Formality and geometric structures

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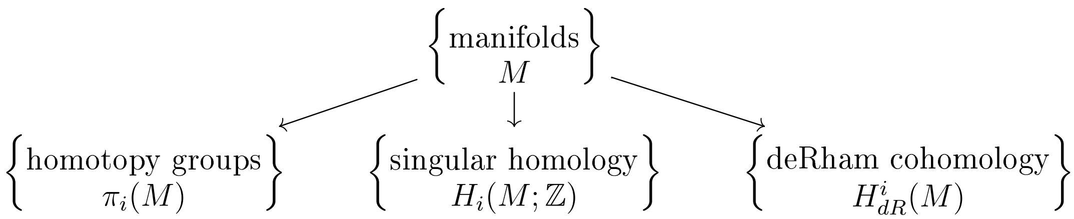
Contents

Introduction

Suppose we are given a pair of manifolds (or more generally topological spaces, or homotopy classes of spaces), and we are interested in ways to distinguish them from each other. Of course we have different ways to do this, for instance: the homotopy groups, the (singular) homology, or the ~~deRham~~ cohomology.



But although the homotopy groups have a high theoretical value ~~(they are very geometrically defined~~ and hence are really useful if one wants information about basic geometry of the space), apart from π_1 they are notoriously hard to calculate, even in the easiest examples like the sphere. On the other hand the homology and cohomology groups are quite easy to calculate (there are gadgets like excision, Mayer-Vietoris and an actual full description of the values on spheres), but we lose a certain theoretical value (surely one can use it to distinguish spaces, but what does an element of the homology group *represent geometrically*?). So it is a natural question to ask whether there are ways to retrieve information on the homotopy groups from (co)homology. The most important example is a quite fundamental result by Hurewicz.

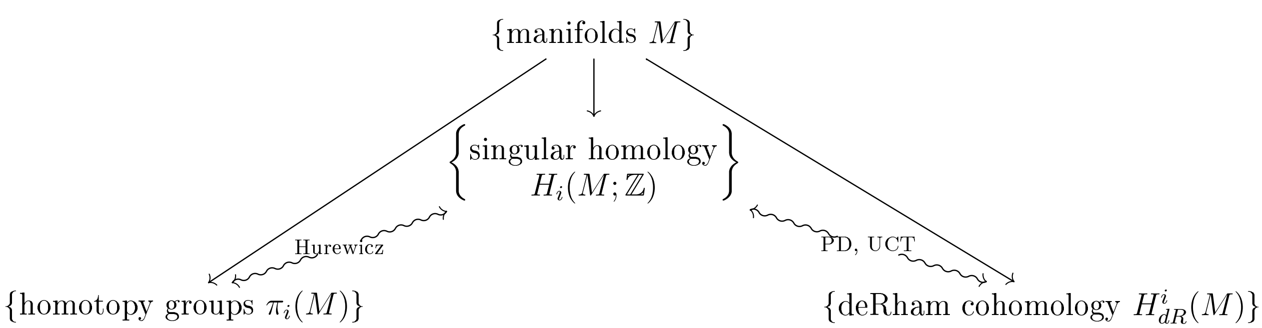
Theorem 0.1 (Hurewicz) The first non-zero $\pi_i(M)$ and first non-zero $H_i(M; \mathbb{Z})$ $(i > 0)$ occur at the same index, and for this index n the Hurewicz map $\pi_n(M) \to H_n(M; \mathbb{Z})$ is an isomorphism (if $n \neq 1$) and an isomorphism $(\pi_1(M))_{\text{ab}} \to H_1(M; \mathbb{Z})$ (if $n = 1$). $\diamond$

We can also connect homology and cohomology of a manifold, like Poincare duality and the universal coefficient theorem

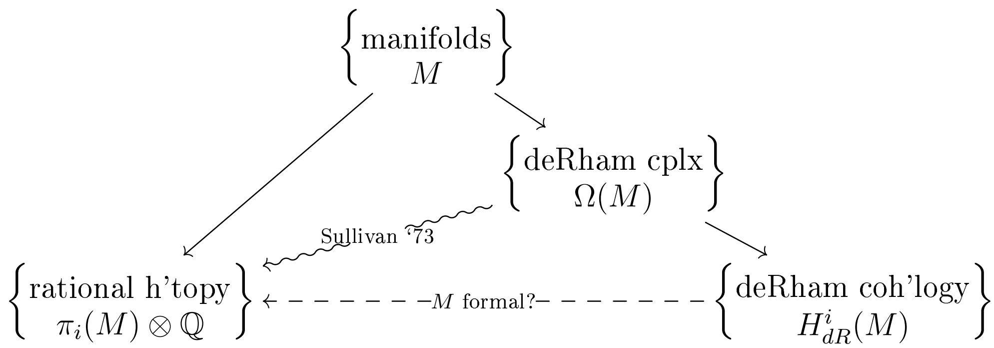
Theorem 0.2 (Poincare duality) If M is an oriented closed n-dimenisonal manifold then $H^k_{dR}(M) \cong H_{n-k}(M; \mathbb{R})$. $\diamond$

Theorem 0.3 $H^i_{dR}(M) \cong \text{Hom}_{\mathbb{Z}}(H_i(M, \mathbb{Z}), \mathbb{R})$ $\diamond$

So we're in the following situation:



We should note however, that the Hurewicz theorem stops giving information at an early stage (indeed it only connects the first non-zero entries), and we would like to have an arrow going to all the homotopy groups, not just the first non-trivial one. This may be too optimistic, but if we restrict to only the torsion free part, and insert the full deRham-complex in the middle, Sullivan [22] provided us with a way to go.



Now, ideally we would like an arrow from the deRham cohomology to the rational homotopy that gives us roughly the same result as going directly from the deRham complex to the rational homotopy. This is what will be called formality of M, and will be our main topic. It turns out that one can use geometric structures to deduce that certain classes of manifolds are formal. This leads to two justifications for trying to find formal (and non-formal) manifolds: firstly the rational homotopy of formal manifolds is fully contained in the cohomology, making them interesting from a topological point of view, and secondly the fact that admitting a certain geometrical structure can imply formality, showing that a manifold is not formal is an efficient way to prove that it can *not* admit certain structures.

In the first part of the text we will be setting up the language to formulate (and give a sketch of the proof) of the theorem by Sullivan. After that, starting with Kähler manifolds, we will discuss a few types of geometric structures that are interesting with respect to formality. The discussion of a few of these structures will be used as a starting point for certain more broad theorems which will (or at least try to) solve the formality question.

Overview

The thesis will be structured as follows.

In Section 1 we will set up the theory of differential graded algebras, starting with the definition and some basic examples (Section 1.1). From one of the examples we will define the notion of a minimal algebra and minimal models (Section 1.2), which replaces an algebra by a more workable one which shares important properties. In this part we will also make precise the theorem of Sullivan.
To settle some technical question (for instance showing the uniqueness of minimal models), we will also discuss Sullivan algebra (Section 1.3) and relative algebras (Section 1.4). In the end we will encounter a class of algebras which has preferable properties with respect to minimal models and cohomology, namely formal algebras (Section 1.5). A goal of the text is to find ways to distinguish formal algebras and formal manifolds (manifolds M for which $\Omega(M)$ is a formal algebra) from the non-formal ones. As a start we describe some algebraic characterizations of formality (Section 1.6).

The theory of CDGA's is well understood in the context of rational homotopy theory, in that there is a one-to-one correspondence between rational homotopy types and isomorphism classes of minimal algebras. We will discuss elements of this in Section 2. For instance we will describe a topological counterpart to de Rham complex (Section 2.1), and show that the algebraic theory of it is equivalent to the one of the deRham complex (Section 2.2). Using this broader point of view we will give a sketch of the proof of Sullivan's result (Section 2.3). The topological viewpoint allows to not only make algebraic models out of spaces, but also make spaces out of algebraic models (Section 2.4), which we will use to show that passing to the algebraic models preserves pushout-like properties (Section 2.5).

One of the main goals is to formulate formality of manifolds in terms of geometric structures that a manifold can admit. Indeed using the differential algebra that comes with certain geometric structures, one can massage the deRham-complex to show that it is formal. In Section 3 we will give some examples of this. The prototypical example are Kähler manifolds, and we will describe a result by Deligne, Griffiths, Morgan and Sullivan [5] that show that Kähler manifolds are formal, using the so-called dd^c-lemma (Section 3.1). We will also show that Lie groups are formal, and the proof of this generalizes to show that all symmetric spaces are formal (Section 3.2). An alternative proof of this fact leads us to the notion of geometrical formality, which deals with the connection of harmonic forms as formality. We will briefly discuss this notion, and show that it is a far stronger notion which comes with a few obstructions (Section 3.3).
We will end this section by describing the path along we will pursue our search of geometric structures that induce formality. In particular we introduce (Section 3.4) the notion of special holonomy, which gives us a short list of geometric structures that are central in our discussion.

Following this path, we first discuss quaternionic Kähler manifolds in Section 4. We do this in two parts, first doing some geometry in Section 4.1, to show that a quaternionic Kähler manifold is the base of a fibration with formal total space and formal fiber (namely S^2). With this we then look at a result of Amman and Kapovitch [1] concerning fibrations with formal fiber (Section 4.2), discussing both the proof in the general case (in which we leave some black boxes concerning spectral sequences), and proving it in the case where the fiber is an even dimensional sphere.

Then after quaternionic Kähler manifolds come Joyce-manifolds (or also G_2-manifolds). We will not discuss a proof that all Joyce-manifolds are formal (at the moment such a proof is not known to the author). What we will do is mention a few papers describing ways to mass produce Joyce-manifolds (Section 5.1) and discuss the central feature that these constructions have. In particular they produce manifolds that come around as the gluing of two formal manifolds along a formal intersection. Because of this we will use Section 5 to discuss formality of manifolds that arise in this way.
We start by using Poincaré duality to greatly simplify the question of formality for orientable (in particular simply connected) manifolds (Section 5.2). With this at hand we present new results, showing that together with a simple cohomological condition gluing formal manifolds along a formal intersection yields a formal manifold (Section 5.3).
The prototypical use of the theorem is showing that the connected sum of two formal manifolds is formal. In order to use our new result, we first need to show that $M\backslash\{*\}$ is formal if M is (which we will do). To generalize this further we will finish by presenting some partial results to the question whether $M\backslash S$ is formal if M is formal (Section 5.4).

1 Differential graded algebras

1.1 Definition

As mentioned in the overview we want to develop an algebraic language in which to do topology. To this end we set up the theory of differential graded algebras. The language is aimed to put the usual deRham-complex of differential forms (or equivalently the algebra of rational forms of any topological space) in a general framework, as to effectively extract algebraic data out of them. With the deRham-complex in mind, we define them as follows:

Definition 1.1

a)

- A vector space V is called a *graded vector space* if there are linear subspaces $\{V^i\}_{i\in\mathbb{N}}$ such that $V = \oplus_{i\in\mathbb{N}} V^i$. The element $v \in V^i$ are called *homogeneous of degree* i, the latter of which is denoted by $|v| = \deg(v) = i$.
- A morphism between graded vector spaces V and W is a linear map $f\colon V \to W$ such that $f(V^i) \subset W^i$.

b)

- A graded vector space A which carries an algebra structure is called a *graded algebra* if $V^i \cdot V^j \subset V^{i+j}$.
- A morphism between two graded algebras A and B is a linear map $\phi\colon A \to B$ which is both a morphism of graded vector spaces and a morphism of algebras (i.e. $\phi(a)\cdot\phi(b) = \phi(a\cdot b)$).

c)

- A graded algebra is called *commutative* if $ab = (-1)^{|a||b|}ba$ for a and b homogeneous elements.
- Passing from maps of graded algebras to commutative graded algebras we do not pose any more properties on them (indeed we don't add structure to preserve).

d)

- A graded vector space V is called a *cochain complex* if there is a linear map $d\colon V \to V$ s.t. $d(V^i) \subset V^{i+1}$ subject to the relation $d^2 = 0$. The elements in $\ker(d)$ are called *closed*, while elements in $\mathrm{im}(d)$ are called *exact*.
- A morphism of cochain complexes (V, d_V) and (W, d_W) is a linear map $f\colon V \to W$ such that $d_W \circ f = f \circ d_V$.

e)

- A cochain complex A which also is a graded algebra is called a *differential graded algebra (DGA)* if $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ (the *Leibniz identity*) for all homogeneous a and b.
- A morphism between two DGA's A and B is a linear map $\phi\colon A \to B$ that is both a morphism of cochain complexes and a morphism of graded algebras.

f)

- A differential graded algebra which is commutative as a graded algebra is called a *commutative differential graded algebra (CDGA)*

- Just as going from graded algebras to commutative graded algebras, we do not pose new properties when going from maps of DGA's to maps of CDGA's. ◇

Remark 1.2 This definition can be phrased for vector spaces over any field. Since we're mostly interested in deRham-complexes of manifolds, in this section we will work out the theory for vector spaces over $\mathbb{R}$, but everything works verbatim for any field of characteristic 0. In the next chapter we will delve a bit into rational topology, hence there most things will be defined over $\mathbb{Q}$. ◇

Now resembling the familiar procedure of going from singular (co)chains to singular (co)homology or going from the deRham-complex to deRham-cohomology, we can take the cohomology of a cochain complex.

Definition 1.3 Given a cochain complex (A, d) we can construct the graded vector space $H(A, d)$, called the *cohomology* of (A, d), setting:

$$H^i(A, d) = \frac{\ker(d\colon A^i \to A^{i+1})}{\mathrm{im}(d\colon A^{i-1} \to A^i)}$$ ◇

Note that d factors to the cohomology, where it vanishes for obvious reasons. In particular we get $(H(A, d), 0) \in \mathbf{Ch}^{\geq 0}$. Further even, any addional structure we have put on (A, d) descends to one on $H(A, d)$. Indeed, an algebra structure on (A, d) that is compatible with d (i.e. satisfies Leibniz), descends to $H(A, d)$ (precisely because of Leibniz), and $H(A, d)$ is commutative if (A, d) is. Furhermore any cochain map $f\colon (A, d) \to (B, d)$ induces a map $H(f) = f^*\colon H(A, d) \to H(B, d)$, also preserving any additional structure. This establishes H as a functor from $\mathbf{Ch}^{\geq 0}$ to $\mathbf{Ch}^{\geq 0}$, restricting to one from **dga** to **dga** and one from **cdga** to **cdga** in a canonical way, and in the following we will interpret the cohomology of a CDGA always as a CDGA in its own right.

Throughout this text we will encounter the notion of 'finite type' algebras and spaces. With the definition of cohomology in hand we can make precise what we mean by that:

Definition 1.4 A CDGA (A, d_A) is called of *finite type* if $H(A, d_A)$ is finitely generated as algebra (in particular $H^i(A, d_A)$ is always finite dimensional). A space X is called of *finite type* if $H^i(X; \mathbb{Q})$ is finite dimensional for all i (note that for most spaces the cohomology is trivial after some finite degree, so in that case we get that $H(X; \mathbb{Q})$ is also finitely generated). ◇

The notion of cohomology also enables us to define an important class of morphisms, namely the quasi-isomorphisms.

Definition 1.5 A map $f\colon (A, d) \to (B, d)$ of chain complexes (or DGA's, or CDGA's) is a *quasi-isomorphism* if $H(f)\colon H(A, d) \to H(B, d)$ is an isomorphism. ◇

Now that we have the abstract framework in position, we fit deRham-complex in it:

Example 1.6 For a manifold M, the ~~deRham-complex~~ $\Omega(M)$ together with the wedge product and the exterior derivative becomes a CDGA. Furthermore, for any smooth map $F\colon M \to N$, the pull-back $F^*\colon \Omega(N) \to \Omega(M)$ becomes a morphism of CDGA's.
Furthermore any homotopy equivalence (and even any weak homotopy equivalence) induces a quasi-isomorphism. ◇

Example 1.7 Fix a volume form $\alpha \in \Omega^n(S^n)$ and consider the map $\phi\colon H(S^n) = \mathrm{span}(1, [\alpha]) \to \Omega(S^n)$ sending 1 to the constant function $1 \in \Omega^0(S^n)$ and sending $[\alpha]$ to α. This map then is a quasi-isomorphism between $H(S^n)$ and $\Omega(S^n)$. ◇

Example 1.8 Suppose we are given a graded vector space V. We can then form the free CGA ΛV defined by taking the quotient of the tensor algebra by relations of the form $a \otimes b - (-1)^{ij} b \otimes a$ where $a \in V^i$ and $b \in V^j$ (i.e. we only impose relations to make ΛV graded commutative). Note that as an algebra $\Lambda V = \mathrm{Symmetric}(V^{\mathrm{even}}) \otimes \mathrm{Exterior}(V^{\mathrm{odd}})$. Of course, together with a map $d\colon \Lambda V \to \Lambda V$, satisfying $d^2 = 0$ and Leibniz, ΛV becomes a CDGA. The important remark here is that, since an element in ΛV is a polynomial in elements of V, by Leibniz any differential is induced by its restriction $d\colon V \to \Lambda V$.
Given a basis $\{x_i\}$ of V, we will sometimes build up ΛV with only information about the x_i's. Indeed given a set $\{x_i\}$ together with $|x_i| \in \mathbb{N}$ and $d(x_i)$ a polynomial in the elements of $\{x_i\}$ such that everything works out (i.e. the degree of dx_i adds up to $|x_i| + 1$, imposing Leibniz d^2 becomes zero, et cetera), the structure of ΛV for $V = \mathrm{span}(\{x_i\})$ is fixed. ◇

Remark 1.9 To add to the last point made, from time to time we will be a bit sloppy with notation in the case that we have a basis for V. Then we will sometimes not write ΛV or $(\Lambda V, d)$ but something like $\Lambda\{x_i\}$ or $\langle \{x_i\}; |x_i| = ..., dx_i = ...\rangle$. ◇

1.2 Minimal models

Having the language of free objects as described in the last example, we can define another important class of CDGA's, the minimal ones:

Definition 1.10 A CDGA $(\Lambda V, d)$ as described above is called *minimal* if

- $V = \oplus_{n\geq 1} V^n$ (i.e. $V^0 = \{0\}$) and $d(V) \subset \Lambda^{\geq 2} V$ (where $\Lambda^{\geq 2}$ means wedges of length at least 2).
- V admits a homogeneous basis $\{v_\alpha\}$ indexed by a well-ordered set such that $|v_\alpha| \leq |v_\beta|$ if $\alpha \leq \beta$ and $dv_\alpha \in \Lambda\{v_\beta\}_{\beta<\alpha}$. ◇

Remark 1.11 Most of the time we will restrict to algebras for which H^1 vanishes, which for minimal algebras translates to $V^1 = 0$. Then the second point is vacuous. To see this, note that if we have a homogeneous basis of V, ordered by degree, for a v in this basis $|dv| = |v| + 1$. Since there are no elements of the basis of degree 1, this means that dv is a product of elements of degree smaller than the degree of v.
So since we will only work with simply connected algebras, we will ignore the second part

in what follows (although in the general setting we should really not forget about it!). ◇

The nomenclature of the word minimal here is explained by the fact that the second statement implies that none of the generators is exact, hence we have a minimal number of generators needed to generate the cohomology.

Definition 1.12 Given a CDGA A, a *minimal model* for A is a minimal algebra M together with a quasi-isomorphism $\phi\colon M \to A$. ◇

So a minimal model of A is somehow a way to encode all the relevant information of A with a minimal amount of algebraic relations. Note also that since ϕ has to be a quasi-isomorphism, we get that $H^0(A) = \mathbb{R}$. Any CDGA with this property is called *connected*.

Example 1.13 Consider the odd-dimensional sphere S^{2n-1} with a volume form α. We define ΛV with as generating set one element v in degree $2n-1$ with $dv = 0$. Note that since the order of v is odd, we have $\Lambda V = \mathrm{span}(1, v)$. Then defining $\phi\colon (\Lambda V, d) \to \Omega(S^{2n-1})$ by $\phi(v) = \alpha$, we get a quasi-isomorphism, establishing $(\Lambda V, d)$ as a minimal model of $\Omega(S^{2n-1})$. ◇

Example 1.14 Now consider even-dimensional the sphere S^{2n} with a volume form α. If we do the same as before we get the right cohomology up to degree $4n-1$, but note that since now the order of v is even, ΛV becomes a polynomial algebra on v with zero differential, hence ΛV also has cohomology in degree $4n$, $6n$, $8n$, et cetera generated by v^2, v^3 and v^4 respectively. To cope with this problem we artificially make v^2 exact, by setting an extra generator w in degree $4n-1$, together with $dw = v^2$ and $\phi(w) = 0$ (from this point on in this example we have $V = \mathrm{span}\{v, w\}$). Then $\phi\colon (\Lambda V, d) \to \Omega(S^{2n})$ becomes a quasi-isomorphism, and we have a minimal model for $\Omega(S^{2n})$. ◇

Example 1.15 If one considers the cohomology algebras $H(S^n)$ of the spheres, and tries to find minimal models for them, it dawns that the result is the same. This is no coincidence. Indeed for such a minimal model $\psi\colon M \to H(S^n)$, composing with the map ϕ from Example 1.7 we get a minimal model $\phi \circ \psi\colon M \to \Omega(S^n)$ for $\Omega(S^n)$. In the next theorem we will show uniqueness of minimal models, so it was to be expected that the minimal models coincide. ◇

Example 1.16 We recall that $H(\mathbb{C}P^n) = \mathbb{R}[\alpha]/(\alpha^{n+1})$ with $\alpha = [\omega]$ for some symplectic form $\omega \in \Omega^2(\mathbb{C}P^n)$ (for instance the Fubini-Study form). So if we want to make a minimal model of $\Omega(\mathbb{C}P^n)$ we first need a closed generator v in degree 2, but then $d(v^{n+1}) = 0$. Hence we need to make v^{n+1} exact, thus we add a generator w in degree $2n+1$ with $dw = v^{n+1}$. Then setting $\phi(v) = \omega$ and $\phi(w) = 0$ yields a quasi-isomorphism $\phi\colon (\Lambda V, d) \to \Omega(\mathbb{C}P^n)$ which is the minimal model of $\Omega(\mathbb{C}P^n)$. ◇

Now one might wonder whether all CDGA's admit a minimal model, and, if they do, how many. The first question is always answered positively, while it may happen that there are CDGA's with multiple non-isomorphic minimal models. Both to have a simpler proof of the first fact, while also having an unique minimal model, we restrict to CDGA's A such that

$H^1(A) = 0$, these are called *simply connected*.

Theorem 1.17 Let A be a simply connected CDGA,

a) There is a minimal model $\phi\colon M \to A$ of A.

b) If there are two minimal modes $M_1 \xrightarrow{\phi_1} A \xleftarrow{\phi_2} M_2$ of A, there is an isomorphism $\psi\colon M_1 \to M_2$ such that $\phi_1 \simeq \phi_2 \circ \psi$. ◇

Remark 1.18 The notion of homotopy that we refer to in the second part of the theorem will be defined in the next section. Also we will prove the second part of the theorem only in the next section. ◇

Proof. At this point we only prove existence. We construct models $\phi\colon (\Lambda V, d) \to A$ with $(\Lambda V, d)$ minimal such that $H^i(\phi)$ is an isomorphism for $0 \leq i \leq n$, in such a way that we get from the n-step to the $(n+1)$-step by adding a few extra elements to V in degree n and $n+1$. Then taking the union of all these additions (or equivalently one can take the direct limit under the inclusions) we get a minimal model for A.

We can make the induction basis for any of $n = 0$, $n = 1$ or $n = 2$, but we'll settle on the last case. In that case we set $V^2 = H^2(A)$, with $d = 0$, and $\phi\colon \Lambda V \to A$ such that $[\phi(v)] = v$ for all $v \in V^2$ (i.e. find a section of $A^2 \cap \ker d \to H^2(A)$).

Then suppose we have $\phi\colon \Lambda V \to A$ up to the point that $H^i(\phi)$ is an isomorphism for all $0 \leq i \leq n$. We look at $H^{n+1}(\phi)$, and if it is not an isomorphism already we have to artificially make it surjective and injective. To this end we add generators W to V, by setting $W^n = \ker(H^{n+1}(\phi))$ and $W^{n+1} = \mathrm{coker}(H^{n+1}(\phi))$. We set $d|_{W^{n+1}} = 0$ and $d|_{W^n}$ in such a way that $d(w) \in (\Lambda V)^{n+1}$ is a generator of $w \in H^{n+1}(\Lambda V)$. Furthermore we extend ϕ to W^n in such a way that $\phi(d(w)) = d_A(\phi(w))$. This is possible since $H(\phi)(w) = 0$ and hence for the representative dw of w we have that $\phi(dw) = d_A(a)$, where a can be chosen to depend linearly on w, and we set $\phi(w) = a$. To end we define $\phi|_{W^{n+1}}\colon W^{n+1} \to A^{n+1} \cap \ker d_A$ a section of $A^{n+1} \cap \ker d_A \to H^{n+1}(A) \to \mathrm{coker} H^{n+1}(\phi)$.

At this point it is straightforward to check that $H^i(\phi)$ is an isomorphism for $i \leq n+1$. To see this note that we don't add closed or exact elements in $\Lambda(V \oplus W)$ in degree $\leq n$ so the extended ϕ satisfies that $H^i(\phi)$ is an isomorphism for $0 \leq i \leq n$ simply because the original ϕ is. On the other hand by construction $H^{n+1}(\phi)$ becomes an isomorphism. □

The important result by Sullivan, as announced in the introduction, is the following:

Theorem 1.19 (Sullivan '73) Let M be a simply connected manifold. Let $\phi\colon \Lambda V \to \Omega(M)$ be the minimal model of $\Omega(M)$. There is a bilinear non-degenerate pairing $V^k \times (\pi_k(M) \otimes \mathbb{R}) \to \mathbb{R}$, in particular the rank of $\pi_k(M) \otimes \mathbb{R}$ as an $\mathbb{R}$-vector space (or equivalently the rank of $\pi_k(M) \otimes \mathbb{Q}$ as a $\mathbb{Q}$-vector space) equals the number of generators of V in degree k. ◇

This theorem has a far-reaching implication on the calculation of the rational homotopy groups of manifolds. Normally when you want to do this, one needs to do calculations with the Postnikov tower of the manifold, which tends to be quite a nasty job. Instead when one has a grip on the deRham-complex of the manifold, calculating the minimal model is a far easier task. Essentially the theorem states that the information contained in the deRham-complex is the same as in the rational Postnikov-tower, and therefore it doesn't come as a surprise that this pops up in the proof (which we will not discuss at this time).

Combining the theorem with Examples 1.13, 1.14 and 1.16 one concludes:

Corollary 1.20

$$\pi_i(S^{2n}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i = 2n, 4n-1 \\ 0 & \text{else} \end{cases}$$
$$\pi_i(S^{2n-1}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i = 2n-1 \\ 0 & \text{else} \end{cases}$$
$$\pi_i(\mathbb{C}P^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i = 2, 2n+1 \\ 0 & \text{else} \end{cases}$$ ◇

1.3 Sullivan algebras

We still want to have a feasable language to proof the uniqueness of the minimal model. Sullivan algebras do this for us, and hence we digress for a bit in that direction. This will yield a proof of the uniqueness of the minimal model, and also gives us a few other interesting results on the side. We closely follow chapter 12 of the book by Félix, Halperin and Thomas [7].

Definition 1.21 A *Sullivan algebra* is a CDGA of the form $(\Lambda V, d)$ where:

a) $V = \oplus_{p \geq 1} V^p$

b) There are subspaces $V_{(i)}$ of V such that $V_{(0)} \subset V_{(1)} \subset \cdots \subset V = \cup_{i=0}^{\infty} V_{(i)}$ such that d maps $V_{(k)}$ into $\Lambda V_{(k-1)}$ (and in particular $d = 0$ on $V_{(0)}$).

Further, a *Sullivan model* of a CDGA A is a quasi-isomorphism $\phi\colon \Lambda V \to A$ with ΛV a Sullivan algebra. ◇

Clearly any minimal CDGA is a Sullivan algebra, and hence we'll also use the terms minimal Sullivan algebra and minimal Sullivan model.

An important result of using the language of Sullivan algebra's is that, up to a certain level, a Sullivan algebra will not notice that a quasi-isomorphism is not invertible, in that you can lift a map with a Sullivan algebra as domain through a (surjective) quasi-isomorphism.

Lemma 1.22 ~~**(lifting lemma)**~~ Let $A \xrightarrow{\eta} C \xleftarrow{\psi} \Lambda V$ be two maps of CDGA's with η a surjective quasi-isomorphism and ΛV a Sullivan algebra. Then there is a map $\phi\colon \Lambda V \to A$, called the *lift of* ψ, with the property that $\eta \circ \phi = \psi$. ⋄

Proof. We will define ϕ on V, inductively on the spaces V_k (which are defined such that $V_{(n)} = \oplus_{k=0}^{n} V_k$). Also we will call all the differentials d, notwithstanding which algebra we're working in.

Suppose that ϕ is defined on $V_{(k-1)}$ (and hence $\Lambda V_{(k-1)}$) and let $v \in V_k$. Since $dv \in \Lambda V_{(k-1)}$ (since ΛV is a Sullivan algebra), ϕdv is defined, and furthermore $(d \circ \phi \circ d)v = \phi(d^2 v) = 0$ and $\eta(\phi \circ d)v = (\psi \circ d)v = d(\psi v)$. Since η is a surjective quasi-isomorphism, we find an $a \in A$ such that $da = (\phi \circ d)v$ and $\eta a = \psi v$. Furthermore this a can be chosen to depend linearly on v. We set $\phi(v) = a$. Note that the seeming lack of induction base can be taken care of by starting on $V_{-1} = \{0\}$ with $\phi = 0$ on V_{-1}. □

So we see that maps from a Sullivan algebra lift through a quasi-isomorphism. We want this lift to be somehow unique. For this we need the notion of homotopy.

Definition 1.23 Let $\Lambda(t, dt)$ be the free algebra generated by t in degree 0 with formal differential. Elements of $\Lambda(t, dt)$ can be written as $P(t) + Q(t)dt$ with P and Q polynomials, with $d(P(t) + Q(t)dt) = P'(t)dt$. This algebra comes with maps $\epsilon_i \colon \Lambda(t, dt) \to \mathbb{R}$ for $i = 0, 1$ defined by $\epsilon_i(t) = i$ and $\epsilon_i(dt) = 0$.
Let $(\Lambda V, d)$ be a Sullivan algebra and $\phi_0, \phi_1 \colon \Lambda V \to A$ be two maps of CDGA's. A *homotopy* from ϕ_0 to ϕ_1 is a map $\Phi \colon \Lambda V \to A \otimes \Lambda(t, dt)$ such that $(\mathrm{id} \cdot \epsilon_i) \circ \Phi = \phi_i$ for $i = 0, 1$. ⋄

Note that we can define the notion of homotopy for maps starting from any CDGA, but to prove that the notion is transitive, we need to use the lifting lemma and so the domain should be a Sullivan algebra.

Similar to homotopic continuous maps, the notion of homotopy induces equal maps in cohomology. When the codomain is also a Sullivan algebra, there is even more to be said. To wit, let $\phi \colon \Lambda V \to \Lambda W$ be a morphism of Sullivan algebras. The *linear part of* ϕ is a linear map $Q\phi \colon V \to W$ defined by the fact that $\phi v - Q\phi v \in \Lambda^{\geq 2} W$ (i.e. we delete all the wedges and just take the pure elements from W). Under mild assumptions this Q is also equal for homotopic maps. This yields the following results:

Proposition 1.24 Let ΛV be a Sullivan algebra.

a) If $\phi_0, \phi_1 \colon \Lambda V \to A$ are homotopic maps, then $H(\phi_0) = H(\phi_1)$.

b) ~~Suppose that ΛV is minimal with $H^1(\Lambda V) = 0$, and ΛW is another minimal Sullivan~~ algebra. If $\phi_0, \phi_1 \colon \Lambda V \to \Lambda W$ are homotopic maps, then $Q\phi_0 = Q\phi_1$. ⋄

Proof. **a)** ~~When we think back to the proof that homotopic continuous maps induce the same map in homology, we remember that building the prism operator from the homotopy was the important point. So we~~ want to find ~~a~~ $h \colon \Lambda V \to A$ such that $\phi_1 - \phi_0 = dh + hd$. We'll do this by dissecting Φ. Note that elements of $\Lambda(t, dt)$ can be written as $u = \lambda_0 + \lambda_1 + \lambda_2 dt + x + dy$ with $x, y \in I$, where I is the ideal generated by $t(1-t)$. Then writing $\Phi \colon \Lambda V \to A \otimes \Lambda(t, dt)$

we get:

$$\Phi(z) = \phi_0(z) + (\phi_1(z) - \phi_0(z))t - (-1)^{|z|}h(z)dt + \Omega(z)$$

where $\Omega(z)$ is the part in $A \otimes (I \oplus d(I))$. This defines a map h which will have the property $\phi_1 - \phi_0 = dh + hd$. Then with this it is immediate that $H(\phi_1) - H(\phi_0) = 0$.

b) First note that under the assumptions, we have $V^1 = 0$. Indeed, writing $V = \cup_k V_{(k)}$ with $d\colon V_{(k)} \to \Lambda^{\geq 2} V_{(k-1)}$, we see that if $V^1_{(k-1)} = 0$ then $d = 0$ on $V^1_{(k)}$ since d maps $V^1_{(k)}$ to $V^1_{(k-1)} \cdot V^1_{(k-1)}$, so all elements of $V^1_{(k)}$ are closed. Since d maps into $\Lambda^{\geq 2} V$ it follows that no non-zero elements of $V^1_{(k)}$ can be exact. So we see that all non-zero elements of $V^1_{(k)}$ induce non-zero elements in cohomology. But since $H^1(\Lambda V) = 0$, we conclude that $V^1_{(k)}$ cannot have non-zero elements.

