Master's Thesis

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Singularities in Spacetimes with Torsion

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Abstract

We study the singularity theorems of Hawking and Penrose in spacetimes with non-vanishing torsion (Einstein-Cartan theory). Assuming that test particles move on geodesics we manage to generalize those theorems to the case of totally anti-symmetric torsion. In doing so we derive the generalized Raychaudhuri equation for timelike geodesics and arbitrary torsion. For totally anti-symmetric torsion we find that the vorticity of a geodesic is directly related to the torsion. We find that the theorems are not directly extendable to spacetimes that have non totally anti-symmetric torsion; other arguments are needed. However, we formulate a construction that enables one to create a null geodesically incomplete spacetime with vectorial torsion starting from a spacetime that is null geodesically incomplete with respect to the Levi-Civita connection.

The geometric assumptions of the singularity theorems can be translated to assumptions on the matter content of spacetime via the equations of motion of the theory. We studied the two ways of deriving these equations. It turns out that for totally anti-symmetric torsion the metric formalism, in which one takes the metric and torsion as dynamical variables and assumes metric compatibility, is equivalent to the metric-affine formalism, in which one takes the metric and connection as dynamical variables and does not assume metric compatibility. Matter in the Standard model induces totally anti-symmetric torsion. We also generalize the Bianchi identity and conservation of the energy-momentum tensor in general relativity to Einstein-Cartan theory.

We examine the singularity theorems in FLRW spacetimes (assuming vanishing torsion). We argue that one should not use the general definition of a singularity in these spacetimes but should define a singularity as a comoving geodesic that is incomplete. We prove theorems for FLRW spacetimes about the relation between conjugate points and a singularity. In particular we show that for a class of singular spacetimes all points on certain geodesics are conjugate to the point at the singularity. Also when every point on a geodesic is conjugate to a certain point, one must have a singularity. Lastly we show under which condition a geodesic in an FLRW spacetime with flat spacelike three-surfaces has conjugate points.

Contents

Introduction

Einstein's theory of general relativity has been around for over a 100 years now. There has been a lot of experimental evidence that proves the validity of this theory on macroscopic scales. Within the solar system (weak field limit), precession of perihelia of planets, deflection of light and gravitational redshift can be explained with general relativity. The theory has also been tested on larger scales, by explaining gravitational lensing effects that can be observed in distant astrophysical sources and for stronger fields, by explaining periapsis precession of pulsars around other stars.

Some intriguing objects predicted by general relativity are singularities. Loosely speaking they are points where spacetime stops being a manifold and that can be reached by objects or light rays in a finite amount of time. Massive particles and photons move on timelike geodesics and null geodesics respectively, so a more rigorous definition of a singularity is that there exists a non-spacelike geodesic for which the affine parameter cannot be extended to infinity (we call a geodesic incomplete if that is the case). Solutions in general relativity that are singular are for instance the Schwarzschild metric, the Reissner-Nordström metric, the Kerr metric and the Kerr-Newman metric which all describe black holes. The big bang at the beginning of the universe is also an example of a singularity. Although these singular solutions theoretically exist, it is not immediately clear that they really exist in our physical universe. If one considers the collapse of a star to a black hole, in principle it would be possible that rotation of that star counteracts the collapse and keeps the singularity from forming. This changed with the theorems of Hawking and Penrose [1, 2, 3]. They proved theorems for general relativity that under reasonable conditions on the matter content of the universe singularities will always form. In the example of the collapse of the star, the theorems state that it is impossible to prevent the singularity from forming once an event horizon is formed. The presence of an event horizon (or a black hole) can be observed from the interaction with the matter around it. The gas falling into a black hole will form a disk-like structure (accretion disk) because of angular momentum conservation. Due to friction within the disk angular momentum is transported outward such that matter can fall into the black hole. This releases potential energy causing the gas near the black hole to become so hot that it emits enough radiation to be detected by telescopes. Many of the more energetic phenomena in the universe are explained by accretion disks of black holes. The active galactic nuclei are such phenomena. It is now widely accepted that nearly all galaxies have a super massive black hole at the center (masses $10^5 - 10^9$ $\mathrm{M_{Sun}}$, where $\mathrm{M_{Sun}}$ is a solar mass) (e.g. [4]). Recently gravitational waves have been detected directly by the Advanced Ligo team [5]. Those waves were generated from a black hole merger, so besides being the observation of gravitational waves, this was also the first direct detection of a binary black hole merger.

Besides its successes, general relativity is not complete yet. On large scales, cosmological observations imply that roughly 95 percent of the universe that is measured via its gravitational interaction must be dark matter or dark energy, which basically means that it is unknown what it is. If the Standard model is complete, general relativity has to be adapted on large scales to account for this extra energy. If general relativity should be trusted, there is a need for extra particles to be added to the Standard model. A second problem appears when one wants to combine general relativity with another main pillar of modern physics: quantum mechanics. Both theories have to be applied at the same time when one considers large energy on small scales, for instance when one studies black holes. Unfortunately, we are still far from a concrete theory that unifies relativity with quantum mechanics.

The above mentioned sources of trouble with general relativity motivate to examine modifications of the theory in such a way that its predictions on the solar system scale do not change. In general relativity the metric $g_{\mu\nu}$ is fixed by the matter content of spacetime. The connection $\Gamma^{\rho}_{\mu\nu}$ is fully expressed in terms of the metric by requiring metric compatibility and vanishing torsion $S^{\rho}{}_{\mu\nu} = 2\Gamma^{\rho}_{[\mu\nu]}$. One modification one can make in general relativity is to loosen these restrictions. Metric compatibility makes sense from a physical point of view: the inner product of two vectors that do not change, should stay the same. Vanishing torsion however, cannot be motivated in such a way. This has been put to zero originally because torsion only modifies gravity on small scales [6], scales that so far are out of reach of experiments. Because of the problems that appear when combining general relativity with quantum mechanics, modification on small scales is exactly what we are looking for, which motivates to examine the theory with non-vanishing torsion. In general relativity, the geometric quantity that describes curvature is coupled to the energy-momentum tensor of matter, which describes the distribution of energy. One can couple torsion to the spin tensor of matter:

$$\Pi_\rho{}^{\mu\nu} = \frac{-4}{\sqrt{-\det g}} \frac{\delta S_{\mathrm{m}}}{\delta S^\rho{}_{\mu\nu}}, \tag{1}$$

where S_{m} is the matter action. It turns out that the spin tensor is only non-vanishing (for matter from the Standard model) for particles with spin, therefore we can say it describes the distribution of spin. We want to stress that torsion is not coupled to the spin of macroscopic objects as planets, it is merely coupled to some intrinsic property of matter. The theory which has torsion and the metric as dynamical variables was first proposed by E. Cartan [7, 8] and in other works [9, 10] and it has the name Einstein-Cartan theory.

In this thesis we will study what happens with the singularity theorems of Hawking and Penrose when one considers Einstein-Cartan theory. One question that needs to be answered is what trajectories we want to consider for the definition of a singularity. In general relativity we use the trajectories of massive particles and light as motivation to define a singularity as an incomplete non-spacelike geodesic. It is not clear that when torsion is non-vanishing particles and light still move on geodesics. In the literature there have been two opinions:

1. particles and light move on geodesics, e.g. [11, 12];
2. particles and light move on curves of maximal length, e.g. [6, 13].

