in fact contractible [20]. It is a straightforward exercise to check that given a non-degenerate two-form ω and a Riemannian metric g, there exists an ω-compatible J inducing g if and only if ω is self-dual with respect to g and satisfies $|\omega| = \sqrt{2}$.

Remark 2.3.11. If the almost complex structure J of an almost Kähler manifold X is integrable, then X is called *Kähler*: it is also a complex manifold, and its complex structure is compatible with the symplectic structure.

Now, because $J^2 = -1$ we can split TX into its $\pm i$-eigenspaces. If we extend J complex linearly to the complexifications $TX \otimes_{\mathbb{R}} \mathbb{C}$ and $T^*X \otimes_{\mathbb{R}} \mathbb{C}$ we can also split these bundles, and hence decompose all complex differential forms into types. Writing $T^{*1,0}X$ and $T^{*0,1}X$ for these $+i$- and $-i$-eigenspaces respectively, we denote for $p, q \in \mathbb{N}$

$$\bigwedge^{p,q} T^*X = \left(\bigwedge^{p} T^{*1,0}X\right) \wedge \left(\bigwedge^{q} T^{*0,1}X\right). \tag{2.3.20}$$

We then have denote the space of (p, q)-forms on X by $\Omega^{p,q}(X) = \Gamma\left(\bigwedge^{p,q} T^*X\right)$. For example, we have

$$\begin{aligned} \Omega^1(X; \mathbb{C}) &= \Omega^{1,0}(X; \mathbb{C}) \oplus \Omega^{0,1}(X; \mathbb{C}), \\ \Omega^2_+(X; \mathbb{C}) &= \mathbb{C}\omega \oplus \Omega^{2,0}(X; \mathbb{C}) \oplus \Omega^{0,2}(X; \mathbb{C}). \end{aligned} \tag{2.3.21}$$

Furthermore, we see that ω is of type $(1, 1)$ with respect to J. Now define $\partial, \overline{\partial}$ by projection. Given $\alpha \in \Omega^{p,q}(X; \mathbb{C})$, set

$$\partial\alpha = (d\alpha)^{p+1,q}, \qquad \overline{\partial}\alpha = (d\alpha)^{p,q+1} \tag{2.3.22}$$

Only if J is integrable do we have $d = \partial \oplus \overline{\partial}$, as $d\alpha$ in general can also have $(p-1, q+2)$ and $(p+2, q-1)$ components. We set

$$\overline{N}_J\alpha = \frac{1}{4}(d\alpha)^{p-1,q+2}, \qquad N_J\alpha = \frac{1}{4}(d\alpha)^{p+2,q-1}. \tag{2.3.23}$$

This $N_J : \Omega^{p,q} \to \Omega^{p+2,q-1}$ is called the *Nijenhuis tensor* of J. Alternatively, we can define $N_J : TX \otimes TX \to TX$ by

$$N_J(Y,Z) = \frac{1}{4}\left([Y,Z] + J[JY,Z] + J[Z,JY] - [JY,JZ]\right), \qquad Y,Z \in \mathcal{X}(X). \tag{2.3.24}$$

One can readily see that N_J is $C^\infty(X)$-bilinear, skew-symmetric and complex anti-linear. In other words, N_J satisfies $N_J(fY,Z) = fN_J(Y,Z)$, $N_J(Y,Z) = -N_J(Z,Y)$ and $N_J(JY,Z) = N_J(Y,JZ) = -JN_J(Y,Z)$. Furthermore, we have the famous Newlander-Nirenberg theorem.

Theorem 2.3.12 (Newlander-Nirenberg)**.** *An almost complex structure J is integrable if and only if $N_J = 0$.*

Other equivalent ways of stating J is integable is by $d = \partial + \overline{\partial}$, or $\overline{\partial}^2 = 0$. Indeed, the Nijenhuis tensor measures how far $\overline{\partial}$ is from squaring to zero.

Lemma 2.3.13. *Let $a \in \Omega^{1,0}(X)$ be given. Then $(da)^{0,2} = a \circ N_J$.*

Proof. Let $Y,Z \in \mathcal{X}(X)$ be arbitrary. Note that given a two-form $\tau \in \Omega^2(X)$ we have

$$\tau^{0,2}(Y,Z) = \frac{1}{4}\left(\tau(Y,Z) - \tau(JY,JZ) + i\tau(JY,Z) + i\tau(Y,JZ)\right). \tag{2.3.25}$$

Now, by definition of the differential we have

$$da(Y,Z) = L_X(a(Z)) - L_Y(a(Y)) + a([Y,Z]). \tag{2.3.26}$$

The lemma follows simply by substituting this in equation (2.3.25) and noticing that a lot of the terms cancel. □

Lemma 2.3.14. *For $f \in \Omega^0(X;\mathbb{R})$ we have $\overline{\partial}^2 f = -\partial f \circ N_J$.*

Proof. This follows from Lemma 2.3.13 by setting $a = \overline{\partial} f$. □

Now let $E \to X$ be a complex vector bundle equipped with a Hermitian metric, and let $P_E \to X$ be its associated unitary frame bundle. Then a unitary connection $B \in \mathcal{A}(P_E)$ gives rise to a covariant derivative $d_B : \Omega^0(X;E) \to \Omega^1(X;E)$, and using J we can decompose it into its complex linear and anti-linear parts,

$$\partial_B : \Omega^0(X;E) \to \Omega^{1,0}(X;E), \qquad \overline{\partial}_B : \Omega^0(X;E) \to \Omega^{0,1}(X;E), \tag{2.3.27}$$

explicitly as $\partial_B s = \frac{1}{2}\left(d_B + id_B s \circ J\right)$ and $\overline{\partial}_B s = \frac{1}{2}\left(d_B - id_B s \circ J\right)$ for $s \in \Omega^0(X;E)$. For later use we prove the following result, measuring to what extent the composition $\overline{\partial}_B^2$ corresponds to the $(0,2)$-part of the curvature.

Lemma 2.3.15. *We have for $s \in \Omega^0(X;E)$*

$$\overline{\partial}_B \overline{\partial}_B s = F_B^{0,2} s - (\partial_B s) \circ N_J. \tag{2.3.28}$$

Proof. By Lemma 2.3.13 we have for $a \in \Omega^{1,0}(X)$ that

$$(d_B a)^{0,2} = a \circ N_J. \tag{2.3.29}$$

Because of this, we compute

$$\overline{\partial}_B \overline{\partial}_B s = \left(d_B \overline{\partial}_B s\right)^{0,2} = (d_B d_B s)^{0,2} - (d_B \partial_B s)^{0,2} = F_B^{0,2} s - \partial_B s \circ N_J. \tag{2.3.30}$$

□

Given a two-form α, let $\Lambda\alpha$ denote contraction with ω. In other words, set $\Lambda\alpha = \langle \alpha, \omega \rangle$. We have the following Weitzenböck formula relating the two Laplacians $\overline{\partial}_B^* \overline{\partial}_B$ and $d_B^* d_B$.

Lemma 2.3.16. *Let (X, ω, J) be an almost Kähler four-manifold, and let $E \to X$ be a Hermitian line bundle with a unitary connection B. Then for any $s \in \Omega^0(X; E)$ we have*

$$\overline{\partial}_B^* \overline{\partial}_B s = \frac{1}{2}\left(d_B^* d_B s - i\Lambda F_B s\right). \tag{2.3.31}$$

Proof. By definition we have $J = i * \omega\wedge$ on one-forms. Hence

$$\overline{\partial}_B^* \overline{\partial}_B s = \overline{\partial}_B^* \frac{1}{2}\left(d_B s + i * (\omega \wedge d_B s)\right) = \frac{1}{2} d_B^* \left(d_B s + i * (\omega \wedge d_B s)\right). \tag{2.3.32}$$

Now recall that $d_B^* * = - * d_B$, so that because ω is closed this is also equal to

$$\frac{1}{2}\left(d_B^* d_B s - i * d_B(\omega \wedge d_B s)\right) = \frac{1}{2}\left(d_B^* d_B s - i * (\omega \wedge d_B^2 s)\right) = \frac{1}{2}\left(d_B^* d_B s - i * (\omega \wedge F_B s)\right). \tag{2.3.33}$$

Looking at the definition of the Hodge star operator, we now see that $*\omega\wedge = \Lambda$ on two-forms, so that we are done. □

2.4 Elliptic operators

In this section we discuss the notion of a differential operator between two vector bundles to be elliptic. A comprehensive treatment is provided in [14]. Recall first the following fact from linear algebra. If $T : V \to W$ is a linear map between two finite-dimensional vector spaces V and W, we can consider the dimensions of its kernel and cokernel. These obviously depend on what T we picked, but their difference does not:

$$\dim \ker(T) - \dim \operatorname{coker}(T) = \dim V - \dim W. \tag{2.4.1}$$

This observation is naturally extended to (separable) infinite-dimensional Hilbert spaces.

Definition 2.4.1. A map $T : V \to W$ between Hilbert spaces is called *Fredholm* if it has closed range and its kernel $\ker T$ and cokernel coker$T$ are finite dimensional. If this is true, its *index* $\operatorname{ind}(T)$ is defined by

$$\operatorname{ind}(T) = \dim \ker(T) - \dim \operatorname{coker}(T). \tag{2.4.2}$$

We now consider the analogous notion for operators between sections of vector bundles.

Definition 2.4.2. An operator $D : \Gamma(E) \to \Gamma(F)$ between sections of two vector bundles E and F over X is called *Fredholm* if its kernel $\ker(D)$ and cokernel $\operatorname{coker}(D)$ are finite-dimensional. If this is true we define its *index* $\operatorname{ind}(D)$ to be

$$\operatorname{ind}(D) = \dim\ker(D) - \dim\operatorname{coker}(D). \tag{2.4.3}$$

If we have equipped our bundles with metrics such that we can speak of formal adjoints, we of course have $\operatorname{coker}(D) = \ker(D^*)$. An important fact of Fredholm operators is that their index is invariant under perturbations. In other words, if one were to take a family of Fredholm operators parameterized by some connected topological space, all of their indices would agree. There is a broad theory on determining Fredholmness of operators and calculating their index. A key concept is that of ellipticity. Let $\pi : T^*X \to X$ be the natural projection. Let E and F be vector bundles of rank n over X, and note that we can pullback them to bundles π^*E, π^*F over T^*X.

Definition 2.4.3. A linear operator $D : \Gamma(E) \to \Gamma(F)$ is called a *differential operator of order at most* $k \in \mathbb{N}$ if it decreases supports and in local coordinates is given by

$$D = \sum_{0 \le |\alpha| \le k} f_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{2.4.4}$$

for some matrix-valued functions f_α.

In the above definition we use multi-index notation, meaning that we use n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and write $|\alpha| = \sum_i \alpha_i$ and $x^\alpha = (x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})$.

Definition 2.4.4. Given a differential operator D of order at most k, its *principal symbol* $\sigma(D) : \pi^*E \to \pi^*F$ is a bundle map locally given for $\xi \in T_x^*X$ by

$$\sigma(D)(\xi) = i^k \sum_{|\alpha|=k} f_\alpha \xi^\alpha : E_x \to F_x. \tag{2.4.5}$$

The operator D is called *elliptic* if for all $x \in X$ and nonzero $\xi \in T_x^*X$ the map $\sigma(D)(\xi) : E_x \to F_x$ is an isomorphism.

There is a way to define the principal symbol invariantly, i.e. without going to a local description of D. One of the main reasons for introducing the notion of ellipticity is that several differential operators encountered in geometry satisfy this condition, but moreover that we have the following result.

Theorem 2.4.5. *Let X be a compact manifold and $D : \Gamma(E) \to \Gamma(F)$ a first order elliptic differential operator between vector bundles E and F over X. Then D is Fredholm.*

2.5 Sobolev spaces

In this section we briefly go over the theory of Sobolev spaces as we will need it. For more details, see [5, Appendix] and references therein. Let (X, g) be a compact Riemannian manifold, and let $E \to X$ be either an $\mathrm{O}(m)$- or $\mathrm{U}(m)$-bundle over X, equipped with a connection A. Recall that we can extend ∇_A to the space of E-valued k-forms $\Omega^k(X; E)$ as per equation (2.2.13). Given $s \in \Gamma(E)$ we then have $\nabla_A^k s = (\nabla_A \circ \cdots \circ \nabla_A)s \in \Omega^k(X; E)$.

Definition 2.5.1. The kth *Sobolev norm* $\|\cdot\|_{p,k}$ of exponent $p > 1$ on $\Gamma(E)$ is defined by

$$\|s\|_{p,k}^p = \sum_{i=0}^{k} \int_X \left|\nabla_A^i s\right|^p \, \mathrm{d}\mu \qquad \text{for } s \in \Gamma(E), \tag{2.5.1}$$

where ∇_A^0 is understood to be the identity on $\Gamma(E)$.

This indeed defines a norm on $\Gamma(E)$.

Definition 2.5.2. The *Sobolev space* $L_k^p(E)$ is the completion of $\Gamma(E)$ with respect to the norm $\|\cdot\|_{p,k}$.

Remark 2.5.3. If we were to change the Riemannian metric g on X, the choice of fiber metric on E or the connection A on E, we would get norms $\|\cdot\|_{p,k}$ on $\Gamma(E)$ which are all equivalent. In other words, the Sobolev space $L_k^p(E)$ would be the same as our choice does not affect the completion process.

We then naturally have the following result.

Proposition 2.5.4. *The space $L_k^p(E)$ is a Banach space for all p and k and a Hilbert space for $p = 2$.*

Note that by definition the space $\Gamma(E)$ of smooth sections of E lies dense in any Sobolev space $L_k^p(E)$. The main reason for introducing these Sobolev spaces is that functional analysis tells us the following three key theorems are true.

Theorem 2.5.5 (Sobolev embedding)**.** *Let $n = \dim X$. Then for $k - \frac{n}{p} > \ell$ there is a continuous embedding*

$$L_k^p(E) \hookrightarrow C^\ell(E), \tag{2.5.2}$$

where $C^\ell(E)$ is the space of C^ℓ-sections of E.

The above theorem has an immediate corollary.

Corollary 2.5.6. *Let s be such that $s \in L_k^p(E)$ for some fixed p and all k. Then s is smooth.*

Proof. Clearly s is smooth if and only if it lies in $C^\ell(E)$ for all ℓ. Hence let ℓ be given. Then an application of Theorem 2.5.5 with $k = \ell + \lfloor \frac{n}{p} \rfloor + 1$ implies the result. □

Theorem 2.5.7 (Sobolev multiplication)**.** *Let E, F be two* $\mathrm{O}(m)$*- or* $\mathrm{U}(m)$*-bundles over X. Then for $k - \frac{n}{p} > 0$ there is a continuous multiplication*

$$L_k^p(E) \otimes L_k^p(F) \to L_k^p(E \otimes F). \tag{2.5.3}$$

Remark 2.5.8. A more elaborate version of Sobolev multiplication states that there is a continuous multiplication

$$L_k^p(E) \otimes L_\ell^q(F) \to L_m^r(E \otimes F), \tag{2.5.4}$$

where $0 \leq m \leq \min\{k, \ell\}$ is such that

$$0 < \frac{m}{n} + \left(\frac{1}{p} - \frac{k}{n}\right) + \left(\frac{1}{q} - \frac{\ell}{n}\right) \leq \frac{1}{r} \leq 1. \tag{2.5.5}$$

Theorem 2.5.9 (Rellich's lemma)**.** *The inclusion*

$$L_{k+1}^p(E) \hookrightarrow L_k^p(E) \tag{2.5.6}$$

is compact for all p and k. In other words, a sequence $\{s_i\}$ which is bounded in L_{k+1}^p has a convergent subsequence in L_k^p.

There is one more result which will be of importance to us, having to do with elliptic operators between Sobolev spaces.

Theorem 2.5.10 (Elliptic regularity)**.** *Let $D : L_k^p(E) \to L_k^p(F)$ be a first order elliptic differential operator between Sobolev spaces of sections. Then there exists a constant $C > 0$ such that for any $s \in \Gamma(E)$ we have*

$$\|s\|_{p,k+1} \leq C(\|Ds\|_{p,k} + \|s\|_{p,k}). \tag{2.5.7}$$

Moreover, when D has trivial kernel we instead have

$$\|s\|_{p,k+1} \leq C\|Ds\|_{p,k}. \tag{2.5.8}$$

2.6 Hilbert manifolds

In this section we discuss some basic results of the theory of Hilbert manifolds. Recall that an ordinary (finite-dimensional) manifold of dimension n is a topological space X which is Hausdorff, second countable and locally Euclidean. This last condition can also be phrased by saying that it is *modelled* on $\mathbb{R}^n$. Hilbert manifolds are a generalization of this concept in that they allow for topological spaces which are locally modelled on Hilbert spaces instead. Throughout, let V denote a Hilbert space, i.e. a vector space equipped with an inner product $\langle \cdot, \cdot \rangle_V$ for whose norm it is complete. Almost all of the below results are true if one replaces the word Hilbert by Banach everywhere, but all Banach manifolds we will encounter will in fact be Hilbert. More information on Hilbert manifolds pertaining to Seiberg-Witten theory can be found in [27, Appendix A & B].

Definition 2.6.1. A map $F : V \to W$ between Hilbert spaces is said to be *Fréchet differentiable* if for each $x \in V$ there exists an open neighbourhood U of x such that there is a bounded linear operator $A_x : V \to W$ satisfying

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - A_x\|_W}{\|h\|_V} = 0. \tag{2.6.1}$$

The operator A_x is called the *Fréchet derivative* of F at x, and is also denoted by dF_x.

Remark 2.6.2. One can readily check that if one takes V and W to be standard Euclidean Hilbert spaces that the Fréchet derivative agrees with the ordinary definition of a derivative.

Recall that a map between manifolds is smooth if it is locally smooth, i.e. written in coordinate charts it is infinitely differentiable. We now have a similar notion of a map between Hilbert spaces being infinitely Fréchet differentiable, which we will again refer to as smoothness.

Definition 2.6.3. A *Hilbert manifold* Y modelled on V is a paracompact topological space Y such that each point has a neighbourhood diffeomorphic to an open set in V, along with a choice of maximal atlas of smooth transition functions.

We say a map $F : Y \to Z$ between Hilbert manifolds is smooth if it is locally infinitely Fréchet differentiable. In the same manner we can extend the notions of regular and critical values of a map $F : Y \to Z$.

Definition 2.6.4. Given a smooth map $F : Y \to Z$, a point $z \in Z$ is called a regular value of F if for each $y \in F^{-1}(z)$ we have that dF_y is surjective and has a right inverse. The point z is called a *critical value* of F if it is not a regular value of F.

Because Hilbert manifolds are generally infinite dimensional, one must be careful in trying to extend theorems which hold true for maps between finite-dimensional manifolds to this setting. It turns out that there are analogous statements for maps which are Fredholm.

Definition 2.6.5. A map $F : Y \to Z$ between Hilbert manifolds is *Fredholm* if it is differentiable and for each $y \in Y$ its differential dF_y is Fredholm.

Consider now some continuously differentiable Fredholm map $F : Y \to Z$. We can consider for each $y \in Y$ the index $\mathrm{ind}(dF_y) \in \mathbb{Z}$. Because dF is assumed to be continuous and $\mathbb{Z}$ is discrete, we see that the index is locally constant. Assuming that Y is connected we then define the index of F to be

$$\mathrm{ind}(F) = \mathrm{ind}(dF_y), \tag{2.6.2}$$

for any $y \in Y$. Recall now Sard's theorem in the finite-dimensional case: for any smooth map between manifolds its set of critical values has measure zero. There is an analogous statement for Fredholm maps between Hilbert manifolds due to Smale [29].

Theorem 2.6.6 (Sard-Smale)**.** *Let $F : Y \to Z$ be a Fredholm map between Hilbert manifolds such that Y is connected. Then the set of regular values of F is dense in Z.*

Remark 2.6.7. The standard formulation of the theorem is to say that the set of regular values of F is *residual* in Z. Here, a subset $A \subset Z$ is residual if it is the countable intersection of open dense sets. Because Z is a metric space, residual sets are dense by the Baire category theorem. We will also say that a *generic* point $z \in Z$ is regular.

Theorem 2.6.8 (Inverse function)**.** *Let $F : Y \to Z$ be a smooth map between Hilbert manifolds and let $z \in Z$ be a regular value of F. Then $M = f^{-1}(z) \subset Y$ is a (possibly empty) smooth Hilbert submanifold. If F is Fredholm, then M is finite-dimensional with* $\dim M = \operatorname{ind}(F)$*.*

Proof. A proof using the analogue of the implicit function theorem can be found [27, Appendix B]. If $\operatorname{ind}(F)$ is negative then M will be empty. □

In particular, the above two theorems together show that given a Fredholm map $F : Y \to Z$, a generic point $z \in Z$ will give rise to a finite dimensional Hilbert manifold $F^{-1}(z)$ of dimension $\operatorname{ind}(F)$.

Chapter 3

Spin geometry

In this chapter we develop just enough of the theory of spin geometry as is required for our purposes. In particular, we will quickly restrict ourselves to the four-dimensional case whenever this is convenient. A more thorough introduction to the subject of spin geometry can be found, for example, in [11, 18].

3.1 The Spin and Spin^c groups

In this section we discuss the Lie groups $\mathrm{Spin}(n)$ and $\mathrm{Spin^c}(n)$. We will give explicit descriptions of these groups in the case where $n = 4$, which will be the only relevant case for us. Moreover, we will mention several group homomorphisms which will be used in the next section. We mostly follow the treatment in [22].

Definition 3.1.1. For $n > 1$ a positive integer, the *spin group* in dimension n $\mathrm{Spin}(n)$ is the non-trivial double cover of $\mathrm{SO}(n)$.

The group $\mathrm{Spin}(n)$ is a compact Lie group of dimension $\frac{n(n-1)}{2}$ and is connected for $n \geq 2$ and simply connected for $n \geq 3$. Letting $\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ denote this double cover, these groups fit in the short exact sequence of Lie groups

$$1 \to \{\pm 1\} \to \mathrm{Spin}(n) \xrightarrow{\rho} \mathrm{SO}(n) \to 1. \tag{3.1.1}$$

We see that $\mathrm{Spin}(n)$ is a central extension of $\mathrm{SO}(n)$ and because $\mathrm{SO}(n)$ has fundamental group $\mathbb{Z}_2$ for $n > 2$ it has a unique double cover.

Corollary 3.1.2. *The Lie algebra of* $\mathrm{Spin}(n)$ *is* $\mathfrak{so}(n)$.

We can be more explicit when $n = 4$. The important fact we use is that $\mathbb{R}^4$ carries a natural multiplication through its identification with the quaternions $\mathbb{H}$. Let us identify $\mathbb{R}^4$ and $\mathbb{H}$ with a subspace V of complex 2×2-matrices, i.e.

$$\mathbb{R}^4 \cong \left\{ A \in \mathrm{M}(2, \mathbb{C}) \;\middle|\; A = \begin{pmatrix} x + iy & z + it \\ -z + it & x - iy \end{pmatrix}, x, y, z, t \in \mathbb{R} \right\} \cong \mathbb{H}. \tag{3.1.2}$$

This subspace V is generated by the matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{3.1.3}$$

One can readily check that these matrices indeed satisfy the defining relations for the quaternions,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}. \tag{3.1.4}$$

Note that we have

$$\det A = x^2 + y^2 + z^2 + t^2 = \langle A, A \rangle, \tag{3.1.5}$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product. From this we also readily see that we can identify SU(2) with the subspace of unit vectors in V

$$\mathrm{SU}(2) \cong \{A \in V : \langle A, A \rangle = 1\} \subset V. \tag{3.1.6}$$

From this, the group Spin(4) is the direct product of two copies of SU(2),

$$\mathrm{Spin}(4) = \mathrm{SU}_+(2) \times \mathrm{SU}_-(2) = \left\{ \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} \middle| A_\pm \in \mathrm{SU}_\pm(2) \right\}. \tag{3.1.7}$$

The aforementioned covering map is realized by the representation $\rho : \mathrm{Spin}(4) \to \mathrm{GL}(V)$ given by

$$\rho(A_+, A_-)(B) = A_- B (A_+)^{-1} \qquad (A_+, A_-) \in \mathrm{Spin}(4), B \in V. \tag{3.1.8}$$

We then have

$$\langle A_- B (A_+)^{-1}, A_- B (A_+)^{-1} \rangle = \det \left(A_- B (A_+)^{-1} \right) = \det B = \langle B, B \rangle, \tag{3.1.9}$$

as by definition both A_+ and A_- have determinant one. The representation ρ thus preserves the inner product, and hence $\rho : \mathrm{Spin}(4) \to \mathrm{SO}(4)$. It is now readily checked that ρ is surjective, and because both Spin(4) and SO(4) are compact Lie groups of dimension 6 we immediately get that ρ induces an isomorphism of their Lie algebras. Furthermore, the kernel K of ρ is a finite subgroup and ρ is a covering map. Because as a manifold we have $\mathrm{SU}(2) \cong S^3$, we see that $\mathrm{Spin}(4) \cong S^3 \times S^3$ and hence it is simply connected because S^3 is. Now consider the exact sequence on the level of homotopy groups

$$\pi_1 (\mathrm{Spin}(4), I) \to \pi_1 (\mathrm{SO}(4), I) \to K \to 0. \tag{3.1.10}$$

Because we know that $\pi(\mathrm{SO}(4), I) = \mathbb{Z}_2$ we see from this that $K \cong \mathbb{Z}_2$, and more explicitly we have $K = \{(I, I), (-I, -I)\}$.

Definition 3.1.3. For $n > 0$ a positive integer, the *Spin^c-group* in dimension n $\mathrm{Spin}^c(n)$ is the unique non-trivial double cover of $\mathrm{SO}(n) \times \mathrm{U}(1)$ for which the preimage of each factor is connected.

An alternative definition of $\mathrm{Spin}^{\mathrm{c}}(n)$ would be

$$\mathrm{Spin}^{\mathrm{c}}(n) \cong \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{U}(1), \tag{3.1.11}$$

i.e. as a twisted product of $\mathrm{Spin}(n)$ with $\mathrm{U}(1)$ through the diagonal action of $\mathbb{Z}_2$.

Remark 3.1.4. It is clear that $\mathrm{Spin}(n)$ sits inside $\mathrm{Spin}^{\mathrm{c}}(n)$ through a natural inclusion $i : \mathrm{Spin}(n) \hookrightarrow \mathrm{Spin}^{\mathrm{c}}(n)$.