Now let Φ be the homotopy from ϕ_0 to ϕ_1. Then since $V = V^{\geq 2}$ it follows that Φ maps V into $\Lambda^{>0} W \otimes \Lambda(t, dt)$ and hence Φ maps $\Lambda^{\geq k} V$ into $\Lambda^{\geq k} W \otimes \Lambda(t, dt)$. Dividing on both sides by $\Lambda^{\geq 2}$ we get a linear map

$$\overline{\Phi}\colon \Lambda^{>0} V / \Lambda^{\geq 2} V \to (\Lambda^{>0} W / \Lambda^{\geq 2} W) \otimes \Lambda(t, dt).$$

Both the domain and codomain have induced cochain structures and $\overline{\Phi}$ preserves this induced structure. Since ΛV and ΛW are minimal, the image of $\Lambda^{>0}$ under d lies in $\Lambda^{\geq 2}$ and hence the induced differential on $\Lambda^{>0}/\Lambda^{\geq 2}$ is 0. Furthermore $\Lambda^{>0} V = V \oplus \Lambda^{\geq 2} V$ and similarly for W so we can see $\overline{\Phi}$ as a linear cochain map:

$$\overline{\Phi}\colon (V, 0) \to (W, 0) \otimes \Lambda(t, dt)$$

Under this identification we have $Q\phi_i = (\mathrm{id} \cdot \epsilon_i) \circ \overline{\Phi}$ for $i = 0, 1$. Since $\overline{\Phi}$ preserves cocycles, its image lies in the cocycles of $(W, 0) \otimes \Lambda(t, dt)$. A simple consideration shows that these cocycles have to live in $(W \otimes \mathbb{R} \cdot 1) \oplus (W \otimes \mathbb{R}[t] \cdot dt)$. Since ϵ_0 and ϵ_1 act the same on $\mathbb{R} \cdot 1$ and $\mathbb{R}[t] \cdot dt$, we conclude that $Q\phi_0 = Q\phi_1$. □

Using the notion of homotopy we can refine the lifting lemma. Roughly speaking, the lifting lemma implies that the map $\eta_\#\colon \mathbf{cdga}(\Lambda V, A) \to \mathbf{cdga}(\Lambda V, C)$ is surjective. It turns out that at the level of homotopy the map is bijective (even when η is any quasi-isomorphism, not necessarily a surjective one).

Proposition 1.25 ~~(lifting lemma)~~ Let $A \xrightarrow{\eta} C \xleftarrow{\psi} \Lambda V$ be two maps of CDGA's, with η a quasi-isomorphism and ΛV a Sullivan algebra. Then there is a morphism $\phi\colon \Lambda V \to A$ such that $\eta \circ \phi \simeq$ ~~η~~. Furthermore ϕ is unique up to homotopy, that is, the map $\eta_\#\colon [\Lambda V, A] \to [\Lambda V, C]$ is a bijection. ◇

Proof. We will only proof this for the case that η is surjective, in which case we can directly use the lifting lemma. When η is not surjective, one uses a construction philosophically similar to the use of a mapping cylinder to factor η up to homotopy as a zigzag of two surjective quasi-isomorphisms.

By use of the lifting lemma we get a ϕ such that $\eta\phi = \psi$, so clearly $\eta_\#$ is surjective, and we have to show injectivity. For this we consider $C \otimes \Lambda(t, dt) \xrightarrow{(\mathrm{id}_C \cdot \epsilon_0, \mathrm{id}_C \cdot \epsilon_1)} C \times C \xleftarrow{\eta\times\eta} A \times A$ to construct a fiber product $(C \otimes \Lambda(t, dt)) \times_{C\times C} (A \times A)$. With this the map $(\eta \otimes \mathrm{id}, \mathrm{id}_A \cdot \epsilon_0, \mathrm{id}_A \cdot \epsilon_1)\colon A \otimes \Lambda(t, dt) \to (C \otimes \Lambda(t, dt)) \times_{C\times C} (A \times A)$ is a surjective quasi-isomorphism.

Suppose we find $\phi_0, \phi_1\colon \Lambda V \to A$ such that $\eta\phi_0 \simeq \psi \simeq \eta\phi_1$, and let Ψ be a homotopy from $\eta\phi_0$ to $\eta\phi_1$. We lift the morphism (Ψ, ϕ_0, ϕ_1) through $(\eta \otimes \mathrm{id}, \mathrm{id}_A \cdot \epsilon_0, \mathrm{id}_A \cdot \epsilon_1)$ and we get a map $\Phi\colon \Lambda V \to A \otimes \Lambda(t, dt)$, which will be a homotopy from ϕ_0 to ϕ_1. □

Remark 1.26 In what follows we will refer to Proposition 1.25 as the ~~'lifting lemma'~~, instead of Lemma 1.22. Essentially they state the same results, but Proposition 1.25 is always applicable at the cost of only having commutativity up to homotopy (which we can deal with). ◇

Combining the two previous propositions we get the punchline of minimal Sullivan algebras and quasi-isomorphisms

Corollary 1.27 A quasi-isomorphism between minimal Sullivan algebras with vanishing first degree cohomology is an isomorphism. ◇

Proof. Let $\psi\colon \Lambda V \to \Lambda W$ be a quasi-isomorphism as described. Then the surjectivity of the previous proposition gives a morphism $\phi\colon \Lambda W \to \Lambda V$ such that $\psi\phi \simeq \mathrm{id}$. Then $\psi\phi\psi \simeq \psi$ and hence by injectivity in the preceding proposition we get $\phi\psi \simeq \mathrm{id}$. Now the second-last proposition yields that $Q(\phi)$ and $Q(\psi)$ are inverse isomorphisms to each other, and hence in particular $Q(\psi)\colon V \to W$ is surjective. Fix $w \in W^k$, and let $v \in V$ such that $Q\psi(v) = w$. Then $\psi(v) \in w + \Lambda W^{\leq k-1}$ and hence $W^k \subset \mathrm{im}\psi + \Lambda W^{\leq k-1}$. By induction it follows that W^k and hence $\Lambda W^{\leq k}$ is contained in $\mathrm{im}\psi$, and so ψ is surjective.

Now that ψ is surjective, we can choose ϕ such that $\psi\phi = \mathrm{id}$, and hence ϕ is injective, and since $H(\psi)H(\phi) = H(\mathrm{id}) = \mathrm{id}$, ϕ is also a quasi-isomorphism. Then since $\psi\phi\psi = \psi$ we get that $\phi\psi \simeq \mathrm{id}$, but then running the same argument as above again with the roles of ϕ and ψ interchanged we find that ϕ is surjective, and hence an isomorphism with ψ being the inverse isomorphism. □

With this in hand, proving uniqueness of minimal models is not hard:

Proof of Theorem 1.17.b). First using the lifting lemma we find a map $\psi\colon M_1 \to M_2$ such that $\phi_1 \simeq \phi_2 \circ \psi$. Then looking in cohomology we have $H(\phi_1) = H(\phi_2) \circ H(\psi)$, and since $H(\phi_1)$ and $H(\phi_2)$ are isomorphisms, so is $H(\psi)$. Thus we see that ψ is a quasi-isomorphism between simply connected minimal algebras. Then the previous corollary implies that ψ is an isomorphism. □

1.4 Relative algebras and models

In the big picture 'spaces are CDGA's' one can lay parallels between free algebras and minimal models on one side, and procedures like cellular approximation on the other side, in that both are a way to replace a structure with an easier structure while preserving invariants. In somewhat the same spirit there are two more, relative, notions that can also be phrased in algebraic form. Those are relative CW-complexes (where one adds cells to a base space) and mapping cylinders (where one can factor any map as an embedding and a quasi isomorphism). To that end we introduce relative algebras and relative models.

Definition 1.28 A *relative Sullivan algebra* is a CDGA of the form $(B \otimes \Lambda V, d)$ where

a) $(B, d) = (B \otimes 1, d)$ is a sub DGA (called the *base*), and $H^0(B) = \mathbb{R}$.

b) $1 \otimes V = V = \oplus_{p \geq 1} V^p$.

c) $V = \cup_{k=0}^{\infty} V_{(k)}$ where $V_{(0)} \subset V_{(1)} \subset \cdots$ is an increasing sequence of graded subspaces such that $d\colon V_{(0)} \to B$ and $d\colon V_{(k)} \to B \otimes \Lambda V_{(k-1)}$. ◇

Definition 1.29 Let $\phi\colon (B, d) \to (C, d)$ be a CDGA-map such that $H^0(B) = \mathbb{R}$. A *Sullivan model for* ϕ is a quasi-isomorphism $m\colon (B \otimes \Lambda V, d) \to (C, d)$, such that $(B \otimes \Lambda V, d)$ is a relative Sullivan algebra with base (B, d) and $m|_B = \phi$. ◇

Definition 1.30 A relative Sullivan algebra $(B \otimes \Lambda V, d)$ is *minimal* if $\operatorname{im} d \subset B^+ \otimes \Lambda V + B \otimes \Lambda^{\geq 2} V$. A *minimal ~~Sullivan~~ model* for $\phi\colon (B, d) \to (C, d)$ is a Sullivan model $(B \otimes \Lambda V, d) \to (C, d)$ such that $(B \otimes \Lambda V, d)$ is minimal. ◇

Remark 1.31 If we restrict to the case $B = \mathbb{R}$ and the canonical map $\phi\colon \mathbb{R} \to (C, d)$, we recover the definitions of Sullivan algebra, Sullivan model of (C, d) and minimal Sullivan model. ◇

Proposition 1.32 A morphism $\phi\colon (B, d) \to (C, d)$ of connected CDGA's such that $H^1(\phi)$ is injective, has a minimal Sullivan model. ◇

Proof. For simplicity we restrict to the case that B and C are simply connected and that $H^2(\phi)$ is injective. In this case we proceed like in the construction of the minimal model. We construct spaces $V_{(n)}$ concentrated in degrees $\geq n$, together with a $d\colon V_{(n)} \to B \otimes \Lambda V_{(n)}$ and $m\colon V_{(n)} \to C$. We do this in such a way that extending m to be ϕ on B, and using preservation of the product gives a map $m\colon B \otimes \Lambda V_{(n)} \to C$ such that $H^i(m)$ is an isomorphism for all $i \leq n$.

To start we set $V_{(2)} = V_{(2)}^2 = \operatorname{coker} H^2(\phi)$ with $d(V_{(2)}) = 0$ and m a section of $C^2 \cap \ker d \to H^2(C) \to \operatorname{coker} H^2(\phi)$. Note that since $H^2(\phi)$ is injective, we don't need to kill the kernel, and hence in this way $H^2(m)$ becomes an isomorphism.

Then supposing that we have constructed $V_{(n)}$, we set $V_{(n+1)} = V_{(n)} \oplus \ker H^{n+1}(m) \oplus$

coker$H^{n+1}(m)$ with the kernel in degree n and the cokernel in degree $n+1$. We set d on the corkernel 0 and m on the cokernel a section of $C^{n+1} \cap \ker d \to H^{n+1}(C) \to \text{coker}H^{n+1}(m)$. Further d on $\ker H^{n+1}(M)$ should map in $B \otimes \Lambda V_{(n)}$ such that dw represents w, and at last m on the added kernel should be such that $m(dw) = d_C(m(w))$. That we can arrange d and m on $\ker H^{n+1}(M)$ such that this holds follows in the same way as in the construction of the minimal model.

To then show that we get a minimal Sullivan model is similar to the end of the construction of the minimal model. □

1.5 Formal algebras

Up to this point for all algebras we have encountered, the minimal model of the algebra and the minimal model of the cohomology are the same. This is of course no coincidence since up to this point all algebras were quasi-isomorphic to their cohomology. One can wonder whether this always happens. Unfortunately it doesn't, as we see in the next example.

Example 1.33 The Heisenberg group H is the Lie group consisting of 3×3-matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \qquad (a, b, c \in \mathbb{R})$$

We note that the inverse of such a matrix is given by

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & ab-c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$$

From this it follows that the matrices with integer coefficients form a closed subgroup $H_{\mathbb{Z}}$.

Consider the following minimal algebra

$$A = \langle x, y, z \| x| = |y| = |z| = 1, dx = dy = 0, dz = xy \rangle$$

This is the minimal model of the deRham-complex of the quotient $H/H_{\mathbb{Z}}$. As a vector space it is spanned by x, y, z, xy, xz, yz and xyz. It follows that all spanning vectors are closed, except for z for which $dz = xy$. So we get that:

$$H(A) = \text{span}_{\mathbb{R}}([x], [y], [xz], [yz], [xyz])$$

If we name $[x] = \alpha$, $[y] = \beta$, $[xz] = \gamma$ and $[yz] = \delta$, we get the following generator-relation-description of $H(A)$:

$$H(A) = \langle \alpha, \beta, \gamma, \delta | \alpha\delta + \beta\gamma = \alpha\beta = \alpha\gamma = \beta\delta = \gamma^2 = \delta^2 = \gamma\delta = 0 \rangle$$

Following the construction we described in the proof of Theorem 1.17 for making the minimal model $\mathcal{M}$ of $H(A)$, we set:

$$\mathcal{M}_{(1)} = \langle v_1, v_2 || v_1| = |v_2| = 1, dv_1 = dv_2 = 0\rangle$$

where v_1 is to represent α and v_2 is to represent β. Then for the next step we get a closed $v_1 v_2$ we want to be exact. On the other hand we also need a representative for the closed elements γ and δ in degree 2. So we get:

$$\mathcal{M}_{(2)} = \langle v_1, v_2, w_1, v_3, v_4 || v_1| = |v_2| = |w_1| = 1, |v_3| = |v_4| = 2, dv_i = 0, dw_1 = v_1 v_2\rangle$$

Where v_1, v_2, v_3 and v_4 represent α, β, γ and δ respectively. Since from this point on the minimal model will only grow we see that $H(A)$ has a much bigger minimal model than A. So we see that there is an algebra which is not quasi-isomorphic to its cohomology. $\diamond$

~~**Remark 1.34** One can construct a similar example for a simply connected algebra, by putting x and y in degree 3 and z in degree 5, as we will encounter in Example 4.19.~~ $\diamond$

In the context of Theorem 1.19 this makes life a little bit harder, since we really need to use the whole algebra, we cannot get away with using just the cohomology. On the other hand we saw that there are a few examples of algebras where we can get away with doing this, so it is natural to give this class of algebras a special name:

Definition 1.35 A *weak equivalence* between CDGA's (A, d) and (B, d') is a zigzag of quasi-isomorphisms $(A, d) \xrightarrow{\simeq} (C_1, d_1) \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} (C_n, d_n) \xleftarrow{\simeq} (B, d')$. If such a weak equivalence exists we call (A, d) and (B, d') *weakly equivalent*.
A CDGA (A, d) is called *formal* if it is weakly equivalent to a CDGA $(H, 0)$ with trivial differential. A manifold M is called *formal* is $\Omega(M)$ is a formal CDGA. $\diamond$

Remark 1.36 Of course, if (A, d) is weakly equivalent to some $(H, 0)$, it is weakly equivalent to $(H(A, d), 0)$ Furthermore, a weak equivalence can always be represented by a zigzag containing one algebra in between, namely the minimal model which A and B share.
However, it is more natural to come across zig-zags which are longer than two maps, or zig-zags that do not end at $H(A, d)$ but at some other algebra with zero differential. $\diamond$

Example 1.37 By Example 1.7, all the spheres are formal manifolds. Exhibiting a map $\phi\colon H(\mathbb{C}P^n) \to \Omega(\mathbb{C}P^n)$ which sends $[\omega]$ to ω (for a fixed symplectic form ω) one also finds that $\mathbb{C}P^n$ is formal. $\diamond$

As Theorem 1.17 implies that quasi-isomorphic algebras have isomorphic minimal models (indeed having the same minimal models is equivalent to being weakly equivalent), we get that:

Proposition 1.38 Let A be a formal CDGA, and let $\phi\colon M \to A$ and $\phi'\colon M' \to H(A)$ be minimal models of A and $H(A)$ respectively, then there is an isomorphism $\psi\colon M \xrightarrow{\cong} M'$. $\diamond$

Combining this with Theorem 1.19 we get that:

Corollary 1.39 Let M be a formal and simply connected manifold. Then the rank of $\pi_i(M) \otimes \mathbb{Q}$ as a $\mathbb{Q}$-vector space equals the number of generator of the minimal model of $H(M)$ in degree i. $\diamond$

In particular, we see that for formal manifolds, the torsion free part of the homotopy can be fully determined from the cohomology algebra.

1.6 Characterizing formality

The previous corollary justifies why one is interested in finding formal manifolds among all manifolds. Since the credo is 'spaces are CDGA's', this means that we want to characterize formal CDGA's among all CDGA's. In this section we discuss a few ways to characterize formality of CDGA's.

We introduce the first result by proving that low-dimensional manifolds are formal. Since the reason for this is algebraic in nature, namely Poincaré duality, we generalize this notion to general algebras.

Definition 1.40 A CGA H is called a *Poincaré duality algebra (PDA)* of dimension n if $H^{>n} = 0$, $H^n = \mathbb{R} \cdot \omega$ for some $\omega \in H^n$ and the multiplication induces perfect pairings $H^{n-i} \otimes H^i \to H^n \cong \mathbb{R}$ for $0 \leq i \leq n$. $\diamond$

Example 1.41 For any compact oriented manifold M, its cohomology $H(M)$ is a Poincaré duality algebra (this is what is commonly referred to as the Poincaré duality theorem). $\diamond$

Theorem 1.42 (Miller) Let A be a connected CDGA such that $H(A)$ is a Poincaré duality algebra of dimension D with $H^i(A) = 0$ for $1 \leq i \leq k$. If $D \leq 4k+2$, then A is formal. $\diamond$

Proof. Let $\phi\colon \Lambda V \to A$ be the minimal model of A. We strive to find a quasi isomorphism $\psi\colon \Lambda V \to H(A)$. To that end we split $V = C \oplus N$ where $d(C) = 0$ and $d\colon N \to \Lambda V$ is injective. In the general construction of Theorem 1.17 C corresponds to the generators added to create cohomology, while N corresponds to the generators that kill cohomology.

The most obvious way to make ψ as a quasi-isomorphism is setting $\psi|_{\ker d} = [\phi(-)]$, and then we better set $\psi(N) = 0$ to not spoil it being a quasi-iso. Then we want to extend ψ multiplicatively, but for that to be well-defined we better have $\psi(\ker(d) \cap (N \cdot \Lambda V)) = 0$. Since ϕ is a quasi-isomorphism we conclude that we want to prove that every element $x \in N \cdot \Lambda V \cap \ker(d)$ is exact.

So we inspect C and N. Since $H^i(A) = 0$ for $1 \leq i \leq k$, we have that $C = \oplus_{i \geq k+1} C^i$. Then the first possible unwanted product lives in degree $2k+2$, so we have $N = \oplus_{i \geq 2k+1} N^i$ and hence $N \cdot \Lambda V$ lives in degree $3k+2$ upwards. Consider now a homogeneous element $x \in N \cdot \Lambda V$ such that $dx = 0$. If $|x| \neq D$ we have that $H^{|x|}(\Lambda V) \cong H^{|x|}(A) = 0$ since $3k+2 \geq D-k$, and we conclude that x is exact. If $|x| = D$ we note that this implies that the lowest degree of C is within the range $k+1$ up to $D-k-1$ and in particular within this range there is non-zero cohomology, say $H^j(\Lambda V) \cong H^j(A) \neq 0$. The fact that the product $H^j(A) \otimes H^{D-j}(A) \to H^d(A)$ is a perfect pairing, means that in this case it is in particular surjective. We conclude that the (up to exact elements) unique element $v \in \Lambda V$ such that $[\phi(v)] = \omega$ lives in ΛC.

So if x is not exact, it has to be (up to a scalar) a representative for ω, and hence $x \in \Lambda C + d(\Lambda V) \subset \Lambda C + N \cdot \Lambda V$. Since $N \cdot \Lambda V \cap \Lambda C = 0$, we conclude that the representative for ω that lives in ΛC has to be zero, which is a contradiction. So if $x \in (N \cdot \Lambda V)^D$ is closed, it represents 0 in cohomology, hence it is exact.

We see that every closed element of $N \cdot \Lambda V$ is exact, and hence ψ as described defines a well-defined quasi isomorphism $\Lambda V \to H(A)$, and hence A is formal. □

The topological counterpart to this algebraic statement shows that low-dimensional manifolds are formal, philosophically because 'it is possible to choose representatives in a well-defined way, since there aren't so many choices to make'.

Corollary 1.43 Let M be a k-connected compact manifold for $k \geq 1$. If $\dim(M) \leq 4k+2$, then M is formal. ◇

Proof. If M is k-connected, then by definition $\pi_i(M) = 0$ for $1 \leq i \leq k$, so by Hurewicz we conclude that $H^i(M) = 0$ for $1 \leq i \leq k$. Since $\pi_1(M) = 0$ we conclude that M is orientable and hence satisfies Poincaré duality, i.e. $H(M)$ is a PDA of the same dimension as M. The result now follows immediately from the previous theorem. □

In particular we see that simply connected manifolds up to dimension 6 are formal, so the theory will start to get interesting from dimension 7 onwards (this will become apparant when we will discuss special holonomy in Section 3).

The proof of Theorem 1.42 also exposes a characterization of formality which on the one hand enables us to make efficient statements on formality of algebras constructed out of others, and on the other hand turns out be generalizable to such a degree that deciding on the formality of manifolds will be a significantly less complicated endeavour than it looks at first glance.

Theorem 1.44 Let A be a CDGA. Then A is formal if and only if it has a minimal model $\phi\colon \Lambda V \to A$ such that V splits as $V = C \oplus N$ with $d(C) = 0$ and $d\colon N \to \Lambda V$ injective with the property that any closed element in $N \cdot \Lambda V$ is exact. ◇

Proof. As seen in the proof of Theorem 1.42 given this data we can construct $\psi\colon \Lambda V \to H(A)$ defined by $\psi|_{\ker d} = [\phi(-)]$ and $\psi|_N = 0$.

Now for the converse, we will show that the minimal model of a CDGA H with $d = 0$ as constructed in Theorem 1.17 satisfies this property. Since A is assumed to be formal, the minimal model of A is the same as that of $H(A)$, so the result will follow.

Recalling, we constructed partial minimal models $M_H(n)$ which have the right cohomology up to degree n and then added W^n (which we will call W_1^n in this proof) to kill excess cohomology and W^{n+1} (to be called W_2^{n+1}) to create missing cohomology.
To this end we set $d(W_2) = 0$ and d on W_1 as a section of a surjective map, in particular if we set $C = \oplus W_2^i$ and $N = \oplus W_1^i$ we have that $d(C) = 0$ and $d|_N$ is injective.
The map $\phi\colon M_H \to H$ was set such that $\phi(dw) = d_H(a)$ for elements $w \in W_1$, but since $d_H = 0$, ϕ can be chosen to be 0 on W_1. In particular $\phi(N \cdot \Lambda V) = 0$.

Now suppose we have a closed element $x \in N \cdot \Lambda V$, then $\phi^*([x]) = [\phi(x)] = [0] = 0$. Since ϕ^* is an isomorphism by construction, it follows that $[x] = 0$, i.e. x is exact, which finishes the proof. □

The preceding theorem allows us to determine whether an algebra is formal, by only considering the minimal model of the algebra, and not the minimal model of the cohomology. We also want a similar result the other way around, i.e. to make statements about the formality of A by only considering the minimal model of $H(A)$ and not having to calculate the minimal model of A.

The idea is to consider the minimal model $(\Lambda V, d)$ of $H(A)$ and put some extra structure on it (the result is called the bigraded model). Then from the bigraded model of $H(A)$ we can construct a model (not necessarily minimal) $(\Lambda V, D)$ of A such that $D - d$ is suitably nice with respect to the extra structure on V (this is called the filtered model of A). The main result is then that formality can be read from how close to 0 the difference $D - d$ is.

Theorem 1.45 [10, Prop 3.4, Def 3.5, p.243] Let A be a finitely generated graded commutative algebra with $A^0 = \mathbb{R}$. Then the CDGA $(A, 0)$ admits a minimal model $\rho\colon (\Lambda V, d) \to (A, 0)$ where V is equipped with a lower ~~gradation~~ $V = \oplus_{p\geq 0} V_p$ extended in a multiplicative way to ΛV such that:

a) $d(V_p) \subset (\Lambda V)_{p-1}$, in particular $d(V_0) = 0$ and we get homology of the algebra, together with a ~~bigradation~~ on the cohomology $H(\Lambda V, d) = \oplus_{p\geq 0} H_p(\Lambda V, d)$.

b) $\rho(V_p) = 0$ for $p > 0$.

c) $H_p(\Lambda V, d) = 0$ for $p > 0$ and $\rho^* H_0(\Lambda V, d) \to H(A, 0) = A$ is an isomorphism.

The CDGA $(\Lambda V, d)$ is called the *bigraded model* of the graded algebra A. ◇

Heuristically what we do is that we rank the relations imposed in the algebra, where rank 0 is for generators, rank 1 for products of generators we don't want, rank 2 for products of a rank 1 element with generators that we don't want, et cetera.

Proof. We will restrict A's with $A^1 = 0$, in which case we can tweak the proof of Theorem 1.17, making it more insightful how the lower grading arises. The construction of a minimal model for the non-simply connected case is much more technical, as is the full proof of this theorem, which can be found in the original paper.

When we look back at the proof of Theorem 1.17, basically the only thing we have to do is fix the lower grading on W during the induction step, and fix some sections. For notational purposes we define W'' for the closed part of W (i.e. the cokernel we add) and W' for the non-closed part of W (i.e. the kernel we add with a shift in degree). Since lower grading 0 is reserved for elements generating cohomology, we set $W'' = W''_0$. Furthermore, since by assumption we already have a lower grading on ΛV we can find a filtration on W', induced by d', namely $W'_{(p+1)} = d'^{-1}(\Lambda V)_{\leq p}$. Then, if we find a complementary basis of $W'_{(p)}$ in $W'_{(p+1)}$ we set W'_{p+1} to be the span of this complement, so that $W'_{(p+1)} = W'_{p+1} \oplus W'_{(p)}$, we find the lower grading of W' (note that this start iterating at 1). Also note that the construction with complimentary bases ensures that $d(W'_p) \subset (\Lambda W)_{p-1}$. Furthermore we can look at the definition of ρ' and see that for it we find an a so that $\rho(d'w') = d_A a$. But, since $d_A = 0$, we can always choose $a = 0$, and hence $\rho' = 0$.

In this way we get lower grading 0 for creating cohomology, while lower degree p kills of redundant cohomology in degree $p+1$. Using that as philosophical inspiration, a careful consideration shows that with this construction we indeed get that $H_p(\Lambda V, d) = 0$ for $p \geq 1$ and that $\rho^*|_{H_0}$ is an iso. Since by construction $\rho' = 0$, we see that $\rho(V_p) = 0$ for $p \geq 1$ and similarly by construction $d(V_p) \subset (\Lambda V)_{p-1}$, which finishes the proof. □

Of course one should think of A as the cohomology of some CDGA, and given a CDGA (A, d_A) the bigraded model of the cohomology algebra $H(A, d_A)$ gives a model (not necessarily minimal) of A, with only a slight tweak of the differential.

Theorem 1.46 [10, Thm 4.4, Lem 4.5, Def 4.11, pp.248-252] Let (A, d_A) be a connected CDGA of finite type. Let $\rho\colon (\Lambda V, d) \to (H(A, d_A), 0)$ be a bigraded model of A. Then there exists a differential D on ΛV and a map $\pi\colon (\Lambda V, D) \to (A, d_A)$ such that:

a) $D - d$ decreases the lower degree by at least two: $D - d\colon V_p \to (\Lambda V)_{\leq p-2}$. That is, we can write $D = d + d_2 + d_3 + \cdots$ where d_i is homogeneous of degree $-i$ in the lower degree: $d_i\colon V_p \to (\Lambda V)_{p-i}$. In particular we have $D = 0$ on ΛV_0.

b) $[\pi v] = \rho v$ for all $v \in \Lambda V_0$.

c) π is a quasi-isomorphism.

We will call $\pi\colon (\Lambda V, D) \to (A, d_A)$ a *filtered model* of A with respect to ρ. ◇

Proof. We fix a linear map $\eta\colon H(A) \to \Lambda V_0$ such that $\rho\eta = \mathrm{id}$ (which exists since ρ is a surjective linear map). We construct D and π inductively on $V_0, V_1, \ldots$, starting with base cases V_0, V_1 and V_2.

First of all, by degree reasons, we need to set $D = 0$ on V_0. Then set π on V_0 such that $d_A(\pi(v)) = 0$ and $[\pi(v)] = \rho(v)$ for all $v \in V_0$, and extend π to a homomorphism $\pi\colon (\Lambda V_0, D) \to (A, d_A)$. In particular $[\pi(v)] = \rho(v)$ for all $v \in \Lambda V_0$.
Also for degree reasons we set $D = d$ on V_1, and since $[\pi D v] = [\pi d v] = \rho d v = 0$ for $v \in V_1$

we can extend π to a degree zero linear map $\pi\colon V_1 \to A$ such that $d_A\pi z = \pi dz$.
Next suppose that $v \in V_2$, then $Ddv = d^2v = 0$ since $dv \in \Lambda(V_{\leq 1})$ and the previous, and so $d_A\pi dv = 0$, and we extend D to V_2 by $Dz = dz - \eta([\pi dz])$. Taking all this together we extend D to a derivation on $\Lambda(V_{\leq 2})$. Since for $v \in V_2$ we have $D^2z = dDz = 0$, we have $D^2 = 0$ on $\Lambda(V_{\leq 2})$. Now since η maps into ΛV_0 we have $[\pi\eta\alpha] = \rho\eta\alpha = \alpha$ for all $\alpha \in H(A)$. In particular we get $[\pi Dv] = [\pi dv] - [\pi\eta[\pi dv]] = [\pi dv] - [\pi dv] = 0$ for $v \in V_2$, and so we can extend π to a degree zero linear map $V_2 \to A$ such that $d_A\pi v = \pi Dv$, and we extend again to a homomorphism $\pi\colon (\Lambda(V_{\leq 2}), D) \to (A, d_A)$.

For what comes we need the following claim, on describing D-closed elements as the sum of D-exact elements and induced elements from η:

Claim: Let $\rho\colon (\Lambda V, d) \to (H, 0)$ be the bigraded model for a connected CGA H, and let $\eta\colon H \to \Lambda V_0$ be a linear map such that $\rho\eta = \mathrm{id}$. Suppose $(\Lambda(V_{\leq n}), \overline{D})$ is a CDGA such that $(\overline{D} - d)\colon V_l \to (\Lambda V)_{\leq l-2}$ for $0 \leq l \leq n$. Assume $u \in (\Lambda V)_{\leq n-1}$ is $\overline{D}$-closed. Then for some $v \in (\Lambda V)_{\leq n}$ and some $\alpha \in H$ it holds that: $u = \overline{D}(v) + \eta(\alpha)$.

Proof of the Claim: Let $n = 1$, and $u \in \Lambda V_0$. Then write $u = dv + \eta(\alpha)$ which can be done since ρ^* is an isomorphism, so choose $\alpha = \rho(u)$ and then $[\eta(\alpha) - u] = 0$. But then $dv = \overline{D}v$ so we're done.
Next suppose that the claim holds for $n-1$ and let $u \in (\Lambda V)_{\leq n-1}$ be $\overline{D}$-closed. Write $u = \sum_{j=0}^{n-1} u_j$, $u_j \in (\Lambda V)_j$. Then since $\overline{D}u = 0$, by assumption on the relation between $\overline{D}$ and d we have $du \in (\Lambda V)_{\leq n-3}$ and hence $du_{n-1} = 0$. Using that the bigraded model has trivial homology in higher degree we get $u_{n-1} = dv_n$ for some $v_n \in (\Lambda V)_n$, but then $u - \overline{D}v_n \in (\Lambda V)_{\leq n-2}$ satisfies the hypothesis for $n-1$, and hence $u - \overline{D}v_n = \overline{D}v' + \eta(\alpha)$ and the result follows for $v' - v_n$. □

Now suppose that D and π have been defined on $\Lambda V_{(n)}$ and let $v \in V_{n+1}$. Then by the claim there are $w \in (\Lambda V)_{\leq n-1}$ (which may be chosen depending on v in a linear way) and $\alpha \in H(A)$ such that $D(dv) = Dw + \eta(\alpha)$. Applying π and using that $[\pi\eta\alpha] = \alpha$ we get $\alpha = [\pi\eta\alpha] = [d_A\pi dv] - [d_A\pi w] = 0$ and hence $D(dv - w) = 0$.
We extend D to V_{n+1} by $Dv = dv - w - \eta[\pi(dz - w)]$. Then $[\pi Dv] = 0$ and hence we can extend π to a homomorphism $\pi\colon (\Lambda(V_{\leq n+1}), D) \to (A, d_A)$.

Then by construction we have that $D - d$ is a pertubation of degree 2, and by construction $(\Lambda V, D)$ is a Sullivan algebra, so we only have to show that π^* is an isomorphism. From the Claim it follows that $\eta^*\colon H(A, d_A) \to H(\Lambda V, D)$ is surjective. Indeed if we take $u \in \Lambda V$ to be D-closed, then by the claim we find $v \in \Lambda V$ and $\alpha \in H(A, d_A)$ such that $u = Dv + \eta(\alpha)$ and then $[u] = [\eta(\alpha)] = \eta^*(\alpha)$. Furthermore since $[\pi v] = \rho(v)$ and $\rho\eta = \mathrm{id}$ we have $\pi^*\eta^* = \mathrm{id}$, so η^* must be injective, hence an isomorphism. Then π^*, being the inverse is also an isomorphism. □

Remark 1.47 We will not explicitely need it in what follows, but the bigraded model and filtered model are essentially unique up to isomorphism. ◇

Obviously when A is formal, we can take $D = d$, and so $d_i = 0$ for $i \geq 2$. When we don't know whether A is formal, we can't expect that $(\Lambda V, D)$ is anywhere close to the minimal model, but the d_i's actually contain the information to decide whether A is formal. This comes via a clever bit of obstruction theory.

First note that the d_i actually are derivations, and we may wonder how 'trivial', how 'close to zero', they are. The framework for this is to look at the algebra of derivations. To set up the frame work let (C, d) be a DGA, then $\mathrm{Der}(C)$ is the set of maps $C \to C$ satisfying Leibniz, together with the obvious structure of a Lie algebra (the (anti-)commutator of derivations being a derivation). This comes with a differential $\mathcal{D}$ given by the bracket with d, that is $\mathcal{D}(\theta) = d \circ \theta - (-1)^q \theta \circ d$ where θ is a derivation of degree q.