When torsion is totally anti-symmetric, the set of maximal curves and the set of geodesics are the same, but for more general torsion this is not the case. When one assumes the second case, there is an easy extension of the singularity theorems (e.g. [14]) by using that one can integrate out torsion from the equations (all of this will become more clear in this thesis). However, we will use a more cumbersome approach by directly considering all of the propositions needed to prove the singularity theorems for general relativity and see whether we can generalize them. We do this for two reasons: firstly because this will give a different proof of what already has been done in [14], secondly because we also want to see what happens when one assumes opinion 1). We will give the full proof for totally anti-symmetric torsion and then discuss the case of more general torsion. We will give a way to construct a null geodesically incomplete spacetime with vectorial torsion.

The singularity theorems are geometrical in nature and to see whether their assumptions make sense, we need a translation to the matter content of spacetime. This translation happens via the equations of motion of Einstein-Cartan theory. There are however different ways of deriving these equations. In the metric formalism, the metric and torsion are treated as dynamical variables and metric compatibility is assumed. In the 'more natural' metric-affine formalism the metric and connection are treated as dynamical variables and one has to see whether metric compatibility is enforced by nature via the equations of motion. We will actually see that for totally anti-symmetric torsion the two formalisms are equivalent. We will also see that matter of the Standard model only induces totally anti-symmetric torsion. In general relativity one has the Bianchi identity and conservation of the energy-momentum tensor of matter. We will also derive generalizations of those identities.

The Friedmann-Lemaître-Robertson-Walker (FLRW) metric is the metric that describes a spatially homogeneous, isotropic spacetime. This metric is a good description of our universe, since from experiments as WMAP and Planck, it follows that our universe is spatially homogeneous and isotropic when averaged over large scales. Torsion can be introduced in an FLRW spacetime as an energy density. Since this spacetime has a lot of symmetry it can be used to study the singularity theorems in this spacetime. We will discuss the definition of (initial or big bang) singularities in this spacetime. It actually turns out that we should forget about certain geodesics when looking for incomplete ones, since the particles that follow those geodesics have an energy that becomes arbitrarily large when going back in the past. This means that their energy will become larger than the Planck energy at some time in the past and when that happens, a black hole will form, breaking the description of that particle.

One of the assumptions of all the singularity theorems implies that when a geodesic is complete it has conjugate points. Having conjugate points basically means that there is a one-parameter family of geodesics that leaves from one of those points and comes back to the other point. We will study this occurrence of conjugate points in FLRW spacetimes with and without initial singularity. We prove theorems that show that for a certain class of singular FLRW models all points on certain geodesics are conjugate to the point of the geodesic at the singularity. We also argue that when it happens that all points are conjugate to one point, one must have a singularity. Lastly, we prove a theorem that gives the conditions under which a geodesic in an FLRW spacetime with flat spacelike three-surfaces has conjugate points. After proving these theorems we introduce torsion in FLRW spacetimes and see what happens in combination with a perfect matter fluid and a perfect radiation fluid. We will find that a singularity is avoided.

As far as we know, the following results in this thesis are new:

- The direct generalization of the Einstein-Penrose singularity theorems to totally anti-symmetric torsion, but we want to stress once again that the generalized theorems can also be proven in another way.
- The construction of null geodesically incomplete spacetimes with vectorial torsion.
- A derivation of the Raychaudhuri equation for timelike geodesics and arbitrary torsion. This equation is needed for the proof of the singularity theorems. We find an extra term with respect to the literature, because we use a different approach.
- Section 6.2 about the definition of a singularity in FLRW spacetimes. This section can also be found in our paper [15].
- Section 6.3 in which we prove several theorems relating conjugate points to singularities in FLRW spacetimes.

The thesis is organized as follows. In Chapter 1 we give a review of general relativity, introduce Einstein-Cartan theory and discuss the definition of singularities. In Chapter 2 and 3 we generalize propositions that are needed to prove the singularity theorems for totally anti-symmetric torsion. In Chapter 2 we do this with propositions related to conjugate points and in Chapter 3 with propositions related to the length of geodesics. After that, in Chapter 4 we review the metric and metric-affine formalisms and examine when they are equivalent. We also derive the generalizations of the Bianchi identity and conservation of the energy-momentum tensor in general relativity. In Chapter 5 we finally state the singularity theorems, discuss their assumptions and extendibility to non totally anti-symmetric torsion and give a construction to find null geodesically incomplete spacetimes with vectorial torsion. In Chapter 6 we study singularities in FLRW spacetimes, prove some theorems about the relation between conjugate points and singularities and treat torsion introduced as an energy density as example. We end with a conclusion, where we will also discuss some possibilities for future work.

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Notation and Conventions

- Units are such that $\hbar = c = 1$.
- Lorentzian metrics $g_{\mu\nu}$ will have signature $(-1, 1, 1, 1)$.
- Index summation notation:
$$T_{\dots\mu\dots}Y^{\dots\mu\dots} = \sum_{\mu} T_{\dots\mu\dots}Y^{\dots\mu\dots}.$$
- Symmetrization and anti-symmetrization:
$$\begin{aligned} T_{(\mu\nu)} &= \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right); \\ T_{[\mu\nu]} &= \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right). \end{aligned}$$
- Determinant of the metric $g_{\mu\nu}$ is $\det g$.
- Geodesics are denoted by $\gamma(\tau)$ where τ is an affine parameter. Also
$$\dot{\gamma} = \frac{d}{d\tau}\gamma.$$
- The covariant derivative is denoted by ∇ and the covariant derivative along a curve (or parametrized vector field) parametrized by τ by $\nabla_{\dot{\gamma}} = D_\tau$.
- Riemann curvature tensor
$$R^{\rho}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\lambda} + \Gamma^{\beta}_{\nu\lambda}\Gamma^{\rho}_{\mu\beta} - \partial_\nu \Gamma^{\rho}_{\mu\lambda} - \Gamma^{\beta}_{\mu\lambda}\Gamma^{\rho}_{\nu\beta}.$$
- Torsion:
$$\begin{aligned} S^{\rho}{}_{\mu\nu} &= 2\Gamma^{\rho}_{[\mu\nu]}; \\ S_\nu &= S^{\rho}{}_{\rho\nu}. \end{aligned}$$
- Quantities with respect to the Levi-Civita connection
$$\Gamma^{\rho}_{\mu\nu} = \left\{{}^{\rho}_{\mu\nu}\right\} \equiv \frac{1}{2}g^{\rho\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right)$$
are denoted by $\breve{}$, e.g. $\breve{R}_{\mu\nu}$ are the components of the Ricci tensor with respect to the Levi-Civita connection.
- Energy-momentum tensor:
$$T_{\mu\nu} = \frac{-2}{\sqrt{-\det g}}\frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}}.$$

- Spin tensor:

$$\begin{aligned}\Pi_\rho{}^{\mu\nu} &= \frac{-4}{\sqrt{-\det g}}\frac{\delta S_{\mathrm{m}}}{\delta S^\rho{}_{\mu\nu}};\\ \Pi^\nu &= \Pi_\rho{}^{\rho\nu}.\end{aligned}$$

- Variation of the matter action with respect to the connection:

$$\Delta_\rho{}^{\mu\nu} = \frac{-2}{\sqrt{-\det g}}\frac{\delta S_{\mathrm{m}}}{\delta\Gamma^\rho_{\mu\nu}}.$$

Chapter 1

Review of the Theory

In this chapter we briefly review the theory. We first look at general relativity in Section 1.1 where we introduce a lot of geometric quantities needed in the rest of this thesis. After that we will also consider the matter content of the universe. In Section 1.2 we then introduce Einstein-Cartan theory. Lastly, in Section 1.3 we give the definition of a singularity and compare this definition with other possibilities.