We see that $\mathrm{Spin}^{\mathrm{c}}(n)$ is a central extension of $\mathrm{SO}(n)$ by $\mathrm{U}(1)$. If we let $\rho^{\mathrm{c}} : \mathrm{Spin}^{\mathrm{c}}(n) \to \mathrm{SO}(n) \times \mathrm{U}(1)$ be this double cover, we have $\rho^{\mathrm{c}}([u, \lambda]) = (\rho(u), \lambda^2)$ and these groups fit in the short exact sequence of Lie groups

$$1 \to \{\pm 1\} \to \mathrm{Spin}^{\mathrm{c}}(n) \xrightarrow{\rho^{\mathrm{c}}} \mathrm{SO}(n) \times \mathrm{U}(1) \to 1. \tag{3.1.12}$$

Note furthermore that while $\mathrm{Spin}^{\mathrm{c}}(n)$ is connected for all $n \geq 2$ it is never simply-connected, and in fact has $\pi_1(\mathrm{Spin}^{\mathrm{c}}(n), I) = \mathbb{Z}$ for $n \geq 3$. We again can conclude from the above short exact sequence what the Lie algebra of $\mathrm{Spin}^{\mathrm{c}}(n)$ is.

Corollary 3.1.5. *The Lie algebra of* $\mathrm{Spin}^{c}(n)$ *is* $\mathfrak{so}(n) \oplus i\mathbb{R}$.

Again we can be more explicit when $n = 4$. We now have an explicit description of $\mathrm{Spin}^{\mathrm{c}}(4)$ as

$$\mathrm{Spin}^{\mathrm{c}}(4) = \left\{ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} \middle| \; A_\pm \in \mathrm{SU}_\pm(2), \lambda \in \mathrm{U}(1) \right\}, \tag{3.1.13}$$

where $\mathrm{SU}_\pm(2)$ denote two copies of $\mathrm{SU}(2)$ which we distinguish by their subscript. We see that $\mathrm{Spin}^{\mathrm{c}}(4)$ has dimension 7 and that

$$\mathrm{Spin}^{\mathrm{c}}(4) = \left(\mathrm{SU}_+(2) \times \mathrm{SU}_-(2) \times \mathrm{U}(1)\right)/\mathbb{Z}_2, \tag{3.1.14}$$

through the diagonal action of $\mathbb{Z}_2 \cong K = \{(I, I), (-I, -I)\}$. Perhaps a more symmetric description of $\mathrm{Spin}^{\mathrm{c}}(4)$ then is

$$\mathrm{Spin}^{\mathrm{c}}(4) = \left\{(A_+, A_-) : \; A_\pm \in \mathrm{U}(2), \det(A_+) = \det(A_-)\right\}. \tag{3.1.15}$$

Moreover, the representation ρ described earlier extends to a representation $\rho^{\mathrm{c}} : \mathrm{Spin}^{\mathrm{c}}(4) \to \mathrm{GL}(V)$ given by

$$\rho^{\mathrm{c}}([A_+, A_-, \lambda])(B) = (\lambda A_-)\, B\, (\lambda A_+)^{-1} = A_- B A_+^{-1}. \tag{3.1.16}$$

Furthermore, there is a group homomorphism $\pi : \mathrm{Spin}^{\mathrm{c}}(4) \to \mathrm{U}(1)$ given by

$$\pi([A_+, A_-, \lambda]) = \det(\lambda A_+) = \det(\lambda A_-) = \lambda^2. \tag{3.1.17}$$

Lastly, we will need two further representations of $\mathrm{Spin}^{\mathrm{c}}(4)$. Let W_+ and W_- be two copies of $\mathbb{C}^2$ with the standard Hermitian metric. Then we have actions of $\mathrm{Spin}^{\mathrm{c}}(4)$ on $w_\pm \in W_\pm$ defined by

$$\rho_+^{\mathrm{c}}([A_+, A_-, \lambda])(w_+) = \lambda A_+ w_+, \qquad \rho_-^{\mathrm{c}}([A_+, A_-, \lambda])(w_-) = \lambda A_- w_-. \tag{3.1.18}$$

The representations ρ_+^{c} and ρ_-^{c} are also called the positive and negative spinor representations associated to the complex spin representation ρ^{c}.

3.2 $\mathrm{Spin}^{\mathrm{c}}$-structures and spinor bundles

Let X be an oriented n-dimensional manifold with Riemannian metric g. In this section we discuss how to equip X with a principal $\mathrm{Spin}^{\mathrm{c}}$-bundle. Again we mostly follow the treatment in [22]. Consider the frame bundle $P \to X$, i.e. the principal $\mathrm{GL}(n, \mathbb{R})$-bundle constructed out of transition functions of the tangent bundle TX. Because X is oriented and we have chosen a Riemannian metric, we can reduce the structure group from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{SO}(n)$. Doing this we now speak of P as the *orthonormal frame bundle* of (X, g). Because the space of Riemannian metrics is path-connected and contractible, all such orthonormal frame bundles will be isomorphic.

Definition 3.2.1. Given an oriented n-dimensional Riemannian manifold X, a *Spin*$^{\mathrm{c}}$*-structure* $\mathfrak{c}$ on X is a lift of the orthogonal frame bundle P to a principal $\mathrm{Spin}^{\mathrm{c}}$-bundle $\widetilde{P} \to X$. A *Spin-structure* on X is a lift of P to a principal Spin-bundle.

In other words, the following diagram has to commute.

$$
\begin{array}{ccc}
\widetilde{P} \times \mathrm{Spin}^{\mathrm{c}}(n) & \longrightarrow & \widetilde{P} \\
\downarrow & & \downarrow \\
P \times \mathrm{SO}(n) & \longrightarrow & P \\
 & & \downarrow \\
 & & X
\end{array}
\tag{3.2.1}
$$

We denote the space of all $\mathrm{Spin}^{\mathrm{c}}$-structures of (X, g) by $\mathcal{S}^{\mathrm{c}}(X, g)$. Because of the map $\pi : \mathrm{Spin}^{\mathrm{c}}(n) \to \mathrm{U}(1)$ constructed in equation (3.1.17), we see that given a $\mathrm{Spin}^{\mathrm{c}}$-structure $\widetilde{P}$ we can reduce its structure group to $\mathrm{U}(1)$ through π to obtain a principal $\mathrm{U}(1)$-bundle $\mathcal{L} = \widetilde{P} \times_\pi \mathrm{U}(1)$, which will be called the *determinant line bundle* $\det(\widetilde{P})$ or $\det(\mathfrak{c})$ associated to the $\mathrm{Spin}^{\mathrm{c}}$-structure. Because of the inclusion of $\mathrm{Spin}(n)$ into $\mathrm{Spin}^{\mathrm{c}}(n)$, the following result should not be surprising.

Lemma 3.2.2. *Every* Spin*-structure determines a* Spin^{c}*-structure.*

Proof. If P_{Spin} is a Spin-structure we can use the inclusion $i : \mathrm{Spin}(n) \hookrightarrow \mathrm{Spin}^{\mathrm{c}}(n)$ to reduce the structure group of P_{Spin} to $\mathrm{Spin}^{\mathrm{c}}(n)$. Effectively we are creating a $\mathrm{Spin}^{\mathrm{c}}$-structure $\widetilde{P}$ with trivial determinant line bundle, viz.

$$\widetilde{P} = P_{\mathrm{Spin}} \times_{\mathbb{Z}_2} \mathbb{U}(1), \tag{3.2.2}$$

where $\mathbb{Z}_2$ acts in the same diagonal manner as expressed in equation (3.1.11), and $\mathbb{U}(1)$ is the trivial $\mathrm{U}(1)$-bundle. □

Let us now look at what the construction of a $\mathrm{Spin}^{\mathrm{c}}$-structure means locally, in terms of transition functions. We stated that given a Riemannian metric g on an oriented manifold

X we can reduce the structure group of the frame bundle $P \to X$ from $\mathrm{GL}(n,\mathbb{R})$ to $\mathrm{SO}(n)$. In terms of transition functions, this means that we can trivialize the tangent bundle TX so that there now exists an open cover $\{U_\alpha\}_\alpha$ of X with maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{SO}(n), \tag{3.2.3}$$

satisfying the cocycle condition. A Spin- respectively $\mathrm{Spin}^{\mathrm{c}}$-structure is now a collection of transition functions $\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}(n)$ such that $\rho \circ \widetilde{g}_{\alpha\beta} = g_{\alpha\beta}$ respectively $\rho^{\mathrm{c}} \circ \widetilde{g}_{\alpha\beta} = g_{\alpha\beta}$, which again satisfy the cocycle condition. There now is a nice topological criterion for the existence of $\mathrm{Spin}^{\mathrm{c}}$- and Spin-structures on X in terms of its second Stiefel-Whitney class. First we recall a notion from the theory of Čech cohomology.

Definition 3.2.3. An open covering $\{U_\alpha\}_\alpha$ of X is called *good* if for each choice of $(\alpha_1, \ldots, \alpha_k)$ for any k, the intersection $U_{\alpha_1} \cap \ldots U_{\alpha_2}$ is either empty or diffeomorphic to $\mathbb{R}^n$, i.e. is geodesically contractible.

It can be shown that one can always find good covers. In particular, we can choose a good cover as the trivializing cover for TX considered above.

Lemma 3.2.4. *An oriented manifold X admits a* Spin^c*-structure if and only if $w_2(X)$ is the reduction* mod 2 *of an integral class, i.e. $w_2(X) = c \in H^2(X;\mathbb{Z})$* mod 2 *for some c. It admits a* Spin*-structure if and only if $w_2(X) = 0$.*

Proof. We will first deal with the question of a Spin-structure. Given a good cover $\{U_\alpha\}_\alpha$ trivializing TX, we get maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{SO}(n)$. Because $U_\alpha \cap U_\beta$ is assumed to be contractible, we can surely find lifts $\widetilde{g}_{\alpha\beta}$ mapping into $\mathrm{Spin}(n)$. However, these might not satisfy the cocycle condition. Define now the maps

$$\eta_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathrm{Spin}(n) \tag{3.2.4}$$

by $\eta_{\alpha\beta\gamma} = \widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\gamma\alpha}$. Because the $g_{\alpha\beta}$ satisfy the cocycle condition, we see that each $\eta_{\alpha\beta\gamma}$ maps into $\mathbb{Z}_2$. Unravelling the definition of $\widetilde{g}_{\alpha\beta}$ immediately implies that they satisfy $\eta_{\alpha\beta\gamma} = \eta_{\beta\gamma\alpha} = \eta_{\beta\alpha\gamma}$. We further have that as $x^{-1} = x$ for $x \in \mathbb{Z}_2$ that $\eta_{\alpha\beta\gamma} = \eta^{-1}_{\beta\alpha\gamma}$. Because of this, Čech cohomology theory tells us that the set of all maps $\eta_{\alpha\beta\gamma}$ defines a cohomology class in $H^2(X;\mathbb{Z}_2)$, which is exactly the second Stiefel-Whitney class $w_2(X)$. If X has a Spin-structure, we are able to find a lift $\widetilde{g}_{\alpha\beta}$ satisfying the cocycle condition, which means then that the maps $\eta_{\alpha\beta\gamma}$ are all constant 1. But then $w_2(X) = 0$. Conversely, if $w_2(X) = 0$, then $\{\eta_{\alpha\beta\gamma}\}$ is a coboundary and hence we can find constant maps $\eta_{\alpha\beta}$ mapping into $\mathbb{Z}_2$ such that $\eta_{\alpha\beta}\eta_{\beta\gamma}\eta_{\gamma\alpha} = \eta_{\alpha\beta\gamma}$. Consider now the maps $\widetilde{g}'_{\alpha\beta} = \eta_{\alpha\beta}\widetilde{g}_{\alpha\beta}$. These map into $\mathrm{Spin}(n)$, and we claim these satisfy the cocycle conditions. Indeed, the maps $\eta_{\alpha\beta}$ commute with all $\widetilde{g}_{\gamma\delta}$ by linearity, and we have

$$\begin{aligned} \widetilde{g}'_{\alpha\beta}\widetilde{g}'_{\beta\gamma}\widetilde{g}'_{\gamma\alpha} &= (\eta_{\alpha\beta}g_{\alpha\beta})\,(\eta_{\beta\gamma}g_{\beta\gamma})\,(\eta_{\gamma\alpha}g_{\gamma\alpha}) \\ &= \eta_{\alpha\beta\gamma}\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\gamma\alpha} = 1, \end{aligned} \tag{3.2.5}$$

again because any element of $\mathbb{Z}_2$ is its own inverse.

Now on to Spin^c-structures. In this case, we are looking for a set of lifts $\widetilde{g}_{\alpha\beta}$ of $g_{\alpha\beta}$ mapping into Spin^c satisfying the cocycle condition. For a good cover we can again always find such a lift, but they may not satisfy the cocycle condition. Now assume that $w_2(X) = c \in H^2(X;\mathbb{Z})$ mod 2. Then we can lift the Čech-cocycle $\eta_{\alpha\beta\gamma}$ defined earlier to an integral cocycle $\widetilde{\eta}_{\alpha\beta\gamma}$, with the reduction mod 2 map expressed by the relation $\exp(\pi i \widetilde{\eta}_{\alpha\beta\gamma}) = \eta_{\alpha\beta\gamma}$. One can now pick a partition of unity $\{\psi_\alpha\}_\alpha$ subordinate to our open cover and form the maps $f_{\beta\gamma} = \sum_\alpha \psi_\alpha \widetilde{\eta}_{\alpha\beta\gamma}$. Then setting $\lambda_{\alpha\beta} = \exp(\pi i f_{\alpha\beta})$ we see that these map into $\mathrm{U}(1)$, and one can check then that the maps $h_{\alpha\beta}^{-1}\widetilde{g}_{\alpha\beta}$ define a Spin^c-structure for X. Conversely, note that out of transition functions $\widetilde{g}_{\alpha\beta}$ we can get the transition functions for its determinant line bundle L as the $\mathrm{U}(1)$-valued maps $\pi \circ \widetilde{g}_{\alpha\beta}$. Thus we see from applying (i, π) to $\eta_{\alpha\beta\gamma}$ that we get $w_2(X) + c_1(L) = 0$ mod 2, from which the conclusion follows, as $c_1(L)$ is integral. □

Due to this lemma, we can give another description of a Spin^c-structure. Say we are given a principal $\mathrm{U}(1)$-bundle $\mathcal{L} \to X$ over X with $c_1(\mathcal{L}) \equiv w_2(X) \mod 2$. Then we can obtain a Spin^c-structure on X by letting $\widetilde{P} \to X$ be the principal $\mathrm{Spin}^c(4)$-bundle obtained as the double covering of $P \times \mathcal{L}$ such that the covering map is ρ^c-equivariant for the principal bundle actions. The bundle $\mathcal{L}$ is again the determinant line bundle of $\widetilde{P}$ mentioned earlier. We can in fact describe all possible Spin^c-structures on a four-manifold X.

Theorem 3.2.5. *The map $\mathcal{S}^c(X) \to H^2(X;\mathbb{Z})$ given by $\widetilde{P} \mapsto c_1(\det(\widetilde{P}))$ is a surjection. Moreover, if $H^2(X;\mathbb{Z})$ is torsion-free (e.g. if X is simply connected) it is also injective.*

Proof. Let $\mathfrak{c} \in \mathcal{S}^c(X)$ be a given Spin^c-structure with determinant line bundle $\mathcal{L}$, and let another principal $\mathrm{U}(1)$-bundle $\mathcal{E} \to X$ be given. Consider now the bundle

$$\mathcal{L}' = \mathcal{L} \otimes \mathcal{E}^2. \tag{3.2.6}$$

This is again a principal $\mathrm{U}(1)$-bundle, and furthermore its first Chern class satisfies

$$c_1(\mathcal{L}') = c_1(\mathcal{L}) + 2c_1(\mathcal{E}) \equiv w_2(X) \mod 2. \tag{3.2.7}$$

In other words, $\mathcal{L}'$ again specifies a Spin^c-structure. Looking at a local description of this process it is not hard to show that all Spin^c-structures arise in this manner. The failure of injectivity comes from the fact that different Spin^c-structures might have the same first Chern class. □

We see that in the simply-connected case can identify equivalence classes of Spin^c-structures on X with $H^2(X,\mathbb{Z})$. Because of this theorem, we say that $\mathcal{S}^c(X)$ is an $H^2(X;\mathbb{Z})$-torsor, i.e. a homogeneous space for the action we just described. This action is now shown to be transitive and in the absence of torsion also free, but there is no canonical choice of 'origin', so that we then only get that $\mathcal{S}^c(X)$ is non-canonically isomorphic to $H^2(X;\mathbb{Z})$. We now have the following very nice result for $n = 4$, i.e. for four-manifolds, due to Hirzebruch and Hopf [13].

Theorem 3.2.6. *Let X be an oriented four-manifold. Then X admits a* Spin^c*-structure.*

Proof. By an algebraic topology argument, the second Stiefel-Whitney class $w_2(X)$ always lifts to an integral class $c \in H^2(X;\mathbb{Z})$. Now construct a line bundle L such that $c_1(L) = c$ and pick a Hermitian metric for L. Then consider its principal U(1)-bundle $\mathcal{L}$. More details can be found in [11, Appendix A]. The proof of the lifting of the second Stiefel-Whitney class simplifies when we assume the manifold is simply-connected and compact (simply-connectedness implies that it is orientable). Namely, consider the short exact sequence of coefficient groups

$$1 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{r} \mathbb{Z}_2 \to 1, \tag{3.2.8}$$

where $r : \mathbb{Z} \to \mathbb{Z}_2$ is given by reduction mod 2. This induces a long exact sequence in cohomology,

$$\ldots \to H^2(X;\mathbb{Z}) \xrightarrow{2\cdot} H^2(X;\mathbb{Z}) \xrightarrow{r} H^2(X;\mathbb{Z}_2) \xrightarrow{\beta} H^3(X;\mathbb{Z}) \to \ldots, \tag{3.2.9}$$

where β is called the Bockstein homomorphism. We need to show that $w_2(X) \in \operatorname{im}(r) \subset H^2(X;\mathbb{Z}_2)$. But by exactness of the above sequence, we see that $\operatorname{im}(r) = \ker(\beta)$. However, because X is simply connected we have $H^1(X;\mathbb{Z}) = \{0\}$, and then by compactness Poincaré duality implies that $H^3(X;\mathbb{Z}) = \{0\}$ as well. This in turn implies that $\ker(\beta) = H^2(X;\mathbb{Z}_2)$, so that the conclusion is immediate. □

Now, recall that for $\mathrm{Spin^c}(4)$ we had the very special representations $\rho^{\mathrm{c}}_\pm$ onto the vector spaces $W_\pm \cong \mathbb{C}^2$. Using these and the associated bundle construction from Definition 2.1.10 we can create the spinor bundles, which are one of our main objects of interest.

Definition 3.2.7. Given an oriented Riemannian four-manifold X and a $\mathrm{Spin^c}$-structure $\mathfrak{c} \in \mathcal{S}^{\mathrm{c}}(X)$ with principal $\mathrm{Spin^c}(4)$-bundle $\widetilde{P}$, the *complex plus and minus spinor bundles* $S^\pm(\mathfrak{c})$ are defined by

$$S^\pm(\mathfrak{c}) = \widetilde{P} \times_{\rho^{\mathrm{c}}_\pm} W_\pm. \tag{3.2.10}$$

The *spinor bundle* $S(\mathfrak{c})$ is defined by the direct sum $S(\mathfrak{c}) = S^+(\mathfrak{c}) \oplus S^-(\mathfrak{c})$.

Sections $\psi \in \Gamma(S^\pm(\mathfrak{c}))$ are called *positive* and *negative spinors* respectively. The bundles $S^\pm(\mathfrak{c})$ are the U(2)-bundles associated with the projections $\mathrm{Spin^c}(4) \to \mathrm{U}(2)$ corresponding to the two factors $S\mathrm{U}_\pm(2)$ in $\mathrm{Spin^c}(4)$. This means one can think of them as complex rank 2 vector bundles with Hermitian inner products. Indeed, a complex rank 2 vector bundle initially has structure group $\mathrm{GL}(2,\mathbb{C})$, and the choice of an Hermitian inner product corresponds to a reduction of the structure group to $\mathrm{U}(2) < \mathrm{GL}(2,\mathbb{C})$. The determinant line bundle $\det_{\mathbb{C}}(S^\pm) = L$ is then just the Hermitian line bundle associated to $\mathcal{L}$, as is clear from equation (3.1.15). In particular, both determinant line bundles of $S^+(\mathfrak{c})$ and $S^-(\mathfrak{c})$ are canonically isomorphic. One can also define the spinor bundle in arbitrary dimensions n, but this requires a more abstract description of the representations of $\mathrm{Spin^c}(n)$.

3.3 Clifford multiplication

In this section we introduce an important operation on the spinor bundles. We will create a map called *Clifford multiplication*, which is an action of the cotangent bundle T^*X on S. In other words, we will define a map

$$\mathrm{cl}: T^*X \to \mathrm{End}(S(\mathfrak{c})) = \mathrm{End}(S^+(\mathfrak{c}) \oplus S^-(\mathfrak{c})). \tag{3.3.1}$$

Let us now describe how to define this action. As in equation (3.1.2) we identify $\mathbb{R}^4 \cong \mathbb{H}$ with the subspace V of complex 2×2-matrices. Now consider the map

$$\mathrm{cl}: \mathbb{R}^4 \times W_+ \to W_-, \qquad (x, \psi) \mapsto x\psi, \tag{3.3.2}$$

where the expression $x\psi$ means having the matrix in V corresponding to x act on ψ through ordinary matrix multiplication.

Lemma 3.3.1. *This map* cl *induces a bundle map* $T^*X \otimes S^+(\mathfrak{c}) \to S^-(\mathfrak{c})$.

Proof. Recall that $S^+(\mathfrak{c})$ and $S^-(\mathfrak{c})$ are defined through the associated vector bundle construction out of our given principal $\mathrm{Spin}^c(4)$-bundle. To show this map indeed gives rise to a bundle map, we must check that it commutes with the representations $\rho^c_\pm$ used to define $S^\pm(\mathfrak{c})$, and the representation $\rho^c : \mathrm{Spin}^c(4) \to \mathrm{GL}(V)$. Hence let $p = [A_+, A_-, \lambda] \in \mathrm{Spin}^c(4)$ be given. Then we have

$$\begin{aligned} \mathrm{cl}(\rho^c(p)x, \rho^c_+(p)\psi) &= \mathrm{cl}(A_- x A_+^{-1}, \lambda A_+ \psi) = A_- x A_+^{-1} \lambda A_+ \psi \\ &= \lambda A_- x \psi = \rho^c_-(p) x \psi \\ &= \rho^c_-(p)\mathrm{cl}(x, \psi). \end{aligned} \tag{3.3.3}$$

□

We can similarly define a map $\mathrm{cl}: \mathbb{R}^4 \times W_- \to W_+$ given by $(x, \varphi) = -\overline{x}^t \varphi$. Again one readily checks that this induces a bundle map $T^*X \otimes S^-(\mathfrak{c}) \to S^+(\mathfrak{c})$.

Definition 3.3.2. The *Clifford multiplication map* is defined by $\mathrm{cl}(v)\psi = \mathrm{cl}(v, \psi)$ for $v \in T^*X$, $\psi \in \Gamma(S(\mathfrak{c}))$.

We can now check that this map satisfies the three properties mentioned at the beginning of this section.

Lemma 3.3.3. *The map* cl *satisfies the following three properties.*

1. *Given* $v \in T^*X$ *we have* $\mathrm{cl}(v) \circ \mathrm{cl}(v) = -|v|^2 Id$;
2. *If* $v \in T^*X$ *satisfies* $|v| = 1$, *then* $\mathrm{cl}(v)$ *is a unitary transformation;*
3. *Given* $v \in T^*X$, *its Clifford multiplication maps* $\mathrm{cl}(v) : S^\pm(\mathfrak{c}) \to S^\mp(\mathfrak{c})$.

Proof. Let $v \in T^*X$ be given. By definition it maps positive spinors to negative spinors and vice versa, establishing property 3. Moreover, we have for $\psi \in \Gamma(S^+(\mathfrak{c}))$ that

$$\mathrm{cl}(v) \circ \mathrm{cl}(v)\psi = \mathrm{cl}(v)v\psi = -\overline{v}^t v\psi = -\det(v)\psi = -|v|^2\psi \tag{3.3.4}$$

where we use the local identification $\mathbb{R}^4 \cong V$ and equation (3.1.5). Property 1 follows. That $\mathrm{cl}(v)$ is unitary when $|v| = 1$ is now immediate, so that property 2 is seen to hold as well. □

We can extend Clifford multiplication to a map on two-forms $\mathrm{cl} : \bigwedge^2 T^*X \to \mathrm{End}(S(\mathfrak{c})$ by setting

$$\mathrm{cl}(v \wedge w) = \mathrm{cl}(v)\mathrm{cl}(w) - \mathrm{cl}(w)\mathrm{cl}(v). \tag{3.3.5}$$

Note that Clifford multiplication by two-forms preserves the positive and negative spinors. For later use it will be important to note that we can consider this map after complexification and restriction as a map

$$\mathrm{cl}_+ : \bigwedge\nolimits_+^2 T^*X \otimes \mathbb{C} \to \mathrm{End}(S_+). \tag{3.3.6}$$

We will now determine the matrices in V are associated to this map cl_+. Consider a local trivialization of T^*X given by the orthonormal frame e_1, e_2, e_3, e_4. Then we know that the space of self-dual two-forms $\bigwedge_+^2 T^*X$ is spanned by the elements

$$\alpha_1 = e_1 \wedge e_2 + e_3 \wedge e_4, \qquad \alpha_2 = e_1 \wedge e_3 - e_2 \wedge e_4 \qquad \text{and} \qquad \alpha_3 = e_1 \wedge e_4 + e_2 \wedge e_3. \tag{3.3.7}$$

Lemma 3.3.4. *Under the identification $\mathbb{R}^4 \cong V$, Clifford multiplication cl_+ by the elements α_i is given by the matrices*

$$\mathrm{cl}_+(\alpha_1) = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}, \qquad \mathrm{cl}_+(\alpha_2) = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \qquad \mathrm{cl}_+(\alpha_3) = \begin{pmatrix} 0 & -2i \\ -2i & 0 \end{pmatrix}. \tag{3.3.8}$$

Proof. First recall that by the identification $\mathbb{R}^4 \cong V$, the elements e_i correspond under Clifford multiplication to the matrices given in equation (3.1.3). Because of this, we merely compute

$$\begin{aligned} \mathrm{cl}_+(\alpha_1) &= \mathrm{cl}(e_1 \wedge e_2) + \mathrm{cl}(e_3 \wedge e_4) = (\mathrm{cl}(e_1)\mathrm{cl}(e_2) - \mathrm{cl}(e_2)\mathrm{cl}(e_1)) + (\mathrm{cl}(e_3)\mathrm{cl}(e_4) - \mathrm{cl}(e_4)\mathrm{cl}(e_3)) \\ &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) + \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} - \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}. \end{aligned} \tag{3.3.9}$$

The proofs for α_2 and α_3 are completely analogous. □

Remark 3.3.5. Note in particular that the above lemma shows that self-dual two-forms act by traceless skew-Hermitian endomorphisms. In fact, it shows that by Clifford multiplication $\bigwedge_+^2 T^*X$ and the space of traceless skew-Hermitian endomorphisms of S^+ are isomorphic. Similarly, one can check that the restriction cl_- to $\bigwedge_-^2 T^*X$ gives an isomorphism from this space to the space of trace-free skew-Hermitian endomorphisms of S^-. This type of isomorphism only holds in four dimensions, and will be important in stating the Seiberg-Witten equations.