If $(C, d) = (\Lambda V, d)$ is the bigraded model of a DGA $(A, 0)$, the double gradation of ΛV induces a double gradation on $\mathrm{Der}(\Lambda V, d)$, and we denote by $\mathrm{Der}_p^q(\Lambda V, d)$ the subspace of derivations which increase the higher degree by q and decrease the lower degree by p. In this way we get $\mathcal{D}\colon \mathrm{Der}_p^q(\Lambda V, d) \to \mathrm{Der}_{p+1}^{q+1}(\Lambda V, d)$, and we denote the cohomology of this complex by $H_p^q(\mathrm{Der}(\Lambda V, d))$. This cohomology is the right way of measuring the triviality of the derivations, and the result is the following obstruction theorem.

Theorem 1.48 [1, Thm 2.8, p.2058] Let (A, d_A) be a connected CDGA of finite type. Let $\rho\colon (\Lambda V, d) \to (H(A), 0)$ be a bigraded model of A. Let $\pi\colon (\Lambda V, D) \to (A, d_A)$ with $D = d + d_2 + d_3 + \cdots$ be a filtered model of A with respect to the bigraded model ρ. Then:

a) If $d_j = 0$ for $j < i$, d_i is a closed derivation in $\mathrm{Der}_i^1(\Lambda V, d)$.

b) If $[d_i] = 0$, then there exists an automorphism of ΛV as CGA (not as a CDGA) such that conjugating both the algebra and the differential by this automorphism yields a new filtered model $(\Lambda V, D')$, such that $D' = d + d'_{i+1} + d'_{i+2} + \cdots$.

c) Repeating this process, A is formal if and only if the consecutive sequence of obstruction classes o_i are all 0. ◇

The strength of this theorem is that we don't necessarily need the minimal model of A (of course if we'd have that, we can easily see whether A is formal), but only the minimal model of $H(A)$ which normally is a lot easier to work with.

To strengthen it even further, in the proof of this theorem, we never really use minimality of $(\Lambda V, d)$, and the theorem also works if we let go of this. This leads to the following definition of a 'bigraded model without minimality':

Definition 1.49 Let $(H, 0)$ be a CDGA with zero differential. A *multiplicative resolution* of H is a quasi-isomorphism $\rho\colon (\Lambda V, d) \to (H, 0)$ where V is given a bigrading extended multiplicatively to ΛV such that:

a) $\rho\colon \Lambda V_0 \to H$ is onto,

b) $\rho(V_i) = 0$ for $i > 0$,

c) $H_i(\Lambda V, d) = 0$ for $i > 0$ and $H_0(\rho)\colon H_0(\Lambda V, d) \to H$ is an isomorphism. $\diamond$

Then in the same way as we can talk about a filtered model with respect to a bigraded model, we can also talk about a filtered model with respect to a multiplicative resolution (the definition copies literally here since it does not make reference to minimality of $(\Lambda V, d)$). We do not care for existence of a filtered model with respect to any multiplicative resolution for the moment, but the immediate generelization of Theorem 1.48 is as follows:

Corollary 1.50 [1, Cor 2.9, p.2058] Let (A, d_A) be a connected CDGA. Let $\phi\colon (\Lambda V, d) \to (H(A), 0)$ be a multiplicative resolution of A. Let $\pi\colon (\Lambda V, D) \to (A, d_A)$ with $D = d + d_2 + d_3 + \cdots$ be a filtered model of (A, d_A) with respect to the multiplicative resolution ϕ. If $d_j = 0$ for $j < i$, then d_i is a closed derivation in $\mathrm{Der}^1_i(\Lambda V, d)$ and we denote its cohomology class by o_i. Then (A, d_A) is formal if and only if $o_i = 0$ for all i. $\diamond$

Proof of Theorem 1.48. .
a) If $D = d + d_i + d_{i+1} + \cdots$ the equation $D^2 = 0$ yields $d^2 + dd_i + d_i d + O(i+2) = 0$ where by $O(i+2)$ we mean terms decreasing the lower degree by at least $i+2$ and hence $dd_i + d_i d = \mathcal{D}(d_i) = 0$.
b) Suppose that we are in the case $D = d + d_i + d_{i+1} + \cdots$ with $[d_i] = 0$. That means that there is an element $\phi_i \in \mathrm{Der}^0_{i-1}(\Lambda V, d)$ such that $d_i = d\phi_i - \phi_i d$. We consider the map $e^{-\phi_i}\colon \Lambda V \to \Lambda V$. Here we define the exponential by the usual formula

$$e^{-\phi_i} = \mathrm{id} - \phi_i + \frac{\phi_i^2}{2} - \frac{\phi_i^3}{6} + ...$$

Since ϕ_i decreases the lower degree by $i-1$ and every homogeneous element of ΛV has a finite lower degree, we conclude that this is well-defined. Furthermore by the usual arguments we have that $e^{-\phi_i}$ is a morphism of CGA's with inverse e^{ϕ_i}. If we set $D' = e^{\phi_i} D e^{-\phi_i}$ then obviously the map $e^{-\phi_i}$ becomes an isomorphism $(\Lambda V, D') \to (\Lambda V, D)$.

The important calculation is now:

$$\begin{aligned} D' &= (\mathrm{id} + \phi_i + O(i+1))(d + d_i + O(i+1))(\mathrm{id} - \phi_i + O(i+1)) \\ &= d + (\phi_i d - d\phi_i + d_i) + O(i+1) \end{aligned}$$

Then since by definition $d_i = d\phi_i - \phi_i d$ we conclude that $D' = d + O(i+1)$, i.e. $D' = d + d'_{i+1} + d'_{i+2} + \cdots$.
c) If A is formal then by the discussion above the question is academical, so we show that if the sequence of obstructions vanishes the algebra is formal.

Note we have a zigzag of (quasi-)isomorphisms starting $(A, d_A) \xleftarrow{\pi} (\Lambda V, D_2) \xleftarrow{e^{-\phi_2}} (\Lambda V, D_3) \cdots$. Then since the $e^{-\phi_r} - \mathrm{id}$ decreases the lower degree by at least $r-1$ we have that for any element, the pertubations of $e^{-\phi_r}$ have to vanish at some point, i.e. $\Phi\colon = \cdots e^{-\phi_r} \circ e^{-\phi_{r-1}} \cdots e^{-\phi_2}$ is a well-defined isomorphism of CGA's and since $D_r - d$ decreases the lower degree by at least r it follows that it is an isomorphism of CDGA's from $(\Lambda V, d) \to (\Lambda V, D)$.

So we see that we have a zigzag of (quasi-)isomorphisms $(A, d_A) \xleftarrow{\pi} (\Lambda V, D) \xleftarrow{\Phi} (\Lambda V, d) \xrightarrow{\rho} (H(A), 0)$ and we conclude that A is formal. $\square$

Proof of Corollary 1.50. We didn't use minimality of $\rho\colon (\Lambda V, d) \xrightarrow{\rho} (H(A), 0)$ to construct the zig-zag of (quasi)-isomorphisms $(A, d_A) \xleftarrow{\pi} (\Lambda V, D) \xleftarrow{\Phi} (\Lambda V, d) \xrightarrow{\rho} (H(A), 0)$ in the previous proof, so it still applies when ρ is just a multiplicative resolution.

We conclude that since (A, d_A) is still weakly equivalent to $(H(A), 0)$ it is still formal under this weakened assumption. $\square$

2 Rational homotopy theory and CDGA's

In what we have discussed the main philosophical point is the assignment

$$\{\text{manifolds}\} \to \{\text{minimal CDGA's over } \mathbb{R}\}$$

via the minimal model of the deRham complex. This assignment is not at all surjective (for instance in the simply connected case we need to have Poincaré duality which need not occur in a minimal CDGA), but when we broaden our viewpoint to topological spaces we will find an equivalence of categories between the (naive) homotopy categories of rational homotopy types and the category of isomorphism classes of minimal CDGA's (at least if we impose things like finite type and simply connectedness on both sides).
In particular we will construct a functor from topological spaces to CDGA's over $\mathbb{Q}$, called A_{PL}, which will have the property that $H(A_{\mathrm{PL}}(X)) \simeq H(X;\mathbb{Q})$ and then we associate to a rational homotopy type $[X]$ the minimal model of $A_{\mathrm{PL}}(X)$ (note that a rational homotopy equivalence induces isomorphisms on $H(-;\mathbb{Q})$ and hence quasi-isomorphisms on A_{PL}, so rational equivalent spaces have the same minimal model.). On the other hand, we will build from a minimal algebra a space whose minimal model is the one we started with.

We will show that for manifolds the real and rational theories are equivalent, in that the minimal model defined via A_{PL} is up to extending scalars isomorphic to each other, and also that the notion of formality are equivalent. In fact, we will show that for a manifold X, $A_{\mathrm{PL}}(X)$ and $\Omega^*(X)$ are weakly equivalent in a suitable sense. All this will be done closely following the book by Félix, Halperin and Thomas [7].

In what follows we will consider a CDGA to be defined over a field $\mathbb{K}$ of characteristic 0. Note that the definition of a CDGA as given before does not makes extrinsic reference to the field $\mathbb{R}$ and hence we can safely replace it any other field.

2.1 Simplicial objects and $A_{\mathbf{PL}}$

Definition 2.1 The *simplex category* Δ is defined to have objects $[n] = \{0, ..., n\}$ for every natural $n \geq 0$ together with morphisms which are order preserving maps (nb: not strictly order preserving). $\diamond$

Definition 2.2 For a category $\mathcal{C}$, a *simplicial object in* $\mathcal{C}$ is a contravariant functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. A morphism of simplicial objects is a natural transformation between the two functors. We will denote the category of simplicial objects in $\mathcal{C}$ by $s\mathcal{C}$. $\diamond$

We note that there are two important families of morphisms in the simplex category, namely the maps $\delta^i\colon [n-1] \to [n]$ which are the injections that does not hit the number i and the maps $\sigma^i\colon [n+1] \to [n]$ which are surjections that hit i twice. These satisfy the following relations:

$$\begin{array}{rcll} \delta^j\delta^i & = & \delta^i\delta^{j-1} & \text{if} \quad i<j \\ \sigma^j\sigma^i & = & \sigma^i\sigma^{j+1} & \text{if} \quad i\leq j \\ \sigma^j\delta^i & = & \begin{cases} \delta^i\sigma^{i-1} & \text{if} \quad i<j \\ \text{id} & \text{if} \quad i=j,j+1 \\ \delta^{i-1}\sigma^j & \text{if} \quad i>j+1 \end{cases} \end{array}$$

Since all these maps together generate the morphisms in the simplex category we conclude the following.

Lemma 2.3 A simplicial object in $\mathcal{C}$ is the same as a sequence of objects $\{K_n\}_{n\geq 0}$ in $\mathcal{C}$ together with morphisms $\partial_i\colon K_{n+1}\to K_n$ (called the face maps) for $0\leq i\leq n+1$ and $s_j\colon K_n\to K_{n+1}$ (called the degeneracy maps) for $0\leq j\leq n$ satisfying the identities

$$\begin{array}{rcll} \partial_i\partial_j & = & \partial_{j-1}\partial_i & \text{if} \quad i<j \\ s_is_j & = & s_{j+1}s_i & \text{if} \quad i\leq j \\ \partial_is_j & = & \begin{cases} s_{i-1}\partial_i & \text{if} \quad i<j \\ \text{id} & \text{if} \quad i=j,j+1 \\ s_j\partial_{i-1} & \text{if} \quad i>j+1 \end{cases} \end{array}$$

Furthermore a morphism between simplicial objects $f\colon L\to K$ is the same as a sequence of morphisms $f_n\colon L_n\to K_n$ that commute with the face and degeneracy maps. $\diamond$

Example 2.4 The most used example of simplicial object is the simplicial set of singular chains $S(X)$ of a topological space X. Here we set $S(X)_n=\{\sigma\colon\Delta^n\to X|\,\sigma \text{ continuous}\}$. As face and degenaracy maps we set $\partial_i(\sigma)=\sigma\circ\lambda_i$ and $s_j(\sigma)=\sigma\circ\rho_j$ where $\lambda_i\colon\Delta^n\to\Delta^{n+1}$ sends $(x_0,...,x_n)$ to $(x_0,...,0,...,x_n)$ with the 0 inserted in the i'th slot and $\rho_j\colon\Delta^n\to\Delta^{n-1}$ sends $(x_0,...,x_n)$ to $(x_0,...,x_j+x_{j+1},...,x_n)$.

Furthermore a continuous map $f\colon X\to Y$ gives a map of simplicial sets, by sending σ to $f\circ\sigma$. $\diamond$

We now wish to construct CDGA's in a systematic manner from simplicial sets and simplicial CDGA's. For this consider K a simplicial set and $A=\{A_n\}_{n\geq 0}$ a simplicial CDGA. Note that $\{A_n^p\}_{n\geq 0}$ is a simplicial vector space and in particular a simplicial set for every p. We construct a CDGA $A(K)$ in the following manner:

We set $A^p(K)$ to be the set of morphisms from K to A^p seen as simplicial sets. So an element of $A^p(K)$ is a map Φ that sends $\sigma\in K_n$ to $\Phi(\sigma)\in A_n^p$ such that $\partial_i\Phi=\Phi\partial_i$ and $s_j\Phi=\Phi s_j$.

Addition, scalar multiplication and differential are given as one would expect by $(\Phi+\Psi)(\sigma)=\Phi(\sigma)+\Psi(\sigma)$, $(\lambda\Psi)(\sigma)=\lambda\cdot(\Psi(\sigma))$ and $(d\Psi)(\sigma)=d(\Psi(\sigma))$. Similarly we multiply $(\Phi\cdot\Psi)(\sigma)=\Phi(\sigma)\cdot\Psi(\sigma)$. That we get a CDGA in this way is more or less immediate, ~~and one argues similary to showing that **Set**(S,G) inherits a group structure if G is a group~~.

If $\phi\colon K\to L$ is a morphism of simplicial sets, we set a morphism $A(\phi)\colon A(L)\to A(K)$

by $A(\phi)(\Phi)(\sigma) = \Phi(\phi(\sigma))$. If $\theta\colon A \to B$ is a morphism of simplicial CDGA's, then we get a morphism $\theta(K)\colon A(K) \to B(K)$ where $(\theta(K)\Phi)(\sigma) = \theta(\Phi(\sigma))$. In this way we get a functor $\mathrm{s}\mathbf{Set}^{\mathrm{op}} \times \mathrm{s}\mathbf{cdga} \to \mathbf{cdga}$.

If X is a topological space and A is a simplicial CDGA, we write $A(X)$ for $A(S(X))$, and hence a simplicial CDGA gives a contravariant functor $\mathbf{Top} \to \mathbf{cdga}$.

Now the idea is that given a simplex $\Delta^n \to X$ we can make sense of a 'polynomial form on X' if it formally pulls back to a polynomial differential form on Δ^n and this is precisely what is at the core of the abstract general construction outlined above. To define $A_{\mathrm{PL}}(X)$ we need to define the simplicial CDGA A_{PL}.

Definition 2.5 We set $(A_{\mathrm{PL}})_n$ to be the CDGA generated by elements $\{t_0, ..., t_n, dt_0, ..., dt_n\}$ with $|t_i| = 0$, $|dt_i| = 1$, subject to the relations $\sum t_i = 1$ and $\sum dt_i = 0$. We set the face and degeneracy maps defined by

$$\partial_i(t_k) = \begin{cases} t_k & \text{if} \quad k < i \\ 0 & \text{if} \quad k = i \\ t_{k-1} & \text{if} \quad k > i \end{cases} \qquad s_j(t_k) = \begin{cases} t_k & \text{if} \quad k < j \\ t_k + t_{k+1} & \text{if} \quad k = j \\ t_{k+1} & \text{if} \quad k > j \end{cases}$$ ⋄

If $\mathbb{K} \subset \mathbb{R}$, the CDGA $(A_{\mathrm{PL}})_n$ is the subalgebra of $\Omega^*(\Delta^n)$ consisting of polynomial forms in the coordinates t_i with coefficients in $\mathbb{K}$, hence the name. In this setting we also recognize ∂_i as the pullback by λ_i and s_j as the pullback by ρ_j.

Definition 2.6 We define the *polynomial differential forms*, $A_{\mathrm{PL}}(X)$ (sometimes denoted $A_{\mathrm{PL}}(X, \mathbb{K})$ when we want to stress the field), to be the CDGA obtained by the simplicial CDGA A_{PL} and the simplicial set $S(X)$. ⋄

We see that a polynomial differential p-form $\alpha \in A^p_{\mathrm{PL}}(X)$ is an assignment to every continuous map $\sigma\colon \Delta^n \to X$ a polynomial p-form $\alpha(\sigma) \in (A^p_{\mathrm{PL}})_n$ such that $\alpha(\partial_i\sigma) = \partial_i(\alpha(\sigma))$ and $\alpha(s_j\sigma) = s_j(\alpha(\sigma))$.

The important property of A_{PL} is the following:

Theorem 2.7 [7, Cor 10.10, p.126] There is a natural weak equivalence between $C^*(X; \mathbb{K})$ and $A_{\mathrm{PL}}(X; \mathbb{K})$. In particular there is a natural isomorphism $H^*(X; \mathbb{K}) \cong H(A_{\mathrm{PL}}(X; \mathbb{K}))$.⋄

Rationale. We will not proof this, but what one should have in mind is that one can see a polynomial differential p-form α as a morphism of the p-simplices of X to $\mathbb{K}$ by taking a simplex $f\colon \Delta^p \to X$ and then formally integrating $\alpha(f)$ over Δ^p (note that integrating a polynomial over a simplex is a purely formal algebraic affair). □

2.2 Comparing $A_{\mathbf{PL}}(M;\mathbb{R})$ and $\Omega(M)$

If M is a manifold, we now have constructed two model CDGA's (over $\mathbb{R}$) for M, namely $A_{\mathrm{PL}}(M;\mathbb{R})$ and $\Omega(M)$, and we set up an algebraic theory for $\Omega(M)$. If we want to extend this theory to arbitrary topological spaces, we would better make sure that $A_{\mathrm{PL}}(M;\mathbb{R})$ and $\Omega(M)$ are comparable. Since we're interested in minimal models, which measure weak isomorphism classes, we will show that there is a natural weak equivalence between $A_{\mathrm{PL}}(M;\mathbb{R})$ and $\Omega(M)$.

To do this we first consider the simplicial set of smooth simplices $S^{\infty}(M)$. As one would expect we set $S^{\infty}_n(M) = \{\sigma\colon \Delta^n \to M |\, \sigma \text{ smooth}\}$, together with face and degeneracy maps precomposition by the structure maps of the standard simplex. In this way $S^{\infty}(M)$ is a simplicial subset of $S(M)$.

Next we want to construct a simplicial CDGA to model the deRham-complex. As an element A_{PL} is an abstract entity that pullsback through each simplex to a polynomial form on the standard simplex, we are inclined to define the simplicial CDGA A_{dR} by $(A_{\mathrm{dR}})_k = \Omega(\Delta^k)$ with face and degenracy maps λ_i^* and ρ_j^*.

We note that by the remark before, since any polynomial is in particular smooth we have a canonical inclusion of simplicial CDGA's $A_{\mathrm{PL}}(-;\mathbb{R}) \to A_{\mathrm{dR}}$. This gives a natural morphism $\beta_M\colon A_{\mathrm{PL}}(S^{\infty}(M);\mathbb{R}) \to A_{\mathrm{dR}}(S^{\infty}(M))$. On the other hand the natural inclusion $S^{\infty}(M) \to S(M)$ gives a natural morphism $\gamma_M\colon A_{\mathrm{PL}}(M;\mathbb{R}) \to A_{\mathrm{PL}}(S^{\infty}(M);\mathbb{R})$.

At last consider an element of $A_{\mathrm{dR}}(S^{\infty}(M))$. This is a function Φ which assigns to every smooth simplex σ a smooth form $\Phi(\sigma) \in \Omega(\Delta^{|\sigma|})$, which commutes with face and degeneracy maps. If one considers some smooth form $\alpha \in \Omega(M)$ we can construct an element of $A_{\mathrm{dR}}(S^{\infty}(M))$ by $\Phi_\alpha(\sigma) = \sigma^*(\alpha)$. In particular we get a natural morphism $\alpha_M\colon \Omega(M) \to A_{\mathrm{dR}}(S^{\infty}(M))$. So we are in the following situation:

$$\Omega(M) \xrightarrow{\alpha_M} A_{\mathrm{dR}}(S^{\infty}(M)) \xleftarrow{\beta_M} A_{\mathrm{PL}}(S^{\infty}(M);\mathbb{R}) \xleftarrow{\gamma_M} A_{\mathrm{PL}}(M;\mathbb{R})$$

Theorem 2.8 [7, Thm 11.4,p.135] All three of α_M, β_M and γ_M are quasi-isomorphism and hence $\Omega(M)$ and $A_{\mathrm{PL}}(M;\mathbb{R})$ are weakly equivalent. $\diamond$

An advantage of the somewhat abstract way to defining these CDGA's, is that we can very effectively construct quasi-isomorphism. The idea is that, given two simplicial CDGA's, if we find a morphism between them that induces a quasi isomorphism at every level, then we get a quasi-isomorphism when plugging in an arbitrary simplicial set in either simplicial CDGA. We do need a small technical property of simplicial objects

Definition 2.9 A simplicial object A in a concrete category $\mathcal{C}$ (i.e. a category where the objects are sets with extra structure, think **Set**, **Top**, **cdga**) is called *extendable* if for any $n \geq 1$ and any $I \subset \{0, ..., n\}$, given $\Phi_i \in A_{n-1}$ for every $i \in I$ such that $\partial_i \Phi_j = \partial_{j-1}\Phi_i$ for $i < j$ then there is an element $\Phi \in A_n$ such that $\Phi_i = \partial_i \Phi$ for every $i \in I$. $\diamond$

The important proposition is then:

Proposition 2.10 [7, Prop 10.5,p.119] Suppose $\theta\colon D \to E$ is a morphism of simplicial CDGA's. Assume that $\theta_n\colon D_n \to E_n$ is a quasi-isomorphism for all $n \geq 0$ and that both D and E are extendable. Then for any simplicial set K, $\theta(K)\colon D(K) \to E(K)$ is a quasi-isomorphism. ◇

Then not surprising are the following lemmas:

Lemma 2.11 [7, Lem 10.7,p.123]

a) $A_{\mathrm{PL}}(-;\mathbb{K})_0 = \mathbb{K}\cdot 1$

b) $H(A_{\mathrm{PL}}(-;\mathbb{K})_n) = \mathbb{K}\cdot 1$ for $n \geq 0$

c) Each A^p_{PL} is extendable. ◇

Lemma 2.12 [7, Lem 11.3,p.134]

a) $(A_{\mathrm{dR}})_0 = \mathbb{R}$

b) $H((A_{\mathrm{dR}})_n) = \mathbb{R}$ for $n \geq 0$

c) Each A^p_{dR} is extendable. ◇

This asserts that any comparison between A_{dR} and A_{PL} needs only to be checked on standard simplices. Then to compare $\Omega(M)$ and $A_{\mathrm{dR}}(S^\infty(M))$ and $A_{\mathrm{PL}}(S^\infty(M);\mathbb{R})$ and $A_{\mathrm{PL}}(M;\mathbb{R})$ we really need to compare functors from n-dimensional manifolds to cochain algebras. The keywords here are, as always with manifolds: localizing and gluing, and the result is:

Lemma 2.13 [7, Lem 11.5, p.135] Let $\theta\colon A \to B$ be a natural transformation between two functors from n-dimensional manifolds to cochain algebras. Suppose that:

a) $H(A(\mathbb{R}^n)) = \mathbb{R} = H(B(\mathbb{R}^n))$

b) If U and V are open in M and θ_U, θ_V and $\theta_{U\cap V}$ are quasi-isomorphisms, then so is $\theta_{U\cup V}$

c) If $O = \sqcup_i O_i$ is the disjoint union of open sets, then $\theta_O = \prod_i \theta O_i$.

Then θ_M is a quasi-isomorphism for all smooth n-dimensional manifolds. ◇

Proof. Consider a family of open subsets $V_\lambda \subset M$ that is closed under finite intersection, and such that any open subset of M is the union of some of the V_λ. Given such a family, it is possible to write $M = O \cup W$ where $O = \sqcup_i O_i$ and $W = \sqcup_j W_j$ are both disjoint unions out of elements of the given family. Suppose that θ_{V_λ} is a quasi-isomorphism for any element of the family, then by induction of p so is $\theta_{V_{\lambda_1}\sqcup\cdots\sqcup V_{\lambda_p}}$ and it follows that θ_O, θ_W and $\theta_{O\cap W}$ are quasi-isomorphisms, and hence θ_M is as well.

Now consider U to be open in $\mathbb{R}^n$ and consider the family of open cubes in U, certainly a family as described in the above. Then, by the first assumption, the natural transformation for any open cube is a quasi-isomorphism, and hence, by the above, θ_U is a quasi-isomorphism.

Since the family of open subsets of M diffeomorphic to an open subset of $\mathbb{R}^n$ is a family as described at the start, the two above parts show that θ_M is a quasi-isomorphism. $\square$

We are now ready to prove Theorem 2.8:

Sketch of proof of Theorem 2.8. To see that α_M is a quasi-isomorphism, note that $H(\Omega(\mathbb{R}^n)) = \mathbb{R}$ as is known, while one can show that $H(A_{\mathrm{dR}}(S^\infty(\mathbb{R}^n))) = H(C^*_\infty(\mathbb{R}^n)) = \mathbb{R}$ where C^*_∞ are the smooth singular cochains, but we will not show this as then we'd have to go even a step deeper in the lingo of simplicial objects and functors.

Next note that for $U, V \subset M$ open we have a short exact sequence:

$$0 \to \Omega(U \cup V) \to \Omega(U) \oplus \Omega(V) \to \Omega(U \cap V) \to 0$$

and similarly

$$0 \to A_{\mathrm{dR}}(S^\infty(U) \cup S^\infty(V)) \to A_{\mathrm{dR}}(S^\infty(U)) \oplus A_{\mathrm{dR}}(S^\infty(V)) \to A_{\mathrm{dR}}(S^\infty(U \cap V)) \to 0$$

where $A_{\mathrm{dR}}(S^\infty(U) \cup S^\infty(V))$ is naturally quasi-isomorphic to $A_{\mathrm{dR}}(S^\infty(U \cup V))$ by barycentric subdivision. Then an argument with the long exact sequence shows that $\alpha_{U\cup V}$ is a quasi-iso if α_U, α_V and $\alpha_{U\cap V}$ are.
Finally if $O = \sqcup_i O_i$ then $\alpha_O = \prod_i \alpha_{O_i}$ and hence by Lemma 2.13 α_M is a quasi-isomorphism for all M.

That β_M is a quasi-isomorphism is a direct result of Lemma 2.11, Lemma 2.12 and Proposition 2.10.

Like before we only note (but not show) that $A_{\mathrm{PL}}(S^\infty(\mathbb{R}^n); \mathbb{R}) = \mathbb{R}$. To show that $\gamma_{U\cup V}$ is a quasi isomorphism if γ_U, γ_V and $\gamma_{U\cup V}$ are, note that barycentric subdivision $C_*(U) + C_*(V) \xrightarrow{\simeq} C_*(U \cup V)$ restrict to smooth barycentric subdivision $C^\infty_*(U) + C^\infty_*(V) \xrightarrow{\simeq} C^\infty_*(U \cup V)$. Then by an argument with the long exact sequence in homology we see that if $C^\infty_*(U) \to C_*(U)$, $C^\infty_*(V) \to C_*(V)$, $C^\infty_*(U \cap V) \to C_*(U \cap V)$ are quasi-isos, then so is $C^\infty_*(U \cup V) \to C_*(U \cup V)$. Dually we see that if γ_U, γ_V and $\gamma_{U\cap V}$ are quasi-isomorphisms, then so is $\gamma_{U\cup V}$. Clearly $\gamma_{\sqcup_i O_i} = \prod_i \gamma_{O_i}$ and so by Lemma 2.13 γ_M is a quasi-iso for all M. $\square$

We can now enlarge our theory of formal manifolds to the theory of formal spaces, which will sometimes be handy when we want to use intermediate constructions which may not be manifolds, for instance the wedge of two spaces.

Definition 2.14 A topological space X is called $\mathbb{K}$-*formal* if $A_{\mathrm{PL}}(X; \mathbb{K})$ is a formal CDGA over $\mathbb{K}$. If we do not specify the field, i.e. if we say that X is *formal*, we will mean that it is $\mathbb{Q}$-formal. $\diamond$

At this point we should notice the following:

Lemma 2.15 Let $\mathbb{K}$ be a field of characteristic 0, and let $(\Lambda V, d) \to A_{\mathrm{PL}}(X;\mathbb{Q})$ be a minimal model. Then by extending scalars from $\mathbb{Q}$ to $\mathbb{K}$, we obtain a minimal model $(\Lambda V', d') \to A_{\mathrm{PL}}(X;\mathbb{K})$ where $V' = V \otimes_{\mathbb{Q}} \mathbb{K}$ (and similarly for d'). ◇

Since Theorem 1.44 also applies for fields other than $\mathbb{R}$, we conclude:

Lemma 2.16 For a field $\mathbb{K}$ of characteristic 0, a space X is $\mathbb{K}$-formal if and only if it is $\mathbb{Q}$-formal. ◇

This remark, combined with Theorem 2.8 we conclude:

Proposition 2.17 A manifold M is formal in the sense of Definition 1.35 if and only if it is formal in the sense of Definition 2.14. ◇

2.3 Minimal models and homotopy groups

As mentioned in the previous chapter, there is a connection between the minimal model of $\Omega(M)$ and the rational homotopy groups (Theorem 1.19). In this section we will discuss the construction and proof, which is more topological in flavour, using A_{PL}.

First we discuss a way to assigning to maps between topological spaces (or maps between algebras), maps between the minimal models. Consider two CDGA's A and B with minimal models $m_X\colon \Lambda V \to A$ and $m_Y\colon \Lambda W \to B$ and a map $\phi\colon A \to B$. The lifting lemma (Proposition 1.25) yields that there is an (up to homotopy unique) map $m_\phi\colon \Lambda V \to \Lambda W$ such that $\phi \circ m_X \simeq m_Y \circ m_\phi$. Furthermore the homotopy class of m_ϕ depends only on the homotopy class of ϕ. We call m_ϕ a *Sullivan representative* of ϕ. Note that the linear part $Q(m_\phi)\colon V \to W$ is independent of the choice since homotopic morphisms have the same linear part.

Going one step further, given a continuous map $f\colon Y \to X$ we can take a Sullivan representative of $A_{\mathrm{PL}}(f)\colon A_{\mathrm{PL}}(X;\mathbb{K}) \to A_{\mathrm{PL}}(Y;\mathbb{K})$, and take the linear part $Q(f)\colon = Q(m_{A_{\mathrm{PL}}(f)})$. Suppose for instance that we have a map $f\colon S^k \to X$. Then the k^{th} part of $Q(f)$ defines a map $V^k \to \mathbb{K}$ (where ΛV is the minimal model of X). Note that $Q(f)$ only depends on the homotopy class of f.

To make this precise, let X be a topological space and fix a minimal model $m_X\colon \Lambda V \to A_{\mathrm{PL}}(X;\mathbb{K})$. Furthermore, denote minimal models of S^k by $\langle e; |e| = k, de = 0\rangle$ for k odd and $\langle e, e'; |e| = k, de = 0, de' = e^2\rangle$ for k even. Let $\alpha \in \pi_k(X)$ be a homotopy element, represented by $a\colon S^k \to X$. Then having fixed the minimal model, $Q(a)\colon V^k \to \mathbb{K}\cdot e$ only depends on α. In this way we get a pairing $\langle -; -\rangle\colon V \times \pi_*(X) \to \mathbb{K}$ by:

$$\langle v; \alpha\rangle e = \begin{cases} Q(a)(v) & v \in V^k \\ 0 & \deg v \neq \deg \alpha \end{cases}$$

The theorem by Sullivan is made precise as follows:

Theorem 2.18 [7, Thm 15.11, p.208] Suppose that $H_i(X;\mathbb{K})$ is of finite type. Then the map $\nu_X \colon V \to \mathrm{Hom}_{\mathbb{Z}}(\pi_*(X);\mathbb{K})$, $v \mapsto \langle v; - \rangle$ is a linear isomorphism. ◇

To make the statement ~~even~~ reasonable, we first show that the pairing is bilinear:

Lemma 2.19 The pairing $\langle -; - \rangle$ is bilinear. ◇

Proof. Linearity in V^k is clear by definition (since $Q(a)$ is linear), so we will focus on linearity (really addivity) in the second slot. So let α_0 and α_1 be two elements of $\pi_k(X)$ represented by maps $a_0 \colon S^k \to X$ and $a_1 \colon S^k \to X$. If we denote by j the pinch map $S^k \to S^k \vee S^k$, then $\alpha_0 + \alpha_1$ is represented by the composite $(a_0, a_1) \circ j$.

Denote by i_0 and i_1 the two obvious inclusions $S^k \to S^k \vee S^k$, then the canonical basis $\{w_0, w_1\}$ of $H^k(S^k \vee S^k;\mathbb{K})$ is determined by $i_j^*(w_l) = 1$ if $j = l$ and 0 otherwise. It follows that the minimal of $S^k \vee S^k$ is give by

$$m \colon (\Lambda(e_0, e_1, ...), d) \to A_{\mathrm{PL}}(S^k \vee S^k)$$

where e_0 and e_1 are cocycles of degree k, such that $H(m)[e_i] = w_i$, and where the other generators are in higher degree to kill of any excess products (we will come back to this in 5.3).

A Sullivan representative ϕ_0 for i_0 needs to satisfy $H(\phi_0)[e_0] = [e]$ and $H(\phi_0)[e_1] = 0$ by definition of e_i. Since there are no coboundaries in degree k on either side this means that $\phi_0 e_0 = e$ and $\phi_0 e_1 = 0$, and hence $Q(i_0)e_0 = e$ and $Q(i_0)e_1 = 0$. Similarly $Q(i_1)e_0 = 0$, $Q(i_1)e_1 = e$ and $Q(j)e_0 = e = Q(j)e_1$.