1.1 General Relativity

With the theory of General Relativity Einstein tried to explain gravity by curvature of spacetime. The mathematical model that is used for spacetime is a pair (M, g) where M is a connected four-dimensional smooth manifold and g is a Lorentz metric on M, i.e. a metric with signature (1,3). We will denote the tangent space of M by TM and a local frame of TM induced by a coordinate system, by ∂_μ. For some more background on manifold theory, see the appendix.

Spacetimes (M, g), (M', g') are equivalent when there is a diffeomorphism $\phi : M \to M'$ such that $\phi_* g' = g$. Such a function is also called an isometry. A pair (M', g') is an extension of (M, g) if there is an isometric imbedding $\phi : M \to M'$. We require the model to be inextendible, so if an extension exists we should have that $\phi(M) = M$. This is to ensure that all non-singular points of spacetime are included.

A vector field is a section of the tangent bundle. We will denote the space of vector fields on the manifold by $\mathcal{T}(M)$ and the space of vector fields along a (smooth) curve $\gamma : [\tau_\mathrm{i}, \tau_\mathrm{f}] \to M$ by $\mathcal{T}(\gamma)$. To differentiate vector fields in a coordinate independent way, a connection ∇ is introduced. Let now $\{E_\mu\}$ be a local frame for TM. The well-known Christoffel symbols $\Gamma^\rho_{\mu\nu}$ are then defined as

$$\nabla_{E_\mu} E_\nu = \Gamma^\rho_{\mu\nu} E_\rho. \tag{1.1}$$

Differentiation of a vector field $V = V^\beta \partial_\beta$ along the vector field $X = X^\alpha \partial_\alpha$ yields

$$\nabla_X V = X^\mu \nabla_{\partial_\mu} (V^\nu \partial_\nu) = X^\mu \left(\partial_\mu V^\nu\right) \partial_\nu + X^\mu V^\nu \Gamma^\rho_{\mu\nu} \partial_\rho. \tag{1.2}$$

So in coordinates we get

$$\nabla_\mu V^\rho \equiv (\nabla V)^\rho_\mu = \partial_\mu V^\rho + \Gamma^\rho_{\mu\nu} V^\nu. \tag{1.3}$$

It can be shown that when $p \in M$, $\nabla_X V(p)$ only depends on $X(p)$ and on the value of V in a neighborhood of p. If we differentiate along a curve $\gamma(\tau)$ or a parametrized vector field we will use the notation D_τ for the covariant derivative. A useful construction is parallel translation. We say that a vector field V along a curve γ is parallel along γ if $D_\tau V = 0$. A curve $\gamma(\tau)$ is called a *geodesic curve* when $D_\tau \dot\gamma = f \dot\gamma$ and a curve is called a *geodesic* when

$$D_\tau \dot\gamma = 0, \tag{1.4}$$

or in other words: when its tangent vector field is parallel along the curve. The parametrization τ of a geodesic is defined by (1.4) up to an affine transformation: $\tau' = \alpha\tau + \beta$, where α, β are constants. A geodesic curve can always be reparametrized such that it becomes a geodesic and vice versa. Writing Eq. (1.4) in coordinates one finds a 2nd order linear differential equation and such an equation can always be solved locally. So let $p \in M$ and $X \in T_pM$ then there is a geodesic $\gamma(\tau)$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X$; we denote this geodesic with γ_X. It can be proven that there is a neighborhood N_0 of $0 \in T_pM$ that is diffeomorphic to a neighborhood N_p of p via the exponential map $\exp_p(X) = \gamma_X(1)$. Furthermore, N_p can be chosen such that all points $q, r \in N_p$ can be connected by a unique geodesic that lies in N_p; such a neighborhood will be called a *convex normal neighborhood*.

Using the connection we can define the torsion tensor:

$$S : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M) \quad S(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{1.5}$$

In components we get

$$S^\rho{}_{\mu\nu} = S(\partial_\mu, \partial_\nu)(dx^\rho) = \Gamma^\lambda_{\mu\nu}\partial_\lambda(dx^\rho) - \Gamma^\lambda_{\nu\mu}\partial_\lambda(dx^\rho) = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}. \tag{1.6}$$

Torsion can be decomposed as

$$S^\rho{}_{\mu\nu} = {}^V S^\rho{}_{\mu\nu} + {}^A S^\rho{}_{\mu\nu} + {}^T S^\rho{}_{\mu\nu}, \tag{1.7}$$

where the vectorial part ${}^V S^\rho{}_{\mu\nu}$, totally anti-symmetric part ${}^A S^\rho{}_{\mu\nu}$ and traceless part ${}^T S^\rho{}_{\mu\nu}$ are defined as

$$\begin{aligned} {}^V S^\rho{}_{\mu\nu} &= \frac{1}{3}\left(\delta^\rho_\mu S_\nu - \delta^\rho_\nu S_\mu\right); \\ {}^A S^\rho{}_{\mu\nu} &= g^{\rho\lambda} S_{[\lambda\mu\nu]}; \\ {}^T S^\rho{}_{\mu\nu} &= \frac{2}{3}\left(S^\rho{}_{\mu\nu} - S_{[\mu\nu]}{}^\rho - S_{[\nu}\delta^\rho_{\mu]}\right). \end{aligned} \tag{1.8}$$

Notice that all of these parts define a torsion by itself.

We can also define the Riemann curvature tensor R:

$$R : \mathcal{T}(M) \times \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M) \quad R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z. \tag{1.9}$$

Components are defined as

$$\begin{aligned} R^\rho{}_{\lambda\mu\nu} &= (R(\partial_\mu, \partial_\nu)\partial_\lambda)^\rho \\ &= [\nabla_\mu(\nabla_\nu\partial_\lambda) - \nabla_\nu(\nabla_\mu\partial_\lambda)]^\rho \\ &= \left[\nabla_\mu(\Gamma^\alpha_{\nu\lambda}\partial_\alpha) - \nabla_\nu(\Gamma^\alpha_{\mu\lambda}\partial_\alpha)\right]^\rho \\ &= \partial_\mu\Gamma^\rho_{\nu\lambda} + \Gamma^\beta_{\nu\lambda}\Gamma^\rho_{\mu\beta} - \partial_\nu\Gamma^\rho_{\mu\lambda} - \Gamma^\beta_{\mu\lambda}\Gamma^\rho_{\nu\beta}. \end{aligned} \tag{1.10}$$