3.4 Dirac operators

Let $\widetilde{P} \to X$ be a Spin^c-structure with determinant principal $\mathrm{U}(1)$-bundle $\mathcal{L}$. In this section we discuss the notion of a Dirac operator, mostly following [22]. This will be a differential operator between sections of the spinor bundle $S(\mathfrak{c})$. For this we will first have to construct a connection on $S(\mathfrak{c})$. As X comes with a given Riemannian metric g, its tangent bundle carries a natural connection, namely the Levi-Civita connection ∇_{LC}. We can lift this connection to a connection on its frame bundle. Now let A be a unitary connection on $\mathcal{L}$. Then ∇_{LC} together with A give a connection on the principal $\mathrm{SO}(4) \times \mathrm{U}(1)$-bundle which is the quotient of $\widetilde{P}$ by $\{\pm 1\}$. Now recall that by $\mathrm{Spin}^{\mathrm{c}}(4)$ is the double-cover of $\mathrm{SO}(4) \times \mathrm{U}(1)$, so that they have the same Lie algebras, $\mathfrak{spin}^{\mathrm{c}}(4) = \mathfrak{so}(4) \oplus i\mathbb{R}$. Because of this, this connection in turn induces a unique connection on $\widetilde{P}$, and following the representations $\rho^{\mathrm{c}}_{\pm}$ we then get a connection on the spinor bundle $S(\mathfrak{c})$ which we will denote by ∇_A. In other words, we get a covariant derivative

$$\nabla_A : \Gamma(S(\mathfrak{c})) \to \Gamma(T^*X \otimes S(\mathfrak{c})). \tag{3.4.1}$$

This connection is called the *spin connection*, and it satisfies the following compatibility condition with respect to Clifford multiplication on $S(\mathfrak{c})$ and the Levi-Civita connection. For any vector fields $V, W \in \Gamma(T^*X)$ and section $s \in \Gamma(S(\mathfrak{c}))$ we have the following Leibniz rule

$$\nabla_A(\mathrm{cl}(V)s) = \mathrm{cl}(\nabla_{\mathrm{LC}} V)s + \mathrm{cl}(V)\nabla_A s. \tag{3.4.2}$$

Some important things to note are that ∇_A leaves the bundles $S^+(\mathfrak{c})$ and $S^-(\mathfrak{c})$ invariant, and if we let it induce a connection on either determinant line bundle, we get back our original connection A. It will be useful to note how spin connections made from different connections A and A' on L are related. Recall that by Lemma 2.2.7 the space of $\mathrm{U}(1)$-connections on L is affine over $\Omega^1(X; i\mathbb{R})$.

Lemma 3.4.1. *Let A and $A' = A + a$ for $a \in \Omega^1(X; i\mathbb{R})$ be two $\mathrm{U}(1)$-connections on L. Then we have*

$$\nabla_{A'} = \nabla_A + \frac{1}{2}a. \tag{3.4.3}$$

Proof. This is straightforward, noting that the factor $\frac{1}{2}$ stems from the fact that the map $\pi : \mathrm{Spin}^{\mathrm{c}}(4) \to \mathrm{U}(1)$ used to get back L is given by $\pi([A_+, A_-, \lambda]) = \lambda^2$. □

Using the spin connection ∇_A we can define our Dirac operator.

Definition 3.4.2. Given a unitary connection A on $\mathcal{L}$, the *Dirac operator* D_A is defined to be the map

$$D_A : \Gamma(S^+(\mathfrak{c})) \to \Gamma(S^-(\mathfrak{c})) \tag{3.4.4}$$

given by the composition

$$D_A : \Gamma(S^+(\mathfrak{c})) \overset{\nabla_A}{\to} \Gamma(S^+(\mathfrak{c}) \otimes T^*X) \overset{\mathrm{cl}}{\to} \Gamma(S^-(\mathfrak{c})), \tag{3.4.5}$$

where we use the Clifford multiplication map cl described in the previous section.

It is perhaps instructive to give a local expression for D_A. Given a local orthonormal system of coordinates $\{e_i\}$ and a section $\psi \in \Gamma(S^+(\mathfrak{c}))$, we have

$$D_A(\psi) = \sum_{i=1}^{4} \mathrm{cl}(e_i)\nabla_{A,e_i}\psi. \tag{3.4.6}$$

One can readily check that this expression is indeed invariant, i.e. does not depend on the coordinate system chosen. Moreover, one sees from this that D_A is a linear first order differential operator. Of course, the same defining expression for D_A makes sense to define an operator $\Gamma(S^-(\mathfrak{c})) \to \Gamma(S^+(\mathfrak{c}))$, also denoted by D_A. We can then consider the Dirac operator as a map $\widetilde{D}_A$ between sections of the full spinor bundle $S(\mathfrak{c})$, which in matrix form with respect to the splitting $S = S^+ \oplus S^-$ reads

$$\widetilde{D}_A = \begin{pmatrix} 0 & D_A \\ D_A & 0 \end{pmatrix}. \tag{3.4.7}$$

The fact that Clifford multiplication by one-forms sends positive spinors to negative spinors and vice versa is reflected by the diagonal of this matrix being zero. Now recall that the bundles $S^\pm$ are equipped with metrics through the metric g on TX and the metric on L. Because of this, we can define a formal adjoint to $D_A : \Gamma(S^+(\mathfrak{c})) \to \Gamma(S^-(\mathfrak{c}))$, giving an operator $D_A^* : \Gamma(S^-(\mathfrak{c}) \to \Gamma(S^+(\mathfrak{c}))$. This operator is identical to operator D_A we defined between $\Gamma(S^-(\mathfrak{c}))$ and $\Gamma(S^+(\mathfrak{c}))$ using the local expression.

Lemma 3.4.3. *Let X be compact. Then the Dirac operator $\widetilde{D}_A : \Gamma(S(\mathfrak{c})) \to \Gamma(S(\mathfrak{c}))$ is formally self-adjoint. In other words, for any two sections $\psi, \varphi \in \Gamma(S(\mathfrak{c}))$ we have*

$$\int_X \langle \widetilde{D}_A \psi, \varphi \rangle \, \mathrm{d}\mu = \int_X \langle \psi, \widetilde{D}_A \varphi \rangle \, \mathrm{d}\mu. \tag{3.4.8}$$

Proof. Let $x \in X$ be some point. Trivialize S in some neighbourhood U of x using a local orthonormal frame $\{e_i\}$ of S which is *moving*, i.e. satisfies $\nabla_{A,e_i} e_j = 0$ for all i, j. We then have

$$\langle \widetilde{D}_A \psi, \varphi \rangle(x) = \sum_{i=1}^{4} \langle \mathrm{cl}(e_i) \nabla_{A,e_i} \psi, \varphi \rangle(x). \tag{3.4.9}$$

Now recall that Clifford multiplication by unit vectors is a orthogonal map. As such, we have

$$\langle \mathrm{cl}(e_i) \nabla_{A,e_i} \psi, \varphi \rangle = \langle \mathrm{cl}(e_i)\mathrm{cl}(e_i) \nabla_{A,e_i} \psi, \mathrm{cl}(e_i)\varphi \rangle = -\langle \nabla_{A,e_i} \psi, \mathrm{cl}(e_i)\varphi \rangle, \tag{3.4.10}$$

using the defining expression $\mathrm{cl}(e_i)\mathrm{cl}(e_i) = -|e_i|^2 \mathrm{Id}$. Using this together with the fact that ∇_A is a unitary connection we get by the Leibniz rule for the spin connection that

$$\begin{aligned}
\langle D_a \psi, \varphi \rangle(x) &= -\sum_{i=1}^{4} \langle \nabla_{A,e_i} \psi, \mathrm{cl}(e_i)\varphi \rangle(x) \\
&= \sum_{i=1}^{4} \langle \psi, \nabla_{A,e_i}(\mathrm{cl}(e_i)\varphi) \rangle(x) - \frac{\partial}{\partial e_i} \langle \psi, \mathrm{cl}(e_i)\varphi \rangle(x) \\
&= \sum_{i=1}^{4} \langle \psi, \mathrm{cl}(e_i) \nabla_{A,e_i} \varphi \rangle(x) + \langle \psi, \mathrm{cl}(\nabla_{\mathrm{LC},e_i}(e_i))\varphi \rangle(x) - \frac{\partial}{\partial e_i} \langle \psi, \mathrm{cl}(e_i)\varphi \rangle(x) \\
&= \langle \psi, \widetilde{D}_A \varphi \rangle(x) - \sum_{i=1}^{4} \frac{\partial}{\partial e_i} \langle \psi, \mathrm{cl}(e_i)\varphi \rangle(x).
\end{aligned} \tag{3.4.11}$$

Now we need only notice that $\sum_{i=1}^{4} \frac{\partial}{\partial e_i} \langle \psi, \mathrm{cl}(e_i)\varphi \rangle(x) = d^*\omega(x)$ where ω is the one-form given by $\langle \omega, e_i \rangle = \langle \psi, \mathrm{cl}(e_i)\varphi \rangle$.[1] We can now integrate over X and use Stokes' theorem to finish the proof. □

We can compare the square $D_A^* D_A$ of the Dirac operator $D_A : \Gamma(S^+(\mathfrak{c})) \to \Gamma(S^-(\mathfrak{c}))$ to the vector bundle Laplacian $\nabla_A^* \nabla_A$. Let us first give a local expression for this latter operator.

Lemma 3.4.4. *Let $\{e_i\}$ be a moving orthonormal frame. Then we have*

$$\nabla_A^* \circ \nabla_A(\psi) = -\sum_{i=1}^{4} \nabla_{A,e_i} \circ \nabla_{A,e_i}(\psi). \tag{3.4.13}$$

Proof. Let $\psi \in \Gamma(S(\mathfrak{c}))$ and $\alpha \in \Gamma(T^*X \otimes S(\mathfrak{c}))$ be given. Then by definition of the adjoint we have

$$\langle \nabla_A \psi, \alpha \rangle = \langle \psi, \nabla_A^* \alpha \rangle. \tag{3.4.14}$$

Because ∇_A is unitary, it satisfies

$$d\langle \psi, \varphi \rangle(x) = \langle \nabla_A \psi, \varphi \rangle(x) + \langle \psi, \nabla_A \varphi \rangle(x) \qquad \text{for all } \varphi \in \Gamma(S(\mathfrak{c})). \tag{3.4.15}$$

Now choose $\varphi = *\alpha$ in the above formula. Note that if we then integrate the above expression the left hand side vanishes, so that by definition of the inner product we get

$$\langle \nabla_A \psi, \alpha \rangle = \int_X \langle \nabla_A \psi, *\alpha \rangle(x) = -\int_X \langle \psi, \nabla_A(*\alpha) \rangle(x) = \langle \psi, - * \nabla_A(*\alpha) \rangle. \tag{3.4.16}$$

This implies that $\nabla_A^* = - * \nabla_A *$ when applied to S-valued one-forms. Because of this, we get

$$\nabla_A^* \circ \nabla_A(\psi) = -*\nabla_A \left(\sum_{i=1}^{4} \nabla_{A,e_i}(\psi) * (\mathrm{d}e_i) \right) = -*\sum_{i=1}^{4} \nabla_{A,e_i} \nabla_{A,e_i}(\psi) \mathrm{d}\mu = -\sum_{i=1}^{4} \nabla_{A,e_i} \circ \nabla_{A,e_i}(\psi). \tag{3.4.17}$$

□

The resulting formula we get through this comparison is called *Weitzenböck formula*, and will prove to be very useful later on.

Theorem 3.4.5. *The Dirac operator satisfies*

$$D_A^* D_A = \nabla_A^* \nabla_A + \frac{1}{4} s + \frac{1}{2} \mathrm{cl}(F_A), \tag{3.4.18}$$

where $s : X \to \mathbb{R}$ denotes the scalar curvature of g.

[1] This is because the codifferential d^* is given locally by

$$d^*\omega = -\sum_{i=1}^{4} \iota(e_i) \nabla_{\mathrm{LC},e_i} \omega, \tag{3.4.12}$$

where ι denotes internal contraction.

Proof. Let ψ be a section of $S(\mathfrak{c})$ and let $\{e_i\}$ be a moving orthonormal frame. Then we have by the Leibniz rule and the definition of Clifford multiplication that

$$\begin{aligned} D_A^* D_A \psi &= \left(\sum_{i=1}^4 \mathrm{cl}(e_i) \nabla_{A,e_i} \right) \left(\sum_{j=1}^4 \mathrm{cl}(e_j) \nabla_{A,e_j} \right) \psi \\ &= \sum_{i,j=1}^4 \mathrm{cl}(e_i) \nabla_{A,e_i} \left(\mathrm{cl}(e_j) \nabla_{A,e_j} \psi \right) \\ &= \sum_{i,j=1}^4 \mathrm{cl}(e_i) \left(\mathrm{cl}(\nabla_{\mathrm{LC},e_i} e_j) \nabla_{A,e_j}(\psi) + \mathrm{cl}(e_j) \nabla_{A,e_i} \nabla_{A,e_j} \psi \right) \\ &= \sum_{i,j=1}^4 \mathrm{cl}(e_i) \mathrm{cl}(e_j) \nabla_{A,e_i} \nabla_{A,e_j} \psi \\ &= -\sum_{i=1}^4 \nabla_{A,e_i} \nabla_{A,e_i} \psi + \frac{1}{2} \sum_{i,j=1}^4 \mathrm{cl}(e_i) \mathrm{cl}(e_j) \left(\nabla_{A,e_i} \nabla_{A,e_j} - \nabla_{A,e_j} \nabla_{A,e_i} \right) \psi \\ &= \nabla_A^* \nabla_A \psi + \frac{1}{2} \sum_{i,j=1}^4 \mathrm{cl}(e_i) \mathrm{cl}(e_j) \nabla_A^2(\psi)(e_i, e_j). \end{aligned} \tag{3.4.19}$$

The second term can be seen to split into two parts, one of which is scalar multiplication of ψ by the scalar curvature, and the other is Clifford multiplication by the curvature F_A. See [22, Proposition 5.1.5.] for more details. □

It will be useful later on to know how to relate the Dirac operators of two connections A and A' with each other. Again recall that by Lemma 2.2.7 the space of $\mathrm{U}(1)$-connections on L is affine over $\Omega^1(X; i\mathbb{R})$.

Lemma 3.4.6. *Let A and $A' = A + \alpha$ be two $\mathrm{U}(1)$-connections on L. Then we have*

$$D_{A+\alpha} \psi = D_A \psi + \frac{1}{2} \mathrm{cl}(a) \psi \qquad \text{for } \psi \in \Gamma(S(\mathfrak{c})). \tag{3.4.20}$$

Proof. This readily follows from the local description of the Dirac operator: if ω and A are the connection one-forms for the spin connection we have by equation (3.4.6) and Lemma 3.4.1 that

$$D_A \psi = \sum_i \mathrm{cl}(e_i) \frac{d\psi}{de_i} + \frac{1}{2} A(e_i) \mathrm{cl}(e_i) + \sum_{k<j} \omega_{k,j}(e_i) \mathrm{cl}(e_j e_k) \psi. \tag{3.4.21}$$

Changing A to A' indeed just amounts to changing the connection one-form from A to $A + a$, and we note that by Definition 3.3.2 that

$$\sum_i a(e_i) \mathrm{cl}(e_i) \psi = \mathrm{cl}(a) \psi. \tag{3.4.22}$$

□

Recall that we mentioned that D_A is a linear first order differential operator. It is a natural question to ask whether it is elliptic, so that it is Fredholm and hence has an index. We now show that this is indeed the case, and will in fact give a formula for its index.

Lemma 3.4.7. *Any Dirac operator D_A is elliptic.*

Proof. Consider again the local expression for D_A used in the proof of Lemma 3.4.6. As the highest order term does not involve the connection matrices at all, we see that for $\xi \in T_x^*X$ we have

$$\sigma(D_A)(\xi) = i \sum_j \mathrm{cl}(\xi_j), \tag{3.4.23}$$

where we use the metric to identify T_x^*X and T_xX. In other words, the principal symbol of D_A at ξ is just Clifford multiplication by $i\xi$. By the defining relation for Clifford multiplication this is an isomorphism for all nonzero ξ. □

Remark 3.4.8. Note that the previous proof showed that the principal symbol of D_A is independent of the unitary connection A on L used to define it.

Now because any Dirac operator is elliptic, it is Fredholm. Furthermore, we can calculate its index using the Atiyah-Singer index theorem. This is a very deep theorem expressing the index of any elliptic differential operator on a compact manifold in terms of its principal symbol and topological information of the bundles involved. We will not state the precise formulation of this theorem and instead only give the result of applying it to our situation.

Theorem 3.4.9. *Let X be a compact oriented four-manifold with a Spin^c-structure $\mathfrak{c}$. Then the index of $D_A : \Gamma(S^+(\mathfrak{c})) \to \Gamma(S^-(\mathfrak{c}))$ is*

$$\mathrm{ind}(D_A) = \frac{c_1(\mathcal{L})^2 - \tau(X)}{8}, \tag{3.4.24}$$

where $\tau(X)$ is the signature of X.

Remark 3.4.10. We saw that the principal symbol of D_A does not depend on the connection A. As by the Atiyah-Singer index theorem its index only depends on it principal symbol, this index should be independent of the connection A as well. The above formula shows that this is indeed the case.

3.5 Spin$^{\mathrm{c}}$-structures for a symplectic manifold

In this section we describe how a symplectic manifold X is naturally equipped with a Spin^c-structure. Moreover, we give a more explicit description of the spinor bundles and Dirac operators in this case. This section has two main goals. The first is to elucidate the perhaps abstract definitions of a Spin^c-structure and accompanying spinor bundles and Dirac operators. The second is to develop the necessary formulas for the Dirac operator in this context for when we discuss the Seiberg-Witten invariants for symplectic manifolds in Section 5.6.

This section follows [15].

Let X be a four-manifold equipped with an almost complex structure J. Consider once more the frame bundle P of TX. Through the use of J we can reduce the structure group of P to $\mathrm{U}(2)$. Now note that there is a natural inclusion of $\mathrm{U}(2)$ into $\mathrm{SO}(4)$, which we will denote by $i : \mathrm{U}(2) \hookrightarrow \mathrm{SO}(4)$. Similarly, we have the group homomorphism $\det : \mathrm{U}(2) \to \mathrm{U}(1)$ given by taking the determinant. Looking at the definition of the group $\mathrm{Spin}^{\mathrm{c}}(4)$, it should now be clear that it is possible to lift the map $i \times \det : \mathrm{U}(2) \to SO(4) \times \mathrm{U}(1)$ to a map $j : \mathrm{U}(2) \to \mathrm{Spin}^{\mathrm{c}}(4)$. By reduction of the structure group, we then immediately get the following result.

Lemma 3.5.1. *An almost complex structure J equips X with a canonical Spin^c-structure $\mathfrak{c}_J$.*

Proof. More specifically, the canonical embedding $j : \mathrm{U}(2) \to \mathrm{Spin}^{\mathrm{c}}(4)$ is given by

$$j(A) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \det A & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{A}} \\ 0 & 0 & & \end{pmatrix} \subset \mathrm{U}(2) \times \mathrm{U}(2). \tag{3.5.1}$$

Looking at equation (3.1.15) it is clear that this indeed lands in $\mathrm{Spin}^{\mathrm{c}}(4)$. □

Let us now describe the positive and negative spinor bundles in this situation. It will be convenient to assume X is symplectic with symplectic form ω, which we will do from here on. Now let $\mathfrak{c}$ be any $\mathrm{Spin}^{\mathrm{c}}$-structure on X, and let J be ω-compatible as per Definition 2.3.10. Recall that we can act on the spinor bundle $S(\mathfrak{c})$ with forms through Clifford multiplication. In particular we get a map

$$\mathrm{cl}_+(\omega) : S^+(\mathfrak{c}) \to S^+(\mathfrak{c}). \tag{3.5.2}$$

Now recall that we can choose local coordinates $\{x_i\}$ on X such that the symplectic form looks like

$$\omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4. \tag{3.5.3}$$

By Lemma 3.3.4 we see from this that

$$\mathrm{cl}_+(\omega) = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}. \tag{3.5.4}$$

Because of this we can split $S^+(\mathfrak{c})$ up into a $+2i$- and $-2i$-eigenspace for $\mathrm{cl}_+(\omega)$. Denote the subbundle of $S^+(\mathfrak{c})$ given by the $-2i$-eigenspace by $E \to X$. This is necessarily a line bundle, and hence it will have a first Chern class $c_1(E)$.

Proposition 3.5.2. *The positive and negative spinor bundles $S^{\pm}(\mathfrak{c})$ are given by*

$$\begin{aligned} S^+(\mathfrak{c}) &= \left(\bigwedge^{0,0} T^*X \oplus \bigwedge^{0,2} T^*X\right) \otimes E = E \oplus (K^{-1} \otimes E), \\ S^-(\mathfrak{c}) &= \bigwedge^{0,1} T^*X \otimes E. \end{aligned} \tag{3.5.5}$$

Moreover, Clifford multiplication by an element $v \in T^*(X) \otimes \mathbb{C}$ on a form $\alpha \in \bigwedge^{0,\bullet} T^*X$ is given by

$$\mathrm{cl}(v)(\alpha) = \sqrt{2}(v^{0,1} \wedge \alpha - \iota(\overline{v^{1,0}})\alpha). \tag{3.5.6}$$

Proof. Consider first the case of S^+. From the discussion preceding the theorem it is clear that we must show that the $+2i$-eigenspace of $\mathrm{cl}_+(\omega)$ is isomorphic to $K^{-1} \otimes E$. An isomorphism from $K^{-1} \otimes E$ to this $+2i$-eigenspace can be given by $\frac{1}{2}\mathrm{cl}$, as one then checks that in local coordinates that

$$\frac{1}{2}\mathrm{cl}(\overline{\mathrm{d}z} \wedge \overline{\mathrm{d}w}) = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}. \tag{3.5.7}$$

The isomorphism for S^- is constructed similarly by considering instead $\frac{1}{\sqrt{2}}\mathrm{cl}$. The given relation for Clifford multiplication is then immediate. □

The above lemma can also be proven using the property that the irreducible spin representations ρ_+^{c} and ρ_-^{c} used to construct the spinor bundle are up to isomorphism unique. This line of proof can be found in [23, Section 3.4].

Remark 3.5.3. The Spin$^{\mathrm{c}}$-structure $\mathfrak{c}_J$ we constructed in Lemma 3.5.1 has E trivial, so that we can realize the non-canonical isomorphism $\mathcal{S}^{\mathrm{c}}(X) \cong H^2(X;\mathbb{Z})$ from Theorem 3.2.5 by the map

$$E \mapsto c_1(E). \tag{3.5.8}$$

We immediately see that a Spin$^{\mathrm{c}}$-structure $\mathfrak{c} = \mathfrak{c}_E$ has determinant line bundle

$$L(\mathfrak{c}_E) = \det(S^+(\mathfrak{c}_E)) = K^{-1} \otimes E^2. \tag{3.5.9}$$

In particular, the determinant line bundle of the canonical Spin$^{\mathrm{c}}$-structure associated to J is K^{-1}.

Let us now turn to describing the Dirac operator in this setting. It will turn out that Dirac operator can be neatly expressed in terms of the Cauchy-Riemann operator $\overline{\partial}$, for any almost complex manifold. However, we will only do so in the case of a symplectic manifold. An initial observation is that by Lemma 3.4.7 the Dirac operator has the same symbol as the standard Dolbeault operator $\sqrt{2}(\overline{\partial} + \overline{\partial}^*)$. What this means is that they can only differ as differential operators by a term of order zero. So let $\mathfrak{c}_J$ be the canonical Spin$^{\mathrm{c}}$-structure and let A be any unitary connection on $\det(\mathfrak{c}_J) = K^{-1}$. Out of this we can form the associated spin connection as in Section 3.4. In other words, we get an operator

$$\nabla_A : \Gamma(S(\mathfrak{c})) \to \Gamma(T^*X \otimes S(\mathfrak{c})). \tag{3.5.10}$$

Recall now the splitting of $S^+(\mathfrak{c}_J)$ as $S^+(\mathfrak{c}_J) = \mathbb{C} \oplus K^{-1}$, and let $s_0 \in \Gamma(\mathbb{C}) \cong C^\infty(X;\mathbb{C})$ be the trivial constant section of $\mathbb{C}$ which is equal to 1 everywhere. By definition the spin connection respects the splitting of S into the positive and negative spinors, so that in fact

$$\nabla_A : \Gamma(\mathbb{C} \oplus K^{-1}) \to \Omega^1(X;\mathbb{C}) \oplus \Omega^1(X;K^{-1}). \tag{3.5.11}$$

There now is a preferred connection with which to form the Spin-connection on S.

Lemma 3.5.4. *There exists a unique unitary connection A_0 on K^{-1} such that the associated* Spin-*connection satisfies*

$$\nabla_{A_0} s_0 \in \Omega^1(X; K^{-1}). \tag{3.5.12}$$

Proof. We regard A as being given. Now let $A_0 = A + a$ be any other unitary connection on K^{-1}, where $a \in \Omega^1(X; i\mathbb{R})$ is some imaginary one-form. By Lemma 3.4.1 we have

$$\nabla_{A_0} s_0 = \nabla_A s_0 + \frac{1}{2} a s_0. \tag{3.5.13}$$

Now again decompose the term $\nabla_A s_0$ according to $\Omega^1(X;\mathbb{C}) \oplus \Omega^1(X; K^{-1})$. Then the fact that ∇_A is a metric connection implies that its $\mathbb{C}$-part $(\nabla_A s_0)_{\mathbb{C}}$ is an imaginary one-form multiplied by s_0. From this the solution is seen to be

$$a = -2(\nabla_A s_0)_{\mathbb{C}}. \tag{3.5.14}$$

□

Now, if the manifold X happens to be Kähler, then the term $\nabla_{A_0} s_0$ in fact vanishes identically. This has to do with the fact that for Kähler manifolds the Levi-Civita connection and the Chern connection agree. Hence for Kähler manifolds we further have that $D_{A_0} s_0 = 0$. While the former property does not hold when X is just almost Kähler, the latter property does.