Now since $(a_0, a_1) \circ i_0 = a_0$ and $(a_0, a_1) \circ i_1 = a_1$ we see that $Q(a_0, a_1)v = \langle v; \alpha_0 \rangle e_0 + \langle v; \alpha_1 \rangle e_1$ and composing with $Q(j)$ we conclude that $Q(a)v = (\langle v; \alpha_0 \rangle + \langle v; \alpha_1 \rangle)e$ which is what was to be shown. □

Sketch of proof of Theorem 2.18. We fix $k \geq 2$ and denote $r = r(X)$ the least integer such that $\pi_{r(X)}(X) \neq 0$. To show that $\nu_X^k \colon V_X^k \to \mathrm{Hom}_{\mathbb{Z}}(\pi_k(X), \mathbb{K})$ is an isomorphism we will use induction on $k - r$.
If $k = r$ we know that the Hurewicz homomorphism $\pi_r(X) \to H_r(X;\mathbb{Z})$ is an isomorphism and furthermore that $H_i(X;\mathbb{Z}) = 0$ for $i < k$. Also we see that $H^i(X;\mathbb{K}) = 0$ for $i < k$ and $H^r(X;\mathbb{K}) \cong \mathrm{Hom}(H_r(X;\mathbb{Z}), \mathbb{K})$. The former implies that $V^i = 0$ for $i < k$ and hence $H^r(m_X)$ is an isomorphism $V^r \to H^r(X;\mathbb{K})$.
When we look at the definition of the Hurewicz homomorphism and of ν_X^r we conclude that under these isomorphisms they coincide and hence ν_X^r is an isomorphism.

If $k > r$, note that any space X is weakly equivalent to a CW complex Y via some map $f \colon Y \to X$. For this f the Sullivan representative is an isomorphism between the minimal models of X and Y. Since in particular $Q(f)$ and $\pi_*(f)$ are isomorphisms, by naturality we may assume X to be an CW complex.

Next, let K be an Eilenberg-MacLane space of type $(\pi_r(X), r)$ together with a continuous map $g\colon X \to K$ such that $\pi_r(g) = \mathrm{id}$. By factoring g via a fibration and a homotopy equivalence we may assume that there is a fibration $p\colon X \to K$ such that $\pi_r(p)$ is an isomorphism.
Consider a typical fiber F of p. By the long exact sequence of the fibration we conclude that $\pi_i(F) = 0$ for $i \leq r$, and in partcular $r(F) \geq r(X) + 1$.

By looking at the structure of models of bases, total spaces and fibers of fibrations (the nuances of which we will discuss in Section 4.2), although we will leave this as a complete black box), we are able to identify ν_X^k and ν_F^k via isomorphisms between $\pi_k(X)$ and $\pi_k(F)$ and the k'th parts of the minimal models of X and F. Since $r(F) \geq r(X) + 1$, we have $k - r(F) < k - r(X)$ so by induction on this number we see that ν_F^k is an isomorphism, and we conclude that ν_X^k is an isomorphism. □

2.4 Spatial realization

We will end the part of topological constructions by showing that up to reasonable equivalence, the category of minimal models is the same as the category of topological spaces. This will turn out to be useful in what follows, since we can deduce pushout and pullback properties of squares of CDGA's out of those of squares of topological spaces. To not wit too much on the somewhat more diehard topology, we will only give a brief outline of chapter 17 of Felix, Halperin and Thomas [7], which in its own right is partly based on May [18] and Sullivan [22].

For what comes we will restrict ourselves to simply connected rational CDGA of finite type, so unless otherwise stated we will assume every algebra mentioned to have these properties.

The plan de campagne is as follows.
We will construct a contravariant functor assigning to any algebra (A, d) a CW-complex $|A, d|$ and to any map $(A, d) \to (B, d)$ of Sullivan algebras a continuous map $|\phi|\colon |B, d| \to |A, d|$, such that the functor will have the following properties:

- ~~First~~ for any Sullivan algebra $(\Lambda V, d)$ we get a quasi-isomorphism $(\Lambda V, d) \to A_{\mathrm{PL}}(|\Lambda V, d|)$. In particular if $(\Lambda V, d)$ is minimal, it is the minimal model of its own spatial realization.
- ~~Next~~ any morphism $\phi\colon (\Lambda V, d) \to (\Lambda W, d)$ of Sullivan algebras is a Sullivan representative of $|\phi|\colon |\Lambda W, d| \to |\Lambda V, d|$ with respect to quasi-isomorphisms mentioned above.
- ~~Then two~~ morphisms $\phi, \psi\colon (\Lambda V, d) \to (\Lambda W, d)$ between Sullivan algebras are homotopic if and only if $|\phi|$ and $|\psi|$ are homotopic as continuous maps.
- For any Sullivan representative $\phi\colon (\Lambda V, d) \to (\Lambda W, d)$ of a continuous map $f\colon X \to Y$ between CW-complexes with rational cohomology of finite type, we find maps $h_X\colon X \to |\Lambda W, d|$ and $h_Y\colon Y \to |\Lambda V, d|$ such that $|\phi| \circ h_X \simeq h_Y \circ f$ and h_X and h_Y are rational homotopy equivalences.

Restricting to simply connected CW complexes with rational cohomology of finite type and simply connected Sullivan algebras we will find bijections:

$$\left\{\begin{matrix}\text{rational homotopy}\\ \text{types}\end{matrix}\right\} \to \left\{\begin{matrix}\text{isomorphism classes of}\\ \text{minimal Sullivan algebras}\\ \text{over } \mathbb{Q}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{homotopy classes of}\\ \text{continuous maps}\\ \text{between rational spaces}\end{matrix}\right\} \to \left\{\begin{matrix}\text{homotopy classes of}\\ \text{morphisms of Sullivan algebras}\\ \text{over } \mathbb{Q}\end{matrix}\right\}$$

We will describe the functor $|\cdot|$ as a composition of a contravariant functor $\langle\cdot\rangle$ from CDGA's to simplicial sets, and a covariant functor $|\cdot|$ from simplicial sets to CW-complexes. Most of the time we can do all the constructions over any field $\mathbb{K}$, but in the end we are mostly interested in $\mathbb{K} = \mathbb{Q}$.

Definition 2.20 Let (A, d) be a CDGA, we construct the simplicial set $\langle A, d\rangle$ by setting

$$\langle A, d\rangle_n = \{\sigma\colon (A, d) \to (A_{\mathrm{PL}})_n | \sigma \text{ is a map of CDGA's}\},$$

or in alternative notation $\mathbf{cdga}((A, d), (A_{\mathrm{PL}})_n)$. We set face and generacy maps by $\partial_i\sigma = \partial_i \circ \sigma$ and $s_j\sigma = s_j \circ \sigma$.
If $\phi\colon (A, d) \to (B, d)$ is a morphism of CDGA's then $\langle\phi\rangle\colon \langle B, d\rangle \to \langle A, d\rangle$ is given by $\langle\phi\rangle(\sigma) = \sigma \circ \phi$. ◇

Remark 2.21 ~~If we recall~~ that A_{PL} ~~was~~ a functor from simplicial sets to CDGA's set by s**Set**(K, A_{PL}) we get a natural bijection

$$\mathbf{cdga}((A, d), A_{\mathrm{PL}}(K)) \to \mathrm{s}\mathbf{Set}(K, \langle A, d\rangle)$$

set by $\phi \mapsto f$ with $f(\sigma)(a) = \phi(a)(\sigma)$. In particular $A_{\mathrm{PL}}(\cdot)$ and $\langle\cdot\rangle$ are adjoint functors between CDGA's and simplicial sets. From this we get a canonical morphism $\eta_A\colon (A, d) \to A_{\mathrm{PL}}\langle A, d\rangle$ adjoint to the identity of $\langle A, d\rangle$. ◇

Then to define the functor $|\cdot|$ from simplicial sets to CW-complexes recall the maps $\lambda_i\colon \Delta^{n-1} \to \Delta^n$ and $\rho_j\colon \Delta^{n+1} \to \Delta^n$ where λ_i sends $(x_0, ..., x_n)$ to $(x_0, ..., 0, ..., x_n)$ with the 0 inserted in the i'th slot and ρ_j sends $(x_0, ..., x_n)$ to $(x_0, ..., x_j + x_{j+1}, ..., x_n)$.

Definition 2.22 Let K be a simplicial set. Using K_n as a index set we define $|K|$ to be

$$|K| = (\sqcup_n K_n \times \Delta^n)/\sim$$

where we identify $(\partial_i\sigma, x) \sim (\sigma, \lambda_i x)$ and $(s_j\sigma, x) \sim (\sigma, \rho_j x)$.
Given a map $f\colon K \to L$ of simplicial sets, written as $f = \{f_n\colon K_n \to L_n\}$, the maps $f_n \times \mathrm{id}\colon K_n \times \Delta^n \to L_n \times \Delta^n$ are continuous and preserve $\sim$, so they factor to a map $|f|\colon |K| \to |L|$ in a functorial way. ◇

We also describe an adjoint pair in this case. In particular for a simplicial set K and a $\sigma \in K_n$, the inclusion $\{\sigma\} \times \Delta^n \to \sqcup_n K_n \times \Delta^n$ descends to a map $q_\sigma \colon \Delta^n \to |K|$, which gives an inclusion of simplicial sets $\xi_K \colon K \to S(|K|)$, $\sigma \mapsto q_\sigma$.

On the other hand, given a topological space X we get a map $\sqcup_n S_n(X) \times \Delta^n \to X$ which sends a pair (σ, y) to $\sigma(y)$. This map factors through to the quotient into a map $s_X \colon |S(X)| \to X$.

These maps acts in some sense as inverses to each other:

Proposition 2.23 [7, Prop 17.3, p.242]

a) If X is a simply connected CW-complex, then s_X is a homotopy equivalence.

b) If K is a simplicial set such that $|K|$ is simply connected, then $|\xi_K|$ is a homotopy equivalence. ◇

Combining both functors we arrive at our spatial realization functor from CDGA's to CW-complexes:

Definition 2.24 Let (A, d) be a CDGA. We define its spatial realization by $|A, d| = |\langle A, d\rangle|$. Similarly, we define the spatial realization of a map $\phi \colon (A, d) \to (B, d)$ by $|\phi| = |\langle \phi \rangle|$. ◇

We now give brief outlines of constructions used to prove the properties described at the start, while referring to FHT [7] for the proofs.

First, we describe the quasi-isomorphism $m_{(\Lambda V,d)} \colon (\Lambda V, d) \xrightarrow{\simeq} A_{\mathrm{PL}}(|\Lambda V, d|)$. Note from before the inclusion $\xi \colon = \xi_{\langle \Lambda V, d\rangle} \colon \langle \Lambda V, d\rangle \to S(|\Lambda V, d|)$, one can show that applying A_{PL} to it gives rise to a surjective quasi-isomorphism $A_{\mathrm{PL}}(\xi) \colon A_{\mathrm{PL}}(|\Lambda V, d|) \to A_{\mathrm{PL}}(\langle \Lambda V, d\rangle)$. Then we can apply the lifting lemma (Lemma 1.22) to the canonical map $\eta_{(\Lambda V,d)}$ to obtain the map $m_{(\Lambda V,d)} \colon (\Lambda V, d) \to A_{\mathrm{PL}}(|\Lambda V, d|)$.

Theorem 2.25 [7, Thm 17.10.(ii), p.250] If $\mathbb{K} = \mathbb{Q}$, the map $m_{(\Lambda V,d)}$ is a quasi-isomorphism. ◇

So we see that if we start with a minimal algebra and construct its spatial realization, the minimal model of the space we get is the minimal algebra we started with. Note that we in particular get that $\pi_*(|\Lambda W, d|)$ is dual to W.
We get a similar result for maps between minimal algebras.

Lemma 2.26 [7, Eqn. (17.14), p.255] If $\phi \colon (\Lambda V, d) \to (\Lambda W, d)$ is a morphism of Sullivan algebras, then it is a Sullivan representative of its spatial realization. ◇

Proof. Using adjointsness of $A_{\mathrm{PL}}(\cdot)$ and $\langle \cdot \rangle$ we conclude that $A_{\mathrm{PL}}(\langle \phi \rangle) \circ \eta_{(\Lambda V,d)} = \eta_{(\Lambda W,d)} \circ \phi \colon (\Lambda V, d) \to A_{\mathrm{PL}}(\langle \Lambda W, d\rangle)$. Now using the lifting lemma we can crank this up to $A_{\mathrm{PL}}(|\cdot|)$

to get a square

$$\begin{array}{ccc} (\Lambda V, d) & \xrightarrow{\phi} & (\Lambda W, d) \\ \downarrow{\scriptstyle m_{(\Lambda V,d)}} & & \downarrow{\scriptstyle m_{(\Lambda W,d)}} \\ A_{\mathrm{PL}}(|\Lambda V, d|) & \xrightarrow[A_{\mathrm{PL}}(|\phi|)]{} & A_{\mathrm{PL}}(|\Lambda W, d|) \end{array}$$

which commutes up to homotopy, which is what we wanted to conclude. □

Proposition 2.27 [7, Prop 17.13, p.255] Maps $\phi, \psi \colon (\Lambda V, d) \to (\Lambda W, d)$ between Sullivan algebras are homotopic if and only if $|\phi|$ and $|\psi|$ are homotopic. ◇

Proof. Since homotopic maps have homotopic Sullivan representatives, the preceding lemma shows that if $|\phi|$ and $|\psi|$ are homotopic then ϕ and ψ are homotopic.
For the converse we take a homotopy $\Phi \colon (\Lambda V, d) \to (\Lambda W, d) \otimes \Lambda(t, dt)$ between ϕ and ψ. Since spatial realization preserves products (see Example 1 at page 248 of FHT [7]), we get a map $|\Phi| \colon |\Lambda W, d| \times |\Lambda(t, dt)| \to |\Lambda V, d|$. Then we can identify $|\Lambda(t, dt)|$ with the interval such that $|\Phi|$ becomes a homotopy from ϕ to ψ. □

Now for the last point, let X be a topological space with rational cohomology of finite type and let $m_X \colon (\Lambda W, d) \xrightarrow{\simeq} A_{\mathrm{PL}}(X)$ be a minimal model. Recall that the map $s_X \colon |S(X)| \to X$ is a homotopy equivalence, so there is a homotopy inverse t_X uniquely defined up to homotopy. Furthermore the adjointness of $A_{\mathrm{PL}}(\cdot)$ and $\langle\cdot\rangle$ gives a natural simplicial map $\gamma_X \colon S(X) \to \langle \Lambda W, d\rangle$. We set $h_X = |\gamma_X| \circ t_X \colon X \to |\Lambda W, d|$.

Lemma 2.28 [7, Thm 17.12.(ii), p.254] If $\mathbb{K} = \mathbb{Q}$, then $A_{\mathrm{PL}}(h_X)$ is a quasi-isomorphism. ◇

Corollary 2.29 The map h_X is a rational homotopy equivalence. ◇

Theorem 2.30 [7, Thm 17.15, p.256] Let X and Y be topological spaces with minimal models $m_X \colon (\Lambda W, d) \to A_{\mathrm{PL}}(X)$ and $m_Y \colon (\Lambda V, d) \to A_{\mathrm{PL}}(Y)$. Let $f \colon X \to Y$ be a continuous map with $\phi \colon (\Lambda V, d) \to (\Lambda W, d)$ a Sullivan representative. Then

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle h_X} & & \downarrow{\scriptstyle h_Y} \\ |\Lambda W, d| & \xrightarrow[|\phi|]{} & |\Lambda V, d| \end{array}$$

commutes up to homotopy. ◇

This concludes the main properties of the spatial realization, and realizes for us the motto 'spaces are (minimal) algebras'.

Since we are mainly interested in (simply connected) manifolds, we can also wonder when given a simply connected minimal algebra, there is a simply connected manifold realizing it. Note that asking for simply connectedness means that we in particular need Poincaré duality. Then assuming Poincaré duality, if we look at $4k$-dimensional algebras, the map $H^{2k} \times H^{2k} \to H^{4k} \cong \mathbb{R}$ will be a quadratic form, and extra obstructions with signature and Pontryagin classes arise. We will not discuss these in detail (or even define them properly), but at this point it is reasonable to at least quote the following result by Sullivan, which says that 'necessary conditions for a realization by a simply connected manifold is also sufficient'.

Theorem 2.31 [22, Thm 13.12, p.321] Let $(\Lambda V, d)$ be a minimal algebra (over $\mathbb{R}$) with $V^1 = 0$ whose cohomology algebra is PDA of dimension n. Also we choose cohomology classes $p = \{p_i \in H^{4i}(\Lambda V, d)\}_{i=1,2,\ldots}$.

a) If n is not of the form $4k$, there is a simply connected manifold whose minimal model is $(\Lambda V, d)$ and whose Pontryagin classes correspond to p_i.

b) If n is of the form $4k$, and the signature of the quadratic form over H^{2k} is zero, then there is a simply connected manifold whose minimal model is $(\Lambda V, d)$ and whose Pontryagin classes correspond to p_i if and only if the quadratic form over H^{2k} is congruent over $\mathbb{Q}$ to a quadratic form $\sum_i \pm x_i^2$.

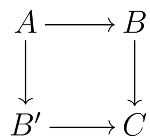
c) If n is of the form $4k$, and the signature of the quadratic form over H^{2k} is nonzero, then there is a simply connected manifold whose minimal model is $(\Lambda V, d)$ and whose Pontryagin classes correspond to p_i if and only if the quadratic form over H^{2k} is congruent over $\mathbb{Q}$ to a quadratic form $\sum_i \pm x_i^2$ and the Pontryagin numbers are numbers satisfying the congruence of a cobordism. $\diamond$

Remark 2.32 As stated before the theorem, we will not describe what congruences of quadratic forms, or Pontryagin classes/numbers, or congruence of a cobordism is. We do note that for a simply connected manifold of dimension $4k$ these are algebraic relations that one can't avoid. The theorem says that these if a minimal model satisfies the these kinds of relations it is enough to have a manifold realizing it.

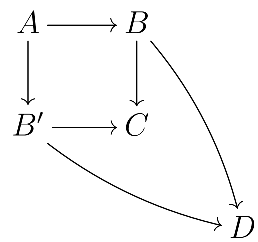
2.5 Spaces and CDGA's, pushouts and pullbacks

An important result of the topological work we did in the last section is that we can jump between pushouts of spaces and pullbacks of CDGA's. Since we need to able to handle things that commute up to homotopy, we will divert from the usual notions of pushout and pullback. Also since we don't need any uniqueness in this context, we will not use the notion of pushout and pullback in the homotopy category or the notion of homotopy pushout and homotopy pullback. The notion we will work with will be called *weak homotopy pullback square* and *weak homotopy pushout square*:

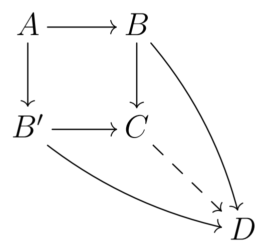
Definition 2.33 Let $\mathcal{C}$ be category with the notion of homotopy. A *weak homotopy pushout* *square* in $\mathcal{C}$ is a square



which commutes up to homotopy, with the property that for every addition

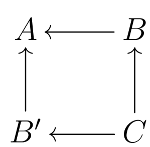


such that the outer square commutes up to homotopy, there is an arrow $C \to D$ such that

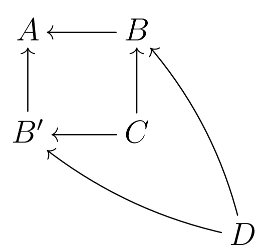


commutes up to homotopy. ◇

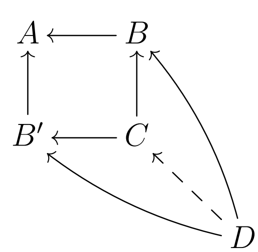
Definition 2.34 Let $\mathcal{C}$ be category with the notion of homotopy. A *weak homotopy pullback square* in $\mathcal{C}$ is a square



which commutes up to homotopy, with the property that for every addition



such that the outer square commutes up to homotopy, there is an arrow $D \to C$ such that



commutes up to homotopy. ◇

The motivating example for this definition is the following:

Lemma 2.35 Suppose that X is a topological space with a cover $X = U \cup U'$ such that there a continuous function $t\colon X \to [0,1]$ with t identically is 0 on an open neighbourhood of $X \backslash U'$ and identically 1 on an open neighbourhood of $X \backslash U$. Then

$$\begin{array}{ccc} U \cap U' & \hookrightarrow & U \\ \downarrow & & \downarrow \\ U' & \hookrightarrow & X \end{array}$$

is a weak homotopy pushout square in **Top**. ◇

Proof. What we really need to prove is that given two functions $f\colon U \to Y$ and $f'\colon U' \to Y$ such that they are homotopic on $U \cap U'$ there is a function $g\colon X \to Y$ such that $g|_U \simeq f$ and $g|_{U'} \simeq f'$.

So suppose we have f and f' together with a homotopy $H\colon U \cap U' \times [0,1] \to Y$ from f to f'. Denote by $\widetilde{U}$ and $\widetilde{U'}$ open neighbourhoods of $X \backslash U'$ and $X \backslash U$ respectively where t is constantly 0 and 1 respectively. Define the closed set $V = X \backslash (\widetilde{U} \cup \widetilde{U'})$. From H we construct the following continuous map:

$$\widetilde{H}\colon V \times [0,1] \cup U \times \{0\} \cup U' \times \{1\} \to Y; \begin{cases} (x,t) \mapsto H(x,t) & \text{if } (x,t) \in V \times [0,1] \\ (x,0) \mapsto f(x) & \text{if } x \in U \\ (x,1) \mapsto f'(x) & \text{if } x \in U' \end{cases}$$

Define $g\colon X \to Y$ by $g(x) = \widetilde{H}(x, t(x))$. It is immediate that $g|_U \simeq f$ and $g|_{U'} \simeq f'$ (indeed outside of the intersection they are equal, while on the intersection we need to 'lift the time coordinate' from $t(x)$ to either 0 or 1, for which we use H). □

Remark 2.36 If X admits partitions of unity subordinated to open covers, any open 2-cover has the property as stated in the last lemma. Indeed if ϕ_U and $\phi_{U'}$ are the partition of unity, then ϕ_U is 0 on an open neighbourhood of $X \backslash U$. So the $\phi_{U'}$ is 1 on the same open neighbourhood of $X \backslash U$, while it is 0 on an open neighbourhood of $X \backslash U'$.

So in particular, the lemma above applies to open 2-covers of paracompact spaces (for instance manifolds). ◇

The main reason to consider these notions is that these kinds of pushouts and pullbacks are preserved by A_{PL}:

Theorem 2.37 If

$$\begin{array}{ccc} Y & \xrightarrow{i} & U \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle j} \\ U' & \xrightarrow[j']{} & X \end{array}$$

is a weak homotopy pushout square in **Top**, then

$$\begin{array}{ccc} A_{\mathrm{PL}}(Y) & \xleftarrow{A_{\mathrm{PL}}(i)} & A_{\mathrm{PL}}(U) \\ {\scriptstyle A_{\mathrm{PL}}(i')}\uparrow & & \uparrow{\scriptstyle A_{\mathrm{PL}}(j)} \\ A_{\mathrm{PL}}(U') & \xleftarrow[A_{\mathrm{PL}}(j')]{} & A_{\mathrm{PL}}(X) \end{array}$$

is a weak homotopy pullback square in **cdga**, at least with respect to additions $A_{\mathrm{PL}}(U') \xleftarrow{\phi'} \Lambda V \xrightarrow{\phi} A_{\mathrm{PL}}(U)$ where ΛV is a minimal CDGA. ⋄

Remark 2.38 Since A_{PL} and Ω^* are naturally weakly equivalent, if Y, U, U' and X are smooth manifolds and i, i', j and j' smoooth maps, a similar statement holds for the induced square with the deRham-complexes with induced maps of pullbacks of differential forms. ⋄

Proof of Theorem 2.37. **NB:** In order not to overdress diagrams, if we have diagrams with A_{PL}-terms in them, and we considers arrows of the form $A_{\mathrm{PL}}(f)$ for f a continuous map, we will dress the arrow with only f.

Consider an addition

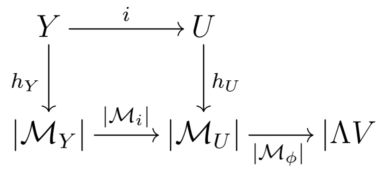
$$\begin{array}{cccc} A_{\mathrm{PL}}(Y) & \xleftarrow{i} & A_{\mathrm{PL}}(U) & \\ {\scriptstyle i'}\uparrow & & \uparrow{\scriptstyle j} & \nwarrow{\scriptstyle \phi} \\ A_{\mathrm{PL}}(U') & \xleftarrow[j']{} & A_{\mathrm{PL}}(X) & \\ & \nwarrow{\scriptstyle \phi'} & & \Lambda V \end{array}$$

such that $A_{\mathrm{PL}}(i) \circ \phi \simeq A_{\mathrm{PL}}(i') \circ \phi'$. Writing $\mathcal{M}$ for minimal models and induced maps between minimal models, by the work of the preceding section we get the square on the level of the spatial realizations

$$\begin{array}{ccc} |\mathcal{M}_Y| & \xrightarrow{|\mathcal{M}_i|} & |\mathcal{M}_U| \\ {\scriptstyle |\mathcal{M}_{i'}|}\downarrow & & \downarrow{\scriptstyle |\mathcal{M}_\phi|} \\ |\mathcal{M}_U| & \xrightarrow[|\mathcal{M}_{\phi'}|]{} & |\Lambda V| \end{array}$$

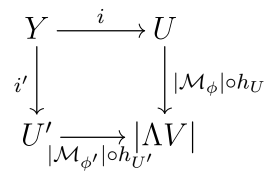
that commutes up to homotopy. To see this, first use that passing to the minimal models is well-defined and functorial up to homotopy, then use the fact that spatial realization is a functor that preserves homotopy.

Consider now the following diagram

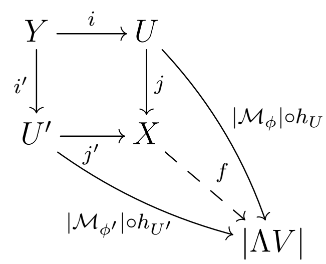


By Theorem 2.30 we know that the square commutes up to homotopy, and hence $|\mathcal{M}_\phi| \circ h_U \circ i \simeq |\mathcal{M}_\phi| \circ |\mathcal{M}_i| \circ h_Y$.

Then starting with $|\mathcal{M}_{\phi'}| \circ |\mathcal{M}_{i'}| \simeq |\mathcal{M}_\phi| \circ |\mathcal{M}_i|$, precomposing with h_Y on both side and using the result above, we conclude that



commutes up to homotopy. Now since we start with a weak homotopy pushout square we get a map $f\colon X \to |\Lambda V|$ such that the following commutes up to homotopy:

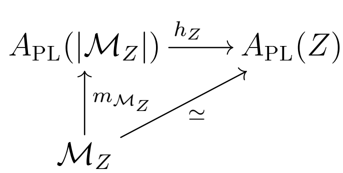


We now claim that the map $A_{\mathrm{PL}}(f) \circ m_{\Lambda V} \colon \Lambda V \to A_{\mathrm{PL}}(X)$ satisfies $A_{\mathrm{PL}}(j') \circ A_{\mathrm{PL}}(f) \circ m_{\Lambda V} \cong \phi'$ and $A_{\mathrm{PL}}(j) \circ A_{\mathrm{PL}}(f) \circ m_{\Lambda V} \cong \phi$.

To see this note that $A_{\mathrm{PL}}(j) \circ A_{\mathrm{PL}}(f) = A_{\mathrm{PL}}(f \circ j) \cong A_{\mathrm{PL}}(h_U) \circ A_{\mathrm{PL}}(|\mathcal{M}_\phi|)$ by the defining property of f.

Now by Lemma 2.26 we know that $\mathcal{M}_\phi$ is a Sullivan representative of $|\mathcal{M}_\phi|$ i.e. $m_{\mathcal{M}_{U'}} \circ \mathcal{M}_\phi \cong A_{\mathrm{PL}}(|\mathcal{M}_\phi|) \circ m_{\Lambda V}$.

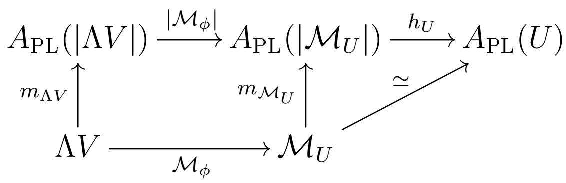
Claim: For any topological space Z the following diagram commutes up to homotopy



where the diagonal arrow is the part of the definition of the minimal model $\mathcal{M}_Z \to A_{\mathrm{PL}}(Z)$.

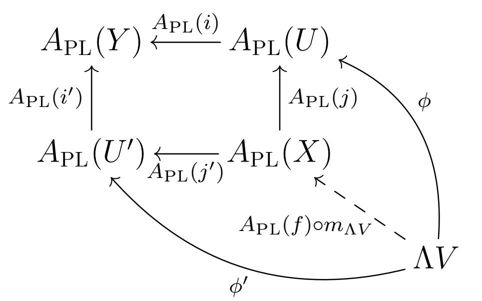
Proof of the claim. This basically follows from unpacking the definition of h_Z and $m_{\mathcal{M}_Z}$ and using the notion of adjointness. This is treated in Theorem 17.12.(i) of FHT [7]. □

So we see that the following diagram commutes up to homotopy:



So we see that $\mathcal{M}_\phi$ is a Sullivan representative of both ϕ and $A_{\text{PL}}(h_U) \circ A_{\text{PL}}(|\mathcal{M}_\phi|) \circ m_{\Lambda V}$, so we conclude that $\phi \simeq A_{\text{PL}}(h_U) \circ A_{\text{PL}}(|\mathcal{M}_\phi|) \circ m_{\Lambda V} \simeq A_{\text{PL}}(f) \circ m_{\Lambda V}$.

Doing the similar thing on the primed-side, we conclude that the following commutes up to homotopy:



which shows that the square is indeed a weak homotopy pushout square with respect to incoming maps from minimal algebras. □

Remark 2.39 We can also construct spatial realizations of algebras which are not minimal, so that the same argument as above applies to any addition $A_{\text{PL}}(U') \leftarrow A \rightarrow A_{\text{PL}}(U)$, but in what follows we don't need this and these spatial realizations are more difficult to construct anyway. ◇

3 Formality & geometric structures: first examples

As described in the introduction there is some connection between geometric structures and formality. For this we will use that such structures are well suited for the algebraic way of doing topology. Indeed geometric structures tend to involve some PDE's, or otherwise special kinds of functions and forms. This will allow us to massage the deRham-complex to simpeler complexes. The prototypical example is the renowned theorem by Deligne, Griffiths, Morgan and Sullivan on Kähler manifolds [5].

3.1 Kähler manifolds

Definition 3.1 A *Kähler manifold* is a manifold M endowed with a symplectic form ω and an integrable almost complex structure J such that $g(v, w) = \omega(v, Jw)$ is a Riemannian metric on M. ◇

Example 3.2 Since S^2 is an oriented surface in $\mathbb{R}^3$ it carries a canonical symplectic form ω. On the other hand since $S^2 \simeq \mathbb{C}P^1$ it carries an integrable ACS J, and these together make S^2 in a Kähler manifold. ◇

The important feature of Kähler manifolds is that they come with a subclass of forms which generate the same cohomology. This allowes us to get a better grip on the deRham-complex. The ingredient that we need is the d^c-operation and the dd^c-lemma.

Lemma 3.3 **(dd^c-lemma)** Let M be a compact Kähler manifold, and let $d^c = J^{-1}dJ$ where J is the induced map on $\Omega(M)$. Then d^c is a real operator such that:

a) $(d^c)^2 = 0$ and $dd^c = -d^c d$.

b) If $\alpha \in \Omega(M)$ is a form such that $d\alpha = d^c\alpha = 0$ and such that $\alpha = d\beta$ or $\alpha = d^c\beta'$. Then $\alpha = dd^c\gamma$ for some $\gamma \in \Omega(M)$. ◇

To prove this lemma, we will sketch some background in the Hodge theory of Kähler manifolds. Recall that for a Riemannian manifold (M, g) one gets an adjoint operator d^* to the exterior differential d via the use of the Hodge star operator. One then defines the Laplacian of the Riemannian manifold by $\Delta = dd^* + d^*d$, and sets the harmonic forms $\mathcal{H}(M, g)$ to be the kernel of the Laplacian. The Hodge theorem is then the observation that

$$\Omega(M) = \mathrm{im}\, d \oplus \mathrm{im}\, d^* \oplus \mathcal{H}(M, g),$$

which is called the Hodge decomposition. From this one can for instance read that the harmonic forms exactly represent cohomology.

If one now has a complex manifold M, using the complex structure the exterior differential $d\colon \Omega^*(M) \to \Omega^{*+1}(M)$ splits as the sum of two maps $\partial\colon \Omega^{*,*}(M) \to \Omega^{*+1,*}(M)$ and $\overline{\partial}\colon \Omega^{*,*}(M) \to \Omega^{*,*+1}(M)$. Equivalent to the definition given above, one can also set d^c to be $d^c = i(\overline{\partial} - \partial)$, from which one immediately sees that it commutes with complex conjugation.

Using the Hodge star, one can construct adjoints ∂^* and $\overline{\partial}^*$ to the operators ∂ and $\overline{\partial}$, and similarly to above make Laplacians $\Delta_\partial = \partial\partial^* + \partial^*\partial$ and $\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. The upshot of doing this is that the complex structure being Kähler has far reaching implications on these Laplacians (and also the other way around).

Lemma 3.4 [12, Prop 3.1.12, p.120] A complex Riemannian manifold (M, g) is Kähler if and only if $2\Delta = \Delta_\partial = \Delta_{\overline{\partial}}$. ◇

So in particular, if we define the ∂-harmonic and $\overline{\partial}$-harmonic forms as $\mathcal{H}_\partial = \ker(\Delta_\partial)$ and $\mathcal{H}_{\overline{\partial}} = \ker(\Delta_{\overline{\partial}})$ we see that the notion of ∂-harmonic, $\overline{\partial}$-harmonic and harmonic coincide for a Kähler manifold.