Contracting the curvature tensor, one can define the Ricci tensor Ric as the tensor with components

$$R_{\lambda\nu} = R^\rho{}_{\lambda\rho\nu}. \tag{1.11}$$

Taking the trace of the Ricci tensor we find the Ricci scalar

$$R = R^\mu{}_\mu = g^{\mu\nu}R_{\mu\nu}. \tag{1.12}$$

In the discussion of the connection so far, we have not been talking about the metric. Since we normally define the angle between two vectors via the metric, it seems natural to require that the 'inner product' does not change under parallel translation along a curve γ. Or in other words we require

$$\frac{d}{d\tau}g(V,W) = g(D_\tau V, W) + g(V, D_\tau W) \tag{1.13}$$

for an arbitrary curve γ and $V, W \in \mathcal{T}(\gamma)$. This property is equivalent to

$$\nabla_X g = 0 \tag{1.14}$$

for all $X \in \mathcal{T}(M)$ and using this equation it is possible to express the Christoffel symbols via the torsion and metric:

$$\Gamma^\rho_{\mu\nu} = \{^{\ \rho}_{\mu\nu}\} - S_{(\mu\nu)}{}^\rho + \frac{1}{2}S^\rho{}_{\mu\nu}, \tag{1.15}$$

where

$$\left\{{}^{\rho}_{\mu\nu}\right\} = \frac{1}{2} g^{\rho\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \tag{1.16}$$

is the Levi-Civita connection. Indeed, using Eq. (1.14) we can write

$$\begin{aligned} 0 &= \frac{1}{2} g^{\rho\alpha} \left(\nabla_\mu g_{\alpha\nu} + \nabla_\nu g_{\alpha\mu} - \nabla_\alpha g_{\mu\nu}\right) \\ &= \left\{{}^{\rho}_{\mu\nu}\right\} - \frac{1}{2} g^{\rho\alpha} \left(\Gamma^\beta_{\mu\alpha} g_{\beta\nu} + \Gamma^\beta_{\mu\nu} g_{\alpha\beta} + \Gamma^\beta_{\nu\alpha} g_{\beta\mu} + \Gamma^\beta_{\nu\mu} g_{\alpha\beta} - \Gamma^\beta_{\alpha\mu} g_{\beta\nu} - \Gamma^\beta_{\alpha\nu} g_{\mu\beta}\right) \\ &= \left\{{}^{\rho}_{\mu\nu}\right\} - \frac{1}{2} \left(g^{\rho\alpha} S^\beta{}_{\mu\alpha} g_{\beta\nu} + \Gamma^\rho_{\mu\nu} + g^{\rho\alpha} S^\beta{}_{\nu\alpha} g_{\beta\mu} + \Gamma^\rho_{\nu\mu}\right) \\ &= \left\{{}^{\rho}_{\mu\nu}\right\} - \frac{1}{2} \left(2\Gamma^\rho_{\mu\nu} - S^\rho{}_{\mu\nu} + S_{\nu\mu}{}^\rho + S_{\mu\nu}{}^\rho\right). \end{aligned} \tag{1.17}$$

Hence

$$\begin{aligned} \Gamma^\rho_{\mu\nu} &= \left\{{}^{\rho}_{\mu\nu}\right\} + \frac{1}{2} \left(S^\rho{}_{\mu\nu} - S_{\nu\mu}{}^\rho - S_{\mu\nu}{}^\rho\right) \\ &= \left\{{}^{\rho}_{\mu\nu}\right\} - S_{(\mu\nu)}{}^\rho + \frac{1}{2} S^\rho{}_{\mu\nu}. \end{aligned} \tag{1.18}$$

Apart from metric compatibility, in General Relativity we also assume that the torsion is zero. So the connection reduces to the Levi-Civita connection, which is completely determined by the metric.

Using the metric, the vectors in T_pM at a point $p \in M$ can be divided into three classes:

Definition 1.1.1. A vector $X \in T_pM$ is called *timelike* if $g(X, X) < 0$, it is called *spacelike* if $g(X, X) > 0$ and it is called a *null vector* if $g(X, X) = 0$. Timelike, spacelike and null curves are then defined as curves γ such that $g(\dot\gamma, \dot\gamma)$ is negative, positive or zero respectively for the tangent vector field along the curve.

Notice that with metric compatibility, the norm of the tangent vector of a geodesic γ is constant:

$$\partial_\tau g\left(\dot\gamma, \dot\gamma\right) = 2g\left(D_\tau \dot\gamma, \dot\gamma\right) = 0. \tag{1.19}$$

This implies that we can define timelike, null and spacelike geodesics. We can also choose (using the freedom of an affine transformation) the parametrization of the geodesic such that $g\left(\dot\gamma, \dot\gamma\right) = -1$ for timelike geodesics, $g\left(\dot\gamma, \dot\gamma\right) = 1$ for spacelike geodesics and $g\left(\dot\gamma, \dot\gamma\right) = 0$ for null geodesics. Defining the length of a timelike curve γ between $\gamma(\tau_0)$ and $\gamma(\tau_1)$ as

$$L(\gamma) = \int_{\tau_0}^{\tau_1} \sqrt{-g(\dot\gamma, \dot\gamma)}\, d\tau, \tag{1.20}$$

we see that for a timelike geodesic with parameter such that $g\left(\dot\gamma, \dot\gamma\right) = -1$, we get that

$$L(\gamma) = \tau_1 - \tau_0, \tag{1.21}$$

which is why we call this parameter τ the proper time, or length of the geodesic.

So far we have only talked about the geometry of spacetime, but in the real world everything is made out of matter. Matter is described via fields on M (for instance a complex function or a vector field) and these fields obey equations which can be expressed as relations between tensors or spinors on M. We will now discuss three postulates for a spacetime:

Local causality

The equations that describe the matter field must be such that in a convex normal neighborhood U a signal between points $p, q \in U$ can only be send when p and q can be connected by a curve that is either timelike or null; such a curve is called non-spacelike.

Local conservation of energy and momentum

For the matter fields one can define a tensor T, the energy-momentum tensor, which depends on the fields, their covariant derivatives and the metric, and which has the properties:

1. T vanishes on an open set U if and only if all the matter fields vanish on U;
2. T obeys the equation $\nabla_\mu T^{\mu\nu} = 0$;
3. $T^{\mu\nu} = T^{\nu\mu}$.

The first property is that every field has a non-zero energy, the second property makes sure that energy and momentum are locally conserved.

Field equations

The following equation holds on M in general relativity:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu}, \tag{1.22}$$

where Λ is the so called cosmological constant (so it is a constant) and G_{N} is Newton's constant. This is the Einstein equation and the predictions of this equation agree, within the experimental errors, with the observations done so far. Notice that in general relativity, we can find the metric when we know the energy-momentum tensor (and have suitable boundary conditions), but the metric influences how particles move, such that the energy-momentum tensor changes. So in general, Eq. (1.22) is hard to solve. Only in situations with a lot of symmetry, this equation can be solved analytically.

1.2 Einstein-Cartan Theory

As mentioned before, in general relativity, the torsion tensor Eq. (1.5), is assumed to be vanishing. The theory with non-vanishing torsion is Einstein-Cartan theory [7, 8]. At the geometric side, the only difference with general relativity is in the connection that is now given by Eq. (1.15). The introduction of torsion is motivated because it modifies gravity only on small scales [6] where we encounter problems when we try to combine general relativity with quantum mechanics. On large scales where general relativity is experimentally confirmed, Einstein-Cartan theory predicts the same as Einstein's theory. In a theory with torsion the singularity at the beginning of the universe can be avoided [16]. Since the torsion is an extra component to the theory, we need an equation that relates the torsion to the matter field. A derivation of the new equations of motion that one gets in Einstein-Cartan theory can be found in Chapter 4.

1.3 Singularities[1]

There are many possible ways to define a singularity in spacetime, a first guess by analogy with electrodynamics would be to define them as a point where the metric tensor is not defined anymore or not differentiable enough. Unfortunately, those points should not be regarded as part of spacetime because one cannot do any experiments when the laws of physics are not defined anymore. This is also why we defined spacetime as smooth and not extendable. Spacetime without those points would be non-singular.