Lemma 3.5.5. *Let (X, ω, J) be an almost Kähler four-manifold and consider the* Spin^c-*structure $\mathfrak{c}_J$. For the unitary connection A_0 on K^{-1} from Lemma 3.5.4 we have*

$$D_{A_0} s_0 = 0. \tag{3.5.15}$$

Proof. The fact that ∇_{A_0} being a spin connection satisfies the Leibniz rule, equation (3.4.2), implies after contracting by Clifford multiplication (denoted by $\widetilde{\mathrm{cl}}$) that

$$D_{A_0}(\mathrm{cl}(\omega)s_0) = \mathrm{cl}(d\omega + d^*\omega)s_0 + \widetilde{\mathrm{cl}}(\mathrm{cl}(\omega)\nabla_{A_0} s_0). \tag{3.5.16}$$

Here the latter term means the following. We have $s_0 \in \Gamma(\mathbb{C})$, and by Lemma 3.5.4 then $\nabla_{A_0} s_0 \in \Gamma(K^{-1} \otimes T^*X)$. Now apply $\mathrm{cl}(\omega)$ to get an element $\mathrm{cl}(\omega) \in \Gamma(S^+(\mathfrak{c}_J) \otimes T^*X)$, and use the contraction $\widetilde{\mathrm{cl}}$ to finally obtain an element of $\Gamma(S^-(\mathfrak{c}_J))$. Continuing, recall that s_0 lies in the $-2i$-eigenspace of $\mathrm{cl}(\omega)$. Furthermore, by Lemma 3.5.4, the term $\nabla_{A_0} s_0$ lies in the $+2i$-eigenspace of $\mathrm{cl}(\omega)$. Lastly, $d\omega = 0$, and because ω is self-dual also $d^*\omega = 0$. We see from this that equation (3.5.16) boils down to

$$D_{A_0}(-2is_0) = \mathrm{cl}(0)s_0 + \widetilde{\mathrm{cl}}(2i\nabla_{A_0} s_0). \tag{3.5.17}$$

But by definition of the Dirac operator we have $\widetilde{\mathrm{cl}}(\nabla_{A_0} s_0) = D_{A_0} s_0$. The claim follows. □

The reason we went through the trouble finding a connection A_0 satisfying the property that $D_{A_0} s_0 = 0$ will now become clear, as we will be able to write the Dirac operator for any Spin^c-structure on X in terms of the Cauchy-Riemann operator. Using Lemma 3.2.5 and Proposition 3.5.2, let $\mathfrak{c}_E = \mathfrak{c}_J \otimes E$ be any Spin^c-structure on X. The determinant line bundle of $\mathfrak{c}_E$ is given by $L = K^{-1} \otimes E^2$, so that any unitary connection $A \in \mathcal{A}(L)$ can be written as $A = A_0 \otimes B^2 = A_0 + 2B$ for B a unitary connection on E. If E is chosen to be trivial, then this twisting by B just amounts to changing the connection on K^{-1}.

Theorem 3.5.6. *Let (X, ω, J) be an almost Kähler four-manifold with a given* Spin^c*-structure* $\mathfrak{c}_E = \mathfrak{c}_J \otimes E$. *Then for all $A \in \mathcal{A}(L)$ given by $A = A_0 + 2B$ we have*

$$D_A = \sqrt{2}(\overline{\partial}_B + \overline{\partial}_B^*). \tag{3.5.18}$$

Proof. It suffices to prove this for the canonical Spin^c-structure $\mathfrak{c}_J$, as the other cases follow formally from the twisting process. Let $\alpha \in \Gamma(\mathbb{C})$ and $\beta \in \Gamma(K^{-1})$ so that $(\alpha, \beta) \in \Gamma(S^+(\mathfrak{c}_J))$. Note that β is self-dual, so that $\mathrm{cl}(d\beta) = \mathrm{cl}(d^*\beta)$. By equation (3.5.16), the fact that $D_{A_0} s_0 = 0$ by Lemma 3.5.5 and the explicit formula for Clifford multiplication given by equation (3.5.6), we can then write

$$D_{A_0}(\alpha + \beta) = D_{A_0}\left(\mathrm{cl}\left(\alpha + \frac{\beta}{2}\right) s_0\right) = \mathrm{cl}\left(d\alpha + \frac{1}{2}(d\beta + d^*\beta)\right) s_0 = \sqrt{2}(\overline{\partial}\alpha + \overline{\partial}^*\beta). \tag{3.5.19}$$

□

Chapter 4

Seiberg-Witten equations and moduli space

At this point we have reviewed the necessary mathematical objects to give the Seiberg-Witten equations. In this chapter we will state them and start following the program of establishing the properties on the wish list mentioned in the introduction. This chapter mostly follows [23].

4.1 The Seiberg-Witten equations

Let X be a compact oriented four-manifold with Riemannian metric g. Let $\mathfrak{c} \in \mathcal{S}^c(X)$ be a Spin^c-structure with determinant line bundle $L = \det(\mathfrak{c})$ having some fixed Hermitian metric h. We saw in Section 3.2 that $\mathfrak{c}$ gives rise to two spinor bundles $S^{\pm}(\mathfrak{c})$, and in Section 3.4 that a unitary connection $A \in \mathcal{A}(L)$ on L gives rise to a Dirac operator $D_A : \Gamma(S^+(\mathfrak{c})) \to \Gamma(S^-(\mathfrak{c}))$ when coupled with the Levi-Civita connection on TX. Let us now introduce the space of variables in which the Seiberg-Witten equations are expressed.

Definition 4.1.1. The Seiberg-Witten *configuration space* $\mathcal{C}(\mathfrak{c})$ is defined by

$$\mathcal{C}(\mathfrak{c}) = \mathcal{A}(L) \times \Gamma(S^+(\mathfrak{c})). \tag{4.1.1}$$

We see that $\mathcal{C}(\mathfrak{c})$ consists of all pairs (A, ψ) where A is a unitary connection on L and ψ is a section of the positive spinor bundle $S^+(\mathfrak{c})$. Note that because $S^+(\mathfrak{c})$ has a Hermitian metric through h, we can identify it with its dual through an anti-complex isomorphism. If we are given $\psi \in \Gamma(S^+(\mathfrak{c}))$ then under this isomorphism its image is denoted by ψ^*. Given $\varphi \in S^+(\mathfrak{c})$ we can then form

$$\varphi \otimes \psi^* \in S^+(\mathfrak{c}) \otimes \big(S^+(\mathfrak{c})\big)^* \cong \mathrm{End}(S^+(\mathfrak{c})). \tag{4.1.2}$$

We then have an U(2)-equivariant map

$$\sigma : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathrm{End}(\mathbb{C}^2) = \mathbb{C}^2 \otimes \mathbb{C}^{2*}, \qquad \sigma(\varphi, \psi) = \varphi \otimes \psi^* - \frac{1}{2}\mathrm{tr}(\varphi \otimes \psi^*)\mathrm{Id}. \tag{4.1.3}$$

This induces a bundle map $\sigma : S^+(\mathfrak{c}) \otimes S^+(\mathfrak{c}) \to \mathrm{End}(S^+(\mathfrak{c}))$. Now $q(\psi) = \sigma(\psi, \psi)$ defines a map

$$q : \Gamma(S^+(\mathfrak{c})) \to \mathrm{End}(S^+(\mathfrak{c})). \tag{4.1.4}$$

Note that $q(\psi)$ is a traceless endomorphism of $S^+(\mathfrak{c})$, so that by Lemma 3.3.4 we get an element of $\Omega^2_+(X; i\mathbb{R})$ after applying cl_+^{-1}. With this in mind we are now able to state the Seiberg-Witten equations.

Definition 4.1.2. The *Seiberg-Witten equations* for $(X, g, \mathfrak{c})$ are given by

$$\begin{aligned} F_A^+ &= \mathrm{cl}_+^{-1} q(\psi), \\ D_A \psi &= 0. \end{aligned} \tag{4.1.5}$$

The first of these is referred to as the monopole equation, while the second is called the Dirac equation.

We see that the monopole equation takes place in the space $\Omega^2_+(X; i\mathbb{R})$ of imaginary self-dual two-forms, while the Dirac equation takes place in the space $\Gamma(S^-(\mathfrak{c}))$ of sections of the negative spinor bundle (recall that D_A sends positive spinors to negative spinors and vice versa). Alternatively, we could have written the monopole equation as $\mathrm{cl}(F_A^+) = q(\psi)$ so that it takes place in the bundle $\mathrm{End}(S^+(\mathfrak{c}))$ of endomorphisms of the positive spinor bundle, as is also commonly found in the literature.

To ensure the space of solutions to the Seiberg-Witten equations satisfies the desirable properties of the wish list mentioned in the introduction, it will be necessary to consider slight perturbations of the equations.

Definition 4.1.3. Given an imaginary self-dual two-form $\eta \in \Omega^2_+(X; i\mathbb{R})$, the *perturbed Seiberg-Witten equations* for $(X, g, \mathfrak{c}, \eta)$ are given by

$$\begin{aligned} F_A^+ &= \mathrm{cl}_+^{-1} q(\psi) + \eta, \\ D_A \psi &= 0. \end{aligned} \tag{4.1.6}$$

Remark 4.1.4. The form η is also called the *perturbation parameter.* Furthermore, we could again have written $\mathrm{cl}(F_A^+) = q(\psi) + \mathrm{cl}(\eta)$ instead.

To study the space of solutions it will be useful to capture the (perturbed) Seiberg-Witten equations in one map F, such that solutions to the Seiberg-Witten equations correspond to pairs in the configuration space such that $F(A, \psi) = 0$.

Definition 4.1.5. Given an imaginary self-dual two-form $\eta \in \Omega^2_+(X; i\mathbb{R})$, the *Seiberg-Witten map* is the map $F : \mathcal{C}(\mathfrak{c}) \to \Omega^2_+(X; i\mathbb{R}) \oplus \Gamma(S^-(\mathfrak{c}))$ defined by

$$F(A, \psi) = (F_A^+ - \mathrm{cl}_+^{-1} q(\psi), D_A \psi). \tag{4.1.7}$$

Similarly, the *perturbed Seiberg-Witten map* is the map F_η given by

$$F_\eta(A, \psi) = (F_A^+ - \mathrm{cl}_+^{-1} q(\psi) - \eta, D_A \psi). \tag{4.1.8}$$

We wish to study the space of all solutions to the Seiberg-Witten equations given a pair $(X, \mathfrak{c})$. However, recall that the bundles in which the equations take place carry groups of automorphisms. Some of these automorphisms leave the Seiberg-Witten equations invariant, and hence send solutions to solutions. Because of this it is more natural to study the space of solutions not related to each other by such an automorphism. Our choice will be to use those automorphisms of the principal Spin^c-bundle $\widetilde{P}$ which cover the identity on the frame bundle of the tangent bundle. But these are easily identified with the automorphisms of the principal U(1)-bundle $\mathcal{L}$ associated to the determinant line bundle. Taking note of Lemma 2.1.21 we then have the following definition.

Definition 4.1.6. The gauge group $\mathcal{G}$ is defined by $\mathcal{G} = C^\infty(X, \mathrm{U}(1)) = \mathrm{Aut}(\mathcal{L})$.

There is a natural action of the gauge group on the configuration space. Note that we can conjugate a given spin connection ∇^A on $S(\mathfrak{c})$ by an element $u \in \mathcal{G}$ to obtain a new connection $u\nabla^A u^{-1}$. It is not hard to show that

$$u\nabla^A u^{-1} = \nabla^{A-2u^{-1}du}, \tag{4.1.9}$$

the factor 2 arising from the fact that the map $\pi : \mathrm{Spin}^c(4) \to \mathrm{U}(1)$ is given by squaring. In other words, the resulting connection is the same as the spin connection induced by $A - 2u^{-1}du$. Similarly, any automorphism u of the principal Spin^c-bundle $\mathfrak{c}$ induces an automorphism on all of its associated bundles and hence in particular on the spinor bundle $S(\mathfrak{c})$. We will not introduce new notation and also write u for the resulting action on $S(\mathfrak{c})$.

Definition 4.1.7. The *gauge action* of $\mathcal{G}$ on $\mathcal{C}(\mathfrak{c})$ is defined by

$$u \cdot (A, \psi) = (A - 2u^{-1}du, u\psi). \tag{4.1.10}$$

Remark 4.1.8. We say two pairs (A, ψ), $(A', \psi') \in \mathcal{C}(\mathfrak{c})$ are *gauge equivalent* if there exists an element $u \in \mathcal{G}$ such that $u \cdot (A, \psi) = (A', \psi')$. It is clear that this is indeed an equivalence relation, and we denote the equivalence class of (A, ψ) under gauge equivalence by $[A, \psi]$.

We would of course like that this gauge action does not interfere with being a solution to the Seiberg-Witten equations, so that the space of solutions is invariant under the gauge action. Let us therefore also define the action of $u \in \mathcal{G}$ on $\Omega^2_+(X; i\mathbb{R}) \oplus \Gamma(S^-(\mathfrak{c}))$. Given $(\eta, \varphi) \in \Omega^2_+(X; i\mathbb{R}) \oplus \Gamma(S^-(\mathfrak{c}))$, we set

$$u \cdot (\eta, \varphi) = (\eta, u\varphi). \tag{4.1.11}$$

Lemma 4.1.9. *The Seiberg-Witten equations are equivariant with respect to the gauge action.*

Proof. We must show that for each $u \in \mathcal{G}$ we have that

$$F_\eta(u \cdot (A, \psi)) = u \cdot F_\eta(A, \psi). \tag{4.1.12}$$

Using Definition 4.1.5 we see that

$$F_\eta(u \cdot (A, \psi)) = (F^+_{u(A)} - \mathrm{cl}_+^{-1} q(u\psi) - \eta, D_{u(A)}(u\psi)). \tag{4.1.13}$$

But now note that we have

$$\begin{aligned} F_{u(A)} &= F_{A-2u^{-1}du} = F_A - 2d\left(u^{-1}du\right) = F_A, \\ q(u\psi) &= \sigma(u\psi, u\psi) = (u\psi)\otimes(u\psi)^* - \frac{|u\psi|^2}{2}\mathrm{Id} = u\overline{u}\psi\otimes\psi^* - |u|^2\frac{|\psi|^2}{2}\mathrm{Id} = q(\psi), \end{aligned} \tag{4.1.14}$$

because by definition u is a map into U(1) and thus satisfies $u\overline{u} = |u|^2 = 1$. Lastly we have by unravelling the definitions that

$$D_{A-2u^{-1}du}(u\psi) = \mathrm{cl}\circ\nabla^{A-2u^{-1}du}(u\psi) = \mathrm{cl}\circ u\nabla^A(u^{-1}u\psi) = u\circ\mathrm{cl}\nabla^A\psi = uD_A\psi, \tag{4.1.15}$$

where we used the fact that Clifford multiplication commutes with all automorphisms of the spinor bundle $S(\mathfrak{c})$. □

In particular, if (A,ψ) is a solution to the Seiberg-Witten equations, we now have for any $u\in\mathcal{G}$ that

$$F_\eta(u\cdot(A,\psi)) = u\cdot F_\eta(A,\psi) = 0, \tag{4.1.16}$$

so that $u\cdot(A,\psi)$ is again a solution to the Seiberg-Witten equations. Having this result, we can now define the object which we will study for the remainder of this thesis.

Definition 4.1.10. Given a Spin^c-structure $\mathfrak{c}$ on X and a perturbation $\eta\in\Omega^2_+(X;i\mathbb{R})$, the *Seiberg-Witten moduli space* $M_\eta(\mathfrak{c})$ is the space of solutions (A,φ) to the Seiberg-Witten equations, modulo gauge transformations.

Given a Spin^c-structure $\mathfrak{c}$, let us introduce the general quotient space $\mathcal{B}(\mathfrak{c})$ defined by

$$\mathcal{B}(\mathfrak{c}) = \mathcal{C}(\mathfrak{c})/\mathcal{G}, \tag{4.1.17}$$

consisting of all gauge equivalence classes $[A,\psi]$ of configuration pairs. We then see that the Seiberg-Witten moduli space is given by the subset of $\mathcal{B}(\mathfrak{c})$

$$M_\eta(\mathfrak{c}) = \{[A,\psi]\in\mathcal{B}(\mathfrak{c})\,|\,F_\eta(A,\psi)=0\}. \tag{4.1.18}$$

In order to study this space and its properties we will first study the space $\mathcal{B}(\mathfrak{c})$. Note that we can naturally identify the tangent space of $\mathcal{C}(\mathfrak{c})$ at any point with the space $\Omega^1(X;i\mathbb{R})\oplus\Gamma(S^+(\mathfrak{c}))$ due to the space of unitary connections being affine, see Lemma 2.2.7. We wish to understand M as a topological space, and hence must equip $\mathcal{B}(\mathfrak{c})$ with a topology. One might consider using smooth connections and sections of the positive spinor bundle, but in fact it is more useful to consider only those in a certain Sobolev space. There are now several choices to be made, but we will continue to follow the treatment in [23] and work with L^2-spaces. This means the spaces we work with are Hilbert manifolds; see also Section 2.5. In particular, recall how given a vector bundle $E\to X$ we let $L^p_k(E)$ denote the space of L^p_k-sections of E. Given a Spin^c-structure $\mathfrak{c}$ we denote the space of unitary L^2_k-connections on its determinant line bundle $L=\det(\mathfrak{c})$ by $\mathcal{A}_k(L)$. Similarly, we denote the space of L^2_k-configurations by

$$\mathcal{C}_k(\mathfrak{c}) = \mathcal{A}_{L^2_k}(L)\times L^2_k(S^+(\mathfrak{c})). \tag{4.1.19}$$

We denote by $\mathcal{G}_k = L^2_k(\mathbb{U}(1))$ the gauge group consisting of all L^2_k-maps from X to $\mathrm{U}(1)$, where $\mathbb{U}(1)$ is the trivial U(1)-bundle over X. Lastly, we denote by $\mathcal{P}_k = L^2_k\left(\bigwedge^2_+ T^*X \otimes i\mathbb{R}\right)$ the space of L^2_k-perturbations. We now choose to work with $\mathcal{C}_2(\mathfrak{c})$, which will again write as $\mathcal{C}(\mathfrak{c})$. Furthermore, we choose to work with the space $\mathcal{G}_3 = \mathcal{G}$ of gauge transformations, and the space $\mathcal{P}_1$ of perturbations. We now equip $\mathcal{B}(\mathfrak{c})$ with the quotient topology and $M_\eta(\mathfrak{c})$ with the induced subspace topology. For $k \geq 2$, denote by $M_{\eta,k}(\mathfrak{c})$ the moduli space of L^2_k-solutions to the perturbed Seiberg-Witten equations modulo L^2_{k+1}-gauge transformations.

Remark 4.1.11. We require one additional degree of regularity on the gauge transformations as their action on the configuration space involves their derivative. Furthermore, we require that the gauge transformations give rise to continuous bundle automorphisms, and for this we need them to be at least L^2_3. In this case we have by the Sobolev multiplication theorem, Theorem 2.5.7, a continuous map $L^2_3(\mathbb{U}(1)) \times L^2_3(\mathbb{U}(1)) \to L^2_3(\mathbb{U}(1))$. Moreover, similar Sobolev multiplication theorems imply that the gauge action on $\mathcal{C}(\mathfrak{c})$ is continuous. We will see shortly in Corollary 4.1.19 that all $M_{\eta,k}(\mathfrak{c})$ for $k \geq 2$ are diffeomorphic.

Proposition 4.1.12. *The configuration space $\mathcal{C}_k(\mathfrak{c})$ is a Hilbert manifold. Moreover, the gauge group $\mathcal{G}_k$ is an infinite-dimensional Hilbert Lie group.*

As said, to understand $M(\mathfrak{c})$ clearly it is important to consider first the quotient $\mathcal{B}(\mathfrak{c})$ of the configuration space by the gauge action. A priori it is not clear that the gauge action is nice. In particular, for the quotient to again be a manifold we need that the action is free. Whether or not this is true is answered by the following lemma.

Lemma 4.1.13. *Let X be connected. The gauge action at a point (A, ψ) is free if and only if $\psi \neq 0$.*

Proof. Let $(A, \psi) \in \mathcal{C}(\mathfrak{c})$ be any point and let $u \in \mathcal{G}$ be given. Recall that

$$u \cdot (A, \psi) = (A - 2u^{-1}du, u\psi). \tag{4.1.20}$$

For u to act trivially on the connection A, we need exactly that $du = 0$, which because X is connected implies that u is a constant map from X to U(1). Such an element clearly acts freely on ψ if and only if $\psi \neq 0$. □

Remark 4.1.14. Pairs $(A, \psi) \in$ where $\psi \neq 0$ are called *irreducible*, and all others are called *reducible*. We see that reducible pairs have stabilizer equal to the subgroup of $\mathcal{G}$ consisting of constant maps to U(1). Note that being (ir)reducible is preserved by the gauge action, so that it makes sense to speak of (ir)reducible points in the quotient space $\mathcal{B}(\mathfrak{c})$.

Given a Spin^c-structure $\mathfrak{c}$, we denote by $\mathcal{B}^*(\mathfrak{c}) \subset \mathcal{B}(\mathfrak{c})$ the space of irreducible gauge equivalence classes of configuration pairs. Similarly, we denote by $M^*(\mathfrak{c})$ the moduli space of irreducible solutions to the Seiberg-Witten equations. We see that the only solutions where the moduli space can be hoped to be well-behaved are the irreducible ones. As such, we wish to have some results regarding how many reducible solutions there are. This will be treated in detail in Section 4.4 on transversality. More precisely, we will show that for so-called *generic* choices of Riemannian metric g and perturbation η the moduli space $M_\eta(\mathfrak{c})$ does not contain reducible solutions. Let us now state the following "slice theorem" result which we will not prove here.

Theorem 4.1.15. *The quotient space $\mathcal{B}(\mathfrak{c}) = \mathcal{C}(\mathfrak{c})/\mathcal{G}$ is Hausdorff. Moreover, there are local slices for the gauge action: each point $(A, \psi) \in \mathcal{C}(\mathfrak{c})$ has an open neighbourhood U containing a smoothly embedded closed submanifold S which is invariant under the stabilizer* Stab *of (A, ψ) such that the map*

$$\mathcal{G} \times_{\mathrm{Stab}} S \to \mathcal{C}(\mathfrak{c}) \tag{4.1.21}$$

is a diffeomorphism onto a neighbourhood of the orbit going through (A, ψ).

Proof. A proof using the inverse function theorem applied to the differential of the gauge action can be found in [23, Section 4.5]. □

Corollary 4.1.16. *The open subset $\mathcal{B}^*(\mathfrak{c}) \subset \mathcal{B}(\mathfrak{c})$ of irreducible orbits is a Hilbert manifold.*

Proof. This follows directly from Theorem 4.1.15 together with the fact that the gauge action is free at irreducible points by Lemma 4.1.13. □

We now turn to the gauge fixing. Essentially, we wish to pick out a preferred representative of a given gauge equivalence class.

Lemma 4.1.17. *Let $A_0 \in \mathcal{A}(L)$ be some smooth unitary connection and fix $k \geq 2$. Then for any connection $A \in \mathcal{A}_k(L)$ there exists a gauge transformation $u \in \mathcal{G}_{k+1}$ such that $u(A) = A_0 + a$ with $a \in iL^2_k(T^*X)$ satisfying $d^*a = 0$.*

Proof. Given A, we know there exists a $b \in iL^2_k(T^*X)$ such that $A = A_0 + b$. Now using the Hodge decomposition equation (2.3.7) we see that we can write

$$b = b_0 + df + d^*\beta, \tag{4.1.22}$$

where b_0 is harmonic, $f \in L^2_{k+1}(X; i\mathbb{R})$ and $\beta \in iL^2_k\left(\bigwedge^2 T^*X\right)$. Now define the gauge transformation u by $u = e^{\frac{1}{2}f}$. Then by definition of the gauge action we have

$$u(A) = A - 2u^{-1}du = A_0 + b_0 + df + d^*\beta - df = A_0 + b_0 + d^*\beta. \tag{4.1.23}$$

In other words, we have $u(A) = A_0 + a$ for $a = b_0 + d^*\beta$. It is clear by definition that $d^*a = 0$. □

Now consider again the perturbed Seiberg-Witten equations $F_\eta(A, \psi) = 0$.

Lemma 4.1.18. *Given an element $\eta \in \mathcal{P}_k$ for $k \geq 1$, for every solution $(A, \psi) \in \mathcal{C}_2(\mathfrak{c})$ there exists a gauge transformation $u \in \mathcal{G}_3$ such that $u \cdot (A, \psi) \in \mathcal{C}_{k+1}(\mathfrak{c})$. In particular, given a smooth perturbation η, any solution (A, ψ) to the perturbed Seiberg-Witten equations is gauge equivalent to a smooth solution.*

Proof. This will be proven for the unperturbed case in Section 4.3 on compactness. More specifically, this follows from Theorem 4.3.8. Essentially, one uses the Seiberg-Witten equations equations and ellipticity of the Dirac operator and $d^+ + d^*$ together with elliptic regularity, Theorem 2.5.10, to gradually increase the degree of regularity of the solution. An adaptation to the perturbed case is immediate. The second statement follows from the Sobolev embedding theorem, Theorem 2.5.5. □

Let η be a smooth perturbation. We can now answer the question whether choosing to work with Sobolev spaces changed the spaces we work with.

Corollary 4.1.19. *For any $k \geq 2$ we have $M_{\eta,2}(\mathfrak{c}) \cong M_{\eta,k}(\mathfrak{c})$. In other words, the smoothly perturbed Seiberg-Witten moduli space is independent of the choice of Sobolev norm on $\mathcal{C}(\mathfrak{c})$ and $\mathcal{G}$.*

Proof. This is immediate from Lemma 4.1.18: every element $[A, \psi] \in M_{\eta,2}(\mathfrak{c})$ in fact sits inside $M_{\eta,k}(\mathfrak{c})$ for all $k \geq 2$, and clearly the converse is also true. □

Remark 4.1.20. We see that instead of considering just smooth solutions by also looking for solutions in the larger Sobolev space we did not gain anything. Regardless, the Sobolev spaces allow for useful analytical tools to be used.