This will be the main argument to prove the dd^c-lemma. We also need the following lemma's:

Lemma 3.5 [12, Lem A.0.12, p.285] On a compact oriented Riemannian manifold (M, g) a form α is harmonic if and only if $d\alpha = d^*\alpha = 0$. ◇

Lemma 3.6 [12, Lem 3.2.5, p.126] On a compact complex Riemannian manifold (M, g) a form α is ∂-harmonic (respectively $\overline{\partial}$-harmonic) if and only if $\partial\alpha = \partial^*\alpha = 0$ (respectively $\overline{\partial}\alpha = \overline{\partial}^*\alpha = 0$). ◇

The proof is then as follows:

Proof of Lemma 3.3. As to part **a)**, this follows as mentioned before from the fact that $d = \partial + \overline{\partial}$ while $d^c = i(\overline{\partial} - \partial)$, together with the fact that $\partial\overline{\partial} = -\overline{\partial}\partial$ (which on it's own follows from $d^2 = 0$).

For part **b)**, first let α be d-closed and d^c-exact, write $\alpha = d^c\gamma$. From the Hodge decomposition we get $\gamma = d\beta + \mathcal{H}(\gamma) + d^*\delta$. Since we work with on a Kähler manifold the harmonic part $\mathcal{H}(\gamma)$ is also ∂- and $\overline{\partial}$-harmonic and in particular ∂- and $\overline{\partial}$-closed, so also d^c-closed. We conclude that $\alpha = d^c\gamma = d^c d\beta + d^c d^*\delta$.
We show that $d^c d^*\delta = 0$. For this we use that α is d-closed, which leads to $0 = d\alpha = dd^c d^*\delta$. Since $dd^c = -d^c d$ leads to $d^*d^c = -d^c d^*$, we conclude that $dd^*d^c\delta = 0$, and hence pairing with $d^c\delta$ we conclude that $||d^*d^c\delta||^2 = (d^*d^c\delta, d^*d^c\delta) = (dd^*d^c\delta, d^c\delta) = 0$, which finishes to proof in this case.

Now let α be d-exact and d^c-closed, then $J(\alpha)$ is d^c-exact and d-closed, so $J(\alpha) = dd^c\beta$ by the first part of the proof, and then $\alpha = (J^{-1}dJ^{-1}d)(J\beta) = (-1)^{|\alpha|-1}(d^c d)(J\beta) = d^c d(\beta')$ with $\beta' = (-1)^{|\alpha|-1}J\beta$. □

Now we look at two induced algebras. Namely we can look at $\Omega^c(M)$, the algebra of d^c-closed forms, which comes with the restriction of d. Furthermore we can look at the d^c-cohomology $H_{d^c}(M)$ of M, which also comes with an induced differential from d. So we have the following maps of CDGA's:

$$(\Omega(M), d) \xhookleftarrow{i} (\Omega^c(M), d) \xrightarrow{p} (H_{d^c}(M), d)$$

The result by Deligne et al. now is as follows:

Theorem 3.7 (DGMS '75) [5, p.270] Both i and p are quasi-isomorphisms and the differential induced by d on $H_{d^c}(M)$ is 0. $\diamond$

Proof. The proof is basically using the dd^c-lemma five times. We show that i^* is surjective. For this let $[\alpha]$ be a cohomology class in $H(M)$ with representative α. Then by the dd^c-lemma applied to $d^c\alpha$ we get that $d^c\alpha = dd^c\omega$ for some ω. Then looking at $\beta = \alpha + d\omega$ we get that $\beta \in \Omega^c(M)$ since $d^c\beta = d^c\alpha + d^c d\omega = d^c\alpha - d^c\alpha = 0$. Furthermore since $d\alpha = 0$, clearly $d\beta = 0$, and we get $[\alpha] = i^*[\beta]$.

To show that i^* is injective let α be a closed form in $\Omega^c(M)$ such that $i^*[\alpha] = 0$, that is $d\alpha = d^c\alpha = 0$ and $\alpha = d\beta$. Then by the dd^c-lemma we get that $\alpha = dd^c\omega$ and since $d^c d^c\omega = 0$ we get that $d^c\omega \in \Omega^c(M)$ and $[\alpha] = [dd^c\omega] = 0 \in H(\Omega^c(M), d)$.

Next we show that the induced differential is 0. To this end let $\alpha \in \Omega^c(M)$, we want to show that $d\alpha$ is d^c-exact. We may use the dd^c-lemma on $d\alpha$, so we get that $d\alpha = d^c d\omega$ for some ω, then since obviously $d^c(d^c\omega) = 0$, we get that $d[\alpha] = 0$.

Then we look at ρ^*. To see that it is surjecitve, let $[\alpha] \in H_{d^c}(M)$ with some d^c-closed representative α. Then the dd^c-lemma applied to $d\alpha$ results in $d\alpha = dd^c\omega$. Then looking at $\beta = \alpha + d^c\omega$ we get that $d\beta = d\alpha + dd^c\omega = d\alpha - d\alpha = 0$ and $d^c\beta = d^c\alpha$, so β is a closed element of $(\Omega^c(M), d)$ satisfying $\rho^*[\beta] = [\alpha]$.

Lastly, to see that ρ^* is injecitve, let α be a d- and d^c-closed form such that $\rho^*[\alpha] = 0 \in H_{d^c}(M)$, i.e. $d\alpha = d^c\alpha = 0$ and $\alpha = d^c\beta$. Then then the dd^c-lemma gives that $\alpha = dd^c\omega$, which means that $[\alpha] = 0 \in H(\Omega^c(M), d)$. $\square$

This yields the following results:

Theorem 3.8 [5, Main Theorem (i), p.270] Every compact complex manifold for which the dd^c-lemma holds is formal. In particular Kähler manifolds are formal. $\diamond$

The result is that we have a viable way to show that a manifold is not Kähler, namely showing that it is not formal. Indeed, the first examples of symplectic manifolds that are not Kähler were found by searching for symplectic manifolds that are not formal.

3.2 Lie groups and symmetric spaces

The next classic class of formal manifolds are Lie groups. These seem to satisfy any sensible topological property, and indeed they are also formal. To show that, we will consider the subalgebra of bi-invariant forms. Since this argument also works for symmetrics spaces, we will set it up a somewhat broader setting.

Definition 3.9 Let M be a manifold with a left G-action. A *left-invariant* form is a differential form $\omega \in \Omega(M)$ such that $g^*\omega = \omega$ for all $g \in G$. Here we identify g with the diffeomorphism $x \mapsto gx$. ◇

Remark 3.10 Since the pullback of differential forms distributes over the wedge product and the differential commutes with the pullback, we see that the subspace of left-invariant forms is a subalgebra of $\Omega(M)$. We denote it by $\Omega_L(M)$, with cohomology $H_L(M)$. ◇

The main result is then:

Theorem 3.11 Let M be a compact manifold with a left action of a compact connected Lie group G. The inclusion $\iota \colon \Omega_L(M) \hookrightarrow \Omega(M)$ induces an isomorphism $H_L(M) \cong H(M)$. ◇

For the proof we need to recall the existence of a useful tool for integration over Lie groups.

Proposition 3.12 [8, Prop 1.29, p.13] On a compact connected Lie group, there exists a bi-invariant volume form. ◇

Proof of Theorem 3.11. Consider the bi-invariant volume form dg of volume 1: $\int_G dg = 1$. We then construct a map $\rho \colon \Omega(M) \to \Omega(M)$ by left-averaging forms:

$$\rho(\omega)(X_1, ..., X_k)(x) = \int_G g^*\omega(X_1, ..., X_k)(x)dg$$

for $\omega \in \Omega^k(M)$, $X_1, ..., X_k \in \mathfrak{X}(M)$ and $x \in M$.

Using the invariance of dg, we see that ρ has the following properties:

- $\rho(\omega) \in \Omega_L(M)$.
- If $\omega \in \Omega_L(M)$ then $\rho(\omega) = \omega$.
- $\rho \circ d = d \circ \rho$.
- $[\omega] = [\rho(\omega)]$. doesn't this need simply connectedness?

From this it follows that $H(\rho)$ is both a left and right inverse to $H(\iota)$, so the latter is an isomorphism. □

Using this theorem, by choosing a suitable action of a Lie group on a fixed compact connected Lie group G we can prove formality. For this we first consider the following proposition, which gives the candidate for the CDGA with trivial differential.

Proposition 3.13 Let G be a Lie group. Any bi-invariant form $\omega \in \Omega_I(G)$ is closed. ◇

Proof. We can see the bi-invariant forms as a subalgebra of the left-invariant forms, the latter of which we can identify with $\wedge\mathfrak{g}^*$. Since the inversion map i acts as -1 on $\mathfrak{g}$, we see that pullback of the inversion map i^* acts as $(-1)^p$ on $\wedge^p\mathfrak{g}^*$ hence also on $\Omega_I^p(M)$. Since i^* commutes with d we have:

$$(-1)^{p+1}d\omega = i^*d\omega = di^*\omega = (-1)^p d\omega$$

So we see that $d\omega = 0$. □

So the inclusion $\Omega_I(M) \hookrightarrow \Omega(M)$ is an inclusion of an algebra with trivial differential. By choosing the right action on G we can use Theorem 3.11 to show that this is a quasi-isomorphism.

Corollary 3.14 Any compact connected Lie group G is formal. ◇

Proof. Consider the left action of $G \times G$ on G set by $(g_1, g_2)g = g_1 g g_2^{-1}$. Clearly any bi-invariant form is left-invariant for this action, reversely by looking at (g', e) and (e, g'^{-1}) we see that a form that is left-invariant for this action is bi-invariant. So then $\Omega_I(G)$ can be realized as an algebra of left-invariant forms for an action satisfying the assumptions of Theorem 3.11. We conclude that the inclusion $\Omega_I(G) \hookrightarrow \Omega(G)$ is a quasi-isomorphism. □

Since we only use left-actions, we can use this proof a little bit more general, namely in a special class of quotient of Lie groups, called symmetric spaces.

Definition 3.15 Let G be a connected Lie group, and $\sigma \in \mathrm{Aut}(G)$ be an involution with fixed point set G^σ. Let $H \subset G^\sigma$ be an open subgroup, we call G/H a *symmetric space*. If G is compact, and H is the connected component of G^σ containing the identity, we call G/H a *symmetric space of compact type*. ◇

Symmetric spaces bundle a large number of examples we already discussed:

Example 3.16 Let G be a Lie group, with σ the inverse-map. Then since $\sigma(\exp(X)) = \exp(-X)$ and since exp is a local diffeomorphism, we see that the identity is an isolated point in G^σ, so if we set $H = \{\mathrm{id}\}$ we get that $G = G/H$ is a symmetric space. If G is compact, then it is symmetric of compact type. ◇

Example 3.17 Consider the group $O(n)$ with $O(n-1)$ as a subgroup where we see $A \in O(n-1)$ as an element of $O(n)$ via $A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$. Consider the automorphism σ of $O(n)$ defined by conjugation with the map $(x_1, x_2, ..., x_n) \mapsto (-x_1, x_2, ..., x_n)$. In matrix form we have:

$$\sigma \begin{pmatrix} \lambda & v^T \\ w & A \end{pmatrix} = \begin{pmatrix} \lambda & -v^T \\ -w & A \end{pmatrix}$$

and in this way we clearly see that $O(n)^\sigma = O(n-1)$, so $O(n)/O(n-1)$ is a symmetric space of compact type. It is well-known that $O(n)/O(n-1) \cong S^{n-1}$, so all spheres are symmetric spaces. ◇

Example 3.18 Similar to the above example, we have $\mathbb{C}P^n \cong U(n+1)/U(n) \times U(1)$ and $\mathbb{R}P^n \cong O(n+1)/O(n) \times O(1)$, which exhibits $\mathbb{C}P^n$ and $\mathbb{R}P^n$ as symmetric spaces. $\diamond$

The proof that compact connected Lie groups are formal generalizes to symmetric spaces, and we derive the result

Theorem 3.19 A symmetric space G/H of compact type is formal. $\diamond$

Proof. Clearly G acts on G/H from the left, and by Theorem 3.11 we have that the inclusion $\Omega_L(G/H) \hookrightarrow \Omega(G/H)$ is a quasi-isomorphism. Then since the action is transitive we can identify $\Omega_L(G/H)$ as a subalgebra of $\Lambda(\mathfrak{g}/\mathfrak{h})^*$. We will show that $\Omega_L(G/H)$ has trivial differential.

Let $\tau\colon G \to G$ be the involution defining G/H, clearly this restricts to an involution $\sigma\colon G/H \to G/H$. Denote by $d\tau\colon TG \to TG$ and $d\sigma\colon T(G/H) \to T(G/H)$ the derivatives. Since τ is an involution and $\mathfrak{h} = \{h \in \mathfrak{g} | d\tau(h) = h\}$ (since H is an open submanifold of the fixed point set of τ), it follows that $d\sigma = -\mathrm{id}$ on $\mathfrak{g}/\mathfrak{h}$ (indeed it either acts as $+\mathrm{id}$ or $-\mathrm{id}$ on parts of $\mathfrak{g}$ and we quotient out the part where it acts as $+\mathrm{id}$). So $\sigma^* = (-1)^p$ on $\Omega_L^p(G/H)$, so since σ^* commutes with d, similarly as before we conclude that $d = 0$. $\square$

3.3 Formality and harmonic forms

Formal manifolds can loosely be described as 'you can choose representatives in a well enough defined manner'. If we throw in geometric structures and specifically Riemannian metrics there is a well known way of finding representatives, namely harmonic forms. Indeed every cohomology class has precisely one harmonic representative. So if we denote the harmonic forms by $\mathcal{H}(M)$ we are inclined to look at the map $(\mathcal{H}(M), 0) \hookrightarrow \Omega(M)$. ~~Clearly this map is a quasi-isomorphism, so look any manifold is formal, well then!~~

Clearly this is not true, in this case since the product of two harmonic forms is not harmonic, so $\mathcal{H}(M)$ is not an algebra, at least not canonically as a subalgebra of $\Omega(M)$. So we are inclined to give a special name to when this does happen:

Definition 3.20 A manifold M is called *geometrically formal* if there is a Riemmanian metric for which the harmonic forms are closed under the wedge product. $\diamond$

Proposition 3.21 Any geometrically formal manifold is formal. $\diamond$

Proof. Choosing a metric for which the harmonic forms are closed under the wedge product, we have that $(\mathcal{H}(M), 0)$ is a CDGA such that the inclusion $(\mathcal{H}(M), 0) \hookrightarrow \Omega(M)$ is a map of CDGA's. Since it is an isomorphism on cohomology, we conclude that M is formal. $\square$

Hence, geometrically formal manifolds bring in the most well known occurence of 'finding representatives' into the world of formality and we wonder what are examples of such spaces and whether there are extra obstructions to geometrical formality.

Example 3.22 If one chooses an invariant metric on a Lie group, the harmonics turn out to be the bi-invariant forms. Hence the harmonic forms are closed under the wedge product. This leads to an alternative proof of Corollary 3.14. More generally, with respect to an invariant metric, any symmetric space is geometrically formal. ⋄

To answer the question whether there are extra obstruction, we refer to a paper by Kotschick for the following result:

Proposition 3.23 [15, Thm 6, p.524] If M is geometrically formal, then $b_k(M) \leq \binom{\dim(M)}{k}$ (here $b_k(M)$ is the k^{th} Betti-number of M). ⋄

This result allows us to find an example of a formal manifold which is not geometrically formal.

Example 3.24 Consider a K3-surface (or *the* K3-surface M since we don't really care about the complex structure), which is a simply connected 4-manifold and hence formal by Theorem 1.42. From the Hodge diamond we see that $H^2(M)$ is 22-dimensional, so since $22 \not\leq \binom{4}{2} = 6$ by the preceding proposition we conclude that M is not geometrically formal. Also note that a K3-surface is Kähler, so also Kähler manifolds are not necessarily geoemtrically formal, even though the proof that they are formal makes heavy use of the metric. ⋄

The topological obstructions strongly restrict manifolds in being geometrically formal, and hence such manifolds are very rare. To make this story even more interesting: apart from pathological examples searching for a metric which confirms that a manifold is geometrically formal is a non-trivial endeavour.

Proposition 3.25 [15, Thm 16, p.530] ~~A compact oriented manifold admits a metric for which a product of harmonic forms is not harmonic if and only if it is not a rational homology sphere.~~ ⋄

3.4 Special holonomy

Up to this point we have seen a few classes of spaces which are formal, but also examples of spaces which are not formal. We now somehow want a systematic way to deal with manifolds to determine which are formal and which are not. As always when one has a few examples for some structure, one tries to combine them to new examples, for instance via products.

Proposition 3.26 If $\phi\colon \mathcal{M} \to A$ and $\psi\colon \mathcal{N} \to B$ are two minimal models, then $\phi \otimes \psi\colon \mathcal{M} \otimes \mathcal{N} \to A \otimes B$ is the minimal model of the tensor product. As a result, if A and B are formal, then $A \otimes B$ is also formal. ⋄

Proof. The first assertion is easy to check from the definitions, using that $H(A \otimes B) =$

$H(A) \otimes H(B)$. To see the second assertion, note that we have quasi-isomorphisms $\phi' \colon \mathcal{M} \to H(A)$ and $\psi' \colon \mathcal{N} \to H(B)$ by formality of A and B, and then $A \otimes B \xleftarrow{\phi \otimes \psi} \mathcal{M} \otimes \mathcal{N} \xrightarrow{\phi' \otimes \psi'} H(A) \otimes H(B) \cong H(A \otimes B)$ is a zig-zag of quasi isomorphisms, so $A \otimes B$ is formal. □

Proposition 3.27 For two manifolds M and N, the projections induce a quasi-isomorphism between $\Omega(M \times N)$ and $\Omega(M) \otimes \Omega(N)$. In particular if M and N are both formal, then $M \times N$ is formal. ⋄

Proof. The first fact is well-known, while the second assertion is a direct corollary of the first fact and the preceding proposition. □

The fact that the product of two formal manifolds is formal is reversable, namely that if a product is formal, then both its factors are.

Proposition 3.28 [8, Ex 2.88, p.93] Let M be a formal manifold, and N a retract, then N is formal. ⋄

Proof. Consider the chain $N \xrightarrow{i} M \xrightarrow{r} N$ where i is the inclusion and r is the retraction such that $ri \simeq \mathrm{id}_Y$. Then by formality of M we have a quasi-isomorphism $\phi \colon \mathcal{M}_M \to H(M)$. Then denoting $\psi \colon \mathcal{M}_N \to \Omega(N)$ the minimal model, we have the following zig-zag $\Omega(N) \xleftarrow{\psi} \mathcal{M}_N \xrightarrow{\mathcal{M}_r} \mathcal{M}_M \xrightarrow{\phi} H(M) \xrightarrow{i^*} H(N)$. Here $i^* \circ \phi \circ \mathcal{M}_r$ is a quasi-isomorphism (since on cohomology level, it is equal to $i^* r^* = \mathrm{id}$). We conclude that N is formal. □

Corollary 3.29 If $M \times N$ is formal, then both M and N are formal. ⋄

Proof. Both M and N are retracts of the product by considering two base points $m \in M$, $n \in N$ and the chains $N \xrightarrow{\mathrm{const}_m \times \mathrm{id}} M \times N \xrightarrow{\mathrm{pr}_2} N$ and $M \xrightarrow{\mathrm{id} \times \mathrm{const}_n} M \times N \xrightarrow{\mathrm{pr}_1} M$. □

So we know how to determine the formality of product-manifolds, and we know that symmetric spaces are formal. If we disregard these cases, the building blocks of what remains are concisely classified.

To formulate this list, we discuss a little bit of Riemmanian geometry. Recall that for a Riemmanian manifold, there is a unique metric compatible, torsion free connection on TM called the Levi-Civita connection. We are interested in the holonomy of this connection, the group of automorphisms induced by parallel transport.

Definition 3.30 We recall that a connection ∇ on a vector bundle $E \to M$ gives rise to parallel transport with respect to paths in M, in particular a path γ going from x to y gives rise to a linear map $T_\gamma \colon E_x \to E_y$. We define the *holonomy group* of (E, ∇) at a point $x \in M$ to be:

$$\mathrm{Hol}_x(\nabla) = \{T_\gamma \in \mathrm{GL}(E_x) |\, \gamma \text{ loop based at } x\}$$ ⋄

Remark 3.31 We note that ~~a~~ conjugating with the parallel transport of a path from x to y gives an isomorphism between $\mathrm{Hol}_x(\nabla)$ and $\mathrm{Hol}_y(\nabla)$.
So if E is of real rank r, we get that we can define $\mathrm{Hol}(\nabla) \subset \mathrm{GL}(r, \mathbb{R})$, well-defined up to conjugation. ◇

Doing this for the case of the Levi-Civita connection we come to the following definition:

Definition 3.32 We define *the holonomy of a Riemmanian manifold* (M, g), $\mathrm{Hol}(M, g)$, to be $\mathrm{Hol}(\nabla)$ for ∇ the Levi-Civita connection of (M, g) on TM. ◇

Remark 3.33 Since the Levi-Civita connection is compatible with the metric, parallel transport is an isometry, so $\mathrm{Hol}(M, g) \subset O(\dim(M))$. ◇

Note that if we have some structure (think of an orientation, a complex structure) on a single tangent space T_xM that is preserved by $\mathrm{Hol}(M, g)$, then we can extend it to the whole tangent space by parallel transport. So for instance if $\mathrm{Hol}(M, g)$ is contained in $\mathrm{SO}(n)$ then we can choose an orientation on a single tangent space and then extend it to the whole manifold, making it orientable.

Now if we disregard symmetric spaces, or spaces that are locally a product, there is not that much possibilities for holonomy groups. Here locally a product for a Riemannian manifold (M, g) means that around any point we find a neighbourhood U such that $(U, g|_U) \cong (M_1, g') \times (M_2, g'')$. We end up with the following list of 'building blocks'.

Theorem 3.34 (Berger's list) [2, Thm 3, p.318] Let (M, g) be a simply connected Riemannian manifold which is not locally a product and not a symmetric space. Then the holonomy group of (M, g) is ~~precisely on~~ the following:

$$\mathrm{SO}(n),\ \mathrm{U}(n),\ \mathrm{SU}(n),\ \mathrm{Sp}(n) \cdot \mathrm{Sp}(1),\ \mathrm{Sp}(n),\ G_2,\ \mathrm{Spin}(7)$$ ◇

A Riemmanian manifold as described above where the holonomy is not $\mathrm{SO}(n)$ (and hence one of the other 6) is called *of special holonomy*. It is often conjectured that all manifolds of special holonomy are formal, and we will discuss the story surrounding this.

We remark that some of these cases we already figured out. In particular, $\mathrm{U}(n)$, $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$ mean that the manifold is Kähler, so these are always formal. On the other hand $\mathrm{SO}(n)$ just means we have an oriented manifold. Since obviously not all manifolds are formal (see Example 1.33), we cannot expect all these elements on this list to give formal manifolds. So the first interesting example is $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, the quaternionic Kähler manifolds.

4 Formality of quaternionic Kähler manifolds

We will discuss the formality of quaternionic Kähler manifolds in two parts. First we will use the geometric information we get out of the holonomy to deduce a topological result: that a quaternionic Kähler manifold is the base of a fibration with nice total space and nice fiber. Then with this in mind we discuss the connection between formality of the base and the total space of a fibration with formal fiber.

4.1 Quaternionic Kähler geometry

We first describe the group $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4n)$. As the notation ~~predicts~~ it involves a combination of the action of $\mathrm{Sp}(1)$ and that of $\mathrm{Sp}(n)$ on $\mathbb{R}^{4n}$. Indeed $\mathrm{Sp}(1)$, seen as unit quaternions, acts on $\mathbb{R}^{4n}$ by seeing the latter as $\mathbb{H}^n$ in a canonical way and then using left scalar multiplication. On the other hand, we have a left action of $\mathrm{Sp}(n)$ on $\mathbb{R}^{4n}$, again by seeing the latter as $\mathbb{H}^n$ and also the former as a subset of $\mathrm{GL}(\mathbb{H}^n)$, the action becoming $A \cdot (v_1, ..., v_n) = (v_1, ..., v_n)\overline{A}^t$.
Combining these we get an action of $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ on $\mathbb{R}^{4n}$, namely:

$$(A, \lambda) \cdot (v_1, ..., v_n) = (\lambda v_1, ..., \lambda v_n)\overline{A}^t$$

The definition is then as follows.

Definition 4.1 The group $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4n)$ are all the automorphisms of $\mathbb{R}^{4n}$ of the form described above. $\diamond$

Remark 4.2 Of course $(-\mathrm{id}, -1)$ acts trivially on $\mathbb{R}^{4n}$ and this is really the only non-trivial relation in here. So as groups $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \cong (\mathrm{Sp}(n) \times \mathrm{Sp}(1))/\mathbb{Z}_2$. $\diamond$

Definition 4.3 A *quaternionic Kähler manifold* is a $4n$-dimensional oriented Riemmanian manifold with holonomy group contained in $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$. $\diamond$

We wish to pull geometrical information of M out of its holonomy group. To that end we can play the game of what linear structure a subgroup of $\mathrm{GL}(n)$ preserves. For instance $\mathrm{SL}(n)$ preserves orientation, $\mathrm{O}(n)$ preserves inner product, $\mathrm{GL}(n, \mathbb{C})$ preserves the complex structure, et cetera.

A non-standard example is $\mathrm{Sp}(n)$. Since elements of $\mathrm{Sp}(n)$ are in particular quaternionic-linear maps, they preserves the triple (I, J, K) of almost complex structures induced by the quaternionic structure of $\mathbb{R}^{4n}$. Now since we also throw $\mathrm{Sp}(1)$ in the mix, the action of $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ does not preserve I, J and K independently, but it does preserve the sphere $\{aI + bJ + cK | (a, b, c) \in S^2\}$ of metric-compatible complex structures of $\mathbb{R}^{4n}$. Indeed these are all metric-compatible complex strucutres of $\mathbb{R}^{4n}$

Now, if one considers a quaternionic Kähler manifold, and is given a sphere of compatible complex structures on one tangent space, using parallel transport, we can transport it to other points. Using the fact that the holomony is contained in $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ the conclusion is that the obtained sphere is independent of chosen path, hence we get a well-defined S^2-bundle over M consisting of the metric-compatible complex structures at each tangent space.

Definition 4.4 Let M be a quaternionic Kähler manifold. The space of metric-compatible fiberwise complex structures on the tangent space is called the *twistor space* and denoted by Z. By the above discussion this is a smooth S^2-bundle over M. ◇

This twistor space has a lot of nice properties. In particular it has a canonical almost complex structure, although M need not have one (i.e. $Z \to M$ need not have a section). To construct this ACS, we see Z as a subbundle of $\mathrm{Aut}(TM)$, on which we have an induced Levi-Civita connection. The action of the Levi-Civita connection and the bundle map $Z \to M$ give a splitting $T_J Z = T_x M \oplus T_z S^2$ in a uniform way (indeed using the the projection and the Levi-Citiva connected we can write TZ as the sum of the tangent bundle of the fiber and the one of the base). Then the TS^2-term inherits an complex structure from the fact that the fiber is a metric sphere, and S^2 being a complex manifold. On the other hand the point $J \in Z \cap \mathrm{Aut}(T_x M)$ is a complex structure on $T_x M$, and so $T_J Z$ carries a canonical complex structure, which is easily seen to vary smoothly with J. We conclude that the twistor space Z is an almost complex manifold. The important take-aways come from the work of Salamon [21]:

Theorem 4.5 [21, Thm 4.1, p.152] The almost complex structure on Z described above is integrable. ◇

Theorem 4.6 [21, Thm 6.1, p.158] Let M be a quaternionic Kähler manifold with strictly positive scalar curavture. Then then the twistor space Z is a Kähler manifold. ◇

So we see that M fits in a fibration $S^2 \hookrightarrow Z \to M$ with formal total space and formal fiber. So to prove formality of (positive curvature) quaternionic Kähler manifolds we will look at the interplay between formality of the base and formality of the total space for such a fibration.

4.2 Formality of fibrations

4.2.1 Elements of spectral sequences

When we want to discuss formality of a fibration $F \hookrightarrow E \to X$, we first ask if we can compare the cohomology of F, E and X. Here we run into technical issues. While the homotopy groups behave nicely with respect to fibrations (there we have a long exact sequence ~~for covers~~), and not nice with respect to covers (there is not something like Mayer-Vietoris for homotopy), for cohomology it is completely the other way around: there is a nice long exact sequence (the Mayer-Vietoris sequence), but for fibrations we need to look at spectral sequences. Since they are central in what is to come, we cannot ~~refrain from mentioning~~ them, but we will mention some crucial results. A more elaborate exposition can be found in the book my McLeary [19].

Definition 4.7 A *differential bigraded module* over a ring R is a collection of R-module $\{E^{p,q}\}$ $(p, q \in \mathbb{Z})$ together with an r linear mapping $d\colon E^{*,*} \to E^{*,*}$ of bidegree $(s, 1-s)$ or $(-s, s-1)$ such that $d \circ d$.

Given such a module, we define its cohomology by:

$$H^{p,q}(E^{*,*}, d) = \ker(d\colon E^{p,q} \to E^{*,*})/\mathrm{im}(d\colon E^{*,*} \to E^{p,q}) \qquad \diamond$$

Definition 4.8 A *spectral sequence (of cohomological type)* is a collection $\{E_r^{*,*}, d_r\}$ (for $r \geq 1$) of differential bigraded modules where d_r has bidegree $(r, 1-r)$ such that for all p, q, r we have $H^{p,q}(E_r^{*,*}, d_r) \cong E_{r+1}^{p,q}$. $\diamond$

Definition 4.9 We call $(E_r^{*,*}, d_r)$ the *r'th page* of the spectral sequence and say that a spectral sequence $\{E_r^{*,*}, d_r\}$ *collapses at the N'th page* if $d_r = 0$ for $r \geq N$. $\diamond$

Remark 4.10 There is a notion of limit $E_\infty^{*,*}$ of a spectral sequence, which we will not discuss here, but we do note that if the sequence collapses at the N'th page, the N'th page is the same as the limit. $\diamond$

Definition 4.11 We call a spectral sequence *of first quadrant* if $E_r^{p,q} \cong 0$ if $p < 0$ or $q < 0$. In this case $d_r\colon E_r^{-r,q+r-1} \to E^{0,q}$ is always 0, and so we get injections $E_{r+1}^{0,q} = \ker(d_r\colon E^{0,q} \to E^{r,q+1-r}) \hookrightarrow E_r^{0,q}$. Taking this to the limit, we get an injection $E_\infty^{0,*} \to E_2^{0,*}$ called the *edge homomorphism*. $\diamond$

The most general situation in which spectral sequences arise is when one has a filtration on a graded module.

Suppose that H^* is a graded R-module and F is an decreasing filtration on H^*. We define

$$E^{p,q}(H^*, F) = F^p H^{p+q} / F^{p+1} H^{p+q}$$

Definition 4.12 A spectral sequence $\{E_r^{*,*}, d_r\}$ is said to converge to H^* if there is a filtration F on H^* such that $E_\infty^{p,q} \cong E^{p,q}(H^*, F)$. $\diamond$

Remark 4.13 Note that when H^* is finite dimensional, this means that $H^s \cong \oplus_{p+q=s} E_\infty^{p,q}$. $\diamond$

Definition 4.14 A *filtered differential graded module* is an R-module A with a grading $A = \oplus_{n \geq 0} A^n$, a d is of degree ± 1 (i.e. $d(A^n) \subset A^{n \pm 1}$), $d \circ d = 0$, and a filtration F that is respected by d $(d(F^p A) \subset F^p A)$. $\diamond$

Remark 4.15 In this setting the cohomology $H(A, d)$ also inherits a filtration by $F^p H(A, d) = \mathrm{im}(H(F^p A, d) \to H(A, d))$. $\diamond$

The main point is then:

Theorem 4.16 [19, Thm 2.6, p.33] For any filtered differential graded module (A, d, F^*) there is a spectral sequence $\{E_r^{*,*}, d_r\}$ such that $E_1^{p,q} \cong H^{p+q}(F^pA/F^{p+1}A)$ and that converges to $H(A, d)$. ◇

Another important example is the Leray-Serre spectral sequence.

Theorem 4.17 [19, Thm 5.2, p.135] Let R be a commutative ring with unit. Suppose $F \hookrightarrow E \to B$ is a fibration where B is simply connected and F is connected. Then there is a spectral sequence $\{E_r^{*,*}, d_r\}$ converging to $H^*(E; R)$ such that

$$E_2^{p,q} \cong H^p(B; R) \otimes H^q(F; R)$$

Furthermore, the edge homomorphism $H^*(E) \to H^*(F)$ is just the pullback via the fiber inclusion. ◇

Remark 4.18 There is also a version when B is not simply-connected, but then the second page involves cohomology with something called local coefficients. ◇

So we see that the cohomology of the total space of a fibration is related to the product of $H^*(B)$ and $H^*(F)$, but only after taking an arbitrary amount of cohomologies.

So we see that the cohomology of fibrations is somewhat subtle, and this has consequences for formality. Since we know that for a formal space F the product $M \times F$ is formal if and only if M is formal, seeing a fibration as a 'twisted' product we may expect a somewhat similar result. However this is not true.

Example 4.19 Consider the algebra $M = \langle x, y, z \,||x| = |y| = 3, |z| = 5, dx = dy = 0, dz = xy\rangle$, which has a spatial realisation a non-trivial S^5-bundle over $S^3 \times S^3$. Similar to Example 1.33 we see that this algebra (and hence its spatial realisation) is not formal, while S^5 and $S^3 \times S^3$ are. ◇

The reason for failing is that the cohomology of this bundle is not interpretable well enough in terms of the cohomology of S^5 and $S^3 \times S^3$. In the example above the Leray Serre spectral sequence has a non-trivial differential on the 6th page, and this is precisely why the cohomology differs enough from the cohomology of the product. This should serve as a justification of why we want the spectral sequence to collapse at the second page. It isn't too hard to see that the spectral sequence collapses at the second page if and only if the edge map is surjective, i.e. if i^* is surjective.