The question of determining of a spacetime is singular now becomes a question whether any singular points have been cut out. You would expect that this can be determined by the completeness of spacetime in any sense. Riemannian geometry, where the metric is positive definite, has two forms of completeness. First of all, one can define a distance function $d(x,y): M \times M \to \mathbb{R}$ where $d(x,y)$ is the greatest lower bound of the length of curves from x to y, where the length of a curve γ, with $\gamma(\tau_0) = x$ and $\gamma(\tau_1) = y$ is defined (notice the analogy with Eq. (1.20)) as

$$L(\gamma) = \int_{\tau_0}^{\tau_1} \sqrt{g(\dot{\gamma}, \dot{\gamma})} d\tau. \tag{1.23}$$

[1] This section is based on section 8.1 of [3].

Spacetime (M, g) is metrically complete if every Cauchy sequence with respect to the distance function converges to a point in M. There also exists a definition based on geodesics. As we mentioned before, for every point $p \in M$, $X \in T_pM$ there exists a geodesic $\gamma : U \to M$, where $U \subset \mathbb{R}$ is a neighborhood of 0, such that $\gamma(0) = p$, $\dot{\gamma}(0) = X$. If we can expand the domain of the geodesic to $\mathbb{R}$ the geodesic is called complete. In Riemannian geometry one has geodesic completeness if and only if one has metric completeness.

It is hard to find an analogy for metric completeness in Lorentzian geometry. One cannot define a distance function as in Riemannian geometry because if one defines the length of curves as in (1.20) this length can be positive and negative. Hence, one is left with geodesic completeness. There actually are 3 different kinds of geodesic completeness: null, spacelike and timelike geodesic completeness. One would hope the three forms of geodesic completeness are equivalent, but that is not the case [17]. One can construct examples that are incomplete in one of the three ways and complete in the other two. From a physical perspective only timelike and null geodesic completeness are really interesting. Massive test particles in general relativity are believed to follow timelike geodesics. So when such a geodesic is incomplete, that means that that test particle reaches the end of its trajectory in a finite time. This seems like a quite objectionable feature of spacetime and that is why such a spacetime can be regarded singular. Photons are believed to follow null geodesics and while the affine parameter of such a geodesic does not have the same interpretation as the affine parameter of a timelike geodesic, one should probably also regard spacetimes that have an incomplete null geodesic as singular. This can be motivated because the Reissner-Nordström solution that describes a charged black hole is timelike complete but not null geodesically complete.

When torsion is non-vanishing in most literature the opinion is that test particles follow extremal curves (curves of maximal length) with their tangent vector parallel transported with respect to the Levi-Civita connection (i.e. [6, 13]). In [12] it is claimed that the trajectories of point particles must be geodesics with respect to the full connection in order to obtain a consistent path integral. In [18] a new variational method is found that indeed yields these trajectories. Unfortunately, it is impossible up to now to falsify one of these claims experimentally. In this thesis, we will mainly use the second opinion in which we consider geodesics with respect to the connection with torsion. However, we will only be able to give full proofs for totally anti-symmetric torsion in which case both sets of trajectories are the same. We now define a singularity in the following way:

Definition 1.3.1. A spacetime has a *singularity* if it contains a non-spacelike incomplete geodesic.

For generality relativity, theorems have been proven [1, 2, 3] that state the existence of these singularities under rather general assumptions. We would like to generalize these results to spacetimes with torsion.

There still are some spacetimes that are geodesically complete, but should be considered singular. For example [19] constructed an example that is geodesically complete but contains an inextendible timelike curve of bounded acceleration and finite length. An observer with a suitable rocket ship can reach a point that is not in the manifold anymore in a finite amount of time. This is why one should actually define a singularity in such a way that also these kind of spacetimes are included. To do that one needs a generalization of the affine parameter to all continuously differentiable curves. This is indeed possible [20, 21], but we will not dive into this since we will not use it in the remainder of this thesis. Of course we have that non-spacelike geodesic incompleteness implies this kind of incompleteness.

As one might remember from a course on general relativity, singularities are mostly seen as points where one of the contractions of the Riemann curvature tensor (e.g. the Ricci scalar) blows up. Unfortunately, the singular point is excluded from the manifold, so it is hard to give a precise definition of this statement. It becomes even harder to prove theorems with such a definition. When one already has incomplete curves, one can use such a curve to give a statement about the blowing up of a contraction of curvature near a certain point (end of the curve). However, the Riemann curvature tensor is not completely determined by its contractions (Penrose pointed out that for plane-wave solutions all these contractions are zero, but curvature is not). Thus in principle it is possible that all contractions are finite but that curvature still blows up. Instead, one can use the blowing up of components of the curvature tensor in a parallel translated basis along a curve as definition for a singularity. In this thesis however, a singularity is defined as a non-spacelike geodesic that is incomplete.

Chapter 2

Conjugate points

It will be important for the proofs of the singularity theorems that families of geodesics that start from one point or start from one surface will start converging again in a sense that will be made precise later by the definition of conjugate points. That is why in this chapter, we study the behavior of families of geodesics and prove that under certain conditions we always have those conjugate points. Everything will be developed in dimension 4, but most of the results can easily be generalized to Lorentzian manifolds of other dimensions. We will mainly try to follow arguments from [3] and [22] (where they derived the results for vanishing torsion) and expand them where needed.

This chapter is organized as follows. We consider conjugate points on timelike geodesics, to a spacelike hypersurface, on null geodesics and to spacelike two-surfaces in Sections 2.1, 2.2, 2.3 and 2.4 respectively. Before that we derive some theory that is needed in all sections. In particular we will derive a generalized Jacobi equation. In Section 2.1 we will start with general torsion, but after that we will assume totally anti-symmetric torsion. We will also assume totally anti-symmetric torsion in Sections 2.2, 2.3 and 2.4. The main results of this chapter are the propositions that we prove (and are needed for the proofs of the singularity theorems in Chapter 5). Also the derivation of the Raychaudhuri equation, Eq. (2.32), for timelike geodesics and torsion is very interesting because as far as we know nobody used this more mathematical approach. This results in an extra term with respect to the literature.

In this thesis we will be using one-parameter families of geodesics that are defined as follows.

Definition 2.0.1. Let $\gamma : [\tau_{\rm i}, \tau_{\rm f}] \to M$ be a non-spacelike geodesic segment. A *variation* of γ is a smooth function $\Gamma : (-\epsilon, \epsilon) \times [\tau_{\rm i}, \tau_{\rm f}] \to M$, such that $\Gamma(0, \tau) = \gamma(\tau)$ for all $\tau \in [\tau_{\rm i}, \tau_{\rm f}]$. The *variation field* of Γ is the vector field $J(\tau) = \partial_w \Gamma(w, \tau)|_{w=0}$ along γ. We say that Γ is a *variation through geodesics* if every curve $\Gamma(w_0, \tau)$ is a geodesic segment. We will often use $T(w, \tau) \equiv \partial_\tau \Gamma(w, \tau)$ and $W(w, \tau) \equiv \partial_w \Gamma(w, \tau)$.