4.2 Dimension

In this section we will prove that at irreducible solutions the moduli space is a smooth manifold of a certain dimension, determined by the Atiyah-Singer index theorem. This section follows [23] and is an instance of the general Kuranishi picture as in [27, Appendix B]. Let $(A, \psi) \in \mathcal{C}(\mathfrak{c})$ be a solution to the Seiberg-Witten equations. We wish to find a local description of the moduli space for a neighbourhood around some point. In such a situation we will always get a deformation complex, also called the fundamental complex, corresponding to the linearization of the Seiberg-Witten equations and the gauge action. Let us see how this arises. Recall how the solutions to the Seiberg-Witten equations are those pairs $(A, \psi) \in \mathcal{C}(\mathfrak{c})$ satisfying $F(A, \psi) = 0$.

Lemma 4.2.1. *The differential dF of F at a pair $(A, \psi) \in \mathcal{C}(\mathfrak{c})$ is given by*

$$dF_{(A,\psi)}(a, \varphi) = \left(d^+a - \mathrm{cl}_+^{-1} dq_\psi(\varphi), \frac{1}{2}\mathrm{cl}(a)\psi + D_A\varphi\right) \qquad \text{for } (a, \varphi) \in \Omega^1(X; i\mathbb{R}) \oplus \Gamma(S^+(\mathfrak{c})), \tag{4.2.1}$$

where dq_ψ is the differential of q at ψ.

Proof. Consider $dF_{(A,\psi)}$ as a 2×2-matrix. We know from Lemma 2.2.8 that the connection $A + a$ has curvature given by $F_{A+a} = F_A + da$, the term $a \wedge a$ vanishing as the Lie algebra $i\mathbb{R}$ of $\mathrm{U}(1)$ is abelian. From this it is clear that the first matrix entry should be d^+a. Similarly, the third matrix entry is indeed $\frac{1}{2}\mathrm{cl}(a)\psi$ because of Lemma 3.4.6. Furthermore, the fourth matrix entry should indeed be $D_A\varphi$ merely by linearity of the Dirac operator. Let us now compute what $dq_\psi(\varphi)$ should be. We have

$$\begin{aligned} q(\psi + \varphi) &= \sigma(\psi + \varphi, \psi + \varphi) = (\psi + \varphi) \otimes (\psi + \varphi)^* - \frac{1}{2}\langle \psi + \varphi, \psi + \varphi \rangle \mathrm{Id} \\ &= \psi \otimes \psi^* - \frac{1}{2}\langle \psi, \psi \rangle \mathrm{Id} + \left(\psi \otimes \varphi^* + \varphi \otimes \psi^* - \frac{1}{2}\left(\langle \psi, \varphi \rangle + \langle \varphi, \psi \rangle\right) \mathrm{Id}\right) \\ &\quad + \varphi \otimes \varphi^* - \frac{1}{2}\langle \varphi, \varphi \rangle \mathrm{Id}. \end{aligned} \tag{4.2.2}$$

From looking at the terms which are linear in φ we see by conjugate symmetry of the inner product on $S(\mathfrak{c})$ that

$$dq_\psi(\varphi) = \frac{\mathrm{d}}{\mathrm{d}t} q(\psi + t\varphi)\Big|_{t=0} = \psi \otimes \varphi^* + \varphi \otimes \psi^* - \frac{1}{2}\left(\langle \psi, \varphi \rangle + \overline{\langle \psi, \varphi \rangle}\right) \mathrm{Id}. \tag{4.2.3}$$

In particular, we see that this is a traceless self-adjoint element of $\mathrm{End}(S(\mathfrak{c}))$ which by cl_+^{-1} is sent to an imaginary self-dual two-form as desired. □

Lemma 4.2.2. *The differential of the gauge action at a pair $(A, \psi) \in \mathcal{C}(\mathfrak{c})$ is given by*

$$f \mapsto (-2df, f\psi) \qquad \text{for } f \in L_3^2(X; i\mathbb{R}). \tag{4.2.4}$$

Proof. Consider a curve $u_t = e^{tf} \subset \mathcal{G}$ for some map $f : X \to i\mathbb{R}$. We then have $u_0 = 1$ and $u_0' = f$. Because of this we can compute that

$$\frac{\mathrm{d}}{\mathrm{d}t} u_t(A)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(A - 2du_t u_t^{-1}\right)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(A - 2d(e^{tf})e^{-tf}\right)\Big|_{t=0} = -2df, \tag{4.2.5}$$

so that the infinitesimal action of the gauge group indeed is as stated. □

We can now capture both the linearization of the gauge action and the Seiberg-Witten equations into a *fundamental complex* L, which reads

$$0 \to L_3^2(X; i\mathbb{R}) \overset{D_1}{\to} L_2^2\left((T^*X \otimes i\mathbb{R}) \oplus S^+(\mathfrak{c})\right) \overset{D_2}{\to} L_1^2\left(\left(\bigwedge\nolimits_+^2 T^*X \otimes i\mathbb{R}\right) \oplus S^-(\mathfrak{c})\right) \to 0, \tag{4.2.6}$$

where the maps D_1 and D_2 are the linearizations of the gauge action and the differential of F respectively, which by Lemma 4.2.2 and Lemma 4.2.1 are given by

$$D_1 = (-2d, (\cdot)\psi) \qquad \text{and} \qquad D_2 = \begin{pmatrix} d^+ & -\mathrm{cl}_+^{-1} dq_\psi(\cdot) \\ \frac{1}{2}\mathrm{cl}(\cdot)\psi & D_A \end{pmatrix}. \tag{4.2.7}$$

Lemma 4.2.3. *At any solution (A, ψ) to the Seiberg-Witten equations the above composition L is a complex.*

Proof. To show L is a complex we must show that the composition $D_2 \circ D_1$ is the zero map. This follows formally from Lemma 4.1.9, but let us show it explicitly. Let $f \in L_3^2(X; i\mathbb{R})$ be given. Then $D_1 f = (-2df, f\psi)$ so that

$$D_2 \circ D_1 f = \left(-2d^+df - \mathrm{cl}_+^{-1} dq_\psi(f\psi), -\mathrm{cl}(df)\psi + D_A(f\psi)\right). \tag{4.2.8}$$

We now clearly have $d^+df = \mathrm{pr}_+ d^2 f = 0$. Furthermore, we have

$$\begin{aligned} dq_\psi(f\psi) &= \psi \otimes (f\psi)^* + (f\psi) \otimes \psi^* - \frac{1}{2}\left(\langle \psi, f\psi \rangle + \overline{\langle \psi, f\psi \rangle}\right) \mathrm{Id} \\ &= (\overline{f} + f)\psi \otimes \psi^* - \frac{1}{2}(\overline{f} + f)|\psi|^2 \mathrm{Id} = 0, \end{aligned} \tag{4.2.9}$$

as f is purely imaginary and thus $\overline{f} = -f$. We see that the first component of the composition indeed vanishes. Now by the Leibniz rule for the Dirac operator we get

$$-\mathrm{cl}(df)\psi + D_A(f\psi) = -\mathrm{cl}(df)\psi + \mathrm{cl}(df)\psi + fD_A\psi = 0, \tag{4.2.10}$$

where we used that $D_A\psi = 0$ as (A, ψ) is a solution to the Seiberg-Witten equations. We conclude that the composition indeed is a complex. □

We can now define the three cohomology groups associated to this complex. They are given by

$$H^0_{(A,\psi)} = \ker D_1, \qquad H^1_{(A,\psi)} = \frac{\ker D_2}{\mathrm{im} D_1}, \qquad H^2_{(A,\psi)} = \mathrm{coker} D_2. \tag{4.2.11}$$

Proposition 4.2.4. *The complex L is in fact elliptic, with Euler characteristic e_L equal to*

$$e_L = \dim H^0_{(A,\psi)} - \dim H^1_{(A,\psi)} + \dim H^2_{(A,\psi)} = \mathrm{ind}(d^* \oplus d^+) - \mathrm{ind}(D_A). \tag{4.2.12}$$

Proof. To show the complex is elliptic, note that we can change the complex under homotopy deformations, as long as we do not change the corresponding symbol sequence. In other words, we can get rid of zeroth order terms. As such, consider the homotopies

$$D_1(t) = (-2d, (1-t)(\cdot)\psi), \qquad D_2(t) = \begin{pmatrix} d^+ & -(1-t)\mathrm{cl}_+^{-1} dq_\psi(\cdot) \\ \frac{1}{2}(1-t)\mathrm{cl}(\cdot)\psi & D_A \end{pmatrix}. \tag{4.2.13}$$

We then have $D_1(0) = D_1$ and $D_2(0) = D_2$, while

$$D_1(1) = (-2d, 0) \qquad \text{and} \qquad D_2(1) = \begin{pmatrix} d^+ & 0 \\ 0 & D_A \end{pmatrix}. \tag{4.2.14}$$

By using this deformation we see that the original elliptic complex becomes the direct sum of the two complexes L_1 and L_2, given respectively by

$$\begin{aligned} &0 \to L^2_3(X; i\mathbb{R}) \overset{-2d}{\to} L^2_2(T^*X \otimes i\mathbb{R}) \overset{d^+}{\to} L^2_1\left(\textstyle\bigwedge^2_+ T^*X \otimes \mathbb{R}\right) \to 0, \\ &0 \to 0 \to L^2_2\left(S^+(\mathfrak{c})\right) \overset{D_A}{\to} L^2_1\left(S^-(\mathfrak{c})\right) \to 0. \end{aligned} \tag{4.2.15}$$

Ellipticity of L_2 follows from Lemma 3.4.7 saying that Dirac operators are elliptic. Ellipticity of L_1 immediately follows from Lemma 2.3.9, noting that the factor -2 is immaterial. We conclude that the original complex is elliptic as well. Taking the adjoint of the first operator $-2d$, let us transform L_1 into

$$L^2_2(T^*X \otimes i\mathbb{R}) \overset{-2d^* \oplus d^+}{\to} L^2_3(X; i\mathbb{R}) \oplus L^2_1\left(\textstyle\bigwedge^2_+ T^*X \otimes \mathbb{R}\right). \tag{4.2.16}$$

Then the index of L_1 is equal to the index of $-2d^* \oplus d^+$ which in turn is equal to $\mathrm{ind}(d^* \oplus d^+)$. We thus see another proof of ellipticity of L_1: the Hodge Laplacian Δ is clearly elliptic, so that $D = d^* + d$ is as well, because $\Delta = D^2$. The Euler characteristic e_L of our original complex is now the sum of their respective Euler characteristics e_{L_1} and e_{L_2}, which are given by

$$e_{L_1} = \mathrm{ind}(d^* \oplus d^+), \qquad e_{L_2} = -\mathrm{ind}(D_A). \tag{4.2.17}$$

□

We can now see the interpretation of these cohomology groups. They measure exactly the obstructions at a solution (A, ψ) in order for the quotient space to be well-behaved at that point.

Proposition 4.2.5. *A solution (A,ψ) is irreducible if and only if $H^0_{(A,\psi)}$ is trivial. Furthermore, if both $H^0_{(A,\psi)}$ and $H^2_{(A,\psi)}$ vanish, the moduli space $M(\mathfrak{c})$ is a smooth manifold at $[A,\psi]$, of dimension equal to* $\dim H^1_{(A,\psi)} = \operatorname{ind}(D_A) - \operatorname{ind}(d \oplus d^*)$.

Proof. The first statement is immediate as $f \in \ker D_1$ if and only if $df = 0$ and $f\psi = 0$. To see the second statement, note that $H^2_{(A,\psi)}$ vanishing implies that dF is surjective at (A,ψ), so that the claim follows by the inverse function theorem, Theorem 2.6.8. More specifically, the inverse function theorem implies that a neighbourhood of the equivalence class $[A,\psi]$ is a smooth Hilbert submanifold inside $B^*(\mathfrak{c})$ with tangent space equal to $H^1_{(A,\psi)}$. Its dimension is given by $\operatorname{ind}(dF) = \dim H^1_{(A,\psi)}$. □

Remark 4.2.6. The space $H^1_{(A,\psi)}$ is called the *Zariski tangent space* of $M(\mathfrak{c})$ at (A,ψ). The space $H^2_{(A,\psi)}$ is called the *obstruction space* at (A,ψ). We say an irreducible solution (A,ψ) is *smooth* if its obstruction space vanishes. We see from the above proposition that the Zariski tangent space *is* the tangent space of the moduli space at smooth irreducible solutions.

It is clear that we will want to be able to decide when a solution (A,ψ) is irreducible and smooth. More specifically, we want results saying that we can make it so that *all* solutions to the Seiberg-Witten equations are both irreducible and smooth, so that then the entire moduli space is a smooth manifold. As was mentioned earlier, this will be done in Section 4.4 on transversality. The main idea is that given a Riemannian metric we can usually perturb the Seiberg-Witten equations using a *generic* perturbation, i.e. one from some open dense set in the space $\Omega^2_+(X;i\mathbb{R})$ of all perturbations, such that all solutions are irreducible and smooth.

Having established that the moduli space will have tangent space equal to the Zariski tangent space at smooth irreducible points, a natural question is to ask what the dimension of the moduli space is. We can determine this by merely determining the dimension of the space $H^1_{(A,\psi)}$. We see from Proposition 4.2.5 that this dimension, as long as $H^0_{(A,\psi)}$ and $H^2_{(A,\psi)}$ both vanish, is given by the index of D_A minus the index of the operator $d^* \oplus d^+$. Let us therefore calculate the index of these operators.

Proposition 4.2.7. *The index of D_A is equal to $\frac{1}{4}\left(c_1(L)^2 - \tau(X)\right)$. The index of $d^* \oplus d^+$ is equal to $-(1 + b_2^+(X) - b_1(X))$.*

Proof. Note that for this calculation we care about the *real* index of the operators. As the real index is just twice the complex index, the first statement follows from Lemma 3.4.9. To determine the index of $d^* \oplus d^+$, consider again the result of Lemma 2.3.9. We see from this that

$$\begin{aligned} \ker(d^* \oplus d^+) &= \mathcal{H}^1(X;\mathbb{R}), \\ \operatorname{coker}(d^* \oplus d^+) &= \ker(d) \oplus \operatorname{coker}(d^+) = \mathcal{H}^0(X;\mathbb{R}) \oplus \mathcal{H}^2_+(X;\mathbb{R}). \end{aligned} \tag{4.2.18}$$

Now note that $\dim \mathcal{H}^0(X;\mathbb{R}) = b_0 = 1$, so that

$$\operatorname{ind}(d^* \oplus d^+) = \ker(d^* \oplus d^+) - \dim\operatorname{coker}(d^* \oplus d^+) = b_1(X) - 1 - b_2^+(X). \tag{4.2.19}$$

□

Note that the above proposition in particular implies that the dimension of the Zariski tangent space is the same at all smooth irreducible points. After this calculation it is now easy to conclude what the dimension of the moduli space is.

Theorem 4.2.8. *At a smooth irreducible point $[A, \psi] \in M(\mathfrak{c})$ the moduli space $M(\mathfrak{c})$ is a smooth manifold of dimension*

$$d(\mathfrak{c}) = \frac{1}{4}\left(c_1(L)^2 - (2\chi(X) + 3\tau(X))\right), \tag{4.2.20}$$

where χ and τ are the Euler characteristic and signature respectively.

Proof. Note firstly that at a smooth irreducible point we have

$$d(\mathfrak{c}) = \dim H^1_{(A,\psi)} = -e = \mathrm{ind}(D_A) - \mathrm{ind}(d^* \oplus d^+), \tag{4.2.21}$$

as was mentioned above Proposition 4.2.7. Because of this, we can compute by Proposition 4.2.7 that

$$d(\mathfrak{c}) = \mathrm{ind}(D_A) - \mathrm{ind}(d^* \oplus d^+) = \frac{1}{4}(c_1(L)^2 - \tau(X)) + 1 + b_2^+(X) - b_1(X). \tag{4.2.22}$$

Now note that by equation (2.3.13) we have

$$\frac{1}{2}(\chi(X) + \tau(X)) = \frac{1}{2}(2 - 2b_1(X) + b_2(X) + b_2^+(X) - b_2^-(X)) = 1 + b_2^+(X) - b_1(X), \tag{4.2.23}$$

so that indeed

$$d(\mathfrak{c}) = \frac{1}{4}\left(c_1(L)^2 - (2\chi(X) + 3\tau(X))\right). \tag{4.2.24}$$

□

To finish this section we state a corollary of this theorem in the case where the Spin^c-structure $\mathfrak{c}$ comes from an almost complex structure. The main observation is the following result mentioned earlier as Theorem 2.1.15.

Lemma 4.2.9. *Let X be a four-manifold with an almost complex structure J. Then we have*

$$c_1(TX; J)^2 = 2\chi(X) + 3\tau(X). \tag{4.2.25}$$

So now let $\mathfrak{c}_J$ be the canonical Spin^c-structure associated to J. Because we know by Section 3.5 that its determinant line bundle $L(\mathfrak{c}_J)$ is just K_X^{-1}, the dual of the canonical line bundle, the following is immediate.

Corollary 4.2.10. *Given an almost complex structure, the dimension $d(M(\mathfrak{c}_J))$ of the Seiberg-Witten moduli space for its canonical* Spin^c*-structure $\mathfrak{c}_J$ is equal to zero near smooth irreducible points.*

Proof. This is immediate from the dimension formula in Theorem 4.2.8 together with Lemma 4.2.9. □

4.3 Compactness

In this section we deal with the compactness of the moduli spaces. As we will see, the situation is much nicer in Seiberg-Witten theory than in Donaldson theory: indeed, the moduli spaces $M(\mathfrak{c})$ are compact for any Spin^c-structure $\mathfrak{c}$, in stark contrast with Donaldson theory. The exact statement of the compactness theorem is the following.

Theorem 4.3.1. *Let $\mathfrak{c}$ be a Spin^c-structure on a compact oriented four-manifold X. For every fixed Riemannian metric g and perturbation η, the moduli space $M_\eta(\mathfrak{c})$ is compact.*

To prove this theorem we follow the treatment in [23]. We use an a priori bound on the solutions obtained through the Weitzenböck formula together with the gauge fixing idea of Lemma 4.1.17. Throughout, a Spin^c-structure $\mathfrak{c}$ will be fixed. We focus first on the unperturbed Seiberg-Witten equations $F(A, \psi) = 0$. Note that it will suffice to show that the moduli space is sequentially compact, as it is a metric space.

Lemma 4.3.2. *Let (A, ψ) be a smooth solution to the Seiberg-Witten equations. Let $\{e_i\}$ be a moving orthornormal frame. Then for all $x \in X$ we have*

$$\mathrm{Re}\left(\langle \nabla_A^* \nabla_A(\psi(x)), \psi(x)\rangle\right) = \frac{1}{2}\Delta(|\psi(x)|^2) + \sum_i |\nabla_{e_i}(\psi(x))|^2, \tag{4.3.1}$$

where $\Delta = -\sum_i \frac{\partial^2}{\partial e_i^2}$ is the Laplacian on functions.

Proof. Note firstly that by compatibility of the connection with the metric we have

$$\begin{aligned} -\sum_i \frac{\partial^2}{\partial e_i^2}\langle \psi(x), \psi(x)\rangle = -\sum_i \left(\langle \nabla_{e_i} \circ \nabla_{e_i}(\psi(x)), \psi(x)\rangle + 2\langle \nabla_{e_i}(\psi(x)), \nabla_{e_i}(\psi(x))\rangle\right) \\ + \sum_i \langle \psi(x), \nabla_{e_i} \circ \nabla_{e_i}(\psi(x))\rangle. \end{aligned} \tag{4.3.2}$$

Because of this, we have

$$\begin{aligned} \Delta(|\psi(x)|^2) + 2\sum_i |\nabla_{e_i}(\psi(x))|^2 &= -\sum_i \left(\langle \nabla_{e_i} \circ \nabla_{e_i}(\psi(x)), \psi(x)\rangle + \langle \psi(x), \nabla_{e_i} \circ \nabla_{e_i}(\psi(x))\rangle\right) \\ &= \langle \nabla_A^* \circ \nabla_A(\psi(x)), \psi(x)\rangle + \langle \psi(x), \nabla_A^* \circ \nabla_A(\psi(x))\rangle \\ &= 2\mathrm{Re}\left(\langle \nabla_A^* \nabla_A(\psi(x)), \psi(x)\rangle\right). \end{aligned} \tag{4.3.3}$$

where we used Lemma 3.4.4 for the local expression of $\nabla_A^* \circ \nabla_A$. □

Now define the quantity s_X^- by

$$s_X^- = \max_{x \in X} \max\{-s(x), 0\}, \tag{4.3.4}$$

where s is the scalar curvature of g. Note that s_X^- is independent of the Spin^c-structure we chose, as it only depends on the Riemannian metric.

Lemma 4.3.3. *Let (A, ψ) be a smooth solution to the Seiberg-Witten equations. Then for all $x \in X$ we have*

$$|\psi(x)|^2 \leq s_X^-. \tag{4.3.5}$$

Proof. The Weitzenböck formula, Lemma 3.4.5, states that

$$D_A^* D_A = \nabla_A^* \nabla_A + \frac{1}{4}s + \frac{1}{2}\mathrm{cl}(F_A). \tag{4.3.6}$$

Assume now that $\psi \neq 0$. Because (A, ψ) is a solution to the Seiberg-Witten equations, we have $D_A\psi = 0$. Furthermore, as $\psi \in \Gamma(S^+(\mathfrak{c}))$ is a positive spinor, we have $\mathrm{cl}(F_A)\psi = \mathrm{cl}(F_A^+)\psi$ as anti-self-dual forms act trivially on $S^+(\mathfrak{c})$ by definition. This means that applying the above to ψ and using the monopole equation we have

$$0 = \nabla_A^* \nabla_A \psi + \frac{s}{4}\psi + \frac{1}{2}\left(\psi \otimes \psi^* - \frac{|\psi|^2}{2}\mathrm{Id}\right)\psi. \tag{4.3.7}$$

Now note that $\frac{1}{2}\left(\psi \otimes \psi^* - \frac{|\psi|^2}{2}\mathrm{Id}\right)\psi = \frac{|\psi|^2}{4}\psi$. Hence by taking the pointwise inner product with ψ at any point $x \in X$ we get

$$\langle \nabla_A^* \nabla_A \psi(x), \psi(x) \rangle + \frac{s(x)}{4}|\psi(x)|^2 + \frac{|\psi(x)|^4}{4} = 0. \tag{4.3.8}$$

In particular, the term $\langle \nabla_A^* \nabla_A \psi(x), \psi(x) \rangle$ is seen to be real. Now let $x_0 \in X$ be such that the function $x \mapsto |\psi(x)|^2$ takes on its maximum. Clearly then we have that $\Delta(|\psi(x_0)|^2) \geq 0$. From this we see by Lemma 4.3.2 above that

$$\langle \nabla_A^* \nabla_A \psi(x_0), \psi(x_0) \rangle = \frac{1}{2}\Delta(|\psi(x_0)|^2) + \sum_i |\nabla_{e_i}(\psi(x_0))|^2 \geq 0, \tag{4.3.9}$$

as both terms on the right hand side are non-negative. By this inequality we see from equation (4.3.8) that at such an x_0 we have

$$\frac{s(x_0)}{4}|\psi(x_0)|^2 + \frac{|\psi(x_0)|^4}{4} \leq 0. \tag{4.3.10}$$

Note now that by assumption $\psi \neq 0$ and hence we must have $|\psi(x_0)|^2 \neq 0$ by definition of x_0. We conclude after dividing by $|\psi(x_0)|^2$ that for any $x \in X$ we have

$$|\psi(x)|^2 \leq |\psi(x_0)|^2 \leq -s(x_0) \leq s_X^-. \tag{4.3.11}$$

□

Remark 4.3.4. Note that to establish the bound in the previous lemma we needed to assume the solution had some initial regularity, in order for the expression $\Delta|\psi|^2$ to make sense. It is in fact possible to establish this bound without this assumption, as can be seen for example in [19, Lemma 3.10]. Nevertheless, the bound we established is invariant under gauge transformations, so that by Lemma 4.1.18 it also holds for all solutions in $\mathcal{C}_2(\mathfrak{c})$.

The previous lemma also immediately implies a uniform bound on the self-dual part of the curvature F_A^+. Namely, we have by the Seiberg-Witten equations that

$$|F_A^+(x)| = \left|\mathrm{cl}_+^{-1} q(\psi(x))\right| = \left|\psi \otimes \psi^* - \frac{1}{2}\mathrm{tr}(\psi \otimes \psi^*)\mathrm{Id}\right| = \frac{1}{2}|\psi(x)|^2 \leq \frac{1}{2}s_X^-. \tag{4.3.12}$$

Similarly by considering the Seiberg-Witten equations and using the Hodge decomposition one can show that there exists a constant $C_1 > 0$ such that

$$\|F_A^+\|_{2,1}^2 \leq C_1. \tag{4.3.13}$$

Lemma 4.3.5. *Let (A, ψ) be a smooth solution to the Seiberg-Witten equations. Then we have*

$$\|\nabla_A \psi\|_{2,0}^2 \leq \frac{(s_X^-)^2}{4}\mathrm{vol}(X). \tag{4.3.14}$$

Proof. Consider again equation (4.3.8) from Lemma 4.3.3, rewritten slightly to

$$\langle \nabla_A \psi(x), \nabla_A \psi(x)\rangle + \frac{s(x)}{4}|\psi(x)|^2 + \frac{|\psi(x)|^4}{4} = 0. \tag{4.3.15}$$

Now integrate this equation over X and use the bound from Lemma 4.3.3. □

Note that inside of the space $\mathcal{H}^1(X; i\mathbb{R})$ of imaginary harmonic one-forms sits a lattice $\Lambda = \mathcal{H}^1(X; 2\pi i\mathbb{Z})$ of harmonic one-forms whose integral over every loop in X is an integer multiple of $2\pi i$. We say an element $h \in \Lambda$ is *periodic* with period in $2\pi i\mathbb{Z}$.

Lemma 4.3.6. *Let $h \in \Lambda$ be a periodic harmonic one-form with period in $2\pi i\mathbb{Z}$. Then there exist a harmonic function $f : X \to S^1$ such that $h = df$.*

Proof. Let $x_0 \in X$ be given. Consider the universal cover $\widetilde{X}$ of X and define a map $\widetilde{f} : \widetilde{X} \to i\mathbb{R}$ as follows. Given $x \in X$, consider a path γ from x_0 to x, and then define

$$\widetilde{f}(x) = \int_\gamma h. \tag{4.3.16}$$

Because of the periodicity of h, this map descends to a map $f : X \to i\mathbb{R}/(2\pi i\mathbb{Z}) \cong S^1$. By definition we have $df = h$, which then implies that f is harmonic. □

We now establish an extension of the gauge fixing lemma, Lemma 4.1.17, which includes a uniform bound.