Then if i^* is surjective, the following theorem will apply:

Theorem 4.20 (Leray-Hirsch) Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration such that $i^* \colon H(E) \to H(F)$ is surjective. Choose a section $s \colon H(F) \to H(E)$. Then the map $p^* \wedge s \colon H(B) \otimes H(F) \to H(E)$ is an isomorphism of graded vector spaces. ◇

Unfortunately $H(E)$ and $H(F) \otimes H(B)$ are generally not isomorphic as CDGA's. The reason is that, while $H(E)$ contains $H(B)$ as a subalgebra via p^*, the forms in the fiber-direction may multiply to something with horizontal direction. This means that we have to twist the algebraic relations of $H(F) \otimes H(B)$ for it to be isomorphic to $H(E)$ as CDGA's, and this complicates the situation.

A way to make sure that i^* is always surjective is to ask that $H(F)$ has no negative degree derivations. Indeed, any non-trivial differential at any page after the first gives such a derivation.
To see this, consider the first page on which there is a non-zero differential, say it is the r'th page. Then, since all the previous differentials are zero, we have $E^r_{p,q} = H^p(B) \otimes H^q(F)$ and we have $d_r^{p,q} \colon E^r_{p,q} \to E^r_{p+r,q-r+1}$, in particular $d_r^{0,q} \colon E^r_{0,q} \to E^r_{r,q-r+1}$ so we get a map $H^*(F) \to H^r(B) \otimes H^{*-r+1}(F)$ that behaves well with respect to products. Choosing a basis $\{b_1, ..., b_n\}$ for $H^r(B)$ we can now consider the map $H^*(F) \to H^{*-r+1}(F)$ which sends $x \in H^*(F)$ to $d_r(x) \in H^r(B) \otimes H^{*-r+1}(F)$ and then picks the coefficient in front of b_1 to land in $H^{*-r+1}(F)$. It is not hard to see that this is a derivation of $H^*(F)$, necessarily of degree $-r+1$. So we see that a non-zero differential on any page starting from the second induces a negative degree derivation.

So we will restrict to fibers for which there are no negative degree derivations in cohomology. For some technical purposes in what follows we mold this in the following way

Definition 4.21 We call a topological space X a F_0-*space* if it has finite dimensional rational cohomology and rational homotopy and positive Euler characteristic. ⋄

These spaces look like good candidates to work with, indeed we have the following conjecture which has been checked for large classes of F_0-spaces.

Conjecture 4.22 (Halperin) If F is a F_0-space then $H(F)$ has no negative degree derivations. ⋄

In what comes we will call a space *Halperin* if it is F_0 and satisfies the Halperin conjecture (i.e. the Halperin conjecture asks for any F_0-space to be Halperin).

The upshot of all this is, that if we have a fibration with a Halperin fiber (and simply connected base), the spectral sequence will collapse after the second page, which means that the cohomology of the total space at least as a graded vector space looks like the tensor product of the cohomology of fiber and base. As an algebra they may differ, but we only get some extra relations for the forms in the fiber direction, and the theory we'll develop will make sure that we can deal with this effectively.

In this section we will study the work of Amman and Kapovitch [1], who, under these assumptions on the fibers, solved the relation between fibrations and formality:

Theorem 4.23 [1, Thm A, p.2051] Let $F \hookrightarrow E \to B$ be a fibration of simply-connected spaces of finite type. Suppose that F is formal and Halperin. Then E is formal if and only if B is formal. ◇

We will use the obstruction theory with bigraded models, multiplicative resolutions and filtered models developed in Section 1.6, which we will extend a little bit in the context of fibrations.

4.2.2 Relative filtered models

In Section 1.6 we showed a way to determine formality of algebras via bigraded and filtered models. What we now want is that given the map $\Omega(B) \to \Omega(E)$ coming from the fibration, we want to use the filtered model of $\Omega(B)$ to construct a model of $\Omega(E)$ on which to do obstruction theory. One may wish that this model is a filtered model with respect to the bigraded model of $H(E)$, but this will not work in general (in fact, it rarely works if the fibration is not trivial). Luckily we can construct it in such a way that it is a filtered model with respect to a multiplicative resolution of $H(E)$, and so we can use the obstruction theory Corollary 1.50 ~~(in fact we introduced this slightly more general statement purely for the fact that it is needed here).~~

So we introduce the relative version of the bigraded and filtered model. Consider a morphism $\phi\colon (H,0) \to (H',0)$ of CDGA's. Let $\rho\colon (\Lambda Z, d) \to (H,0)$ be a bigraded model and let $(\Lambda Z, d) \overset{i}{\hookrightarrow} (\Lambda Z \otimes \Lambda X, d') \overset{p}{\to} (\Lambda X, d'')$ be a minimal relative Sullivan model for $\phi \circ \rho$. Then we are in the situation:

$$\begin{array}{ccccc} (H,0) & \xrightarrow{\phi} & (H',0) & & \\ \uparrow{\scriptstyle \rho} & \nearrow{\scriptstyle \phi\circ\rho} & \uparrow{\scriptstyle \rho'} & & \\ (\Lambda Z, d) & \overset{i}{\hookrightarrow} & (\Lambda Z \otimes \Lambda X, d') & \xrightarrow{p} & (\Lambda X, d'') \end{array}$$

where ρ and ρ' are both quasi-isomorphisms. Similarly to how we put extra structure on the minimal model to make a bigraded model, we put extra structure on a minimal relative model to make a relative bigraded model:

Theorem 4.24 [24, Thm 2.1.3, p.455] The algebra $(\Lambda Y, d')\colon = (\Lambda Z \otimes \Lambda X, d')$ above can be chosen in such a way that we have a lower grading on X and Y such that:

a) It holds that $Z = \oplus_{i\geq 0} Z_i$, $X = \oplus_{i\geq 0} X_i$ and $Y_i = X_i \oplus Z_i$.

b) The differential d' is homogeneous of degree -1 with respect to the lower grading.

c) All maps in the diagram above preserve the lower grading. Here we use the convention that unless otherwise stated CDGA's with trivial differential have the trivial lower grading, where everything has lower degree 0.

d) The morphism $\rho'\colon (\Lambda Y, d') \to (H', 0)$ is a multiplicative resolution. ◇

Proof. Again for simplicity we restrict to the case where $H^1 = H'^1 = 0$ and $\phi|_{H^2}$ is injective. Then we can simply use the construction of the minimal Sullivan model as in the proof of

Proposition 1.32. First note that the grading of the Z-part is simply the grading of the bigraded model.

Then, to put a grading on the X part, we mimic the construction of the bigraded model. In partciular we look at the spaces $V_{(n)}$ described in the proof of **??**, and put the lower grading on there. All the added cokernels will get lower grading 0 (since they generate cohomology), while the kernels that we add will get a bigrading in the way like with the bigraded model (where d and the lower grading of the inductive step induce a filtration, which by choosing bases gives a grading).

Choosing the right sections such that ρ' becomes a multiplicative resolution is completely analogous to the construction of the bigraded model. □

Then for (almost) any CDGA-map, we can take the relative bigraded model of the map on cohomology and tweak it in a systematic manner to create a model of the original map.

Theorem 4.25 [24, Thm 2.2.4, p.457] Let $\alpha\colon (A, d) \to (A', d')$ be a map of CDGA's such that $H^1(\alpha)$ is injective. Let

$$\begin{array}{ccccc} (H(A,\delta),0) & \xrightarrow{\alpha^*} & (H(A',\delta'),0) & & \\ \uparrow{\scriptstyle\rho} & \nearrow{\scriptstyle\alpha^*\circ\rho} & \uparrow{\scriptstyle\rho'} & & \\ (\Lambda Z, d) & \xrightarrow{i} & (\Lambda Z\otimes\Lambda X, d') & \xrightarrow{p} & (\Lambda X, d'') \end{array}$$

be a relative bigraded model of α^*. Let $\pi\colon (\Lambda Z, D) \to (A, d)$ be a filtered model of (A, d) with respect to ρ. Then $\alpha\circ\pi\colon (\Lambda Z, D) \to (A', d')$ admits a minimal relative model

$$\begin{array}{ccccc} (A,\delta) & \xrightarrow{\alpha} & (A',\delta') & & \\ \uparrow{\scriptstyle\pi} & \nearrow{\scriptstyle\alpha\circ\pi} & \uparrow{\scriptstyle\pi'} & & \\ (\Lambda Z, D) & \xrightarrow{i} & (\Lambda Z\otimes\Lambda X, D') & \xrightarrow{p} & (\Lambda X, D'') \end{array}$$

where $D' - d'$ decreases the lower grading by at least 2, i.e. $D' - d'\colon (\Lambda Z\otimes\Lambda X)_p \to (\Lambda Z\otimes\Lambda X)_{\leq p-2}$.
Then $(\Lambda Z, D) \overset{i}{\hookrightarrow} (\Lambda Z\otimes\Lambda X, D') \overset{p}{\twoheadrightarrow} (\Lambda X, D'')$ is called the *relative filtered model of* α. ◇

Proof. We need to construct π' and D' on Z and X. On Z we just take $D' = D$ and $\pi' = \alpha\circ\pi$. Then on X the construction is entirely analogue to the proof of Theorem 1.46. □

Now we consistenly have the fiber ΛX getting dragged around, and we want to apply the construction to the case $\Omega(B) \to \Omega(E)$, which also has a part $\Omega(E) \to \Omega(F)$ coming with it. The question then is whether we can connect ΛX to F. Since we have a Halperin fiber F we have some control over the spectral sequences that are involved, and so in this case it indeed works.

Theorem 4.26 Let $F \hookrightarrow E \xrightarrow{f} B$ be a fibration of path-connected spaces where B is simply-connected and F is Halperin such that $H^*(F)$ has finite type. Let $f_{\mathbb{Q}} \colon M_B \to M_E$ be the induced map of minimal models. Then $f_{\mathbb{Q}}$ admits a relative filtered model $(\Lambda Z, D) \overset{i}{\hookrightarrow} (\Lambda Z \otimes \Lambda X, D') \xrightarrow{p} (\Lambda X, D'')$ such that $(\Lambda X, D'')$ is a filtered model of M_F.
In particular $D''_1 = d''$ where d'' is the fiber differential in the bigraded model of $f^* \colon H^*(B) \to H^*(E)$ and D''_1 denotes the part of D'' decreasing the lower degree by 1. ◇

The proof relies on arguments using the spectral sequence of a double complex coming out of $\Lambda Z \otimes \Lambda X$. Since for the scope of this thesis we wish to not delve too deep into the theory of spectral sequences, we leave this theorem as a black box.

4.2.3 S^{2n}-bundles

The theory simplifies significantly when we consider S^{2n}-bundles (S^{2n} being the easiest example of a Halperin space), which is after all the case we're interested in for $n = 1$.

Since $H(S^{2n})$ has no negative degree derivations, it follows that for every fibration $S^{2n} \xrightarrow{i} E \xrightarrow{p} B$ the pullback $i^* \colon H(E) \to H(S^{2n})$ is surjective. Hence we can use Leray-Hirsch.

Recall that Leray-Hirsch implied that $H(E)$ is isomorphic as a graded vector space to $H(B) \otimes H(F)$, but as CDGA's we can only get as far as saying that $H(B)$ is a subalgebra. This meant that to get an isomorphism of CDGA's, the algebraic relations concerning $H(F)$ have to be twisted.
However, when we deal with S^{2n} we know that $H(S^{2n})$ has only one generator x in degree $2n$, and we only need to determine the square of this generator as an element of $H(E)$. Since $H(E) \cong H(B) \otimes H(S^{2n})$ as graded vector space we find $a_{2n} \in H^{2n}(B)$ and $a_{4n} \in H^{4n}(B)$ such that $x^2 = a_{2n}x + a_{4n}$. So we see that as CDGA's we get:

$$H(E) \cong H(B)[x]/(x^2 - a_{2n}x - a_{4n})$$

Under the isomorphism, the map p^* corresponds to the inclusion of $H(B)$ into $H(B)[x]/(x^2 - a_{2n}x - a_{4n})$ while the map i^* the projection $H(B)[x]/(x^2 - a_{2n}x - a_{4n}) \to \mathbb{R}[x]/(x^2)$ sending $H^{>0}(B)$ to 0 and x to x.

Now consider the bigraded model $\phi \colon (\Lambda X, d) \to H(B)$ of $H(B)$. To calculate the relative bigraded model of p^*, we really need to find the relative bigraded model of $H(B) \hookrightarrow H(B)[x]/(x^2 - a_{2n}x - a_{4n})$, in particular we need to find a commutative diagram:

$$\begin{array}{ccc} H(B) & \hookrightarrow & H(B)[x]/(x^2 - a_{2n}x - a_{4n}) \\ \uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \phi'} \\ (\Lambda X, d) & \hookrightarrow & (\Lambda X \otimes \Lambda Z, d') \end{array}$$

where ϕ' is a quasi-isomorphism and Z having a bigrading such that **d)** of Theorem 4.24 holds. It suffices to construct Z in this case, since $\phi'|_{\Lambda X}$ is already fixed. The clue is that because we have a very explicit expression of $H(E)$, it is easy to see what we 'miss'.

In particular we get $Z = \operatorname{span}(y, z)$ with y in degree ${}^{2}_{0}$ (i.e. higher degree 2 and lower degree 0) and z in degree ${}^{3}_{1}$, with $d'(y) = 0$, $\phi'(y) = x$ and $\phi'(z) = 0$. We want to use z to eliminate the non-zero product representing $x^2 - a_{2n}x - a_{4n}$. For this find α_{2n} and α_{4n} in ΛX such that $a_i = \phi(\alpha_i)$ and then set $d'(z) = y^2 - \alpha_2 y - \alpha_4$.

Note that we can project down to $(\Lambda Z, d'')$ to get precisely the algebra of the bigraded model of $H(S^{2n})$ even in a fashion that the following diagram is commutative

$$\begin{array}{ccccc} H(B) & \hookrightarrow & H(B)[x]/(x^2 - a_{2n}x - a_{4n}) & \longrightarrow & \mathbb{R}[x]/(x^2) \\ \uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \phi'} & & \uparrow{\scriptstyle \phi''} \\ (\Lambda X, d) & \hookrightarrow & (\Lambda X \otimes \Lambda Z, d') & \longrightarrow & (\Lambda Z, d'') \end{array}$$

where ϕ'' is the actual quasi-isomorphism of the bigraded model of $H(S^{2n})$.

Now we step up to the relative filtered model of $p^* \colon \Omega(B) \to \Omega(E)$. For this we start with a filtered model $\pi \colon (\Lambda X, D) \to \Omega(B)$ and we want to find D' and $\pi' \colon (\Lambda X \otimes \Lambda Z, D') \to \Omega(E)$ such that $D' - d'$ decreases the lower degree by at least two and that π' is a quasi-isomorphism such that the following diagram commutes

$$\begin{array}{ccc} \Omega(B) & \xrightarrow{p^*} & \Omega(E) \\ \uparrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \pi'} \\ (\Lambda X, D) & \hookrightarrow & (\Lambda X \otimes \Lambda Z, D') \end{array}$$

It is clear that we need to define $D'|_{\Lambda X} = D$ and $\pi'|_{\Lambda X} = p^* \circ \pi$. Then when we define D' on ΛZ we note that since $Z_{\geq 2} = 0$ for degree reasons $D' = d'$ on Z and hence on ΛZ. In order to define π' on ΛZ we fix a representative $\beta \in \Omega(E)$ for $x \in H(E)$. The previous remarks imply that we have $[\beta^2 - p^*\pi\alpha_{2n} \wedge \beta - p^*\pi\alpha_{4n}] = 0$ so we find $\gamma \in \Omega^{4n-1}(E)$ such that that $\beta^2 - p^*\pi\alpha_{2n} \wedge \beta - p^*\pi\alpha_{4n} = d\gamma$. Then we set $\pi'(y) = \beta$ and $\pi'(z) = \gamma$. Then it is immediate to check that on cohomology π' is just like ϕ', so it follows that π' is a quasi-isomorphism.

Then, since $i^*(\beta)$ is a representative for $x \in H(S^{2n})$, and $i^*(\gamma) = 0$ by dimension reasons, we have that the following diagram commutes

$$\begin{array}{ccccc} \Omega(B) & \xrightarrow{p^*} & \Omega(E) & \xrightarrow{i^*} & \Omega(S^{2n}) \\ \uparrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \pi'} & & \uparrow{\scriptstyle \pi''} \\ (\Lambda X, D) & \hookrightarrow & (\Lambda X \otimes \Lambda Z, D') & \longrightarrow & (\Lambda Z, d'') \end{array}$$

where π'' is the minimal model of $\Omega(S^{2n})$ sending y to $i^*(\beta)$ and z to 0. So we see that when we deal with an S^{2n}-bundle we can find very explicit constructions to end up in the situation of Theorem 4.26. In particular, we find a model of the total space which acts reasonably like the product, something that the minimal model will not do in general. In this way we can use rough generalizations of the statement that the product of formal manifolds is formal, to prove Theorem 4.23 in this case.

Indeed we write $D = d + d_2 + \cdots$ and $D' = d' + d_2' + \cdots$. Then since $D' = d'$ on ΛZ we see that $d_i' = d_i \otimes 1$. So we see that $d_j' = 0$ for $j < i$ if and only if $d_j = 0$ for $j < i$.

Now suppose that B is formal and $d_j' = 0$ for $j < i$. Then $d_j = 0$ for $j < i$ and hence d_i is exact, say $d_i = \mathcal{D}(\theta)$, then it is immediate that $d_i' = \mathcal{D}(\theta \otimes 1)$, hence d_i' is exact. And thus by Theorem 1.48 it follows that E is formal.

On the other hand if E is formal and $d_j = 0$ for $j < i$ then we see that $d_i' = \mathcal{D}(\theta')$. It is then not so hard to see that $d_i = \mathcal{D}(\theta'|_{\Lambda X \otimes 1})$, and hence d_i is exact and again by Theorem 1.48 we conclude that B is formal.

4.2.4 Proof of Theorem 4.23

We now discuss the proof in the general case.

Proof of Theorem 4.23. Let $F \hookrightarrow E \xrightarrow{f} B$ be a Serre fibration where E, F and B are simply-connected and of finite type. Suppose further that $H^*(F)$ is finite dimensional, E and F are formal and that F is Halperin.

Let $f^*\colon H(B) \to H(E)$ be the induced map on cohomology and let

$$\begin{array}{ccccc} (H(B),0) & \xrightarrow{f^*} & (H(E),0) & & \\ \uparrow\rho & \nearrow^{f^*\circ\rho} & \uparrow\rho' & & \\ (\Lambda Z, d) & \xrightarrow{i} & (\Lambda Z \otimes \Lambda X, d') & \xrightarrow{p} & (\Lambda X, d'') \end{array}$$

be a relative bigraded model. Let M_B and M_E be the minimal models of B and E and let $m_f\colon M_B \to M_E$ be a Sullivan representative of f. Let

$$\begin{array}{ccccc} M_B & \xrightarrow{m_f} & M_E & & \\ \uparrow\pi & \nearrow^{m_f\circ\pi} & \uparrow\pi' & & \\ (\Lambda Z, D) & \xrightarrow{i} & (\Lambda Z \otimes \Lambda X, D') & \xrightarrow{p} & (\Lambda X, D'') \end{array}$$

be a filtered model of m_f.

Decompose $D = d + d_2 + d_3 + \cdots$ and $D' = d' + d_2' + d_3' + \cdots$. Then since $(\Lambda Z, D) \overset{i}{\hookrightarrow} (\Lambda Z \otimes \Lambda X, D') \xrightarrow{p} (\Lambda X, D'')$ is a relative Sullivan algebra, we have $d_i'|_{\Lambda Z \otimes 1} = d_i$.

Since F is Halperin, Theorem 4.26 implies that $(\Lambda X, d'')$ is a bigraded model of $H(F)$ and

$(\Lambda X, D'')$ is a filtered model of F. ~~Since F is on top of that~~ also formal, it follows by [6] that $X_i = 0$ for $i \geq 2$ (which is clear for the sphere S^2, which is still the case we're most interested in), and hence for degree reasons $(D' - d')|_{1\otimes\Lambda X} = 0$ and hence $D'' = d''$.

We now want to connect the obstruction classes $d_i \in \mathrm{Der}^1_i(\Lambda Z, d)$ of E and $d'_i \in \mathrm{Der}^1_i(\Lambda Z \otimes \Lambda X, d)$ of B. The most direct way to connect these is via the space $\mathrm{Der}^1_i(\Lambda Z, \Lambda Z \otimes \Lambda X)$, the set of derivations from ΛZ to $\Lambda Z \otimes \Lambda X$, which makes sense since ΛZ acts on $\Lambda Z \otimes \Lambda X$. Composing with the inclusion $\Lambda Z \hookrightarrow \Lambda Z \otimes \Lambda X$ gives a map $j\colon \mathrm{Der}^1_i(\Lambda Z, \Lambda Z) \to \mathrm{Der}^1_i(\Lambda Z, \Lambda Z \otimes \Lambda X)$, while restricting from $\Lambda Z \otimes \Lambda X$ to $\Lambda Z \otimes 1 = \Lambda Z$ gives a map $j'\colon \mathrm{Der}^1_i(\Lambda Z \otimes \Lambda X, \Lambda Z \otimes \Lambda X) \to \mathrm{Der}^1_i(\Lambda Z, \Lambda Z \otimes \Lambda X)$. The last two ~~alineas~~ show that $j'(d'_i) = j(d_i)$ for every i. Furthermore it shows that if d_j for $j < i$, then $d'_j = 0$ for $j < i$.

Just like with 'ordinary' derivations, the space $\mathrm{Der}(\Lambda Z, \Lambda Z \otimes \Lambda X)$ has a differential given by $\mathcal{D}(\theta) = d' \circ \theta - (-1)^k \theta \circ d$ where θ is of degree k. The morphisms j respectively j' commute with the differential (since $(\Lambda Z \otimes \Lambda X, d')$ is a relative Sullivan algebra), and hence induce maps on cohomology.

Combining all, if $d_j = 0$ for $j < i$ we know that $d'_j = 0$ for $j < i$. Then d'_i is closed and since E is formal, it is exact. Also $j'(d'_i) = j(d_i)$ is exact, so the only thing left to consider is whether j is injective on cohomology. This is the body of the next lemma, which we will discuss next, and finishes the proof. □

Lemma 4.27 [1, Lem 3.2, p.2063] In the situation above, the map $j^*\colon H^*(\mathrm{Der}(\Lambda Z, \Lambda Z)) \to H^*(\mathrm{Der}(\Lambda Z, \Lambda Z \otimes \Lambda X))$ is injective. ◇

Proof. Consider the filtration on $\mathrm{Der}(\Lambda Z, \Lambda Z \otimes \Lambda X)$ where $F^p(\mathrm{Der}(\Lambda Z, \Lambda Z \otimes \Lambda X))$ consists of derivations which increase the ΛZ-degree by at least p. The filtration is invariant under the differential $\mathcal{D}$. We consider the spectral sequence $E^{*,*}_*$ induced by this filtration.

We consider the first few pages of the sequence. We have $E_0 = \mathrm{Der}(\Lambda Z, \Lambda Z \otimes \Lambda X) \cong \mathrm{Der}(\Lambda Z, \Lambda Z) \otimes \Lambda X$ (where the p-index is on the first term, and the q-index is on the second), with differential $d_0 = d''$. It follows that $E_1 = \mathrm{Der}(\Lambda Z, \Lambda Z) \otimes H(\Lambda X, d'')$. Since ΛX is a model for F, we have $H(F) \cong H(\Lambda X, d'')$ and since F is of finite type, the latter is finite dimensional. In particular the first page is bounded, so the sequence converges in a finite amount of steps. Then d_1 works on the $\mathrm{Der}(\Lambda Z, \Lambda Z)$-part, namely via the differential of the latter cochain complex, and we get $E_2^{p,q} = H^p(\mathrm{Der}(\Lambda Z, \Lambda Z)) \otimes H^q(\Lambda X, d'')$.

Then supposing that $d_r \neq 0$ for some $r \geq 2$ we can construct negative degree derivations on $H(\Lambda X, d'') \cong H(F)$. The fact that F is Halperin in particular implies that these derivations don't exist, so we conclude that $d_r = 0$ for $r \geq 2$, and so the spectral sequence collapses after the second page, which shows that the edge homomorphism is injective.

Similar to the Serre spectral sequence of the fibration, carefully looking at all the surjections of which the edge homomorphism is made out of, it follows that j^* is the edge homomorphism. Again, since we do not wish to spend too much time and work on spectral sequence, we leave this as a black box. In either case, we conclude that j^* is injective. $\square$

5 Formality and Mayer-Vietoris

5.1 Examples of G_2-manifolds

Now that we know that all quaternionic Kähler manifolds are formal, we continue on Berger's list to manifolds with holonomy G_2. Together with Spin(7) this is one of the exceptional cases, not coming in a family ranging over different dimensions, but only dealing with the 7-dimensional case.

Definition 5.1 Consider the following 3-form ϕ_0 on $\mathbb{R}^7$

$$\phi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}$$

where we write dx_{ijk} for $dx_i \wedge dx_j \wedge dx_k$. We define G_2 to be the subgroup of $\mathrm{GL}(7, \mathbb{R})$ which preserves ϕ_0. Then G_2 is a compact, finite dimensional, simply connected and 14-dimensional Lie group. Furthermore G_2 preserves the Euclidean metric and the orientation, so $G_2 \subset \mathrm{SO}(7)$. $\diamond$

Remark 5.2 We are inclined to reserve the name G_2-manifolds for manifolds with a metric with holonomy equal or contained in G_2. In literature this name is usually reserved for a manifold M together with a 3-form and a metric that together induce a torsion free G_2-structure on M (in the context of geometric structures). However (c.f. [13, Prop 2.6.5, p.40]) M being a G_2-manifold, i.e. admitting a G_2-structure like this, is equivalent to M admitting a torsion-free connection that has holonomy contained in G_2. Then since G_2 is a subgroup of $O(7)$ any such connection is a Levi-Civita connection of a metric (c.f. [13, Prop 3.1.5, p.45]). We conclude that M being a G_2-manifold as used in the literature is equivalent to M having a metric with holonomy contained in G_2.

We do however divert a bit from this notion, and we reserve the term G_2-manifold for a 7-manifold with a metric that has holonomy **equal to** G_2. $\diamond$

Unlike for Kähler manifolds, or quaternionic Kähler manifolds, there are no known nice topological or geometrical tools to determine formality of G_2-manifolds (as compared with the dd^c-Lemma or the twistor fiber bundle). Indeed it is still an open question whether manifolds with holonomy G_2 are formal. What is known about their topology and what will be of importance for our exposition is best summarized in the following:

Theorem 5.3 [13, Thm 10.2.8, p.247] Suppose M is a G_2-manifold. Then M is orientable and $\pi_1(M)$ is finite. $\diamond$

In particular we see that $H^1(M; \mathbb{R}) = 0$ and M satisfies Poincaré duality, so for all our intents and purposes we can consider a G_2-manifold to be simply connected (because we do not care about torsion since we work over $\mathbb{Q}$ or $\mathbb{R}$).

The facts that G_2-manifolds have 1 dimension too many for Miller (Theorem 1.42) to work, and that we don't know enough about their topology or geometry, we cannot do much more than looking at examples that we know.

Luckily there has been a lot of fruitful work in finding examples of G_2-manifolds, mostly done by Donaldson, Joyce [13][14], Kovalev [16], Kovalev-Lee [17] and Corti-Haskins-Nordström-Pacini [3].

We will not go into the details of the constructions as this would be outside of the intended scope of this thesis, but we will give an outline, which will serve as a justification for the things we will discuss in this final chapter.

The common idea is that one takes two so called Calabi-Yau manifolds (6-dimesional Kähler manifolds with certain desirable properties), take the product with S^1, and find in both submanifolds of the form $\mathbb{R}_{>0} \times S^1 \times S^1 \times K3$, along which both Calabi-Yau manifolds are glued.

We know that Kähler manifolds are formal, and so is S^1 (if we allow ourselves outside the realm of simply connected manifolds for the moment). On the other hand a $K3$-surface is formal (for once because it is Kähler, but it is also simply connected of dimension 4), as is $\mathbb{R}_{>0}$ because it is contractible. Since products of formal manifolds are formal, we see that we find G_2-manifolds made up of 2 parts which are formal, glued along a formal intersection.

So we set out to try to prove formality of spaces made out of two formal parts.

5.2 s-formality

As we saw back in Theorem 1.42, Poincaré duality can take us a long way in determining formality. From the discussion there we got an algebraic characterization of formality (Theorem 1.44), where we find that an algebra is formal if its minimal model ΛV splits as $V = C \oplus N$ with $d(C) = 0$, $d|_N$ injective and whenever an element of $N \cdot \Lambda V$ is closed it is exact. The result easily generalizes to a notion which weakens formality to some low dimensional problem. This notion is due to Fernández and Muñoz [9].

Definition 5.4 A CDGA is called *s-formal* if it admits a minimal model $(\Lambda V, d)$ for which the following holds: for $i \leq s$ the space V^i splits as $V^i = C^i \oplus N^i$ where

a) $d(C^i) = 0$

b) $d\colon N^i \to (\Lambda V)^{i+1}$ is injective

c) Whenever an element of $N^{\leq s} \cdot \Lambda(V^{\leq s})$ is closed in ΛV then it is exact.

Accordingly we call a manifold M *s-formal* if $\Omega(M)$ is s-formal. ◇

Remark 5.5 We should note that s-formality does not imply that $\Lambda(V^{\leq s})$ is formal, which would imply that we always find a primitive in $\Lambda(V^{\leq s})$. ◇

Remark 5.6 For obvious reasons if $H^i(A) = 0$ for $i \geq s$ then A is formal if and only if it is s-formal. If A is also simply connected, we may even lower this to $(s-1)$-formality. This in particular implies to manifolds of finite dimension. ◇

Remark 5.7 Also for obvious reasons, if an algebra is s-formal ($s \in \mathbb{N} \cup \{\infty\}$), then it is also s'-formal for all $s' \leq s$, where by ∞-formality we just mean formality. ◇

Similarly to proving Miller's theorem (Theorem 1.42) using Poincaré-duality, we use Poincaré-duality to show that philosophically speaking 'everything that could go wrong, goes wrong in the first half of dimensions'. Miller's theorem will turn out be a simple consequence of this.

Theorem 5.8 [9, Thm 3.1, p.8] Let A be a connected CDGA such that $H(A)$ is a Poincaré duality algebra of dimension $D \leq 2n$ that is of finite type, then A is formal if and only if it is $(n-1)$-formal. ◇

Proof. By the previous remarks, one side is completely obivious, so we show that $(n-1)$-formality will imply formality. To do this we show that A is $(n+r-1)$-formal for all $r \geq 0$ by induction on r. This will in particular show that A is D-formal, which, again by previous remarks, will show that A is formal. The induction base is clear, since we assume A to be $(n-1)$-formal.

Now suppose that A is $(n+r-1)$-formal, we show that A is $(n+r)$-formal. Consider a minimal model $(\Lambda V, d)$ that is $(n+r-1)$-formal. Since $(\Lambda V, d)$ is minimal, we can find a basis $\{x_1, x_2, ...\}$ of V^{n+r} such that $dx_j \in \Lambda(V^{\leq n+r-1} \oplus \langle x_1, ..., x_{j-1}\rangle)$ (in the simply connected case we can just take any ordered basis). We set V_i by

$$V_i = V^{\leq n+r-1} \oplus \operatorname{span}(x_1, ..., x_i)$$

for $i \geq 1$ and we set $V_0 = V^{\leq n+r-1}$. Note that the V_i's are nested and the union of them is $V^{\leq n+r}$.

We wish to rearrange $V^{\leq n+r}$ such that $(\Lambda V, d)$ becomes $(n+r)$-formal. To do this we will find for every $i \geq 1$ a $\psi_i \in \Lambda V_{i-1}$ such that $\widehat{x}_i = x_i - \psi_i$ gives a new set of generators such that the space

$$\widehat{V}_i = V^{\leq n+r-1} \oplus \operatorname{span}(\widehat{x}_1, ..., \widehat{x}_i)$$

satisfies the properties of $(n+r)$-formality in that it splits as $\widehat{V}_i = \widehat{C}_i \oplus \widehat{N}_i$ with $d(\widehat{C}_i)$ closed, $d\colon \widehat{N}_i \to \Lambda V$ injective and the space $\widehat{N}_i \cdot \Lambda\widehat{V}_i \subset \Lambda V$ satisfies that every closed element is exact. Note that $\Lambda\widehat{V}_i = \Lambda V_i$. Doing this inductively (with $i = 0$ as trivial induction base), we conclude that we can deform ΛV into a $(n+r)$-formal minimal model (since V^{n+r} is finite dimensional).

Now that the strategy is clear, we start by looking at the map

$$V^{n+r} \xrightarrow{d} \Lambda V^{\leq n+r} \xrightarrow{q} \frac{\Lambda V^{\leq n+r}}{d(\Lambda V^{\leq n+r-1})}$$

Without loss of generality we may assume $\{x_1, ..., x_p\}$ to be a basis of the kernel of the above map.

Let $1 \leq i \leq p$ and assume that we have fixed $\widehat{V}_{i-1}$. For x_i it follows that dx_i lies in $d(\Lambda V^{\leq n+r-1})$, i.e. we find ψ_i such that $dx_i = d\psi_i \in \Lambda V^{\leq n+r-1}$. Set $\widehat{x}_i = x_i - \psi_i$ such that $d\widehat{x}_i = 0$. We put

$$\widehat{C}_i = \widehat{C}_{i-1} \oplus \mathrm{span}(\widehat{x}_i) = C^{\leq n+r-1} \oplus \mathrm{span}(\widehat{x}_1, ..., \widehat{x}_i)$$

and

$$\widehat{N}_i = \widehat{N}_{i-1} = N^{\leq n+r-1}$$

We check that any closed element of $\widehat{N}_i \cdot \Lambda \widehat{V}_i$ is exact in ΛV. So consider an element $\eta \in \widehat{N}_i \cdot \Lambda \widehat{V}_i$. By the definition of $\widehat{C}_i$ and $\widehat{N}_i$ we can write

$$\eta = \eta_0 + \eta_1 \widehat{x}_i + \cdots + \eta_k \widehat{x}_i^k$$

with $\eta_i \in \widehat{N}_{i-1} \cdot \Lambda \widehat{V}_{i-1}$. If we assume that η is closed, by closedness of $\widehat{x}_i$ we get:

$$d\eta_0 + d\eta_1 \cdot \widehat{x}_i + \cdots + d\eta_k \cdot \widehat{x}_i^k = 0$$

hence all the η_i's are closed. By induction hypothesis on $\widehat{V}_{i-1}$ we see that all the η_i's are exact, and then again by closedness of $\widehat{x}_i$ we see that η is exact.