Proposition 2.0.1. *If a variation Γ is a variation through geodesics then its variation field $J(\tau)$ obeys the Jacobi equation:*

$$D_\tau^2 J + D_\tau S(J, \dot\gamma) + R(J, \dot\gamma)\dot\gamma = 0. \tag{2.1}$$

Proof. Starting with a variation through geodesics, we have that

$$D_\tau T = 0. \tag{2.2}$$

It can be shown (the proof in [23] is also valid in this case) that for a smooth vector field V along γ

$$D_w D_\tau V - D_\tau D_w V = R(W, T)V. \tag{2.3}$$

Using this we find that

$$\begin{aligned} 0 &= D_w D_\tau T \\ &= D_\tau D_w T + R(W, T)T \\ &= D_\tau D_\tau W + D_\tau S(W, T) + R(W, T)T, \end{aligned} \tag{2.4}$$

where in the last line we have used that:

$$D_w T - D_\tau W = S(W,T) + [W,T] = S(W,T). \tag{2.5}$$

If we evaluate this at $w = 0$ we find that:

$$D_\tau^2 J + D_\tau S(J,\dot\gamma) + R(J,\dot\gamma)\dot\gamma = 0. \tag{2.6}$$

□

The name Jacobi equation is usually given to the equation for vanishing torsion, but we will keep the same name. It can be shown that if $J(\tau)$ is a vector field along γ that obeys Eq. (2.1), there exists a variation through geodesics with J as variation field [23].

Definition 2.0.2. A vector field J along a geodesic $\gamma : [\tau_{\mathrm{i}}, \tau_{\mathrm{f}}] \to M$ that satisfies the Jacobi equation

$$D_\tau^2 J + D_\tau S(J,\dot\gamma) + R(J,\dot\gamma)\dot\gamma = 0 \tag{2.7}$$

will be called a *Jacobi field*.

Notice that Eq. (2.1), when written in coordinates, is a linear system of second order differential equations. Hence, we know that a unique solution is defined for all initial conditions $J(\tau_{\mathrm{i}})$ and $D_\tau J(\tau_{\mathrm{i}})$ and therefore the Jacobi fields form an eight dimensional subspace of $\mathcal{T}(\gamma)$, the vector fields along γ.

Using Eq. (2.5) we also have that

$$D_\tau W = -S(W,T) + D_w T \tag{2.8}$$

and evaluating at $w = 0$ yields

$$D_\tau J = -S(J,\dot\gamma) + \nabla_J T|_{w=0} \tag{2.9}$$

which in coordinates is equal to

$$\begin{aligned}(D_\tau J)^\rho &= -S^\rho{}_{\mu\nu}\dot\gamma^\nu J^\mu + J^\mu \nabla_\mu T^\rho|_{w=0} \\ &= \left(S^\rho{}_{\nu\mu}\dot\gamma^\nu + \nabla_\mu T^\rho|_{w=0}\right) J^\mu. \end{aligned} \tag{2.10}$$

2.1 Timelike Geodesic

Assume now that γ is timelike. Let $N(\gamma(\tau))$ denote the 3 dimensional subspace of $T_{\gamma(\tau)}M$ that consists of vectors orthogonal to $\dot\gamma(\tau)$ and denote by $P : T_{\gamma(\tau)}M \to N(\gamma(\tau))$ the projection onto this space. Let

$$N(\gamma) = \bigsqcup_\tau N(\gamma(\tau)) \tag{2.11}$$

denote the orthogonal bundle. We will now make the statement made in the beginning of this chapter about geodesics that start converging again more precise.

Definition 2.1.1. If γ is a timelike geodesic segment joining $p, q \in \gamma$, p is said to be *conjugate* to q along γ if there exists a Jacobi field J along γ with non-vanishing orthogonal part $P(J)$ such that $P(J)$ is zero at p and q.

If we now look at the Jacobi equation, Eq. (2.1), we see that the component of the Jacobi field parallel to $\dot\gamma$ has no influence on the components perpendicular to $\dot\gamma$. That is why the projected Jacobi field also obeys the equation

$$D_\tau^2(PJ) + D_\tau PS(PJ,\dot\gamma) + R(PJ,\dot\gamma)\dot\gamma \ = \ 0. \tag{2.12}$$

Let V be a vector field that lives in $N(\gamma)$ and obeys Eq. (2.12). Then

$$J = V - \int_{\tau_{\mathrm{i}}}^{\tau} g(S(V,\dot\gamma),\dot\gamma)d\tau'\dot\gamma \tag{2.13}$$

is a Jacobi field with $V = PJ$. That is why we will also refer to vector fields that live in $N(\gamma)$ and obey Eq. (2.12) as Jacobi fields.

To simplify notation, we introduce

$$\begin{aligned} R_\gamma(v) &= R\left(v, \dot\gamma(\tau)\right)\dot\gamma(\tau); \\ S_\gamma(v) &= S\left(v, \dot\gamma(\tau)\right). \end{aligned} \tag{2.14}$$

Let $A:\ N(\gamma) \to N(\gamma)$ be a smooth tensor field. Since $g\left(R\left(v, \dot\gamma(\tau)\right)\dot\gamma(\tau), \dot\gamma(\tau)\right) = 0$ we can define maps $R_\gamma A:\ N(\gamma(\tau)) \to N(\gamma(\tau))$, $PS_\gamma A:\ N(\gamma(\tau)) \to N(\gamma(\tau))$ by

$$R_\gamma A(\tau)(v) = R_\gamma\left(A(\tau)(v)\right); \tag{2.15}$$

$$PS_\gamma A(\tau)(v) = P\left[S_\gamma\left(A(\tau)(v)\right)\right]. \tag{2.16}$$

To study conjugate points we will introduce Jacobi tensors:

Definition 2.1.2. A smooth $(1,1)$ tensor field $A:\ N(\gamma) \to N(\gamma)$ is called a *Jacobi tensor field* if it satisfies

$$D_\tau^2 A + D_\tau(PS_\gamma A) + R_\gamma A = 0, \tag{2.17}$$

$$\mathrm{Ker}(A(\tau)) \cap \mathrm{Ker}(D_\tau A(\tau)) = \{0\} \tag{2.18}$$

for all $\tau \in [\tau_\mathrm{i}, \tau_\mathrm{f}]$. Here $\mathrm{Ker}(A(\tau))$ is the kernel of $A(\tau)$.

If $V \in N(\gamma)\backslash\{0\}$ is a parallel transported vector field along γ, i.e. $D_\tau V = 0$, and $A(\tau)$ a Jacobi tensor field, define $J(\tau) = A(\tau)V(\tau)$. Substitution in Eq. (2.12) gives

$$\begin{aligned} D_\tau^2(J(\tau)) + D_\tau PS_\gamma(J(\tau), \dot\gamma) + R_\gamma(J(\tau), \dot\gamma)\dot\gamma &= D_\tau(D_\tau(A)V + AD_\tau V) + D_\tau(PS_\gamma A)V \\ &\quad + PS_\gamma A D_\tau V + R_\gamma A V \\ &= D_\tau^2(A)V + D_\tau(PS_\gamma A)V + R_\gamma A V \\ &= \left(D_\tau^2(A) + D_\tau(PS_\gamma A) + R_\gamma A\right)V \\ &= 0. \end{aligned} \tag{2.19}$$

Hence $J(\tau)$ is a Jacobi field. Condition (2.18) guarantees that J is non-trivial. Therefore A can be seen as describing different families of geodesics at the same time. In the end we will define a Jacobi tensor field that describes all solutions to Eq. 2.12 that vanish at $\gamma(\tau_\mathrm{i})$ to see under what conditions we have a conjugate point to $\gamma(\tau_\mathrm{i})$ but we first derive some more general theory for a Jacobi tensor field.