Lemma 4.3.7. *Let $A_0 \in \mathcal{A}(L)$ be some smooth unitary connection and let $k \geq 2$ be given. Then there exists constants $C, K > 0$ depending on A_0 and k such that for any connection $A \in \mathcal{A}_k(L)$ there exists a gauge transformation $u \in \mathcal{G}_{k+1}$ such that $u(A) = A_0 + a$ with $a \in iL_k^2(T^*X)$ satisfying $d^*a = 0$, and*

$$\|a\|_{2,k}^2 \leq C\|F_A^+\|_{2,k}^2 + K. \tag{4.3.17}$$

Proof. By Lemma 4.1.17 and write $A = A_0 + a$ for some imaginary one-form a satisfying $d^*a = 0$. As in the proof of Lemma 4.1.18, we can now write the Seiberg-Witten equations for (A_i, ψ_i) as

$$\begin{cases} D_{A_0}\psi &= -\frac{1}{2}\mathrm{cl}(a)\psi, \\ d^+a &= \mathrm{cl}_+^{-1}q(\psi) - F_{A_0}^+. \end{cases} \tag{4.3.18}$$

Now consider the Hodge decomposition of $a = (h, \beta)$ where h is harmonic and β is co-closed. We immediately get that

$$\|a\|_{2,k}^2 \leq \|h\|_{2,k}^2 + \|\beta\|_{2,k}^2. \tag{4.3.19}$$

In other words, we can bound the norm of a by considering h and β separately. First consider β. Note that $d^*\beta = 0$ and $d^+\beta = F_A^+ - F_{A_0}^+$. We saw in Section 4.2 that $d^* \oplus d^+$ is elliptic and its kernel consists of precisely the harmonic one-forms. Now restrict it to the orthogonal complement of the harmonic one-forms, so that it has trivial kernel. By elliptic regularity, Theorem 2.5.10, we then have

$$\|\beta\|_{2,k}^2 \leq C\|(d^* \oplus d^+)\beta\|_{2,k}^2 = C\|F_A^+ - F_{A_0}^+\|_{2,k}^2 \leq C\left(\|F_A^+\|_{2,k}^2 + K_1\right), \tag{4.3.20}$$

where $K_1 = \|F_{A_0}^+\|_{2,k}^2$ is a fixed constant. Now consider the harmonic part h. As the quotient of $\mathcal{H}^1(X; i\mathbb{R})$ by the lattice Λ is a compact torus, there exist a constant K_2 such that we can decompose h into harmonic forms as

$$h = h_1 - h_2, \qquad \text{with} \qquad \|h_1\|_{2,k+1}^2 \leq K_2, \tag{4.3.21}$$

and h_2 with periods in $2\pi i\mathbb{Z}$. By Lemma 4.3.6 we can write $h_2 = df$ for some harmonic function f. Applying the gauge transformation $u = e^{\frac{1}{2}f}$ we see that we may assume that $f = 0$. But by combining the established bounds we get

$$\|a\|_{2,k}^2 \leq K_2 + C\left(\|F_A^+\|_{2,k}^2 + K_1\right). \tag{4.3.22}$$

Setting $K = K_2 + CK_1$ establishes the result. $\square$

With this out of the way we can finally show compactness of the moduli space.

Theorem 4.3.8. *Let $\mathfrak{c}$ be a* Spin^c*-structure on a compact oriented four-manifold X. Then the unperturbed moduli space $M(\mathfrak{c})$ is compact. In other words, any sequence $\{(A_i, \psi_i)\}$ of solutions to the Seiberg-Witten equations possesses, possibly after applying gauge transformations, a convergent subsequence.*

Proof. Again choose a fixed smooth unitary connection $A_0 \in \mathcal{A}(L)$ and using Lemma 4.1.17 write $A_i = A_0 + a_i$ with $d^*a_i = 0$. Consider first an arbitrary solution (A, ψ) written as $A = A_0 + a$ with $d^*a = 0$. We wish to show that (A, ψ) lies in L_k^2 for all $k \geq 2$, which we will do by elliptic bootstrapping. Note that by the estimate of Lemma 4.3.3, we have that $\sup_{x \in X} |\psi(x)|^2 < \infty$. In other words we have $\psi \in C^0$. Furthermore, Lemma 4.3.5 gives us a bound on $\|\nabla_A \psi\|_{2,0}^2$. Now by Lemma 4.3.7 together with the bound from equation (4.3.13) we see that

$$\|a\|_{2,2}^2 \leq CC_1 + K. \tag{4.3.23}$$

In other words, a is bounded in L^2_2. This now gives us an L^2_1-bound on ψ, taken with respect to the fixed connection A_0. We wish to increase the degree of regularity of ψ. The Dirac equation reads

$$D_{A_0}\psi = -\frac{1}{2}\mathrm{cl}(a)\psi. \tag{4.3.24}$$

By the Sobolev multiplication theorem, Theorem 2.5.7, the above implies that $D_{A_0}\psi$ is bounded in L^4. But now recall that Dirac operators are elliptic. Let p_{A_0} denote the projection onto the orthogonal complement of the kernel of D_{A_0}. Then from elliptic regularity, Theorem 2.5.10, we get that $p_{A_0}\psi$ is bounded in L^4_1. On the other hand, ψ is bounded in C^0 and hence in L^2, so that $(1-p_{A_0})\psi$, the projection of ψ onto the kernel of D_{A_0}, is also bounded in L^2, and hence in C^∞. These two bounds then give an L^4_1-bound on ψ. The Sobolev multiplication $L^2_2 \otimes L^4_1 \to L^3_1$ applied to equation (4.3.24) then implies that $D_{A_0}\psi$ is bounded in L^3_1, so that again using elliptic regularity we see that ψ is bounded in L^3_2. We continue this type of argument using the Sobolev multiplication $L^2_2 \otimes L^3_2 \to L^2_2$, so that $D_{A_0}\psi$ is bounded in L^2_2 and hence by elliptic regularity ψ is bounded in L^2_3. Now, the Seiberg-Witten equation

$$F_A^+ = \mathrm{cl}_+^{-1} q(\psi), \tag{4.3.25}$$

together with the Sobolev multiplication $L^2_3 \otimes L^2_3 \to L^2_3$ implies that F_A^+ is bounded in L^2_3. By our choice of gauge this means that a is bound in L^2_4. We are now basically done: if by induction for $k \geq 3$ we have established L^2_3-bounds for a and ψ, then equation (4.3.24) and the Sobolev multiplication $L^2_k \otimes L^2_k \to L^2_k$ implies that $D_{A_0}(\psi)$ is bounded in L^2_k and hence ψ is bounded in L^2_{k+1}. But then the Seiberg-Witten equation again gives that F_A^+ is bounded in L^2_k, so that a is bounded in L^2_{k+1} as well. In other words, we conclude that a and ψ are bounded in L^2_k for all $k \geq 2$.

Now return to our sequence. The above proof shows that each (A_i, ψ_i) lies in L^2_k for all $k \geq 2$. Then Rellich's lemma, Theorem 2.5.9, implies that $\{(A_i, \psi_i)\}$ has a convergent subsequence which converges in all L^2_k. Then the Sobolev embedding theorem, Theorem 2.5.5, implies that it in fact converges in C^k for all k. □

This argument can be repeated for the perturbed moduli spaces.

Theorem 4.3.9. *Let $\eta \in L^2_3(\bigwedge^2_+ T^*X \otimes i\mathbb{R})$ be any perturbation. Then the moduli space $M_\eta(\mathfrak{c})$ is compact.*

Proof. Again we wish to show that any solution is bounded in L^2_k for all k, so that they are smooth. The only thing we must adapt is the a priori bounds obtained in Lemma 4.3.3. In this case, the Weitzenböck formula gives us that

$$0 = \nabla_A^*\nabla_A\psi + \frac{s}{4}\psi + \frac{|\psi|^2}{4}\psi + \mathrm{cl}(\eta)\psi. \tag{4.3.26}$$

Hence by taking the pointwise inner product with ψ at some maximum $x_0 \in X$ of $|\psi|^2$ we get

$$\frac{s(x_0)}{4}|\psi(x_0)|^2 + \frac{|\psi(x_0)|^4}{4} + \mathrm{Re}\left(\langle \mathrm{cl}(\eta(x_0))\psi(x_0), \psi(x_0)\rangle\right) \leq 0. \tag{4.3.27}$$

We now realise that this last term has norm bounded by $|h(x_0)|\,|\psi(x_0)|^2$. As again we must have $|\psi(x_0)|^2 \neq 0$ we can divide to get

$$\frac{s(x_0)}{4} + \frac{|\psi(x_0)|^2}{4} - |h(x_0)| \leq 0, \qquad \text{or} \qquad |\psi(x_0)|^2 \leq 4|h(x_0)| - s(x_0). \tag{4.3.28}$$

Because of this, we clearly have for any $x \in X$ that

$$|\psi(x)|^2 \leq |\psi(x_0)|^2 \leq \max\{\max_{y \in X} (4|h(y)| - s(y))\,, 0\}. \tag{4.3.29}$$

This bound on ψ now plays the role of the bound used in Lemma 4.3.3. The rest of the argument is then identical to that of the previous result. □

4.4 Transversality

Let X be a closed oriented Riemannian four-manifold with a given Spin^c-structure $\mathfrak{c}$. In this section we discuss results which ensure that the Seiberg-Witten moduli space $M(\mathfrak{c})$ is a finite-dimensional manifold. We mostly follow [23].

In light of the results of Section 4.2, more specifically Theorem 4.2.8, we see that we must limit the amount of reducible solutions to the Seiberg-Witten equations. Recall that these were configurations (A, ψ) such that $\psi = 0$. Furthermore, we would like to ensure that all irreducible solutions are smooth, i.e. their second obstruction space $H^2_{(A,\psi)}$ vanishes. Both of these questions will be solved by taking the required data to form the Seiberg-Witten equations to be *generic*, i.e. to lie in some dense subset of the set of all possible data.

We will first turn to the second question. For this it will be necessary to use the perturbed version of the Seiberg-Witten equations, which for convenience reads

$$\begin{aligned} F_A^+ &= \mathrm{cl}_+^{-1} q(\psi) + \eta, \\ D_A\psi &= 0. \end{aligned} \tag{4.4.1}$$

Here, recall that η is a given imaginary self-dual two form. We wish to employ the Sard-Smale theorem, Theorem 2.6.6, and hence must again turn to using Sobolev spaces. We wish for our perturbations to be at least continuous, hence will work with the space $\mathcal{P}_3$ of L^2_3-perturbations. This then forces us to choose configurations to be in $\mathcal{C}_4(\mathfrak{c})$ and gauge transformations to be in $\mathcal{G}_5$. With this we again return to the Seiberg-Witten map F from Definition 4.1.5, but now in a slightly different guise.

Definition 4.4.1. The *parametrized Seiberg-Witten map* F is given by

$$F : \mathcal{C}_4(\mathfrak{c}) \times \mathcal{P}_3 \to L^2_3\left(\bigwedge\nolimits_+^2 T^*X \otimes i\mathbb{R} \oplus S^-(\mathfrak{c})\right), \qquad F(A, \psi, \eta) = (F_A^+ - \mathrm{cl}_+^{-1} q(\psi) - \eta, D_A\psi). \tag{4.4.2}$$

In order to apply the Sard-Smale theorem, we must ensure that F is a Fredholm map, which means that its differential is everywhere Fredholm. Let us now study the surjectivity of dF. It turns out that it is surjective at all irreducible configurations.

Lemma 4.4.2. *Let $(A,\psi) \in \mathcal{C}_4(\mathfrak{c})$ with $\psi \neq 0$ be an irreducible configuration and let $\eta \in \mathcal{P}_3$. Then dF is surjective at (A,ψ,η).*

Proof. It is immediate from Lemma 4.2.1 that the differential dF is given by

$$dF_{(A,\psi,\eta)} = \begin{pmatrix} d^+ & -\mathrm{cl}_+^{-1} dq_\psi(\cdot) & -1 \\ \frac{1}{2}\mathrm{cl}(\cdot)\psi & D_A & 0 \end{pmatrix}. \tag{4.4.3}$$

First, note that if we restrict dF to the tangent space of $\mathcal{P}_3$ it is given by the map $h \mapsto (-h, 0)$. Clearly this map is surjective onto the first factor, i.e. onto $L_3^2\left(\bigwedge_+^2 T^*X \otimes i\mathbb{R}\right)$: for $h \in L_3^2\left(\bigwedge_+^2 T^*X \otimes i\mathbb{R}\right)$ we have $dF_{(A,\psi,\eta)}(0,0,-h) = (h,0)$. Now consider the restriction of dF to the tangent space of $\mathcal{C}_4(\mathfrak{c})$. We see that this map is given by

$$dF_{(A,\psi,\eta)}(a,\varphi) = \left(d^+a - \mathrm{cl}_+^{-1} dq_\psi(\varphi), \frac{1}{2}\mathrm{cl}(a)\psi + D_A\varphi\right). \tag{4.4.4}$$

As we have already ensured surjectivity onto the first factor through the perturbations h, let us consider the projection P of this map onto the second factor, i.e. the map

$$P:\ (a,\varphi) \mapsto \frac{1}{2}\mathrm{cl}(a)\psi + D_A\varphi. \tag{4.4.5}$$

We would be done if we could show this map to be surjective. We argue by contradiction. Suppose it is not surjective, then there exists a $\xi \in L_3^2(S^-(\mathfrak{c}))$ which is orthogonal to the image of P, i.e.

$$\langle P(a,\varphi), \xi\rangle = 0 \qquad \text{for all } (a,\varphi). \tag{4.4.6}$$

Take $a = 0$ for the moment. Then the above reads $\langle D_A\varphi, \xi\rangle = 0$ for all φ, which by self-adjointness of the Dirac operator (see Lemma 3.4.3) means that $\langle \varphi, D_A\xi\rangle = 0$. But by non-degeneracy of the inner product this implies that ξ lies in the kernel of D_A. We see that from equation (4.4.6) we now get that

$$\int_X \langle \frac{1}{2}\mathrm{cl}(a)\psi, \xi\rangle \mathrm{d}\mu = 0 \qquad \text{for all } a. \tag{4.4.7}$$

Now let x be a given point where $\psi(x) \neq 0$. Then because ψ is continuous by Sobolev embedding, there exists some neighbourhood U in X containing x where ψ does not vanish. We now trivialize the bundles T^*X and S^- on some (possibly smaller) neighbourhood V of x and note that by the injectivity relation $\mathrm{cl}(a)^2 = -|a|^2\mathrm{Id}$ for Clifford multiplication we get an isomorphism between T^*V and $S^-(\mathfrak{c})|_V$, given by

$$b|_V \mapsto \mathrm{cl}(b|_V)\ \psi|_V\,. \tag{4.4.8}$$

In particular, we can let a be such that $\frac{1}{2}\mathrm{cl}(a)\psi\big|_V = \varphi|_V$. We then get from equation (4.4.7) that

$$0 = \int_V \langle \frac{1}{2}\mathrm{cl}(a)\psi, \xi\rangle \mathrm{d}\mu = \int_V |\xi|^2 \mathrm{d}\mu. \tag{4.4.9}$$

But this implies that $\xi \equiv 0$ on V. By the so-called unique continuation theorem (see e.g. [2]) proven using elliptic regularity, Theorem 2.5.10, this together with ellipticity of D_A implies that ξ vanishes on the entirety of X. We have arrived at our desired contradiction and conclude that dF is surjective. □

Consider now the parameterized moduli space $PM(\mathfrak{c})$, given by the quotient of the space $\mathcal{C}(\mathfrak{c}) \times \mathcal{P}$ under the action of the gauge group $\mathcal{G}$ acting trivially on the second factor.

Theorem 4.4.3. *The parameterized moduli space of $PM^*(\mathfrak{c})$ of irreducible solutions is a smooth submanifold of $\mathcal{B}_4^*(\mathfrak{c}) \times \mathcal{P}_4$. The projection π onto $\mathcal{P}_4$ is smooth, and the inverse image $\pi^{-1}(\eta)$ is given by the irreducible moduli space $M_\eta^*(\mathfrak{c})$ of the Seiberg-Witten equations perturbed by η.*

Proof. From Lemma 4.4.2 and an application of the inverse function theorem, Theorem 2.6.8, we see that the space

$$\{(A, \psi, \eta) \in \mathcal{C}_4^*(\mathfrak{c}) \times \mathcal{P}_3 : \, F(A, \psi, \eta) = 0\} \tag{4.4.10}$$

is a smooth submanifold. Hence by taking the quotient with respect to the gauge action we see that $PM^*(\mathfrak{c})$ is as well. □

Furthermore, we immediately get that for fixed η we have

$$\ker dF_{([A,\psi],\eta)} \cong H^1_{(A,\psi)} \cong T_{(A,\psi)} M_\eta(\mathfrak{c}), \qquad \operatorname{coker} dF_{([A,\psi],\eta)} \cong H^2_{(A,\psi)}. \tag{4.4.11}$$

In particular, dF is Fredholm of index $\frac{1}{4}\left(c_1(L)^2 - (2\chi(X) + 3\tau(X))\right)$ as per Theorem 4.2.8. We can now settle the question of ensuring smoothness of irreducible solutions.

Theorem 4.4.4. *Let (X, g) be a compact oriented Riemannian four-manifold with a given* Spin^c*-structure $\mathfrak{c}$ with determinant line bundle L. Then for a generic choice of perturbation $\eta \in \mathcal{P}_3$, the irreducible moduli space $M_\eta^*(\mathfrak{c})$ is a (possibly empty) smooth submanifold of $\mathcal{B}_4^*(\mathfrak{c})$ of dimension*

$$d(M(\mathfrak{c})) = \frac{1}{4}\left(c_1(L)^2 - (2\chi(X) + 3\tau(X))\right). \tag{4.4.12}$$

Proof. By an application of the Sard-Smale theorem, Theorem 2.6.6, to the Fredholm map

$$\pi : \, PM^*(\mathfrak{c}) \to \mathcal{P}_4, \tag{4.4.13}$$

we conclude that the of regular values η of π is dense. But for such an η, this means that dF_η is surjective and hence its cokernel vanishes everywhere. But this means that the second obstruction space $H^2_{(A,\psi)}$ vanishes for all irreducible solutions $(A, \psi) \in M_\eta^*(\mathfrak{c})$. Because the second obstruction space now vanishes for all irreducible solutions, the theorem follows from Theorem 4.2.8. □

We now turn to controlling the amount of reducible solutions. Let us first consider the unperturbed Seiberg-Witten equations $F(A, \psi) = 0$. Recall that a reducible point is a configuration pair $(A, \psi) \in \mathcal{C}(\mathfrak{c})$ with $\psi = 0$. For this to be a solution to the unperturbed Seiberg-Witten equations, we see that we must have

$$F_A^+ = 0. \tag{4.4.14}$$

In other words, A must be an ASD connection. Similarly, for it to be a solution to the perturbed Seiberg-Witten equations $F_\eta(A, \psi) = 0$ for $\eta \in \Omega^2_+(X; i\mathbb{R})$, we see that we must have

$$F_A^+ = \eta. \tag{4.4.15}$$

But we also know that the curvature F_A of A is a harmonic two-form with $[F_A] = -2\pi i c_1(L)$ in cohomology, where L is the determinant line bundle of $\mathcal{C}(\mathfrak{c})$, see Theorem 2.2.9. Let us determine when equation (4.4.15) has solutions. Recall from Definition 2.3.5 that the notion of self-duality depends on the Riemannian metric, which we assume to be fixed for now.

Lemma 4.4.5. *Given an imaginary self-dual two-form $\eta \in \Omega^2_+(X; i\mathbb{R})$, there exists a connection $A \in \mathcal{A}(L)$ such that $F_A^+ = \eta$ if and only if η lies in an affine subspace of $\Omega^2_+(X; i\mathbb{R})$ of codimension $b_2^+(X)$.*

Proof. Let $A_0 \in \mathcal{A}(L)$ be any unitary connection. Then by Lemma 2.2.7 any other connection $A \in \mathcal{A}(L)$ can be written as $A = A_0 + a$ for $a \in \Omega^1(X; i\mathbb{R})$, and its curvature is given by

$$F_A = F_{A_0} + da. \tag{4.4.16}$$

Taking the self-dual parts, we see that $F_A^+ = F_{A_0}^+ + (da)^+$. Now define a map $p : \Omega^1(X; i\mathbb{R}) \to \Omega^2_+(X; i\mathbb{R})$ by

$$p(a) = F_{A_0}^+ + d^+ a. \tag{4.4.17}$$

We then see that $F_A^+ = \eta$ has a solution if and only if $\eta \in \operatorname{im}(p) \subset \Omega^2_+(X; i\mathbb{R})$. Now recall the results from Hodge theory mentioned in Section 2.3, more specifically Lemma 2.3.9. This lemma implies that $\operatorname{im}(p)$ has orthogonal complement equal to the space $\mathcal{H}^2_+(X)$ of self-dual harmonic forms, which has dimension $b_2^+(X)$. □

The lemma above allows us to conclude that we can avoid reducible solutions to the Seiberg-Witten equations by perturbation, if we make the assumption that $b_2^+(X)(X)$ is positive. Namely, given a Riemannian metric g we see that there will only be reducible solutions after perturbing by an η lying in a space of codimension $b_2^+(X)$. If $b_2^+(X) > 0$, then generically a choice of perturbation will miss this space. If on the other hand $b_2^+(X) = 0$, then for any perturbation there will be reducible solutions. Combining this with Theorem 4.4.4 we can now conclude the following.

Theorem 4.4.6. *Let (X, g) be a compact oriented Riemannian four-manifold with $b_2^+(X) > 0$, and let $\mathfrak{c} \in \mathcal{S}^c(X)$ be given. Then for a generic choice of perturbation $\eta \in \mathcal{P}_3$, the moduli space $M_\eta(\mathfrak{c})$ is a (possibly empty) smooth submanifold of $\mathcal{B}_4^*(\mathfrak{c})$ of dimension*

$$d(\mathfrak{c}) = \frac{1}{4}\left(c_1(L)^2 - (2\chi(X) + 3\tau(X))\right). \tag{4.4.18}$$

Proof. In light of Theorem 4.4.4 we need only establish that the we can choose our perturbation such that there are no reducible solutions. But by the discussion above this theorem this can clearly be done if $b_2^+(X) > 0$. □

4.5 Orientability

In this section we deal with establishing the orientability of the moduli space, and in fact how to orient it. We mostly follow [23]. It will turn out that the orientation of the moduli space boils down to orienting the so-called determinant line bundle of the linearization of F, which in turn is accomplished by orienting the vector space

$$H^0(X;\mathbb{R}) \oplus H^1(X;\mathbb{R}) \oplus H^2_+(X;\mathbb{R}). \tag{4.5.1}$$

Let us now describe how this comes about. Let N be an n-dimensional manifold and let $E \to N$ be some rank k-bundle over N. Recall then that an *orientation* of E is given by a choice of non-vanishing section of the top exterior power of its tangent bundle, i.e. its determinant line bundle $\det(E) = \bigwedge^n E$,

$$s \in \Gamma(\det(E)), \qquad s(x) \neq 0 \text{ for all } x \in N. \tag{4.5.2}$$

An orientation for N is then an orientation for its tangent bundle, i.e. we choose $E = TN$ in the above. In the case that N is connected and orientable, it suffices to orient E at one point, as one can then extend this choice to an orientation of the entire manifold by parallel transport after choosing a connection on E. Recall now from Section 4.2 that at smooth irreducible points $[A, \psi]$ we had an explicit description of the tangent space of the moduli space, it being equal to the Zariski tangent space. This description involved the linearizations of the Seiberg-Witten map F and the gauge action, which turned out to be Fredholm. There is a general way to orient the bundles arising in this situation. See also [5, Section 5.1].

Definition 4.5.1. Given a Fredholm map $F : V \to W$ between two Hilbert spaces, its determinant line $\det\operatorname{ind}(F)$ is defined to be

$$\det\operatorname{ind}(F) = \bigwedge^{\text{top}} \ker(F) \otimes \bigwedge^{\text{top}} \operatorname{coker}(F)^*. \tag{4.5.3}$$

Here top indicates that one should take the respective top exterior powers of the subspaces $\ker(F) \subset V$ and $\operatorname{coker}(F) \subset W$.

Now suppose we instead have not just one but instead a whole family $\mathcal{F} = \{F_t\}$ of Fredholm maps $F_t : V \to W$ parametrized by some parameter $t \in T$, where T is a manifold. We can then consider for each fixed t the determinant line $\det\operatorname{ind}(F_t)$. However, we wish to instead consider this family of subspaces as a line bundle over T. A priori this seems to be ill-defined, as the dimensions of the kernels $\ker(F_t)$ and cokernels $\operatorname{coker}(F_t)$ may vary for different t. However, there is a natural construction showing that as their difference $\operatorname{ind}(F_t)$ is independent of t one can combine their determinant lines into a bundle.

Lemma 4.5.2. *Given a family $\mathcal{F} = \{F_t\}$ of Fredholm maps parameterized by a manifold T, there is a well-defined line bundle $\det\operatorname{ind}(\mathcal{F}) \to T$ called the* determinant line bundle *of the family, whose fiber at $t \in T$ is given by the vector space $\det\operatorname{ind}(F_t)$.*

Even more is true: if $\mathcal{F}$ and $\mathcal{F}'$ are two homotopic families of Fredholm operators both parameterized by T, then their determinant line bundle are isomorphic. The isomorphism is given by the homotopy, up to multiplication by some positive function. This is shown by considering the entire homotopy as a family of Fredholm operators over the product $T \times I$ and then noting that any line bundle over this product must be isomorphic to the product of a line bundle over T with I, as any line bundle over I is trivial. We can now use this general principle to construct a line bundle over the space $B(\mathfrak{c})$ of equivalence classes of configurations (A, ψ).