Now let $i > p$. In this case we set $\widehat{C}_i = \widehat{C}_{i-1}$. We want to find an element $\psi_i \in \Lambda \widehat{V}_{i-1}$ such that putting $\widehat{x}_i = x_i - \psi_i$ and $\widehat{N}_i = \widehat{N}_{i-1} \oplus \mathrm{span}(\widehat{x}_i)$, the decomposition $\widehat{V}_i = \widehat{C}_i \oplus \widehat{N}_i$ has the right properties. Note that the choice of ψ_i does not influence injectivity of d on $\widehat{N}_i$. This follows from the fact that $q \circ d$ from above is injective on $\mathrm{span}(x_{p+1}, ..., x_i)$ and that $\widehat{x}_j = x_j - \psi_j$ with $\psi_j \in \Lambda \widehat{V}_{j-1}$ for $j \leq i$.

For the time being set $N_i = \widehat{N}_{i-1} \oplus \mathrm{span}(x_i)$ and consider $\eta \in N_i \cdot \Lambda(\widehat{V}_{i-1} \oplus \mathrm{span}(x_i))$ a closed element. As before we can write $\eta = \eta_0 + \eta_1 x_i + \cdots + \eta_k x_i^k$. We distinguish three cases:

Case (1) $k = 0$. In this case $\eta = \eta_0$ with η_0 a closed element of $\widehat{N}_{i-1} \cdot \Lambda \widehat{V}_{i-1}$. By the induction hypothesis it follows that η_0 and hence η is exact.

Case (2) $k \geq 2$. Note that by graded commutativity this means that the degree of x_i is even. Since $\deg x_i \geq n$ we have $\deg \eta \geq 2n$. If either $k > 2$ or $k = 2$ and $\deg x_i > n$ we have $\deg \eta > 2n \geq D$ so since $H^{>D}(\Lambda V) = 0$ it follows that η is exact. The remaining case is that $k = 2$ and $\deg x_i = n$, and we may assume that $\deg \eta = 2n$ and $2n = D$ (since otherwise by the same argument as above it is exact). In this case η_2 comes from $(\Lambda \widehat{V}_{i-1})^0$ which is just the ground field and we write $\eta = \eta_0 + \eta_1 x_i + \lambda x_i^2$ with λ a non-zero scalar.

Now we get $0 = d\eta = (d\eta_0 + \eta_1 \cdot dx_i) + (d\eta_1 + 2\lambda dx_i)x_i$, so $d\eta_1 + 2\lambda dx_i$ is closed. Since $\eta_1 \in (\Lambda \widehat{V}_{i-1})^n$ we can write $\eta_1 = a + b$ with $a \in \Lambda V^{\leq n-1}$ and $b \in \mathrm{span}(x_1, ..., x_{i-1})$ (simply by degree reasons the x_i's cannot be mixed by elements of $\Lambda V^{\leq n-1}$). It follows that

$d(2\lambda x_i - b) = -da$ and hence $2\lambda x_i - b$ lies in the kernel of $q \circ d$ from a while back. Since $2\lambda x_i - b$ carries terms of $x_{p+1}, ..., x_i$ this is a contradiction.

Note that these two cases still work whenever we change x_i with $x_i - \psi_i$ with $\psi_i \in (\Lambda \widehat{V}_{i-1})^{n+r}$.

Case (3) $k = 1$. In this case we have $\eta = \eta_0 + \eta_1 x_i$ with $\eta_0 \in \widehat{N}_{i-1} \cdot \Lambda(\widehat{V}_{i-1})$ and $\eta_1 \in \Lambda \widehat{V}_{i-1}$. Closedness of η implies that η_1 is closed. If $\deg \eta > D$ then again since $H^{>D}(\Lambda V) = 0$, η is exact.

We will now fix $\deg \eta = D$, and then $\deg \eta < D$ will turn out to be automatically fixed. If $\deg \eta = D$, then η_1 is closed of degree $D' \colon = D - n - r$. To show that η is exact consider the set of closed $z_j \in (\Lambda \widehat{V}_{i-1})^{D'}$ such that there exists a $\kappa_j \in (\widehat{N}_{i-1} \cdot \Lambda \widehat{V}_{i-1})^D$ such that $z_j \cdot x_i + \kappa_j$ is closed. Choosing a representing generator $\omega \in \Lambda V^D$ of $H^d(\Lambda V) \cong \mathbb{K}$, this means that we find $\xi_j \in \Lambda V^{D-1}$ and a scalar λ_j such that

$$z_j \cdot x_i + \kappa_j = \lambda_j \omega + d\xi_j \qquad (\star)$$

We want to achieve that $\lambda_j = 0$ for all j. First, given a fixed z_j, suppose that we have two expressions $z_j \cdot x_i + \kappa_j = \lambda_j \omega + d\xi_j$ and $z_j \cdot x_i + \kappa'_j = \lambda'_j \omega + d\xi'_j$. Then

$$\kappa_j - \kappa'_j = (\lambda_j - \lambda'_j)\omega + d(\xi_j - \xi'_j)$$

is closed and lives in $\widehat{N}_{i-1} \cdot \Lambda \widehat{V}_{i-1}$, so by induction hypothesis it is exact. It follows that $\lambda_j = \lambda'_j$. So if we can find one specific expression $(\star)$ with $\lambda_j = 0$ we know that it is always 0.

This also means that we can restrict to a basis of z_j's that satisfy $(\star)$ (indeed, $(\star)$ is linear). If $[z_j] = 0$ then $z_j = d\phi$ with $\phi \in (\Lambda V)^{D'-1}$. Clearly one can take $\phi \in N^{\leq D'-1} \cdot \Lambda(V^{D'-1})$ (indeed otherwise $\phi \in \Lambda(C^{D'-1})$ and then $d\phi = 0$, so $\phi = 0$ would also suffice). Then

$$z_j \cdot x_i + \kappa_j = d\phi \cdot x_i + \kappa_j = d(\phi \cdot x_i) - (-1)^{D'-1} \phi \cdot dx_i + \kappa_j$$

from which it follows that $\phi \cdot dx_i + (-1)^{D'} \kappa_j \in \widehat{N}_{i-1} \cdot \Lambda(\widehat{V}_{i-1})$ is closed and hence exact by the induction hypothesis. It follows that $z_j \cdot x_i + \kappa_j$ is exact, so $\lambda_j = 0$. From this it follows that if $\eta = \eta_0 + \eta_1 \cdot x_i$ is closed with η_1 exact, then η is exact.

What rests is studying a collection z_j such that $[z_j]$ is a basis of $\{[z] | z \text{ satisfies } (\star)\}$. By Poincaré duality we can find a $[\psi_i] \in H^{n+r}(\Lambda V)$ such that $[z_j] \cdot [\psi_i] = \lambda_j [\omega]$ for all j. Such a closed element $\psi_i \in (\Lambda V)^{n+r}$ must lie in $\Lambda(V^{<n+r}) \oplus \mathrm{span}(\widehat{x}_1, ..., \widehat{x}_p)$ since $(\Lambda V)^{n+r} = \Lambda(V^{<n+r}) \oplus \mathrm{span}(\widehat{x}_1, ..., \widehat{x}_p) \oplus \mathrm{span}(x_{p+1}, x_{p+2}, ...)$ and if it had a component in the last part, it would not be closed. It then follows that $[z_j \cdot \widehat{x}_i + \kappa_j] = [d\xi_j] = 0$.

We conclude that whenever $z \cdot \widehat{x}_i + \kappa$ is closed for z closed in $\Lambda(\widehat{V}_{i-1})^{D'}$ and $\kappa \in (\widehat{N}_{i-1} \cdot \Lambda \widehat{V}_{i-1})^D$, then $z \cdot \widehat{x}_i + \kappa$ is exact.

Now to fix $\deg \eta = D$, we set $\widehat{N}_i = \widehat{N}_{i-1} \oplus \mathrm{span}(\widehat{x}_i)$. By the construction then thge case

$\deg \eta = D$ is taken care of, so we now only have to deal with $\deg \eta < D$. So consider $\eta = \eta_0 + \eta_1 \cdot \widehat{x}_i$ of degree less than D. We have that $[\eta_1] \in H^p(\Lambda V)$ with $p < D'$. If $H^{D'-p}(\Lambda V) = 0$ then by Poincare duality $H^{n+r+p}(\Lambda V) = 0$ so $[\eta] = 0$ (note that D' was defined as $D - n - r$ so $D' - p + n + r + p = D$). If $H^{D'-p}(\Lambda V) \neq 0$ consider an arbitrary $[w] \in H^{D'-p}(\Lambda V)$ and consider

$$[w][\eta] = [w \cdot \eta_0 + w \cdot \eta_1 \cdot \widehat{x}_i]$$

We have that $w \cdot \eta_0 \in \widehat{N}_{i-1} \cdot \Lambda \widehat{V}_{i-1}$ and $w \cdot \eta_1$ closed of degree D'. By the above we conclude that $[w] \cdot [\eta] = 0$. Since $[w]$ was arbitrary by Poincaré duality we conclude that $[\eta] = 0$, which finishes the proof. □

Corollary 5.9 A connected, compact, oriented manifold M of dimension $2n - 1$ or $2n$ is formal if and only if it is $(n - 1)$-formal. ◇

Proof. This is an immediate consequence from the previous theorem since $H(M)$ is PDA of the same dimension as the manifold. □

We can now also give a shorter proof of Miller's theorem:

Alternative proof of Theorem 1.42. Since $H(A)$ is a PDA of dimension $\leq 4k + 2$ we need to show that A is $(2k)$-formal. Since $H^i(A) = 0$ for $1 \leq i \leq k$ we have $C = \oplus_{i \geq k+1} C^i$ and hence $N = \oplus_{i \geq 2k+1} N^i$ as we saw before. So the check we have to do is vacuous since $N^{\leq 2k} = 0$. Hence A is $(2k)$-formal, hence by the previous theorem it is formal. □

Example 5.10 By similar arguments as in the last proof, we see that any simply connected manifold is 2-formal, while the theorem states that a 7-manifold is formal if and only if it is 3-formal. Since $N^{\leq 3} = N^3$, heuristically we can say that formality of a 7-manifold is determined by the product $H^2 \times H^2 \to H^4$ (for instance if this is injective, $N^3 = 0$ and 3-formality is again a triviality). ◇

We are also interested in what kind of maps preserve s-formality. In the case of formality, the maps are quasi-isomorphisms (more or less by definition of formality). Since s-formality only deals with the first part of the minimal model, we may expect that we only need a map between algebras to be an isomorphism on low-degree cohomology, and this is indeed the case.

Proposition 5.11 Let $\phi\colon A \to B$ be a map between two CDGA's. If A is s-formal and $H^i(\phi)$ is an isomorphism for $i \leq s + 1$, then B is s-formal. ◇

Proof. Let $m_A\colon \Lambda V \to A$, $m_B\colon \Lambda W \to B$ be minimal models. By the assumption on ϕ we can identify $W^{\leq s}$ with $V^{\leq s}$ such that the following diagram commutes:

$$\begin{array}{ccc} V^{\leq s} & \xrightarrow{m_A} & A \\ \downarrow \cong & & \downarrow \phi \\ W^{\leq s} & \xrightarrow{m_B} & B \end{array}$$

Since all maps preserve products, we can also identify $\Lambda(V^{\leq s})$ with $\Lambda(W^{\leq s})$ such that the following commutes:

$$\begin{array}{ccc} \Lambda(V^{\leq s}) & \xrightarrow{m_A} & A \\ \downarrow \cong & & \downarrow \phi \\ \Lambda(W^{\leq s}) & \xrightarrow{m_B} & B \end{array}$$

Now suppose we have a closed element y of $N_W^{\leq s} \cdot \Lambda(W^{\leq s})$, then its corresponding element x of $N_V^{\leq s} \cdot \Lambda(V^{\leq s})$ is exact by assumption on A. It follows that $m_B^*[y] = \phi^* m_A^*[x] = \phi^* m_A^*(0) = 0$. Since m_B is a quasi-isomorphism we see that $[y] = 0$, i.e. y is exact. □

Remark 5.12 Strictly we only need that $N_W \subset N_V$ under this identification. So it would also suffice that ask for $H^i(\phi)$ to be an isomorphism for $i \leq s$ and surjective for $i = s + 1$.

5.3 Building formal manifolds by gluing

As justified by the large number of G_2-manifolds obtained by gluing formal manifolds, we set out to find a result which at least contains the phrases 'if U, V and $U \cap V$ are formal' and 'then $U \cup V$ is formal'. Unfortunately the most naive guess (namely that for $\{U, V\}$ an open cover we don't need further assumptions) turns out to be false. Not surprisingly we will once more use our favorite non-formal space, Example 4.19.

Example 5.13 Consider the fibration $S^5 \hookrightarrow X \xrightarrow{p} S^3 \times S^3$ where X has minimal model $\langle x, y, z; |x| = |y| = 3, |z| = 5, dx = dy = 0, dz = xy \rangle$ and hence is not formal. Consider an open cover $S^3 = D^3_- \cup D^3_+$ of disks. We set $U = p^{-1}(D^3_- \times S^3)$ and $U' = p^{-1}(D^3_+ \times S^3)$, such that $X = U \cup U'$ is an open cover.

Clearly $U \xrightarrow{p} D^3_- \times S^3$ and $U' \xrightarrow{p'} D^3_+ \times S^3$ are still fibrations with fiber S^5. Furthermore $U \cap U'$ fibers over $(D^3_- \cap D^3_+) \times S^3$, also with fiber S^5. But notice that D^3_- and D^3_+ are contractible, while $D^3_- \cap D^3_+$ is homotopy equivalent to S^2. In particular U and U' are homotopy equivalent to an S^5-fibration over S^3, while $U \cap U'$ is homotopy equivalent to an S^5-fibration over $S^2 \times S^3$.

Since an S^5-bundle over S^3 is a 3-connected space of dimension 8, by Theorem 1.42 it is formal. In particular we see that U and U' are formal. For $U \cap U'$ we see that the minimal model contains the minimal model of $S^2 \times S^3$ as a subalgebra, while it has a generator in degree 5 coming from the fiber S^5. So the minimal model of $U \cap U'$ has the form

$$\langle x, y, z, t; |x| = 2, |y| = |z| = 3, |t| = 5, dx = dy = 0, dz = x^2, dt = \ldots \rangle$$

Since dt has to be degree 6, it has to be a scalar multiple of yz, but since yz is not closed, the only way for ddt to be 0 is that $dt = 0$. In particular we see that $U \cap U'$ has the same minimal model as $S^2 \times S^3 \times S^5$, which is a formal space.

We see that we can cover X with two opens U and U' so that all three of U, U' and $U \cap U'$ are formal, while X is not. ◇

So we are justified to ask for extra conditions, and we may wonder what would do the job for us. Let us first look at some cases where formal spaces do glue to a formal space.

Example 5.14 Let X and Y be topological spaces with minimal models $(\Lambda V, d)$ and $(\Lambda W, d)$. Consider their wedge sum $X \vee Y$ with minimal model $(\Lambda Z, d)$. The inclusions $X \hookrightarrow X \vee Y$ and $Y \hookrightarrow X \vee Y$ induce maps $(\Lambda Z, d) \to (\Lambda V, d)$ and $(\Lambda Z, d) \to (\Lambda W, d)$ respectively, and we can glue these together to a map $(\Lambda Z, d) \to (\Lambda V \oplus_{\mathbb{Q}} \Lambda W, d)$ where $\Lambda V \oplus_{\mathbb{Q}} \Lambda W$ is obtained from the the direct sum $\Lambda V \oplus \Lambda W$ by identifying units and setting all cross products to be 0 (it is easy to see that then the direct sum of two maps into ΛV and ΛW respectively is again a map of CDGA's).

Since the inclusions have retractions, $H^*(\phi)$ is surjective (indeed just construct a section of $H^*(\phi)$ out of the retractions). On the other hand $H^*(\Lambda V \oplus_{\mathbb{Q}} \Lambda W) \cong H^*(\Lambda V) \oplus_{\mathbb{Q}} H^*(\Lambda W)$ and then since ΛV and ΛW are minimal models of X and Y respectively we get $H^*(\Lambda V) \oplus_{\mathbb{Q}} H^*(\Lambda W) \cong H^*(X) \oplus_{\mathbb{Q}} H^*(Y) \cong H^*(X \vee Y)$. So if $H^i(X)$ and $H^i(Y)$ are finite dimensional for all i the fact that $H^*(\phi)$ is surjective is equivalent to ϕ being a quasi-isomorphism.

In particular we see that the minimal model $X \vee Y$ is equal to the minimal model of $(\Lambda V, d) \oplus_{\mathbb{Q}} (\Lambda W, d)$. ◇

Using this fact, the following is an easy consequence:

Theorem 5.15 If X and Y are formal topological spaces such that $H^i(X)$ and $H^i(Y)$ are finite dimensional for all i, then $X \vee Y$ is formal. ◇

Proof. By the example above, we need to show that if $(\Lambda V, d)$ and $(\Lambda W, d)$ are formal minimal models, then $\Lambda V \oplus_{\mathbb{Q}} \Lambda W$ is also formal. It is not hard to see that the minimal model of the latter algebra is $\Lambda(V \oplus W \oplus N'')$ where N'' first kills the closed cross-products of V and W, then then repeatedly kills closed cross-products of itself with V and W.

In particular if we write $V = C \oplus N$ and $W = C' \oplus N'$ as usual, we see that $\Lambda V \oplus_{\mathbb{Q}} \Lambda W$ has a minimal model $\Lambda V''$ with $V'' = C \oplus C' \oplus N \oplus N' \oplus N''$ where any closed element of $N'' \cdot \Lambda V''$ is exact by construction on N''.

Furthermore any closed element of $N \cdot \Lambda V''$ is exact because either we have elements of $V' \cdot N''$ in there, in which case it is exact by construction of N'', or we have an closed element of $N \cdot \Lambda V$ which is exact since we assume X to be formal.

Similarly any closed element of $N' \cdot \Lambda V''$ is exact, and we conclude that our minimal model of $\Lambda V \oplus_{\mathbb{Q}} \Lambda W$ satisfies the conditions of Theorem 1.44, and hence $X \vee Y$ is formal. □

As ~~a nice~~ example of this, we show that wedges of spheres have torsion free parts in their homotopy in arbitrary high dimension.

Example 5.16 Consider the space $S^n \vee S^n$ for n odd ~~(to stay in the realm of simply connected spaces we better take~~ $n \geq 3$; the case n even, or $S^m \vee S^n$ for $m \neq n$ is similar but more tedious). By the previous we know that $S^n \vee S^n$ is formal, so let us calculate the minimal of $H^*(S^n \vee S^n)$.

First we need two closed generators v_1, v_2 in degree n. Then since n is odd we don't need to kill v_1^2 and v_2^2 (indeed by graded commutivity they are zero), but we do need to kill $v_1 v_2$. So let us add a generator w_1 in degree $2n-1$ such that $dw_1 = v_1 v_2$, and we have fixed the cohomology up to degree $3n-1$. Indeed in degree $3n-1$ we have the two closed products $w_1 v_1$ and $w_1 v_2$ which we don't want, so let us add generators w_2 and w_3 in degree $3n-2$ such that $dw_2 = w_1 v_1$ and $dw_3 = w_1 v_2$. The next problems are then the products $w_2 v_1$ and $w_3 v_2$ in degree $4n-2$, so we add extra generators w_4 and w_5 in degree $4n-3$.

Repeating this procedure we get that the minimal model of $S^n \vee S^n$ has closed generators v_1 and v_2 in degree n, a generator w_1 in degree $2n-1$ such that $dw_1 = v_1 v_2$, generators w_2 and w_3 in degree $3n-2$ such that $dw_2 = w_1 v_1$, $dw_3 = w_1 v_2$ and then for $j \geq 2$ generators w_{2j} and w_{2j+1} in degree $(2+j)n-(j+1)$ such that $dw_{2j} = w_{2j-2} v_1$ and $dw_{2j+1} = w_{2j-1} v_2$.

By Theorem 2.18 we conclude that:

$$\pi_i(S^n \vee S^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}^2 & i = n, 3n-2, 4n-3, 5n-4, ... \\ \mathbb{Q} & i = 2n-1 \\ 0 & \text{else} \end{cases}$$

Using the results on the wedge sum, we can obtain similar results for the connected sum of manifolds similar to the wedge sum.

Example 5.17 Consider M, N two compact simply connected n-dimensional manifolds. We construct $M\#N$ by taking disks around the basepoints of M and N, deleting the basepoint and gluing the two punctured disks inside-out in an orientation preserving way (i.e. choose embeddings of the punctures disk where one embedding preserves orientation and the other reverses orientation, and then glue them via a diffeomorphism which switches boundary components).

Consider the pinch map $q\colon M\#N \to M \vee N$, which gives a map $m_q\colon \mathcal{M}_{M\vee N} \to \mathcal{M}_{M\#N}$. If ω_M and ω_N represent the volume forms of M and N in $\mathcal{M}_{M\vee N}$ respectively, we have that $q^*(\omega_M)$ and $q^*(\omega_N)$ are in the same cohomology class (namely that of the volume form of $M\#N$). This, together with arguments with the Mayer-Vietoris sequence which show that $H^{<n}(M\#N) \cong H^{<n}(M \vee N)$, shows that the map

$$\phi\colon (\mathcal{M}_{M\vee N} \otimes \Lambda x, d) \to \mathcal{M}_{M\#N}$$

with $dx = \omega_M - \omega_N$, has the property that $H^{\leq n}(\phi)$ is an isomorphism. Since $H^{>n}(M\#N) = 0$, similar to the examples above what we now have to do is add further generators $\{x_i\}$ in degrees above n such that $H^{>n}(\mathcal{M}_{M\vee N} \otimes \Lambda(x, x_i), d) = 0$.

Theorem 5.18 [8, Thm 3.13, p.115] If M and N are formal simply connected compact n-dimensional manifolds, then $M\#N$ is formal. $\diamond$

Proof. We will show that $\mathcal{M}_{M\vee N}\otimes\Lambda(x, x_i)$ is quasi-isomorphic to a CDGA with 0 differential. First note that from the quasi-isomorphism $\sigma\colon \mathcal{M}_{M\vee N} \to \mathcal{M}_M \oplus_{\mathbb{Q}} \mathcal{M}_N$, by tensoring with $\Lambda(x, x_i)$ we obtain a quasi-isomorphism

$$\sigma'\colon (\mathcal{M}_{M\vee N}\otimes\Lambda(x, x_i), d) \xrightarrow{\simeq} (\mathcal{M}_M \oplus_{\mathbb{Q}} \mathcal{M}_N \otimes \Lambda(x, x_i), D)$$

where $D(x) = \sigma d(x)$ and $D(x_i) = (\sigma\otimes 1)d(x_i)$. Then, using the quasi-isomorphism $\theta\colon \mathcal{M}_M \oplus_{\mathbb{Q}} \mathcal{M}_N \to H^*(M) \oplus_{\mathbb{Q}} H^*(N)$ obtained from the formality of M and N, we get a quasi-isomorphism

$$(\mathcal{M}_M \oplus_{\mathbb{Q}} \mathcal{M}_N \otimes \Lambda(x, x_i), D) \xrightarrow{\simeq} (H^*(M) \oplus_{\mathbb{Q}} H^*(N) \otimes \Lambda(x, x_i), \widetilde{D})$$

where $\widetilde{D} = \theta\circ D$. Then projecting onto $H^*(M) \oplus_{\mathbb{Q}} H^*(N)/(\omega_M - \omega_N)$ yields a quasi-isomorphism

$$(H^*(M) \oplus_{\mathbb{Q}} H^*(N) \otimes \Lambda(x, x_i), \widetilde{D}) \xrightarrow{\simeq} (H^*(M) \oplus_{\mathbb{Q}} H^*(N)/(\omega_M - \omega_N), 0)$$

which shows that $M \vee N$ is formal. $\square$

The common feature of the two examples is that the topology of the intersection is very simple. Indeed for the wedge sum it is just a point, while for the connected sum it is a codimension-1 sphere. In particular the first cohomology groups of the intersection are 0, which ~~makes sure~~ that the connecting homomorphism in the Mayer-Vietoris sequence is also 0, which ~~makes sure that~~ the inclusions of the two parts together give isomorphisms on the first few cohomology groups.

Since we know, by the discussion of s-formality, that formality of a manifold is contained in the first half of the minimal model, we might guess that making sure that the connecting homomorphism is trivial for the first half of terms will take us somewhere. This turns out to be almost true, up to a nifty technical issue, which we discuss first.

Definition 5.19 Let A, B be formal CDGA's with minimal models M_A and M_B respectively. A map $f\colon A\to B$ is called *formal* if the Sullivan representatives of f and f^* agree, that is, there is a map $m_f\colon M_A \to M_B$ such that the following commutes up to homotopy:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \simeq\uparrow & & \uparrow\simeq \\ M_A & \xrightarrow{m_f} & M_B \\ \simeq\downarrow & & \downarrow\simeq \\ H(A) & \xrightarrow[f^*]{} & H(B) \end{array}$$

Furthermore, we call a map $f\colon X \to Y$ of spaces formal if $A_{\mathrm{PL}}(f)\colon A_{\mathrm{PL}}(Y) \to A_{\mathrm{PL}}(X)$ is formal. ◇

Note that this is really a non-trivial thing to ask:

Example 5.20 Consider the algebras $A = \langle x, y; |x| = 2k, |y| = 4k-1, dx = 0, dy = x^2\rangle$ and $B = \langle z; |z| = 4k-1, dz = 0\rangle$, the minimal models of S^{2k} and S^{4k-1} respectively. Consider the map $f\colon A \to B$ that sends x to 0 and y to z, this is the Sullivan representation of the Hopf map $S^{4k-1} \to S^{2k}$. Note that $f^*\colon H(A) \to H(B)$ is the trivial map (sending the generator $[x]$ to 0).

Note that A and B come with maps $\phi\colon A \to H(A)$ and $\psi\colon B \to H(B)$ sending x to $[x]$, y to 0 and z to $[z]$. If f were to be formal, the following map would commute up to homotopy:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\phi} \\ H(A) & \xrightarrow[f^*=0]{} & H(B) \end{array}$$

which means that we find a homotopy $F\colon A \to H(B) \otimes \Lambda(t, dt)$ from the zero map to $\phi \circ f$. Note that we only have to define F on x and y such that it works out with d. First, for degree reasons we have to send x to 0 (indeed $H(B) \otimes \Lambda(t, dt)$ does not contain element of degree $2k$). On the other hand we have that $F(y) = [z] \otimes P(t)$ for some polynomial P such that $P(0) = 0$ and $P(1) = 1$.

Then we need that $dF(y) = F(x)^2$ so that we can extend multiplicatively to a map of CDGA's. But now note that $dF(y) = [z] \otimes P'(t)dt$ since P is not consnat $dF(y) \neq 0$, while $F(x)^2 = 0$, so the homotopy F cannot exist. We conclude that f is not formal. ◇

Then the first Ansatz to a main result is then:

Theorem 5.21 Suppose

$$\begin{array}{ccc} A & \xleftarrow{g'} & B' \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f'} \\ B & \xleftarrow[f]{} & C \end{array}$$

is a weak homotopy pullback square of CDGA's (at least with respect to maps with minimal algebras as domains). Suppose that $H^i(C)$ is finite dimensional for $i \leq s+1$ and that $f^* \oplus (f')^*\colon H^i(C) \to H^i(B) \oplus H^i(B')$ is injective for $i \leq s+1$. If A, B and B' are formal algebras and g and g' are formal maps, then C is s-formal. ◇

Note that for an admissable cover $M = U \cup U'$, injectivity of $H^i(M) \to H^i(U) \oplus H^i(U')$ is equivalent to surjectivity of $H^{i-1}(U) \oplus H^{i-1}(U') \to H^{i-1}(U \cap U')$ by use of the Mayer-Vietoris sequence indeed both are equivalent to the vanishing of the connecting homomorphism $\delta\colon H^{i-1}(U \cap U') \to H^i(M)$. Then translating the algebraic theorem to manifolds we get the following corollary:

Corollary 5.22 Let M be a simply connected compact n-dimensional manifold together with an open cover $M = U \cup U'$ such that U, U' and $U \cap U'$ are formal and the inclusions $U \cap U' \hookrightarrow U, U'$ are formal. If $H^i(U) \oplus H^i(U') \to H^i(U \cap U')$ is surjective for $i \leq \lceil \frac{n}{2} \rceil$ then M is formal. $\diamond$

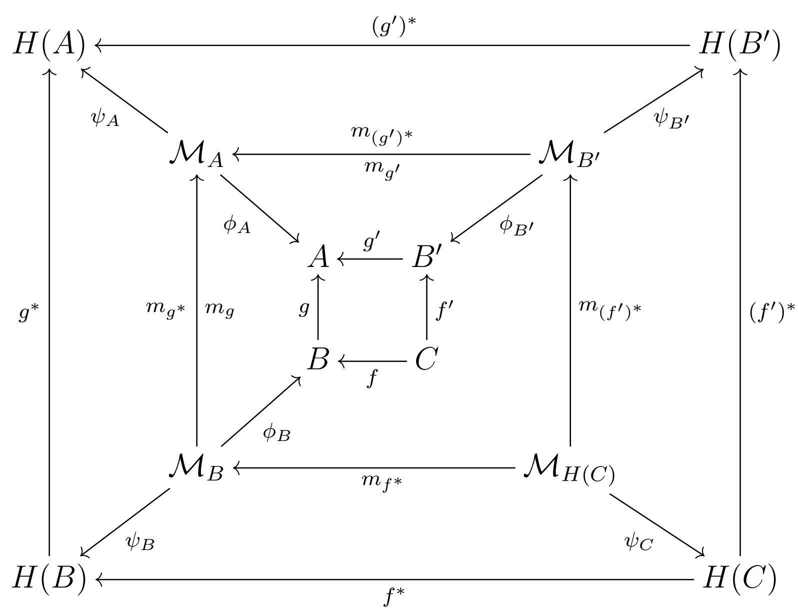
Proof. First by Lemma 2.35 and Remark 2.36 the square of inclusions is a weak homotopy pushout square. Then by Theorem 2.37 we know that A_{PL} applied to the square of inclusions gives us a weak homotopy pullback square of algebras. Then by the formality assumptions on U, U' and $U \cap U'$, this square of algebras satisfies the formality assumptions of Theorem 5.21. Furthermore by the remarks above, the assumpion on the maps on cohomology of the subspaces are making sure that we also satisfy the cohomological assumptions needed for Theorem 5.21 for $s = \lceil \frac{n}{2} \rceil$.

So we may apply Theorem 5.21 on the square

$$\begin{array}{ccc} A_{\mathrm{PL}}(U \cap U') & \longleftarrow & A_{\mathrm{PL}}(U') \\ \uparrow & & \uparrow \\ A_{\mathrm{PL}}(U) & \longleftarrow & A_{\mathrm{PL}}(M) \end{array}$$

to conclude that M is $\lceil \frac{n}{2} \rceil$-formal. Then by Corollary 5.9 it follows that M is formal. $\square$

Proof of Theorem 5.21. Consider the following diagram



Here $H(A) \xleftarrow{\psi_A} \mathcal{M}_A \xrightarrow{\phi_A} A$, $H(B) \xleftarrow{\psi_B} \mathcal{M}_B \xrightarrow{\phi_B}$ and $H(B') \xleftarrow{\psi_{B'}} \mathcal{M}_{B'} \xrightarrow{\phi_{B'}} B'$ are pairs of adapted minimal models, $\mathcal{M}_{H(C)} \xrightarrow{\psi_C} H(C)$ is the minimal model of $H(C)$, while the maps between the minimal models are Sullivan representatives of the outer and inner maps. Since

g and g' are formal maps, we can choose Sullivan representatives for g^* and $(g')^*$ such that $m_{g^*} = m_g$ and $m_{(g')^*} = m_{g'}$.

Note that since assume the inner square to be a weak homotopy pullback, in particular the inner square is commutative up to homotopy, and hence the same applies to the middle square (since the middle square consists of Sullivan representatives). Also the outer square commutes since it consists of the induced maps from the inner square. Lastly by definition of the Sullivan representatives all the little squares all around commute up to homotopy.