In a similar way as in Eq. (2.10) we can write for a Jacobi tensor field A that

$$D_\tau A^\rho{}_\lambda = \left(S^\rho{}_{\nu\mu}\dot\gamma^\nu + \nabla_\mu T^\rho|_{w=0}\right)A^\mu{}_\lambda. \tag{2.20}$$

This T here is not well defined, it depends on which Jacobi field we consider. We will only use this equation and the following analysis to motivate a definition, but we will not use it in the rest of the thesis. The tensor A expresses how nearby geodesics change, for instance the separation of these curves from γ and some sort of volume that is marked out by nearby geodesics. The expression within the brackets,

$$S^\rho{}_{\nu\mu}\dot\gamma^\nu + \nabla_\mu T^\rho|_{w=0}, \tag{2.21}$$

can therefore be seen as the rate these characteristics change. We have that

$$S^\rho{}_{\nu\mu}\dot\gamma^\nu + \nabla_\mu T^\rho|_{w=0} = D_\tau A^\rho{}_\lambda\left(A^{-1}\right)^\lambda{}_\mu, \tag{2.22}$$

so that motivates the definition of the tensor

$$B_A = (D_\tau A)\,A^{-1} \tag{2.23}$$

at points where $\det A \neq 0$. Then we can define the following quantities:

Definition 2.1.3. Let A be a Jacobi tensor field and $B_A = (D_\tau A)\,A^{-1}$ at points where $\det A \neq 0$

1. The *expansion* θ_A is
$$\theta_A = \mathrm{tr}(B_A). \tag{2.24}$$
2. The *vorticity tensor* ω_A is
$$\omega_A = \frac{1}{2}(B_A - B_A^\dagger). \tag{2.25}$$
3. The *shear tensor* σ_A is
$$\sigma_A = \frac{1}{2}(B_A + B_A^\dagger) - \frac{\theta_A}{3}I, \tag{2.26}$$
where I is the identity matrix.

Notice that
$$B_A = \omega_A + \sigma_A + \frac{\theta_A}{3}I. \tag{2.27}$$

Lemma 2.1.1. *Singular points of a Jacobi tensor field A are precisely given by points where $|\theta_A| \to \infty$.*

Proof. We find that
$$\theta_A \ = \ \mathrm{tr}\left((D_\tau A)\, A^{-1}\right) = (\det A)^{-1}\mathrm{tr}\left((D_\tau A)\,\mathrm{adj}(A)\right) = (\det A)^{-1}\sum_{i,j}(D_\tau A)_{ij} C_{ij}, \tag{2.28}$$
where $\mathrm{adj}(A)$ is the adjungate of A and C_{ij} is the (i,j)th cofactor of A (Cramer's rule). Now:
$$\begin{aligned}\partial_\tau(\det A) &= \sum_{i_1,\dots,i_n} \partial_\tau\left(\epsilon_{i_1,\dots i_n} A_{1i_1}A_{2i_2}\dots A_{ni_n}\right)\\ &= \sum_{i_1,\dots,i_n}\left[\partial_\tau\left(A_{1i_1}\right)\left(\epsilon_{i_1,\dots i_n}A_{2i_2}\dots A_{ni_n}\right) + \dots + \partial_\tau\left(A_{ni_n}\right)\left(\epsilon_{i_1,\dots i_n}A_{1i_1}\dots A_{(n-1)i_{n-1}}\right)\right]\\ &= \sum_{i,j}\left(\partial_\tau A_{ij}\right)C_{ij},\end{aligned} \tag{2.29}$$
where $\epsilon_{i_1,\dots i_n}$ is the Levi-Civita symbol. Using an orthonormal parallel transported basis E_i (i.e. $g(E_i,E_j) = \delta_{ij}$) for $N(\gamma)$ we find that
$$\partial_\tau A_{ij} = \partial_\tau g\left(A(E_j), E_i\right) = g\left((D_\tau A)\,(E_j), E_i\right) = (D_\tau A)_{ij}\,. \tag{2.30}$$
With Eqs. (2.28), (2.29) and (2.30) we get
$$\theta_A = (\det A)^{-1}\partial_\tau(\det A) \tag{2.31}$$
and since $\det(A)$ does not depend on the coordinate system chosen, this holds in every coordinate system. We approach a singular point of A exactly when $\det A$ will go to zero. The Jacobi tensor field A obeys Eq. (2.17), which is a system of linear differential equations in the components of A. This implies that A is well-defined as long as the Riemann curvature tensor and torsion are well defined. Therefore we cannot have that $\partial_\tau(\det A)$ blows up. This implies that when θ_A blows up, we have a singular point. Since $\theta_A = \partial_\tau\left(\log(\det A)\right)$ and singular points of A should be isolated, we have a singular point exactly when $|\theta_A| \to \infty$. □

Proposition 2.1.1. The derivative of the expansion of a Jacobi tensor field A is given by
$$\dot\theta_A = -\mathrm{Ric}(\dot\gamma(\tau),\dot\gamma(\tau)) - \mathrm{tr}\left(D_\tau(PS_\gamma)\right) - \mathrm{tr}(PS_\gamma\omega_A) - \mathrm{tr}(PS_\gamma\sigma_A) - \frac{\theta_A}{3}\mathrm{tr}(PS_\gamma) - \mathrm{tr}(\omega_A^2) - \mathrm{tr}(\sigma_A^2) - \frac{\theta_A^2}{3}. \tag{2.32}$$

Proof. Differentiating $I = A^{-1}A$ we find that

$$0 = D_\tau(A^{-1})A + A^{-1}D_\tau A. \tag{2.33}$$

Hence

$$\begin{aligned} D_\tau B_A &= D_\tau(A'A^{-1}) = \left(D_\tau^2 A\right)A^{-1} - (D_\tau A)\,A^{-1}\,(D_\tau A)\,A^{-1} \\ &= -R_\gamma - D_\tau(PS_\gamma A)A^{-1} - B_A B_A \\ &= -R_\gamma - D_\tau(PS_\gamma) - PS_\gamma B_A - B_A B_A, \end{aligned} \tag{2.34}$$

where we have used Eq. (2.17). Using Eq. (2.27) we then have that

$$\begin{aligned} \dot\theta_A &= \mathrm{tr}(D_\tau B_A) \qquad (2.35)\\ &= -\mathrm{tr}(R_\gamma) - \mathrm{tr}((PS_\gamma)') - \mathrm{tr}(PS_\gamma B_A) - \mathrm{tr}(B_A B_A) \\ &= -\mathrm{tr}(R_\gamma) - \mathrm{tr}\left(D_\tau(PS_\gamma)\right) - \mathrm{tr}\left[PS_\gamma\left(\omega_A + \sigma_A + \frac{\theta_A}{3}I\right)\right] - \mathrm{tr}\left[\left(\omega_A + \sigma_A + \frac{\theta_A}{3}I\right)^2\right] \\ &= -\mathrm{tr}(R_\gamma) - \mathrm{tr}\left(D_\tau(PS_\gamma)\right) - \mathrm{tr}(PS_\gamma\omega_A) - \mathrm{tr}(PS_\gamma\sigma_A) - \frac{\theta_A}{3}\mathrm{tr}(PS_\gamma) - \mathrm{tr}(\omega_A^2) - \mathrm{tr}(\sigma_A^2) - \frac{\theta_A^2}{3} \end{aligned}$$