So now let X be a compact oriented four-manifold and $c \in \mathcal{S}^c(X)$ a Spin^c-structure. Let $(A, \psi) \in C(\mathfrak{c})$ be given and consider again the linearizations of the gauge action and the Seiberg-Witten map F, as computed in Lemma 4.2.2 and Lemma 4.2.1

$$D_1 = (-2d, (\cdot)\psi), \qquad D_2 = \begin{pmatrix} d^+ & -\mathrm{cl}_+^{-1} dq_\psi \\ \frac{1}{2}\mathrm{cl}(\cdot)\psi & D_A \end{pmatrix}. \tag{4.5.4}$$

Before, in determining a local description of the moduli space, we considered the complex formed out of their composition. We showed in Lemma 4.2.4 that it was elliptic as long as (A, ψ) is a solution to the Seiberg-Witten equations. However, when (A, ψ) is not a solution this decomposition is not necessarily a complex as $D_2 \circ D_1 \neq 0$ in general. Therefore, instead consider the operator D given by

$$D = (D_2, D_1^*) : L_2^2((T^*X \otimes i\mathbb{R}) \oplus S^+(\mathfrak{c})) \to L_1^2\left(\left(\bigwedge\nolimits_+^2 T^*X \otimes i\mathbb{R}\right) \oplus S^-(\mathfrak{c})\right) \oplus L_1^2(X; i\mathbb{R}). \tag{4.5.5}$$

For exactly the same reason as in the proof of Lemma 4.2.4, this new operator D is elliptic and hence Fredholm. As before, we deform D through a homotopy $D(t) = (D_2(t), D_1^*(t))$, where $D_2(t)$ and $D_1^*(t)$ are again given by

$$D_2(t) = \begin{pmatrix} d^+ & -(1-t)\mathrm{cl}_+^{-1} dq_\psi \\ \frac{1}{2}(1-t)\mathrm{cl}(\cdot)\psi & D_A \end{pmatrix}, \qquad D_1^*(t) = (-2d^* + (1-t)(\cdot)\psi). \tag{4.5.6}$$

We see that $D_2(0) = D_2$ and $D_1^*(0) = D_1^*$, while $D(1)$ is given by $d^+ + D_A + (-2d^*)$. Consider now the homotopy family of operators $\mathcal{D}(t) = \{D_{(A,\psi)}(t)\}$ parameterized by all points $(A, \psi) \in \mathcal{C}(\mathfrak{c})$. Through the determinant line bundle construction of Lemma 4.5.2 we get a line bundle $\det \mathrm{ind}(\mathcal{D}(t))$ over $\mathcal{C}(\mathfrak{c}) \times I$. Now because the gauge group $\mathcal{G}$ acts on $\mathcal{C}(\mathfrak{c})$ we can extend this action to one on $\mathcal{C}(\mathfrak{c}) \times I$ which is trivial on the I factor. Because this action is clearly linear on each fiber of the bundle $\det \mathrm{ind}(\mathcal{D}(t))$, we can quotient at all irreducible configurations to obtain a bundle O over

$$\mathcal{B}^*(\mathfrak{c}) \times I = (\mathcal{C}^*(\mathfrak{c}) \times I)/\mathcal{G} \subset (\mathcal{C}(\mathfrak{c}) \times I)/\mathcal{G}. \tag{4.5.7}$$

Note that we have to restrict ourselves to the irreducible configurations to ensure the resulting space is a manifold. We shall show that we can orient this bundle.

Proposition 4.5.3. *The bundle $O \to \mathcal{B}^*(\mathfrak{c}) \times I$ is trivial, with an orientation for O being determined by an orientation for*

$$H^0(X; \mathbb{R}) \oplus H^1(X; \mathbb{R}) \oplus H^2_+(X; \mathbb{R}). \tag{4.5.8}$$

Proof. Let $O_t : \mathcal{B}^*(\mathfrak{c}) \times \{t\}$ for $t \in I$ denote the bundles obtained by fixing a value of t. Then by the discussion below Lemma 4.5.2 we know that all bundles O_t are isomorphic. Hence it suffices to show just one of the O_t is trivial. Therefore, consider O_1. Recall that $D_{(A,\psi)}(1)$ is given by $d^+ + D_A + (-2d^*)$. In particular, the family of Fredholm operators now splits as the sum of the constant operator $d^+ + (-2d^*)$ and D_A. Looking at the definition of the determinant line bundle this means that $\det \operatorname{ind}(\mathcal{D}(1))$ is given by the tensor product of the constant line given by $d^+ + (-2d^*)$ and the determinant line bundle of the D_A. This decomposition carries over to the quotient as the action by $\mathcal{G}$ does not mix the target spaces of the operator $\mathcal{D}(t)$. But now note that the determinant line bundle of D_A is complex, as the operators D_A are complex linear. Thus this part carries a natural orientation as a real line bundle, coming from the complex structure on it. To orient the constant line given by $d^+ + (-2d^*)$, note that

$$\ker(d^+ - 2d^*) \cong H^1(X;\mathbb{R}), \qquad \operatorname{coker}(d^+ - 2d^*) \cong H^2_+(X;\mathbb{R}) \oplus H^0(X;\mathbb{R}). \tag{4.5.9}$$

Indeed, the factor -2 is immaterial, so instead consider $d^+ + d^*$. Then the result is immediate from Proposition 4.2.7. Hence after choosing an orientation for $H^0(X;\mathbb{R}) \oplus H^1(X;\mathbb{R}) \oplus H^2_+(X;\mathbb{R})$, we get an orientation for $\det \operatorname{ind}(d^+ - 2d^*)$ and hence for O. □

We can now use this result to orient the moduli space $M(\mathfrak{c})$ at smooth irreducible points.

Theorem 4.5.4. *The open subset $M^*(\mathfrak{c})$ of smooth irreducible points of $M(\mathfrak{c})$ is an orientable manifold, with an orientation being determined by an orientation for $H^0(X;\mathbb{R})$, $H^1(X;\mathbb{R})$ and $H^2_+(X;\mathbb{R})$.*

Proof. By Proposition 4.5.3, a choice of orientation for these three spaces gives us an orientation for the bundle $O_0 \to \mathcal{B}^*(\mathfrak{c}) \times \{0\} \cong \mathcal{B}^*(\mathfrak{c})$. Moreover, at smooth irreducible points $[A,\psi] \in M(\mathfrak{c})$ the cokernel of $D_{(A,\psi)}$ vanishes, while $\ker(D_{(A,\psi)}) \cong T_{[A,\psi]}M(\mathfrak{c})$. Because an orientation for $M(\mathfrak{c})$ is given by an orientation for

$$\det(TM(\mathfrak{c})) = \bigwedge^{\text{top}} TM(\mathfrak{c}) \cong \bigwedge^{\text{top}} \ker(\mathcal{D}) = \det \operatorname{ind}(\mathcal{D}) = O_0, \tag{4.5.10}$$

we see that we have indeed given an orientation for the moduli space at smooth irreducible points in this way. □

In order to make the choice of orientation explicit, we would have to make a choice of convention of *how* to orient the determinant line bundle O_0 and hence the moduli space. There are several different choices one can make, but the standard one is the following.

Definition 4.5.5. Given a compact oriented four-manifold X, a *homology orientation* for X is a choice of orientation for the ordered line

$$H^0(X;\mathbb{R}) \otimes \left(\bigwedge^{b_1(X)} H^1(X;\mathbb{R})\right)^* \otimes \left(\bigwedge^{b_2^+(X)} H^2_+(X;\mathbb{R})\right). \tag{4.5.11}$$

Remark 4.5.6. Varying the choice of Riemannian metric does not change the choice of orientation even though the space of self-dual forms changes: this space is also a maximal definite subspace of the intersection form Q_X, and the space of such subspaces is connected and contractible. In other words, a homology orientation does not depend on a Riemannian metric.

4.6 Cohomology

In this section we discuss the cohomology of the moduli space. In particular, we will show the Seiberg-Witten moduli space has a preferred two-dimensional cohomology class which will be used to define the Seiberg-Witten invariants in Section 5.1. This material can be found in [26].

Let (X, g) be a compact oriented Riemannian four-manifold and let $\mathfrak{c} \in \mathcal{S}^c(X)$ be a given Spin^c-structure. Recall that the action of the gauge group $\mathcal{G} = C^\infty(X; \mathrm{U}(1))$ on the configuration space $\mathcal{C}(\mathfrak{c})$ as defined in Definition 4.1.7 is free on pairs $(A, \psi) \in \mathcal{C}(\mathfrak{c})$ with $\psi \neq 0$ by Lemma 4.1.13. Moreover, points where it is not free have stabilizer equal to a copy of U(1).

Definition 4.6.1. Let $x_0 \in X$ be given. The *based gauge group* $\mathcal{G}_0$ is given by

$$\mathcal{G}_0 = \{g \in \mathcal{G} : g(x_0) = 1 \in \mathrm{U}(1)\}. \tag{4.6.1}$$

Of course we have that

$$\mathcal{G}/\mathcal{G}_0 \cong \mathrm{U}(1), \qquad \text{or} \qquad \mathcal{G} = \mathcal{G}_0 \times \mathrm{U}(1). \tag{4.6.2}$$

Now note that the proof of Lemma 4.1.13 shows that $\mathcal{G}_0$ acts freely on $\mathcal{C}(\mathfrak{c})$. Because of this, consider the quotient

$$\mathcal{B}_0(\mathfrak{c}) = \mathcal{C}(\mathfrak{c})/\mathcal{G}_0. \tag{4.6.3}$$

This quotient space is now a smooth Hilbert manifold. As we have by construction that

$$\mathcal{B}_0(\mathfrak{c}) = \mathcal{B}(\mathfrak{c})/\mathrm{U}(1), \tag{4.6.4}$$

we have thus described a principal U(1)-bundle $\mathbb{U}$ over $\mathcal{B}^*(\mathfrak{c})$. Now consider the inclusion

$$i : M^*(\mathfrak{c}) \hookrightarrow \mathcal{B}^*(\mathfrak{c}). \tag{4.6.5}$$

Through this map we can pull back the bundle $\mathbb{U}$ to get a principal U(1)-bundle

$$i^*(\mathbb{U}) \to M^*(\mathfrak{c}). \tag{4.6.6}$$

Being a principal U(1)-bundle, it has a first Chern class. This class is in fact the cohomology class we will use.

Definition 4.6.2. Let (X, g) be a compact oriented Riemannian four-manifold and let $\mathfrak{c} \in \mathcal{S}^c(X)$. We define an element $\mu \in H^2(M^*(\mathfrak{c}); \mathbb{Z})$ by

$$\mu = c_1(i^*(\mathbb{U})). \tag{4.6.7}$$

At this point it is not yet clear that the bundle $\mathbb{U}$ is independent of the chosen base point x_0. However, when X is connected, one can join any two points $x_0, x_1 \in X$ by some smooth path γ. This allows us to construct an isomorphism between the two bundles $\mathbb{U}(x_0)$ and $\mathbb{U}(x_1)$. In particular, we see from this that when X is connected, the class μ is independent of the chosen basepoint.

Remark 4.6.3. The definition of our class μ is analogous to *μ-map* used in Donaldson theory used to construct cohomology classes using the slant product [5]. To explore this analogy, see [26, Section 2.3].

Recall now the definition of a classifying space, Definition 2.1.16. We can use this to establish the homotopy type of $\mathcal{B}^*(\mathfrak{c})$ and hence determine its cohomology. Let H be a group and n a positive integer. Recall that a connected topological space X is called an *Eilenberg-MacLane space* of type $K(H, n)$ if $\pi_k(X) = H$ for $k = n$ and trivial otherwise.

Theorem 4.6.4. *The space $\mathcal{B}^*(\mathfrak{c})$ is a classifying space for the gauge group $\mathcal{G} = C^\infty(X; \mathrm{U}(1))$. In particular, it is homotopy equivalent to $\mathbb{C}P^\infty \times K(H^1(X;\mathbb{Z}), 1)$.*

Proof. Let L denote the determinant line bundle of $\mathfrak{c}$. Recall that $\mathcal{C}(\mathfrak{c}) = \mathcal{A}(L) \times \Gamma(S^+(\mathfrak{c}))$ is an affine space. Because of this it is contractible. Consider now the affine subspace of reducible configurations, which has infinite codimension. Because of this its complement, the open subset $\mathcal{C}^*(\mathfrak{c})$ of irreducible configurations, is also contractible. Hence by definition the quotient space $\mathcal{B}^*(\mathfrak{c}) = \mathcal{C}^*(\mathfrak{c})/\mathcal{G}$ is a classifying space for $\mathcal{G}$. Now consider $\mathcal{G}$ again. One can show that it splits as the product

$$\mathcal{G} = \widetilde{C}^\infty(X; \mathrm{U}(1)) \times H^1(X; \mathbb{Z}), \tag{4.6.8}$$

the first factor denoting the group of all maps from X to $\mathrm{U}(1)$ which are homotopically trivial and the second factor stemming coming from its connected components. We see that the space $\widetilde{C}^\infty(X; \mathrm{U}(1))$ deformation retracts onto the space of constant maps, which is just a copy of $\mathrm{U}(1)$. In particular, it is homotopy equivalent to $\mathrm{U}(1)$. From Example 2.1.18 we know that the classifying space of $\mathrm{U}(1)$ is $\mathbb{C}P^\infty$, so that the claim follows. □

Denote the first Chern class of the bundle $\mathbb{U} \to \mathcal{B}^*(\mathfrak{c})$ by $\mu' \in H^2(\mathcal{B}^*(\mathfrak{c}); \mathbb{Z})$.

Corollary 4.6.5. *The integral cohomology ring of $\mathcal{B}^*(\mathfrak{c})$ is isomorphic to*

$$H^\bullet(\mathcal{B}^*(\mathfrak{c}); \mathbb{Z}) \cong \bigwedge\nolimits^\bullet H^1(X; \mathbb{Z}) \oplus \mathbb{Z}[\mu']. \tag{4.6.9}$$

In other words, $\mathcal{B}^*(\mathfrak{c})$ is homotopic to the product $K(\mathbb{Z}^{b_1}, 1) \times K(\mathbb{Z}, 2)$.

Proof. This follows from Theorem 4.6.4 upon noting that $\dim H^1(X; \mathbb{Z}) = b_1(X)$ and the fact that $\mathbb{C}P^\infty \cong K(\mathbb{Z}, 2)$. We have established that μ' is an element of $H^2(\mathcal{B}^*(\mathfrak{c}); \mathbb{Z})$, and it is not hard to see that it then in fact generates the second cohomology. □

Chapter 5

Seiberg-Witten invariants

In this chapter we finally define the Seiberg-Witten invariant $\mathrm{SW}_X : \mathcal{S}^c(X) \to \mathbb{Z}$, having established all the required properties of the Seiberg-Witten moduli space. After stating its definition we check that it is independent of several choices, and then continue to establish some of its properties. In particular, we prove the theorem by Taubes mentioned in the introduction to this thesis.

5.1 Definition

With the results from the previous chapter, the definition of the Seiberg-Witten invariants is rather straightforward. Let (X, g) be a compact oriented Riemannian four-manifold satisfying $b_2^+(X) > 1$, and let $\mathfrak{c} \in \mathcal{S}^c(X)$ be a Spin^c-structure on X with determinant line bundle L. Furthermore, choose a homology orientation for X, i.e. a fixed choice of orientation for

$$H^0(X;\mathbb{R}) \oplus H^1(X;\mathbb{R}) \oplus H^2_+(X;\mathbb{R}). \tag{5.1.1}$$

We then know that for a generic self-dual 2-form $\eta \in \Omega^2_+(X; i\mathbb{R})$, the moduli space $M_\eta(\mathfrak{c})$ is an oriented compact smooth submanifold of $\mathcal{B}^*(\mathfrak{c})$ of dimension

$$d(M(\mathfrak{c})) = \frac{1}{4}\left(c_1(L)^2 - 2\chi(X) - 3\sigma(X)\right), \tag{5.1.2}$$

which is independent of η. This is the content of Theorems 4.2.8, 4.3.9, 4.4.6 and 4.5.4.

Definition 5.1.1. Consider the above setting. Then the *Seiberg-Witten invariant* of $\mathfrak{c}$ is an integer $\mathrm{SW}_X(\mathfrak{c})$ defined below.

To define the Seiberg-Witten invariant we will distinguish four cases, depending on the virtual dimension $d(M(\mathfrak{c}))$ of the moduli space. These are

- $d(M(\mathfrak{c})) < 0$;
- $d(M(\mathfrak{c})) = 0$;
- $d(M(\mathfrak{c})) > 0$ and is odd;

- $d(M(\mathfrak{c})) > 0$ and is even;

Let us first turn to the case where $d(M(\mathfrak{c})) < 0$. In this case the moduli space is generically empty by Theorem 4.4.6, so we simply set $\mathrm{SW}_X(\mathfrak{c}) = 0$. Now consider the case where $d(M(\mathfrak{c})) = 0$. Because $M_\eta(\mathfrak{c})$ is compact by Theorem 4.3.9, if it is a smooth manifold it consists of finitely many points. Hence let (g, η) be generic in the sense of Theorem 4.4.6. We then set the invariant equal to

$$\mathrm{SW}_X(\mathfrak{c}) = \sum_{m \in M_\eta(\mathfrak{c})} \varepsilon_m, \qquad \varepsilon_m = \pm 1, \tag{5.1.3}$$

with the sign depending on whether the given orientation agrees with the canonical orientation of a finite set. We continue to the cases where the virtual dimension is positive. First assume that $d(M(\mathfrak{c})) > 0$ and is odd. We saw in Section 4.6 that there is no good choice of cohomology class with which one can pair $[M_\eta(\mathfrak{c})]$. Because of this we simply set $\mathrm{SW}_X(\mathfrak{c}) = 0$. Lastly, assume that $d(M(\mathfrak{c})) > 0$ and is even. Again choose (g, η) generic in the sense of Theorem 4.4.6. Then $M_\eta(\mathfrak{c})$ is a finite-dimensional compact submanifold of $\mathcal{B}^*(\mathfrak{c})$, and in particular it has a fundamental class. In this case, recall that by Section 4.6 we have a given cohomology class $\mu \in H^2(M(\mathfrak{c}); \mathbb{Z})$. Given this, we define

$$\mathrm{SW}_X(\mathfrak{c}) = \int_{M_\eta(\mathfrak{c})} \mu^{\frac{1}{2} d(M(\mathfrak{c}))}. \tag{5.1.4}$$

This concludes the definition of the Seiberg-Witten invariant, having covered all possible values of the virtual dimension.

Remark 5.1.2. Looking at the dimension formula for the Seiberg-Witten moduli space in the form

$$d(M(\mathfrak{c})) = b_1(X) - 1 - b_2^+(X) + \frac{c_1^2(L) - \tau(X)}{4}, \tag{5.1.5}$$

one can show that a given Spin^c-structure $\mathfrak{c} \in \mathcal{S}^c(X)$ can only have a nonzero Seiberg-Witten invariant $\mathrm{SW}_X(\mathfrak{c})$ if $b_2^+(X) - b_1(X)$ is odd. Namely, only in this case is the integer $d(M(\mathfrak{c}))$ even.

We now wish to check that the Seiberg-Witten invariant $\mathrm{SW}_X(\mathfrak{c})$ is independent of most of the choices we made, making it truly an invariant of the pair $(X, \mathfrak{c})$.

Lemma 5.1.3. *For a compact oriented four-manifold X with $b_2^+(X) > 1$, the above definition of $\mathrm{SW}_X(\mathfrak{c})$ is independent of the choice of Riemannian metric and perturbation η.*

Proof. Suppose we have chosen two pairs (g_0, η_0) and (g_1, η_1) of Riemannian metrics and perturbations on X. Then we can choose a smooth path of metrics g_t joining g_0 to g_1, and a generic path η of perturbations connecting η_0 and η_1. With this we can form a parameterized moduli space

$$M(\mathfrak{c}, \eta) \subset \mathcal{B}^*(\mathfrak{c}) \times I, \tag{5.1.6}$$

which we know by Section 4.4 to be a smooth compact manifold with boundary. Here we use the assumption $b_2^+(X) > 1$ to ensure that this one-parameter family of moduli spaces does not contain reducible solutions. Having already chosen an orientation for the cohomology spaces $H^1(X;\mathbb{R})$ and $H^2_+(X;\mathbb{R})$, we see that $M(\mathfrak{c},\eta)$ is oriented through the standard orientation on I. Its boundary as an oriented manifold is given by $M(\mathfrak{c},\eta_1) - M(\mathfrak{c},\eta_0)$. But this implies that the fundamental homology classes of these latter two oriented moduli spaces in $\mathcal{B}^*(\mathfrak{c})$ are the same. We conclude then that

$$\int_{M(\mathfrak{c},\eta_1)} \mu^{\frac{1}{2}d(M(\mathfrak{c}))} = \int_{M(\mathfrak{c},\eta_0)} \mu^{\frac{1}{2}d(M(\mathfrak{c}))}, \tag{5.1.7}$$

showing that both values of $\mathrm{SW}_X(\mathfrak{c})$ are the same. □

Corollary 5.1.4. *Let X be a compact oriented four-manifold with $b_2^+(X) > 1$, and let $\mathcal{S}^c(X)$ denote the set of isomorphism classes of* Spin^c*-structures on X. Then Definition 5.1.1 gives a well-defined map*

$$\mathrm{SW}_X : \mathcal{S}^c(X) \to \mathbb{Z}. \tag{5.1.8}$$

Remark 5.1.5. The construction of SW_X in fact shows that it is a diffeomorphism invariant of X. Letting o_X denote the chosen homology orientation for X, this means that given an orientation preserving diffeomorphism $f : X \to Y$ between compact oriented four-manifolds with b_2^+ at least two we have

$$\mathrm{SW}_{X,o_X} \circ f^* = f^* \circ \mathrm{SW}_{Y,o_Y}. \tag{5.1.9}$$

This follows from our discussion in Section 4.6 that $\mathcal{B}^*_X = \mathcal{B}^*(\mathfrak{c})$ has the weak homotopy type of a classifying space for the gauge group, which is a homotopy invariant of X. Because of this f gives an isomorphism between the integral cohomology rings $H^\bullet(\mathcal{B}^*_X;\mathbb{Z})$ and $H^\bullet(\mathcal{B}^*_Y;\mathbb{Z})$.

As was mentioned in the introduction, we can do the same thing for manifolds with $b_2^+(X) = 1$. The definition of SW_X as described above still gives a well-defined map

$$(\mathfrak{c}, g) \mapsto \mathrm{SW}_X(\mathfrak{c}). \tag{5.1.10}$$

In other words, the only difference is that we can not get rid of the dependence on the Riemannian metric g. The proof of Lemma 5.1.3 uses that we can avoid reducible solutions by a generic path between perturbations, but this argument is not possible when $b_2^+(X) = 1$. Indeed, the space of Riemannian metrics is split up into regions divided by so-called walls, where one encounters reducible solutions. There is a *wall-crossing formula* relating the Seiberg-Witten invariants on either side of such a wall. More on this can be found, for example, in [23, Section 6.9].

5.2 Finiteness

In this section we prove that there are only finitely many Spin^c-structures $\mathfrak{c}$ such that $\mathrm{SW}_X(\mathfrak{c})$ is nonzero. In fact, we show that there are only finitely many Spin^c-structures $\mathfrak{c}$ which have non-negative dimensional moduli spaces $d(M(\mathfrak{c}))$ and whose moduli spaces are non-empty. This is done through finding uniform bounds on the curvature of any connection on the corresponding determinant line bundle. Let A be a unitary connection on this line bundle. Recall from Section 2.3 that we can split up the curvature into its self-dual and anti-self-dual parts and that this decomposition is orthogonal with respect to the inner product induced by the metric, i.e.

$$F_A = F_A^+ + F_A^- \qquad \text{and} \qquad \int_X |F_A|^2 \,\mathrm{d}\mu = \int_X |F_A^+|^2 \,\mathrm{d}\mu + \int_X |F_A^-|^2 \,\mathrm{d}\mu. \tag{5.2.1}$$

Now define the quantities $I^+(A)$ and $I^-(A)$ by

$$I^+(A) = \int_X |F_A^+|^2 \,\mathrm{d}\mu, \qquad I^-(A) = \int_X |F_A^-|^2 \,\mathrm{d}\mu. \tag{5.2.2}$$

Theorem 5.2.1. *Let X be a compact oriented four-manifold with $b_2^+(X) > 0$ and let η be a generic perturbation for all of its* Spin^c*-structures. Then there are only finitely many* Spin^c*-structures $\mathfrak{c}$ such that $M(\mathfrak{c})$ is non-empty. In particular for $b_2^+(X) > 1$ we have* $\mathrm{SW}_X(\mathfrak{c}) = 0$ *for all but finitely many $\mathfrak{c}$.*

Proof. Let $\mathfrak{c}$ be given and denote its determinant line bundle by L. The moduli space $M(\mathfrak{c})$ can only be non-empty if $d(M(\mathfrak{c})) \geq 0$, i.e. if

$$c_1(L)^2 \geq 2\chi(X) + 3\tau(X). \tag{5.2.3}$$

Let A be a unitary connection on L. From Theorem 2.2.9 we know that we can represent $c_1(L) \in H^2(X;\mathbb{Z})$ by the two-form $\frac{i}{2\pi}F_A$. From this together with equation (5.2.1) we get that

$$\begin{aligned} I^+(A) - I^-(A) &= -\int_X F_A \wedge F_A = 4\pi^2 \int_X c_1(L) \wedge c_1(L) \\ &= 4\pi^2 c_1(L)^2. \end{aligned} \tag{5.2.4}$$

Hence define the constant C_1 by $C_1 = -4\pi^2\,(2\chi(X) + 3\tau(X))$. We then see that

$$I^-(A) - I^+(A) \leq C_1. \tag{5.2.5}$$

But recall now from Section 4.3, more specifically equation (4.3.12), that if (A,ψ) is a solution to the Seiberg-Witten equations we also have a uniform bound on $I^+(A)$. This equation deals with the unperturbed case, but it is clear that the argument also gives a bound for any fixed perturbation η. In other words, we get an estimate

$$I^+(A) \leq C_2, \tag{5.2.6}$$

for some constant C_2 depending on η (and (X,g)). We can combine these two bounds to get

$$\|F_A\|^2 = I^+(A) + I^-(A) \leq C_1 + 2C_2. \tag{5.2.7}$$

Recalling that $[F_A] = -2\pi i c_1(L)$, we thus see that there exists a constant C such that

$$\|c_1(L)\| \leq C. \tag{5.2.8}$$

In other words, for $M(\mathfrak{c})$ to be non-empty the class $c_1(L)$ must lie in a ball of radius C inside the lattice $H^2(X;2\pi i\mathbb{Z})$. But this means that $c_1(L)$ belongs to a finite set. As from Theorem 3.2.5 we know that there are only finitely many Spin^c-structures with the same first Chern class $c_1(L)$, we are done. □

It is now natural to ask for examples of compact oriented four-manifolds with $b_2^+(X) > 1$ which have a non-zero Seiberg-Witten invariant for some Spin^c-structure giving a moduli space of positive dimension. First we introduce two notions by Kronheimer and Mrowka, which have counterparts in Donaldson theory.