We wish to use the pullback properties of the inner square, and so we wish to find suitable maps $\mathcal{M}_{H(C)} \to B$ and $\mathcal{M}_{H(C)} \to B'$. To this end, note that since the outer square commutes, both $m_{g^*} \circ m_{f^*}$ and $m_{(g')^*} \circ m_{(f')^*}$ are Sullivan representatives of $g^* \circ f^* = (g')^* \circ (f')^*$. Since Sullivan representatives are well-defined up to homotopy, this means that

$$m_{g^*} \circ m_{f^*} \simeq m_{(g')^*} \circ m_{(f')^*}$$

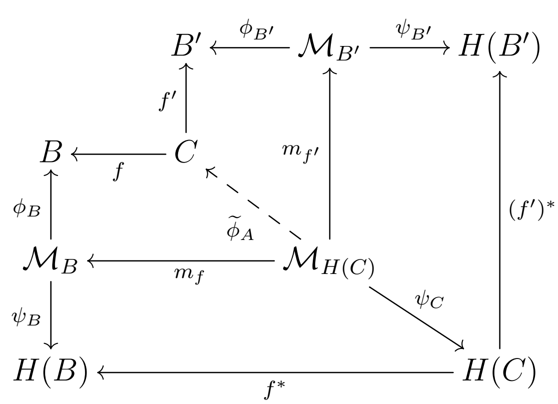
Using the previous note that $m_{g^*} = m_g$ and $m_{(g')^*} = m_{g'}$ and composing with ϕ_A we also see that

$$\phi_A \circ m_g \circ m_{f^*} \simeq \phi_A \circ m_{g'} \circ m_{(f')^*}$$

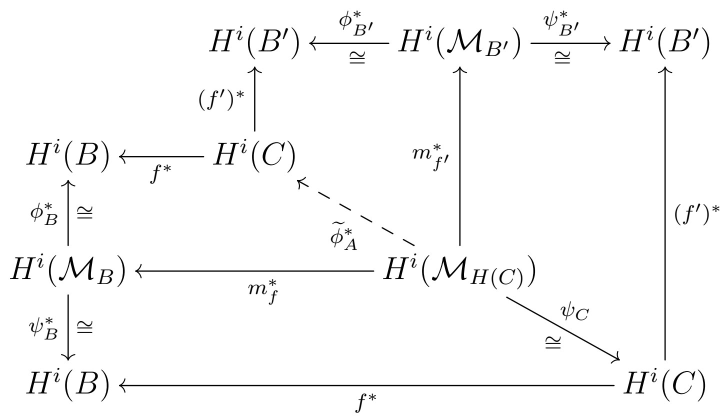
Then we use the fact that the inner-left and inner-upper squares commute up to homotopy (i.e. $\phi_A \circ m_g \simeq g \circ \phi_B$ and $\phi_A \circ m_{g'} \simeq g' \circ \phi_{B'}$), which yields

$$g \circ \phi_B \circ m_{f^*} \simeq g' \circ \phi_{B'} \circ m_{(f')^*}$$

Now we can use the pullback properties of the inner square together with the maps $\phi_B \circ m_{f^*} \colon \mathcal{M}_{H(C)} \to B$ and $\phi_{B'} \circ m_{(f')^*} \colon \mathcal{M}_{H(C)} \to B'$ to find a map $\widetilde{\phi}_A \colon \mathcal{M}_{H(C)} \to C$ such that in the following everything commutes up to homotopy:



Now that we have a map, we see what its properties on cohomology are. So lets apply $H^i(-)$ on the diagram above. Since $H^i(-)$ takes homotopies to equalities between maps, we now get a map where everything commutes:



We will show that $\widetilde{\phi}^*_A$ is an isomorphism for $i \leq s+1$. Since ψ_C is an isomorphism, we know that both $H^i(\mathcal{M}_{H(C)})$ is finite dimensional ($H^i(C)$ is assumed to be finite dimensional for this range of i), so we only need to show that $\widetilde{\phi}^*_A$ is injective on the level of H^i for $i \leq s+1$.

So let

$$x \in \ker(\widetilde{\phi}^*_A\colon H^i(\mathcal{M}_{H(C)}) \to H^i(C))$$

Then $f^*(\widetilde{\phi}^*_A(x)) = 0$, but then using the commutativity of the upperleft square, we get

$$\phi^*_B(m^*_f(x)) = 0$$

so, since ϕ^*_B is an isomorphism, we see that

$$m^*_f(x) = 0$$

Then of course also $\psi^*_B(m^*_f(x)) = 0$ and the the commutativity of the lower-left square yields

$$f^*(\psi_C(x)) = 0$$

Similarly we get that $(f')^*(\psi_C(x)) = 0$, so we see that

$$\psi_C(x) \in \ker(f^* \oplus (f')^*\colon H^i(C) \to H^i(B) \oplus H^i(B'))$$

so by the assumption on $H^i(C) \to H^i(B) \oplus H^i(B')$ we conclude that $\psi_C(x) = 0$, so since ψ_C is an isomorphism we see that $x = 0$, which shows that $\widetilde{\phi}^*_A$ is an isomorphism on H^i for $i \leq s+1$.

Then since $\mathcal{M}_{H(C)}$ is certainly formal (it is a minimal model of a cohomology algebra), by Proposition 5.11 we conclude that C is s-formal. □

5.4 Formality and deleting submanifolds

We would like to ~~put~~ Theorem 5.21 ~~to play~~, for instance ~~to use~~ it for an alternative proof of Theorem 5.18. Since the connected sum of $M\#N$ is covered by opens diffeomorphic to $M\backslash\{*\}$ and $N\backslash\{*\}$, we investigate the formality of such spaces. The result then is:

Proposition 5.23 Let M be a compact simply connected n-dimensional manifold ($n \geq 3$). Then M is formal if and only if $M\backslash\{*\}$ is formal. $\diamond$

Proof. We first prove that $M\backslash\{*\}$ is formal if M is. We will show that the inclusion $M\backslash\{*\} \hookrightarrow M$ induces isomorphisms on H^i for $i < n$. Note that on H^0 this is immediate since both M and $M\backslash\{0\}$ are connected. Indeed consider the Mayer-Vietoris sequence of the cover $M = (M\backslash\{*\}) \cup D$ with D an open disk around $*$. We write S for the intersection of $M\backslash\{*\}$ and D (note that S is homotopy equivalent to S^{n-1}). We get

$$0 \to H^0(M) \to H^0(M\backslash\{*\}) \oplus H^0(D) \to H^0(S) \to H^1(M) \to H^1(M\backslash\{*\}) \oplus H^1(D) \to H^1(S)$$

Note that all the H^0's are rank 1 vector spaces with the map $H^0(M\backslash\{*\}) \oplus H^0(D) \to H^0(S)$ surjective. Since $H^1(D) = 0$ this means that $H^1(M) \to H^1(M\backslash\{*\})$ is an isomorphism, so in particular $H^1(M\backslash\{*\}) = 0$.

Then for $1 < i < n-1$ the Mayer-Vietoris sequence yields

$$H^{i-1}(S) = 0 \to H^i(M) \to H^i(M\backslash\{*\}) \to H^i(S) = 0$$

so the inclusion induces also an isomorphism on H^i for $i < n-1$.

Then for $i = n-1$ we look at the long exact sequence of the pair $(M, M\backslash\{*\})$ in homology with integer coefficients:

$$0 \to H_n(M;\mathbb{Z}) \to H_n(M, M\backslash\{*\};\mathbb{Z}) \to H_{n-1}(M\backslash\{*\};\mathbb{Z}) \to H_{n-1}(M;\mathbb{Z}) \to 0$$

Since M simply connected, it is oriented and hence the first map here is an isomorphism (indeed being oriented in the topological sense means that a generator of $H_n(M;\mathbb{Z})$ gets sent to a generator of $H_n(M, M\backslash\{*\};\mathbb{Z})$ at every point), which means that $H_{n-1}(M\backslash\{*\}) \to H_{n-1}(M)$ is ~~one~~ aswell. Then by the ~~UCT~~ it follows that $H^{n-1}(M) \to H^{n-1}(M\backslash\{*\})$ is also an isomorphism (if we take coefficients in a field, which we do).

Then looking at the end of the Mayer-Vietoris sequence we get:

$$0 \to H^{n-1}(S) \to H^n(M) \to H^n(M\backslash\{*\}) \to H^n(S) = 0$$

Since both $H^{n-1}(S)$ and $H^n(M)$ are rank 1 vector spaces (the latter because of orientability), it follows that $H^n(M\backslash\{*\}) = 0$.

So to recap: we have that the inclusion $M\backslash\{*\} \to M$ induces a map $\Omega(M) \to \Omega(M\backslash\{*\})$ that is an isomorphism on $H^{\leq n-1}$, so by Proposition 5.11, $M\backslash\{*\}$ is $(n-2)$-formal. But since $M\backslash\{*\}$ is simply connected (or at least it has vanishing H^1), and $H^{>n-1}(M\backslash\{*\}) = 0$ we see that $(n-2)$-formality actually implies that $M\backslash\{*\}$ is formal.

For the converse, assume that $M\backslash\{*\}$ is formal. Then M is covered by a disk D around $*$ and $M\backslash\{*\}$. Since D is contractible, it is formal, while by assumption $M\backslash\{*\}$ is. The intersection of the two is homotopy equivalent to S^{n-1}, which is also formal. So by Corollary 5.22, M is formal. $\square$

Remark 5.24 The statement that M is formal if $M\backslash\{*\}$ is is also proven in Theorem 3.12 of Felix, Oprea and Tanré [8] using the bigraded and filtered models. In comparison, the proof using Corollary 5.22 is considerably shorter (although there is a lot of hidden algebra in the background). ◇

Now **assuming that** $S \to M\backslash\{*\}$ ~~$\to S$~~ and $S \to N\backslash\{*\}$ are formal we have the tools to give a shorter proof of Theorem 5.18:

Proof of Theorem 5.18. For the case of 1 and 2 dimensional M and N we are done by Miller, while for $n \geq 3$ we first use the above proposition to see that $M\backslash\{*\}$ and $N\backslash\{*\}$ are formal. Then since $M\#N$ has an open cover of two opens which are diffeomorphic to $M\backslash\{*\}$ and $N\backslash\{*\}$ respectively, while the intersection is homotopy equivalent to S^{n-1}. So we see that $M\#N$ is covered by two formal opens such that the intersection is also formal. We then can readily use Corollary 5.22. □

Remark 5.25 Here is where we run into problems, because there are strong reasons to think that the inclusion of the intersecting sphere $S^{n-1} \to M\backslash\{*\}$ is never a formal map, unless M is a cohomology sphere.

To see this assume for the moment that M is even dimensional, then the minimal model of S^{n-1} has a single closed generator in degree $n-1$ (the case n is odd is similar but more tedious). We know that if M has some intermediate cohomology, the volume form of M is a product in the cohomology of M, and disappears in the cohomology of $M\backslash\{*\}$.
So if $\phi\colon \Lambda(C \oplus N) \to \Omega(M\backslash\{*\})$ is the minimal model of $M\backslash\{*\}$ we have a generator $x \in N^{n-1}$ such that $d\phi(x)$ is the restriction of the volume form to $M\backslash\{*\}$. In particular $\phi(x)$ cannot be extended to M. But this form restricted to the boundary sphere S^{n-1} integrates to something non-zero.

Next consider the map $\mathcal{M}_{M\backslash\{*\}} \to \mathcal{M}_{S^{n-1}}$ induced by the inclusion. Then the fact that we have a generator in N^{n-1} that integrates to something non-zero over S^{n-1}, means that this generator does not go to 0 in $\mathcal{M}_{S^{n-1}}$. In particular in the diagram

$$\begin{array}{ccc} \mathcal{M}_{M\backslash\{*\}} & \longrightarrow & \mathcal{M}_{S^{n-1}} \\ \downarrow & & \downarrow \\ H(M\backslash\{*\}) & \longrightarrow & H(S^{n-1}) \end{array}$$

we have an element of $x \in N^{n-1}$ going to 0 in $H(S^{n-1})$ when going via the lower-left corner, while it doesn't go to 0 when going via the upper-right corner. Similar to the example of the Hopf map, since dx consists of things which map to zero in $H(S^{n-1})$ simply for degree reasons, we cannot homotope the map $\mathcal{M}_{M\backslash\{*\}} \to H(S^{n-1})$ from something sending x to 0 to one which sends x that sends x to something non-zero.

We conclude that the Sullivan representatives of $H(M\backslash\{*\}) \xrightarrow{i^*} H(S^{n-1})$ and $\Omega(M\backslash\{*\}) \xrightarrow{i^*} \Omega(S^{n-1})$ cannot agree. So in particular for a compact simply connected manifold M, the inclusion of the boundary sphere into $M\backslash\{*\}$ can *never* be a formal map, unless M has no intermediate cohomology. ◇

We now wish to generalize the statement "M is formal if and only if $M\backslash\{*\}$ is formal" to something the phrases "$N \subset M$ a closed submanifold", "M is formal" and "$M\backslash N$ is formal". In the discussion that now follows M is a compact simply connected n-dimensional manifold and N is a closed simply connected submanifold.

We will generalize the argument above by using the Mayer-Vietoris sequence applied to the pair $(M\backslash N, U)$ where U is an open neighbourhood of N. Since we want some control over U we assume that N has trivial normal bundle. In that case we find an open neighbourhood U diffeomorphic to $N \times D^r$ (with r the codimension of N) where N sits in there as $N \times \{0\}$. Then the intersection of the two opens is homotopy equivalent to $N \times S^{r-1}$. Now we only have to figure out the right assumptions on the maps in Mayer-Vietoris sequence.

If we want to keep the argument clean and simple more or less the only thing that we can do is showing that $\Omega(M) \to \Omega(M\backslash N)$ has the right properties, i.e. that Proposition 5.11 (or more precisely the proof of Proposition 5.11, c.f. Remark 5.12) applies. We can deduce precisely what is needed for that.

Recall the notation that U is the open neighbourhood of N that is diffeomorphic to $N \times D^r$. Then the following part of the Mayer-Vietoris sequence

$$H^{i-1}((M\backslash N) \cap U) \to H^i(M) \to H^i(M\backslash N) \oplus H^i(U) \to H^i((M\backslash N) \cap U)$$

reduces to a sequence of the form

$$H^{i-1}(N) \oplus H^{i-r}(N) \to H^i(M) \to H^i(M\backslash N) \oplus H^i(N) \to H^i(N) \oplus H^{i-r+1}(N)$$

Here we use that $(M\backslash N) \cap U$ is homotopy equivalent to $N \times S^{r-1}$ and so $H^i((M\backslash N) \cap U) \cong H^i(N \times S^{r-1}) \cong H^i(N) \oplus H^{i-r+1}(N)$ and that U is homotopy equivalent to $N \times D^r$ and so $H^i(U) \cong H^i(N \times D^r) \cong H^i(N)$.

Note that if we take the codimension of N to be high enough ($r \geq 3$ will do), the first part of the Mayer-Vietoris sequence will show that $H^1(M\backslash N) = 0$, so if we know that $H^{>d}(M\backslash N) = 0$ for some d (this d necessairly has to be $\leq n$, but for instance if $N = *$ we saw before that $d = n-1$), we want to show that $M\backslash N$ is $(d-1)$-formal. Hence, we want to show that $H^i(M) \to H^i(M\backslash N)$ is an isomorphism for $i \leq d-1$ and $H^d(M) \to H^d(M\backslash N)$ is surjective.

Consider the map $H^i(N) \to H^i(N) \oplus H^{i-r+1}(N)$. Since this comes from the map $H^i(N \times D^r) \to H^i(N \times S^{r-1})$ it is not hard to see that it is just the inclusion onto the first factor. So if we denote by f the map $H^i(M\backslash N) \to H^i(N)$ and by g the map $H^i(M\backslash N) \to H^{i-r+1}(N)$ we see that

$$[H^i(M\backslash N) \oplus H^i(N) \to H^i(N) \oplus H^{i-r+1}(N)] = \begin{pmatrix} f & \mathrm{id} \\ g & 0 \end{pmatrix}$$

~~Now~~ $H^i(M) \to H^i(M\backslash N)$ surjective is equivalent to finding for every $x \in H^i(M\backslash N)$ a $y \in H^i(N)$ such that (x, y) is in the image of $H^i(M) \to H^i(M\backslash N) \oplus H^i(N)$, i.e. the kernel of $\begin{pmatrix} f & \mathrm{id} \\ g & 0 \end{pmatrix}$. Now simply applying this matrix yields $gx = 0$ and $fx + y = 0$. So we see that $H^i(M) \to H^i(M\backslash N)$ surjective is equivalent to the map $H^i(M\backslash N) \to H^{i-r+1}(N)$ being 0 (indeed then $gx = 0$ has to be satisfied for every x and then just set $y = -fx$).

Now $H^i(M) \to H^i(M\backslash N)$ injective means that **a)** whenever we find a $(0, y)$ in the image of $H^i(M) \to H^i(M\backslash N) \oplus H^i(N)$, we need $y = 0$ and **b)** the map $H^i(M) \to H^i(M\backslash N) \oplus H^i(N)$ should be injective. But then since we have an exact sequence **a)** is equivalent to

$$\begin{pmatrix} f & \mathrm{id} \\ g & 0 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = 0 \Longrightarrow y = 0$$

Again simply writing out the matrix multiplication, we get $y = 0$, so **a)** is immediate. On the other hand **b)** can be easily seen to be equivalent to $H^{i-1}(M\backslash N) \to H^{i-r}(N)$ being surjective.

We conclude that for the argument to work we want that

- $H^i(M\backslash N) \to H^{i-r+1}(N)$ is zero for $i \leq d$
- $H^i(M\backslash N) \to H^{i-r+1}(N)$ is surjective for $i \leq d-2$.

and indeed this is the only way the argument using $\Omega(M) \to \Omega(M\backslash N)$ works. Sadly this means that we need to put quite stringent assumptions on the cohomology of N (indeed it should be 0 for quite a big range). We conclude the following

Theorem 5.26 Let M be a simply connected formal manifold and $N \subset M$ a closed simply connected submanifold of codimension $r \geq 3$ with trivial normal bundle. If $H^{>d}(M\backslash N) = 0$, $H^j(N) = 0$ for $j \leq d - r - 1$ and $H^{d-1}(M\backslash N) \to H^{d-r}(N)$, $H^d(M\backslash N) \to H^{d-r+1}(N)$ are the zero map, then $M\backslash N$ is formal. □

In the case that M is n-dimensional, we know that $H^{>n}(M\backslash N) = 0$ simply for dimension reasons. In this case the cohomological assumptions reduce to $H^j(N) = 0$ for $j \leq n-r-1$ and $H^{n-1}(M\backslash N) \to H^{d-r}(N)$, $H^n(M\backslash N) \to H^{n-r+1}(N)$. However, notice that $n - r = \dim(N)$, so the last assumption is vacuous since $H^{n-r+1}(N) = H^{\dim(N)+1}(N) = 0$. So if we don't know anything about the vanishing of $H^*(M\backslash N)$ in high degree, the theorem reduces to the following:

Corollary 5.27 Let M be a simply connected formal n-dimensional manifold and $N \subset M$ a closed simply connected submanifold of codimension $r \geq 3$ with trivial normal bundle. If $H^j(N) = 0$ for $j \leq \dim(N) - 1$ and $H^{n-1}(M\backslash N) \to H^{\dim(N)}(N)$ is 0, then $M\backslash N$ is formal. □

Outlook and conclusion

pfff

A Appendix on differential geometry

We briefly recap some basic definitions around connections and metrics. We will frequently encounter paths here, so we abbreviate I for $[0, 1]$. We cherry-pick some parts of Crainic [4], and a more thorough exposition can be found there.

A.1 Connections & parallel transport

Definition A.1 Given a vector bundle $E \to M$, a *connection on* E is a bilinear map

$$\nabla \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E);\ (X, s) \mapsto \nabla_X(s)$$

such that

$$\nabla_{fX}(s) = f\nabla_X(s) \quad \& \quad \nabla_X(fs) = f\nabla_X(s) + \mathcal{L}_X(f)s$$

for all $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$ (where by $\mathcal{L}_X(f)$ we mean the Lie derivative of f along X). ◇

Remark A.2 If we are given a local frame $\{e_1, ..., e_n\}$ of E we find a matrix of one forms $\omega^i_j \in \Omega^1(M)$ $(1 \leq i, j \leq n)$ such that

$$\nabla_X(e_j) = \sum_{j=1}^{n} \omega^i_j(X)e_i$$

this is called the connection matrix. ◇

Connections are vehicles to take derivatives of sections of E. To put this to use, let $\gamma \colon I \to M$ be a path in M and let $u \colon I \to E$ be a path above γ (i.e. $u(t) \in E_{\gamma(t)}$).

We can take the derivative of u with respect to ∇. We proceed locally, so suppose we have a frame $\{e_i\}$ of E and write $u(t) = \sum_i u^i(t)e_i(\gamma(t))$. We then set the components $\frac{\nabla u}{dt}$ by

$$\left(\frac{\nabla u}{dt}\right)^i = \frac{du^i}{dt} + \sum_j u^j \cdot (\omega^i_j \circ \dot{\gamma})$$

We should remark that the resulting vector is independent of the chosen frame.

Definition A.3 A path $u \colon I \to E$ is called *parallel (with respect to* ∇*)* if $\frac{\nabla u}{dt} = 0$. ◇

Lemma A.4 [4, Lem 1.21, p.23] Let $E \to M$ be a vector bundle, ∇ a connection on E, and $\gamma \colon I \to M$ be a path in M. Pick $t_0 \in I$ and any $u_0 \in E_{\gamma(t_0)}$. Then there is a unique parallel path $u \colon I \to E$ above γ such that $u(t_0) = u_0$. ◇

Definition A.5 Let $E \to M$ a vector bundle, ∇ a connection on E, $\gamma\colon I \to M$ a path in M, $t_0, t_1 \in I$. *Parallel transport along* γ *(with respect to* ∇*) from* t_0 *to* t_1 is the map

$$T_\gamma^{t_0,t_1}\colon E_{\gamma(t_0)} \to E_{\gamma(t_1)}$$

that picks a vector u_0, takes the path u from the previous lemma, and evaluates u at t_1. $\diamond$

Remark A.6 Since parallel transport is defined by solving a linear PDE, it is not hard to see that $T_\gamma^{t_0,t_1}$ is linear. Also $T_\gamma^{t_0,t_0} = \mathrm{id}$ and $T_\gamma^{t_1,t_2} \circ T_\gamma^{t_0,t_1} = T_\gamma^{t_0,t_2}$. $\diamond$

A.2 Riemmanian manifolds & Levi-Civita connection

Definition A.7 Given a vector bundle $E \to M$, a *metric on* E is a smooth bilinear map $g\colon E \oplus E \to \mathbb{R}$ that restricts to an inner product $g_x\colon E_x \oplus E_x \to \mathbb{R}$ at every point $x \in M$.$\diamond$

Definition A.8 A *Riemmanian manifold* (M, g) is a manifold M together with a metric g on TM. $\diamond$

Definition A.9 For a vector bundle $E \to M$ with a metric g and a connection ∇, we say that *the connection is compactible with the metric* if the parallel transport is an isometry.$\diamond$

Definition A.10 For any connection ∇ on TM we define the *torsion* to be the anti-symmetric bilinear map

$$T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \qquad \diamond$$

Theorem A.11 [4, Thm 1.43, p.38] For any Riemmanian manifold (M, g) there is a unique connection ∇ on TM which is compatible with the metric and is torsion-free ($T_\nabla = 0$). It is called the *Levi-Civita connection of* (M, g). $\diamond$

A.3 Hodge theory of a Riemmanian manifold

Using a Riemannian metric on a compact oriented manifold, there is a structured way to give representatives for cohomology classes, this is called the Hodge theory. Since choosing representatives is a business we're interested in (and indeed we use Hodge theory a few times in the exposition), we discuss the basics here.

First, start with an oriented inner product space V of rank n. Then there is a unique operator $*\colon \Lambda^\bullet V \to \Lambda^{n-\bullet} V$ with the property that for every oriented orthonormal basis $(e_1, ..., e_n)$ we have that

$$*(e_1 \wedge \cdots \wedge e_p) = e_{p+1} \wedge \cdots e_n$$

Then $**$ acts as $(-1)^{p(n-p)}$ on $\Lambda^p V$. Then for an oriented Riemannian manifold (M, g) doing this pointwise to $\Lambda^\bullet T^* M$ clearly takes smooth forms to smooth forms and hence we are justified in making the following definition:

Definition A.12 For an oriented Riemmanian manifold (M, g) of dimension n. The *Hodge star operator* is the operator $*\colon \Omega^\bullet \to \Omega^{n-\bullet}$ that pointwise acts as $*\colon \Lambda^\bullet(T_p^*M, g_p^*) \to \Lambda^{n-\bullet}(T_p^*M, g_p^*)$. ◇

Using the Hodge dual, we can define a codifferential and an important degree 0 operator:

Definition A.13 For an oriented Riemmanian manifold (M, g) of dimension n, we define a codifferential $d^*\colon \Omega^\bullet(M) \to \Omega^{\bullet-1}(M)$ on a k-form ω, $d^*\omega = (-1)^{n(k+1)+1} * d * \omega$. Furthermore we define the *Laplace-Beltrami operator* of (M, g) by $\Delta = dd^* + d^*d\colon \Omega^\bullet(M) \to \Omega^\bullet(M)$.

A differential form $\omega \in \Omega^\bullet(M)$ is called *harmonic* if $\Delta\omega = 0$. We denote the space of harmonic forms of M by $\mathcal{H}^\bullet(M, g)$ ◇

The harmonic forms turn out to be the unique representatives of the cohomology classes:

Theorem A.14 (Hodge) [25, Thm 6.8, p.223; Thm 6.11, p.225] Let (M, g) be a compact oriented Riemmanian-manifold.

(a) A ω is harmonic if and only if $d\omega = d^*\omega = 0$.

(b) The following equation, called the *Hodge decomposition*, holds:

$$\Omega^p(M) = d(\Omega^{p-1}(M)) \oplus \mathcal{H}^p(M, g) \oplus d^*(\Omega^{p+1}(M))$$

(c) Every cohomology class contains a unique harmonic form. That is, the inclusion of cochain complexes $\mathcal{H}^\bullet(M, g) \to \Omega^\bullet(M)$ induces an isomorphism on cohomology. ◇

Remark A.15 If we define an inner product on $\Omega^p(M)$ by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta$$

the harmonic form in a cohomology class $x \in H^p(M)$ is the unique form in x which has minimal norm with respect to this inner product (c.f. Exercise 17-2 on page 464 of Lee [20]). ◇

B Appendix on topology

We briefly recall the definitions of homotopy, homology and cohomology, so that we can formulate some classic results in algebraic topology.

B.1 Definitions of homotopy, homology and cohomology

Definition B.1 Let (X, x_0) be a based space, we define the n^{th} *homotopy group* $\pi_n(X, x_0)$ to be set of continuous maps $(I^n, \partial I^n) \to (X, x_0)$ modulo homotopies $(I^n \times I, \partial I^n \times I) \to (X, x_0)$. If $n \geq 1$ we can put a natural group structure on this where we set $[f]+[g] = [f \star g]$ with

$$(f \star g)(t_1, ..., t_n) = \begin{cases} f(2t_1, ..., t_n) & \text{if } t_1 \leq \frac{1}{2} \\ g(2t_1 - 1, ..., t_n) & \text{if } t_1 \geq \frac{1}{2} \end{cases}$$

This group structure is abelian if $n \geq 2$. ◇

Definition B.2 We define the *standard simplex* Δ^n by

$$\Delta^n = \left\{ (x_0, ..., x_n) \in \mathbb{R}^{n+1} | x_0, ..., x_n \geq 0, \sum_i x_i = 1 \right\}$$

Furthermore we distinguish $\lambda_i \colon \Delta^n \to \Delta^{n+1}$, $\rho_j \colon \Delta^{n+1} \to \Delta^n$:

$$\lambda_i(x_0, ..., x_n) = (x_0, ..., x_{i-1}, 0, x_i, ..., x_n)$$
$$\rho_j(x_0, ..., x_n) = (x_0, ..., x_{j-1}, x_{j+1}, ..., x_n)$$

For a topological space X, we set the simplicial set $S(X)$ by

$$S(X)_n = \{\sigma \colon \Delta^n \to X | \, \sigma \text{ continuous}\}$$

with face and degeneracy maps pre-composition by λ_i and ρ_j.
Then for an abelian group A we define the chain complex $(C_*(X; A), d)$ where $C_n(X; A)$ consists A-linear combinations of $S(X)_n$, and where we define $d \colon C_n(X; A) \to C_{n-1}(X; A)$ by

$$d(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ \lambda_i$$

We define the n^{th} *homology group (with coefficients in A)* by

$$H_n(X; A) = \frac{\ker(d \colon C_n(X; A) \to C_{n-1}(X; A))}{\operatorname{im}(d \colon C_{n+1}(X; A) \to C_n(X; A))}$$ ◇

Definition B.3 Let X be a topological space, R a commutative ring and M an R-module. We define the cochain complex $C^*(X;M)$ to be $(C^*(X;M),\delta) = \mathrm{Hom}_R((C_*(X;R),d),M)$. This means that

$$C^n(X;M) = \mathrm{Hom}_R(C_n(X;R),M)$$

$$\delta = d^*\colon\ \mathrm{Hom}_R(C_n(X;R),M) \to \mathrm{Hom}_R(C_{n+1}(X;R),M)$$

Then we define the n^{th} *cohomology group (with coefficients in M)* to be:

$$H^n(X;M) = \frac{\ker(\delta\colon C^n(X;M) \to C^{n+1}(X;M))}{\mathrm{im}(\delta\colon C^{n-1}(X;M) \to C^n(X;M))}$$

The cohomology comes with a product, called the *cup product*, which can be described as follows. Consider the inclusions $f_p^{p+q}\colon \Delta^p \to \Delta^{p+q}$ sending $\vec{x}$ to $(\vec{x},0,...,0)$ and $g_q^{p+q}\colon \Delta^q \to \Delta^{p+q}$ sending $\vec{x}$ to $(0,...,0,\vec{x})$. Then we define a product $C^p(X;M) \times C^q(X;M) \to C^{p+q}(X;M)$ by

$$(c^p \times c^q)(\sigma) = c^p(\sigma \circ f_p^{p+q}) \cdot c^q(\sigma \circ g_q^{p+q})$$

with respect to this product, δ satisfies a Leibniz-like equation, and so we get a product $H^p(X;M) \times H^q(X;M) \to H^{p+q}(X;M)$. On the level of cohomology this product is graded commutative, while in general it isn't on the level of cochains. ◇

Definition B.4 For a manifold M we endow the space of differential forms with the natural exterior derivative locally defined by

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

together with Leibniz and $d^2 = 0$. In particular we get a cochain complex (Ω^*, d), the *deRham cohomology* of M is then defined by

$$H^k_{dR}(M) = \frac{\ker(d\colon \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{im}(d\colon \Omega^{k-1}(M) \to \Omega^k(M))}$$

The wedge product of forms induces a graded commutative product on $H^*_{dR}(M)$. ◇

B.2 Connections between homotopy, homology and cohomology

Definition B.5 Fix a homeomorphism $f\colon (\Delta^n, \partial\Delta^n) \to (I^n, \partial I^n)$. Then for any map $\sigma\colon (I^n, \partial I^n) \to (X, x_0)$ we get an element of $C_n(X;\mathbb{Z})$, simply by looking at $1 \cdot (\sigma \circ f)$. One can check that σ then becomes a closed singular chain, and the homology class does only depend on the homotopy class of σ. The resulting map

$$h_n\colon \pi_n(X,x_0) \to H_n(X;\mathbb{Z})$$

is called the *Hurewicz map*.

Theorem B.6 (Hurewicz) [11, Thm 4.32, p.366] Let (X, x_0) be a connected based space. The Hurewicz map induces an isomorphism

$$(h_1)_{\mathrm{ab}} \colon \pi_1(X, x_0)_{\mathrm{ab}} \xrightarrow{\cong} H_1(X; \mathbb{Z})$$

If $\pi_1(X, x_0) = 0$ then $\pi_i(X, x_0) = 0$ for $1 \leq i \leq n$ if and only if $H_1(X; \mathbb{Z}) = 0$ for $1 \leq i \leq n$, and in this case the Hurewicz map is an isomorphism

$$h_{n+1} \colon \pi_{n+1}(X, x_0) \xrightarrow{\cong} H_{n+1}(X; \mathbb{Z}) \qquad \diamond$$

Definition B.7 Let R be a principal ideal domain, C_* a levelwise free chain complex and N a R-module. Then we define a map $\Phi \colon H^n(\mathrm{Hom}_R(C_*, N)) \to \mathrm{Hom}_R(H_n(C_*), N)$ by

$$\Phi([f])([c]) = f(c)$$

for f closed in $\mathrm{Hom}_R(C_n, N)$ and c closed in C_n. $\diamond$

Theorem B.8 (Universal coefficient theorem) [11, Thm 3.2, p.195] Let R be principal ideal domain and C_* a levelwise free chain complex. Then the map Φ described above is surjective with kernel isomorphic to $\mathrm{Ext}^1_R(H_{n-1}(C_*), N)$. The resulting short exact sequence

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(C_*), N) \to H^n(\mathrm{Hom}_R(C_*, N)) \xrightarrow{\Phi} \mathrm{Hom}_R(H_n(C_*), N)) \to 0$$

splits non-naturally. $\diamond$

Remark B.9 We will not describe the Ext-functor here, but we do note that for the case $R = \mathbb{Z}$ and N a field $\mathrm{Ext}^1_R(-, N)$ always yields 0. So then the map Φ yields an isomorphism $H^n(\mathrm{Hom}_{\mathbb{Z}}(C_*, N)) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(C_*), N)$. $\diamond$

Definition B.10 Let M be a manifold, and consider the chain complex $C^\infty_*(M)$ consisting of $\mathbb{R}$-linear combinations of smooth simplices. The *deRham homomorphism* is a map $\mathcal{J} \colon \Omega^k(M) \to \mathrm{Hom}(C^\infty_k(M), \mathbb{R})$ set by:

$$\mathcal{J}(\omega)(\sum_i \lambda_i \sigma_i) = \sum_i \lambda_i \int_{\Delta^k} \sigma_i^* \omega$$

Lemma B.11 The inclusion $C^\infty_*(M) \to C_*(M; \mathbb{R})$ induces an isomorphism on homology. $\diamond$

Theorem B.12 (deRham) [20, Thm 18.14] The deRham cohomology induces an isomorphism $H^k_{dR}(M) \to \mathrm{Hom}(H^\infty_k(M), \mathbb{R}) \cong \mathrm{Hom}(H_k(M), \mathbb{R}) \cong H^k(M, \mathbb{R})$. Under this isomorphism the wedge product and the cup product coincide. $\diamond$

Lemma B.13 An orientation on an n-dimensional manifold M is defined by a nowhere vanishing form $\omega \in \Omega^n(M)$, called the *volume form*. The volume form is uniquely defined up to multiplication by a strictly positive smooth function. $\diamond$

Theorem B.14 (Poincaré duality) [25, Thm 6.13, p.226] Let M be a compact oriented n-dimensional manifold with volume form ω. The top-degree cohomology $H^n_{dR}(M)$ is one-dimensional with generator $[\omega]$. Furthermore the product induces perfect pairings

$$\wedge \colon H^k(M) \otimes H^{n-k}(M) \to H^n(M) \cong \mathbb{R}$$

In particular $H^k(M)$ and $H^{n-k}(M)$ are each others dual, and hence have the same rank. $\diamond$

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