Definition 5.2.2. A compact oriented four-manifold X is said to be of *SW-simple type* if $d(M(\mathfrak{c})) = 0$ for all $\mathfrak{c} \in \mathcal{S}^c(X)$. A Spin^c-structure $\mathfrak{c}$ with $\mathrm{SW}_X(\mathfrak{c}) \neq 0$ is called a *basic class.*

The answer to above question is that we do not have any examples of manifolds which are not of SW-simple type yet have a basic class. This in fact led Witten [34] to conjecture the following.

Conjecture 5.2.3. *All compact oriented four-manifolds X with $b_2^+(X) > 1$ are of SW-simple type.*

This conjecture has been shown to hold true for a large class of four-manifolds.

5.3 Positive scalar curvature

Let X be a compact oriented four-manifold. In this section we study the case when X admits a Riemannian metric with positive scalar curvature $s > 0$. In fact, we have

Lemma 5.3.1. *If X admits a Riemannian metric with scalar curvature $s > 0$, then if (A,ψ) satisfies $F(A,\psi) = 0$ we have $\psi = 0$.*

Proof. This is very similar to Lemma 4.3.3 used for compactness. Recall the Weitzenböck formula for the Dirac operator, Lemma 3.4.5. This reads

$$D_A^* D_A = \nabla_A^* \nabla_A + \frac{s}{4} + \frac{1}{2}\mathrm{cl}(F_A). \tag{5.3.1}$$

As again $\mathrm{cl}(F_A) = \mathrm{cl}(F_A^+)$ when acting on ψ, we get

$$0 = D_A^* D_A \psi = \nabla_A^* \nabla_A \psi + \frac{s}{4}\psi + \frac{1}{2}\mathrm{cl}(F_A^+)\psi = \nabla_A^* \nabla_A \psi + \frac{s}{4}\psi + \frac{1}{4}|\psi|^2\psi, \tag{5.3.2}$$

where we used that $\mathrm{cl}(F_A^+)\psi = q(\psi)\psi = \frac{1}{2}\left(\psi \otimes \psi^* - \frac{|\psi|^2}{2}\mathrm{Id}\right)\psi = \frac{|\psi|^2}{4}\psi$. So now take the inner product with ψ and integrate to get

$$0 = \int_X \left(|\nabla_A \psi|^2 + \frac{s}{4}|\psi|^2 + \frac{1}{4}|\psi|^4 \right) \mathrm{d}\mu. \tag{5.3.3}$$

Each of these terms is now necessarily non-negative, hence $\psi = 0$. □

In other words, we see that when $s > 0$ there are only reducible solutions. It is clear that Lemma 5.3.1 still holds if we replace F by F_η whenever η is small enough. Hence combining this with our transversality results we get

Theorem 5.3.2. *Let X be a compact orientable Riemannian four-manifold with $b_2^+(X) > 1$ and $s > 0$. Then $\mathrm{SW}_X \equiv 0$.*

Proof. Choose a Spin^c-structure $\mathfrak{c}$ and let g be a metric with $s > 0$. Note that having positive scalar curvature is an open condition in the space of metrics. What this means is that η can be chosen to be generic in the sense of Theorem 4.4.6 and small enough such that the conclusion of Lemma 5.3.1 still holds. Then by this lemma there are only reducible solutions, and by regularity of g there are only irreducibles. Moreover, g and η are chosen such that the moduli space is a smooth manifold, hence can be used to define the Seiberg-Witten invariants by Lemma 5.1.3. But we now have $M(\mathfrak{c}) = \emptyset$ so that $\mathrm{SW}_X(\mathfrak{c}) = 0$. This holds for all $\mathfrak{c} \in \mathcal{S}^c(X)$, giving the result. □

5.4 Gluing theory

In this section we briefly discuss gluing theory and how it pertains to Seiberg-Witten theory. Consider the following setting. Let X_1, X_2 be compact oriented four-manifolds, and assume that $\Sigma_i \hookrightarrow X_i$ for $i = 1, 2$ are two smoothly embedded closed oriented surfaces of the same genus g. Assume that Σ_i have trivial normal bundle. Then each Σ_i has a neighbourhood $N(\Sigma_i)$ diffeomorphic to $\Sigma_i \times D^2$. Choose an orientation-preserving diffeomorphism $f : \Sigma_1 \to \Sigma_2$ and lift it to an orientation-reversing diffeomorphism $\phi : \partial N(\Sigma_1) \to \partial N(\Sigma_2)$ via conjugation in the normal fiber. Given this ϕ, we can form the following.

Definition 5.4.1. The *fiber sum* $X = X_1 \#_\phi X_2$ of X_1 and X_2 along Σ_i is defined by the gluing

$$X = (X_1 \setminus N(\Sigma_1)) \cup_\phi (X_2 \setminus N(\Sigma_2)), \tag{5.4.1}$$

where ϕ is used to identify the boundaries.

Note that if $g = 0$ then this is just the ordinary connected sum $X = X_1 \# X_2$. Furthermore, note that X can depend on the choice of ϕ. A product theorem is a result which expresses the Seiberg-Witten invariants of X in terms of those of X_1 and X_2. Several types of product theorems have since been found, each covering different assumptions. We mention work by Morgan, Mrowka, Szabó and Taubes [24, 25, 32]. A particular case of their results is the following theorem, which is proven using similar techniques as for its analogue in Donaldson theory.

Theorem 5.4.2. *Let X_1, X_2 be compact oriented four-manifolds with $b_2^+(X_i) > 0$ for $i = 1, 2$. Then their connected sum $X = X_1 \# X_2$ has $\mathrm{SW}_X \equiv 0$.*

Furthermore, one interesting case of gluing is that of blowup, i.e. of taking out a point and replacing it with a copy of $\mathbb{C}P^1$. This amounts to taking the connected sum with $\overline{\mathbb{C}P}^2$. Again there is a result relating the Seiberg-Witten invariants of $X \# \overline{\mathbb{C}P}^2$ and X, showing in particular that there is a bijection between their basic classes. See [26, Section 4.6.2].

5.5 An involution

Let X be a compact oriented four-manifold. In this section we discuss an involution on $\mathcal{S}^c(X)$, the set of Spin^c-structures, and how the Seiberg-Witten invariants of these two are related. This material can be found in [26, 27]. Recall that given a Spin^c-structure $\mathfrak{c}$ we can consider its determinant line bundle $L(\mathfrak{c})$ and take its first Chern class $c_1(L(\mathfrak{c}))$. Now, because the associated spinor bundle $S(\mathfrak{c})$ is defined through a representation, see Definition 3.2.7, we can instead consider the dual to this representation and create another associated vector bundle $S^*(\mathfrak{c})$. It turns out that this then in turn defines a Spin^c-structure $\mathfrak{c}^*$ such that

$$S(\mathfrak{c}^*) = S^*(\mathfrak{c}) \qquad \text{and} \qquad L(\mathfrak{c}^*) = L^*(\mathfrak{c}). \tag{5.5.1}$$

In other words, the operator $\mathfrak{c} \mapsto \mathfrak{c}^*$ defines an involution on $\mathcal{S}^c(X)$, and the first Chern class of the determinant line bundle $L(\mathfrak{c}^*)$ satisfies

$$c_1(L(\mathfrak{c}^*)) = -c_1(L(\mathfrak{c})). \tag{5.5.2}$$

In particular, the dimensions of their Seiberg-Witten moduli spaces $M(\mathfrak{c})$ and $M(\mathfrak{c}^*)$ agree. It is then a natural question to ask how their Seiberg-Witten invariants are related.

Theorem 5.5.1. *Let X be a compact oriented four-manifold with $b_2^+(X) > 1$ and $b_2^+(X) - b_1(X)$ odd as per Remark 5.1.2. Then given a* Spin^c*-structure $\mathfrak{c} \in \mathcal{S}^c(X)$ we have*

$$\mathrm{SW}_X(\mathfrak{c}^*) = (-1)^{\frac{\chi(X)+\tau(X)}{4}} \mathrm{SW}_X(\mathfrak{c}). \tag{5.5.3}$$

5.6 Symplectic manifolds

In this section we study the Seiberg-Witten invariants of symplectic manifolds. We mostly follow [17], which in turn describes [31]. Let (X, ω) be a compact symplectic four-manifold. Choose an almost complex structure J such that (X, ω, J) is almost Kähler as per Definition 2.3.10. We know from Lemma 3.5.1 that an almost complex structure induces a Spin^c-structure $\mathfrak{c}_J$ on X. Furthermore, Proposition 3.5.2 then gives us an explicit description of the spinor bundle for all Spin^c-structures on X. So let $\mathfrak{c}_E = \mathfrak{c}_J \otimes E$ be any Spin^c-structure on X. Recall that its determinant line bundle is equal to $K^{-1} \otimes E^2$. Choose a perturbation $\eta \in \Omega^2_+(X; i\mathbb{R})$ and let $\psi \in \Gamma(S^+(\mathfrak{c}_E))$ be a spinor with components $\psi = (\alpha, \beta)$ with respect to the decomposition

$$S^+(\mathfrak{c}_E) = E \oplus (K^{-1} \otimes E). \tag{5.6.1}$$

We then have by definition of $q(\psi)$ that

$$q(\psi) = \begin{pmatrix} |\alpha|^2 & \alpha\overline{\beta} \\ \overline{\alpha}\beta & |\beta|^2 \end{pmatrix} - \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\overline{\beta} \\ \overline{\alpha}\beta & -\frac{1}{2}(|\alpha|^2 - |\beta|^2) \end{pmatrix}. \tag{5.6.2}$$

Now compare this to the decomposition of the self-dual two forms on X as per equation (2.3.21),

$$\Omega^2_+(X; \mathbb{C}) = \mathbb{C}\omega \oplus \Omega^{2,0}(X; \mathbb{C}) \oplus \Omega^{0,2}(X; \mathbb{C}). \tag{5.6.3}$$

Using Lemma 3.3.4 we see that for a connection A on $K^{-1} \otimes E^2$ the Seiberg-Witten monopole equation for (A, ψ) is equal to

$$\begin{cases} F_A^\omega &= \frac{i}{4}\left(|\alpha|^2 - |\beta|^2\right)\omega + \eta^\omega, \\ F_A^{0,2} &= \frac{1}{2}\overline{\alpha}\beta + \eta^{0,2}. \end{cases} \tag{5.6.4}$$

Now note that we can decompose $A = A_0 \otimes B^2$ with A_0 a connection on K^{-1} and B a connection on E. Choose A_0 such that Theorem 3.5.6 applies. As the curvature of A can be expressed in those of A_0 and B as

$$F_A = F_{A_0} + 2F_B, \tag{5.6.5}$$

we see that equation (5.6.4) becomes

$$\begin{cases} F_B^\omega &= \frac{i}{8}\left(|\alpha|^2 - |\beta|^2\right)\omega - \frac{1}{2}F_{A_0}^\omega + \frac{1}{2}\eta^\omega, \\ F_B^{0,2} &= \frac{1}{4}\overline{\alpha}\beta - \frac{1}{2}F_{A_0}^{0,2} + \frac{1}{2}\eta^{0,2}. \end{cases} \tag{5.6.6}$$

On the other hand, by Theorem 3.5.6 the Seiberg-Witten Dirac equation for (A, ψ) becomes

$$\overline{\partial}_B\alpha + \overline{\partial}_B^*\beta = 0. \tag{5.6.7}$$

Combining the above two observations we see that the Seiberg-Witten equations are equivalent to the system

$$\begin{cases} \overline{\partial}_B\alpha &= -\overline{\partial}_B^*\beta, \\ F_B^\omega &= \frac{i}{8}\left(|\alpha|^2 - |\beta|^2\right)\omega - \frac{1}{2}F_{A_0}^\omega + \frac{1}{2}\eta^\omega, \\ F_B^{0,2} &= \frac{1}{4}\overline{\alpha}\beta - \frac{1}{2}F_{A_0}^{0,2} + \frac{1}{2}\eta^{0,2}. \end{cases} \tag{5.6.8}$$

We now choose a very particular type of perturbation. Namely, let $\eta \in \Omega_+^2(X; i\mathbb{R})$ be given by

$$\eta^{0,2} = F_{A_0}^{0,2}, \qquad \eta^\omega = F_{A_0}^\omega - \frac{i}{4}r\omega, \tag{5.6.9}$$

for some number $r > 0$. Doing this, the Seiberg-Witten equations reduce to

$$\begin{cases} \overline{\partial}_B\alpha &= -\overline{\partial}_B^*\beta, \\ F_B^\omega &= \frac{i}{8}\left(|\alpha|^2 - |\beta|^2 - r\right)\omega, \\ F_B^{0,2} &= \frac{1}{4}\overline{\alpha}\beta. \end{cases} \tag{5.6.10}$$

For use in an upcoming proof we mention the following inequality, referred to as the Peter-Paul inequality.

Lemma 5.6.1. *For $a, b \geq 0$ and $\varepsilon > 0$ we have*

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}. \tag{5.6.11}$$

Proof. This is a trivial corollary of Young's inequality

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}, \tag{5.6.12}$$

proven by setting $a' = a/\sqrt{\varepsilon}$ and $b' = \sqrt{\varepsilon} b$. □

Lemma 5.6.2. *Assume* $\mathrm{SW}_X(\mathfrak{c}_E) \neq 0$*. Then the above system of equations (5.6.10) has at most one solution up to gauge equivalence when* $c_1(E) \cdot [\omega] = 0$ *and* r *is large enough.*

Proof. Certainly if $E = 0$ and $r \geq 0$, this system has a solution with $B = 0$, $\beta = 0$ and α constant such that $|\alpha|^2 = r$. We wish to show this is the only solution if $c_1(E) \cdot [\omega] = 0$. If E has a non-trivial Seiberg-Witten invariant, there is a solution (B, α, β) to the equations. We will omit writing the volume form dvol_g below. By the Weitzenböck formula of Lemma 2.3.16 we get with $s = \alpha$, taking the inner product with α

$$\int_X |d_B \alpha|^2 = \int_X \langle 2\overline{\partial}_B^* \overline{\partial}_B \alpha, \alpha \rangle + i\Lambda F_B |\alpha|^2. \tag{5.6.13}$$

Consider the first term of equation (5.6.13). By the Seiberg-Witten Dirac equation and then adjointness we have

$$\begin{aligned}\int_X \langle 2\overline{\partial}_B^* \overline{\partial}_B \alpha, \alpha \rangle &= 2 \int_X \langle \overline{\partial}_B^* (-\overline{\partial}_B^* \beta), \alpha \rangle = -2 \int_X \langle \overline{\partial}_B^* \overline{\partial}_B^* \beta, \alpha \rangle \\ &= -2 \int_X \langle \beta, \overline{\partial}_B^2 \alpha \rangle.\end{aligned} \tag{5.6.14}$$

By Lemma 2.3.15 and the second monopole equation we then see that this is equal to

$$\begin{aligned}-2 \int_X \langle \beta, \overline{\partial}_B^2 \alpha \rangle &= \int_X \left(-2\langle \beta, F_B^{0,2} \alpha \rangle + 2\langle \beta, N_J(\partial_B \alpha) \rangle \right) \\ &= \int_X \left(-\frac{1}{2}|\alpha|^2 |\beta|^2 + 2\langle \beta, N_J(\partial_B \alpha) \rangle \right).\end{aligned} \tag{5.6.15}$$

Note that ω has length $\sqrt{2}$ as mentioned below Definition 2.3.10. By the first monopole equation we see that the second term of equation (5.6.13) is equal to

$$\int_X i\Lambda F_B |\alpha|^2 = -\frac{1}{4} \int_X (|\alpha|^2 - |\beta|^2 - r)|\alpha|^2. \tag{5.6.16}$$

Combining equations (5.6.15) and (5.6.16) we get from equation (5.6.13) that

$$\int_X |d_B \alpha|^2 = \int_X \left(-\frac{1}{2}|\alpha|^2 |\beta|^2 + 2\langle \beta, N_J(\partial_B \alpha) \rangle - \frac{1}{4}(|\alpha|^2 - |\beta|^2 - r)|\alpha|^2 \right), \tag{5.6.17}$$

which we quickly rewrite to

$$\int \left(|d_B \alpha|^2 + \frac{1}{4}|\alpha|^2 |\beta|^2 + \frac{1}{4}(|\alpha|^2 - r)^2 + \frac{1}{4} r(|\alpha|^2 - r) \right) = \int_X 2\langle \beta, N_J(\partial_B \alpha) \rangle. \tag{5.6.18}$$

But now by Theorem 2.2.9 we have $c_1(E) = \frac{i}{2\pi}[F_B]$, so that

$$c_1(E) \cdot [\omega] = \int_X \frac{i}{2\pi} F_B \wedge \omega = -\frac{1}{8\pi}(|\alpha|^2 - |\beta|^2 - r). \tag{5.6.19}$$

Because N_J is a bounded tensor, by Lemma 5.6.1 there then exists a constant $C > 0$ is a constant not depending on (B, α, β) or on r so that

$$\int_X 2\langle \beta, N_J(\partial_B \alpha)\rangle \leq \int_X \frac{1}{2}|d_B\alpha|^2 + C|\beta|^2. \tag{5.6.20}$$

Now we substitute equations (5.6.19) and (5.6.20) in equation (5.6.18) and rearrange to get

$$\int \left(\frac{1}{2}|d_B\alpha|^2 + \frac{1}{4}|\alpha|^2|\beta|^2 + \frac{1}{4}(|\alpha|^2 - r)^2 + \left(\frac{1}{4}r - C\right)|\beta|^2 \right) \leq 2\pi r c_1(E) \cdot [\omega]. \tag{5.6.21}$$

So now take $r > 4C$. Then we see that $c_1(E) \cdot [\omega] \geq 0$. If we would have $c_1(E) \cdot [\omega] = 0$ then this equation shows again $\beta = 0$, $|\alpha|^2 = r$ and $d_B\alpha = 0$. But then $B = 0$ and $E = 0$. All such solutions are gauge equivalent, so the Seiberg-Witten moduli space consists of one point. □

We now check that the above perturbations $\eta = \eta(r)$ can be made to be regular, so that the associated Seiberg-Witten moduli space is smooth. We follow [26].

Lemma 5.6.3. *Let (X, ω, J) be a compact almost Kähler four-manifold. There exists an $R > 0$ such that the perturbation $\eta(r) = F^{\omega}_{A_0} + F^{0,2}_{A_0} - \frac{i}{4} r\omega$ gives rise to a smooth moduli space $M_\eta(\mathfrak{c}_J)$ for all $r > R$.*

Proof. We know from Theorem 5.6.2 that $M_\eta(\mathfrak{c}_J)$ contains just one point, which has $\psi \neq 0$ and hence does not come from a reducible configuration pair. Hence by the results of Section 4.4 it suffices to check that the linearization dF_η of the perturbed Seiberg-Witten map is surjective at this solution. Recall that the solution is given by $(A, \psi) = (A_0, \alpha_0, \beta_0)$ with $\alpha_0 = \sqrt{r}$ and $\beta_0 = 0$. Let (a, φ) a tangent vector to the configuration space. Decompose (a, ϕ) as

$$(a, \psi) = \left(\frac{i}{\sqrt{2}}(\phi + \overline{\phi}), \alpha, \beta \right), \tag{5.6.22}$$

for $\phi \in \Omega^{0,1}(X; \mathbb{R})$, $\alpha \in \Omega^{0,0}(X)$ and $\beta \in \Omega^{0,2}(X)$ as $S^+(\mathfrak{c}_J) = \bigwedge^{0,0} T^*X \oplus K^{-1}$. After some omitted computations we see from Lemma 4.2.1 and the explicit form of Clifford multiplication given in Proposition 3.5.2 that $dF_\eta(a, \psi) = 0$ amounts to the system

$$\begin{aligned} \sqrt{2}(\overline{\partial}\alpha + \overline{\partial}^*\beta) + i\sqrt{r}\phi &= 0, \\ i\overline{\partial}\phi &= \frac{\sqrt{r}}{4\sqrt{2}}\beta, \\ \overline{\partial}^*\phi &= -\frac{\sqrt{r}i}{4\sqrt{2}}\alpha. \end{aligned} \tag{5.6.23}$$

But now the fact that $D_{A_0} = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$ as per Theorem 3.5.6 gives us that this system amounts to

$$D_{A_0}\zeta = -\frac{\sqrt{r}i}{4}\phi, \qquad D^*_{A_0}\phi = -\frac{\sqrt{r}i}{4}\zeta, \tag{5.6.24}$$

where $\zeta = (\alpha, \beta)$. In other words, we must have

$$D_{A_0}^* D_{A_0} \zeta = -\frac{r}{16}\zeta. \tag{5.6.25}$$

By the Weitzenböck formula, Lemma 3.4.5, this can be rewritten into

$$\left(\nabla_{A_0}^* \nabla_{A_0} + S + \frac{r}{16}\right)\zeta = 0, \tag{5.6.26}$$

where S includes the scalar curvature and Clifford multiplication by F_{A_0}, but at any rate is independent of r. Now choose r large enough so that $S + r$ is a positive definite self-adjoint operator. Then equation (5.6.26) implies that $\zeta = 0$. But then $\phi = 0$, which implies that dF_η is indeed surjective. □

We can now tie our results together to prove a result of Taubes [31] that the canonical Spin^c-structure on a compact almost Kähler four-manifold has Seiberg-Witten invariant equal to 1 modulo 2.

Theorem 5.6.4. *Let (X, ω) be a compact symplectic four-manifold satisfying $b_2^+(X) > 1$ with an almost complex structure J making X almost Kähler. Then $\mathrm{SW}_X(\mathfrak{c}_J) = 1 \mod 2$.*

Proof. By Lemma 5.6.3 we know that the perturbation $\eta = F_{A_0}^\omega + F_{A_0}^{0,2} - \frac{i}{4}r\omega$ is smooth for $r > R$. Now choose r such that $r > \max\{R, 4C\}$ where C is the constant arising in the proof of Lemma 5.6.2. In other words, the Seiberg-Witten moduli space $M_\eta(\mathfrak{c}_J)$ is a compact oriented zero-dimensional manifold, which necessarily consists of a finite number of points. By Lemma 5.6.2 we then see that it in fact consists of just one point, given by the pair $(A, \psi) = (A_0, (\sqrt{r}, 0))$. By the definition of the Seiberg-Witten invariant, this implies that $\mathrm{SW}_X(\mathfrak{c}_J) = 1 \mod 2$. □

Remark 5.6.5. One can show that with a natural choice of orientation one in fact has $\mathrm{SW}_X(\mathfrak{c}_J) = 1$ in the theorem above. For details see [15, 27].

Write $\mathfrak{c}_J = 0$ and $\mathfrak{c}_E = E$. Then the result from Lemma 5.6.2 allows us to prove more. First, note that by Theorem 5.5.1 on involutions of Spin^c-structures we get that

$$\mathrm{SW}_X(E) = \pm \mathrm{SW}_X(K \otimes E^*). \tag{5.6.27}$$

Because of this, we can extend Theorem 5.6.4 to the following statement again due to Taubes [31].

Theorem 5.6.6. *Let (X, ω) be a compact symplectic four-manifold satisfying $b_2^+(X) > 1$. Then $\mathrm{SW}_X(K) = \pm\mathrm{SW}_X(0) = \pm 1$. Moreover, if $\mathfrak{c}_E = \mathfrak{c}_J \otimes E$ is some Spin^c-structure such that $\mathrm{SW}_X(\mathfrak{c}_E) = \mathrm{SW}_X(E) \neq 0$, then*

$$0 \leq c_1(E) \cdot [\omega] \leq c_1(K) \cdot [\omega] \tag{5.6.28}$$

with equality if and only if $E = 0$ or $E = K$.

Proof. We saw in the proof of Lemma 5.6.2 that there is no solution to the Seiberg-Witten equations if $c_1(E) \cdot [\omega] < 0$. But then Theorem 5.6.4 together with equation (5.6.27) implies the result. □

Taubes went on to prove a great deal of other things, equating the Seiberg-Witten invariants for symplectic manifolds with their Gromov invariants [30]. We lastly prove the theorem from the introduction.

Theorem 5.6.7. *Let (X, ω) be a compact symplectic four-manifold with $b_2^+(X) > 1$. Then X does not admit a Riemannian metric of positive scalar curvature. Furthermore, it does not admit a connected sum decomposition $X \cong X_1 \# X_2$ with $b_2^+(X_i) > 0$ for $i = 1, 2$.*

Proof. From Theorems 5.3.2 and 5.4.2 we know that if X would admit either a metric of positive scalar curvature or have such a connected sum decomposition, then $\mathrm{SW}_X \equiv 0$. On the other hand, Theorem 5.6.4 shows that $\mathrm{SW}_X(\mathfrak{c}_J) = 1 \mod 2$ for the Spin^c-structure $\mathfrak{c}_J$ associated to a compatible almost complex structure. □

Corollary 5.6.8. *The manifold $X_{p,q} := p\mathbb{C}P^2 \# q\overline{\mathbb{C}P^2}$ admits an almost complex structure if and only if $p \equiv 1 \mod 2$ and a symplectic structure if and only if $p = 1$.*

Proof. This follows from Theorem 5.6.7 because $X_{p,q}$ admits a metric of positive scalar curvature, as this property is preserved by connected sums. Note that $b_2^+(\mathbb{C}P^2) = 1$ and $b_2^-(\overline{\mathbb{C}P^2}) = 1$ and that b_2^+ and b_2^- behave additively with respect to connected sums. □

Chapter 6

Outlook

We discussed the basic theory for Seiberg-Witten invariants of compact oriented four-manifolds X. We saw that we needed the assumption $b_2^+(X) > 1$ in the section on transversality to ensure the Seiberg-Witten moduli space was indeed generically a smooth manifold and the Seiberg-Witten invariants where independent of the choice of Riemannian metric on X. More can be said in the case where $b_2^+(X) = 1$, and most of the results carry over with slight modifications.

Almost straight from the definition we saw that the Seiberg-Witten invariants are zero for manifolds which admit a metric of positive scalar curvature or a certain connected sum decomposition. On the other hand we could show the canonical Spin^c-structure one can associate to a symplectic manifold has Seiberg-Witten invariant equal to one through picking a smart choice of perturbation. This combination of vanishing and non-vanishing results for the Seiberg-Witten invariants in the presence of additional geometric structure gives rise to a dichotomy between such structures.

As was mentioned in the introduction, many interesting results have been left out of this thesis. One thing of particular interest is if one can prove vanishing or non-vanishing results of the Seiberg-Witten invariants for manifolds admitting other geometric structures than were mentioned in this thesis.

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