because ω_A, σ_A and $\omega_A\sigma_A$ are traceless. Now choose an orthonormal basis $E_i = E_i^\mu\partial_\mu$ of $T_{\gamma(\tau)}M$ (e.g. $g(E_i, E_j) = \eta_{ij}$) such that $E_0 = \dot\gamma$. Then

$$\begin{aligned} \mathrm{tr}(R_\gamma) &= \sum_{i=1}^{3} g(R(E_i, \dot\gamma(\tau))\dot\gamma(\tau), E_i) \\ &= \eta^{ij} g(R(E_j^\mu\partial_\mu, \dot\gamma(\tau))\dot\gamma(\tau), E_i^\nu\partial_\nu) \\ &= \eta^{ij}E_i^\mu E_j^\nu g(R(\partial_\mu, \dot\gamma(\tau))\dot\gamma(\tau), \partial_\nu) \\ &= \eta^{ij}E_i^\mu E_j^\nu R_{\nu\rho\mu\lambda}\dot\gamma(\tau)^\rho\dot\gamma(\tau)^\lambda, \end{aligned} \tag{2.36}$$

where from the second equality sign onwards, we assume the Einstein convention (as usual). From $g_{\mu\nu}E_i^\mu E_j^\nu = \eta_{ij}$ it follows that $E_\nu^k E_j^\nu = \delta_j^k$. This means that $\partial_\mu = E_\mu^i E_i$ and implies $g_{\mu\nu} = E_\mu^i E_\nu^j \eta_{ij}$. Hence $\eta^{ij}E_i^\mu E_j^\nu = g^{\mu\nu}$ and

$$\mathrm{tr}(R_\gamma) = \eta^{ij}E_i^\mu E_j^\nu R_{\nu\rho\mu\lambda}\dot\gamma(\tau)^\rho\dot\gamma(\tau)^\lambda = R^\mu{}_{\rho\mu\lambda}\dot\gamma(\tau)^\rho\dot\gamma(\tau)^\lambda = \mathrm{Ric}(\dot\gamma(\tau), \dot\gamma(\tau)). \tag{2.37}$$

Substitution in Eq. (2.35) yields

$$\dot\theta_A = -\mathrm{Ric}(\dot\gamma(\tau), \dot\gamma(\tau)) - \mathrm{tr}(D_\tau(PS_\gamma)) - \mathrm{tr}(PS_\gamma\omega_A) - \mathrm{tr}(PS_\gamma\sigma_A) - \frac{\theta_A}{3}\mathrm{tr}(PS_\gamma) - \mathrm{tr}(\omega_A^2) - \mathrm{tr}(\sigma_A^2) - \frac{\theta_A^2}{3}. \tag{2.38}$$

□

When $S = 0$, this equation is called the Raychaudhuri equation or Ricatti equation. We will refer to Eq. (2.32) as the generalized Raychaudhuri equation. Notice that Eq. (2.32) is slightly different from the expression found in [24] because we have found the extra term $-\mathrm{tr}(D_\tau(PS_\gamma))$. That is because they considered an expansion of $\nabla_\mu\dot\gamma_\rho$ while we took into account the extra torsion term (see Eq. (2.22)).

To examine if there are conjugate points to $\gamma(\tau_\mathrm{i})$ along γ we will construct a specific kind of Jacobi tensor field that describes all Jacobi fields that live in $N(\gamma)$, obey Eq. (2.12) and vanish at $\gamma(\tau_\mathrm{i})$. Let us again introduce a parallel transported orthonormal frame $\{E_\mu\}$, $\mu = 0, 1, 2, 3$, along γ such that $E_0 = \dot\gamma$ (we can do this because $\dot\gamma$ is timelike). Let $J_i(\tau)$, $i \in \{1, 2, 3\}$, be the Jacobi field with $J_i(\tau_\mathrm{i}) = 0$ and $D_\tau J_i(\tau_\mathrm{i}) = E_i(\tau_\mathrm{i})$. Let A be the tensor such that the components in the basis $E_\mu$are given by

$$\begin{aligned} A^k{}_l(\tau) &= (PJ_l(\tau))^k; \\ A^0{}_0 = A^k{}_0 = A^0{}_l &= 0, \end{aligned} \tag{2.39}$$

for $k, l = 1, 2, 3$.

Lemma 2.1.2. *The tensor field A as constructed in Eq. 2.39 is a Jacobi tensor field and a point $\gamma(\tau_0)$ is conjugate to $\gamma(\tau_i)$ if and only if $|\theta_A| \to \infty$ for $\tau \to \tau_0$.*

Proof. From the way this tensor is constructed, it is clear that it represents a tensor field on $N(\gamma)$. It also describes all Jacobi fields that live in $N(\gamma)$, obey Eq. (2.12) and vanish at $\gamma(\tau_\mathrm{i})$ since solutions of Eq. (2.12) are uniquely determined once we know the $V(\tau_\mathrm{i})$ and $D_\tau V(\tau_\mathrm{i})$ of a solution V at $\gamma(\tau_\mathrm{i})$. If we let $v \in N(\gamma(\tau))$ for a $\tau \in [\tau_\mathrm{i}, \tau_\mathrm{f}]$ and expand v to a parallel vector field V along γ such that $V(\tau) = v$, we have that

$$\left(D_\tau^2 A + D_\tau(PS_\gamma A) + R_\gamma A\right)(V) = D_\tau^2(AV) + D_\tau(PS_\gamma AV) + R_\gamma AV = 0 \tag{2.40}$$

since AV is a Jacobi field. Suppose now that $v \in \mathrm{Ker}(A(\tau)) \cap \mathrm{Ker}(D_\tau A(\tau))$ for some $\tau \in [\tau_\mathrm{i}, \tau_\mathrm{f}]$. Again expanding it to a parallel vector field V such that $V(\tau) = v$, we find that $J = AV$ is a Jacobi field such that $J(\tau) = D_\tau J(\tau) = 0$ and this implies that $J = 0$ which is impossible for $v \neq 0$ since the column vectors of A describe linearly independent Jacobi fields so we cannot have that $J = \lambda^i J_i = 0$ when not all λ^i vanish. Hence A is a Jacobi tensor. Notice that we also have that $A(\tau_\mathrm{i}) = 0$. The span of the columns of this matrix forms the three dimensional subspace of Jacobi fields that vanish at $\gamma(\tau_\mathrm{i})$. This means that if $\gamma(\tau_0)$ for some $\tau_0 \in [\tau_\mathrm{i}, \tau_\mathrm{f}]$ is conjugate to $\gamma(\tau_\mathrm{i})$ along γ, that $A(\tau_0)$ has to be singular. Suppose now that $A(\tau_0)v = 0$ for some $\tau_0 \in [\tau_\mathrm{i}, \tau_\mathrm{f}]$, $v \in N(\gamma(\tau_0))$, $v \neq 0$. Let V be the unique parallel vector field along γ such that $V(\tau_0) = v$, then $J(\tau) = A(\tau)V(\tau)$ is a non-trivial Jacobi field that vanishes at τ_i and $\gamma(\tau_0)$. So points conjugate to $\gamma(\tau_\mathrm{i})$ are exactly the points along $\gamma(\tau)$ where $\det(A(\tau)) = 0$ and we have seen that those points are precisely the points where $|\theta_A| \to \infty$. □

Totally Anti-Symmetric Torsion

From now on in this chapter and in Chapter 3 we will assume a totally anti-symmetric torsion (the other parts of expansion (1.7) vanish): in coordinates we assume $S_{\rho\mu\nu} = -S_{\mu\rho\nu}$. We already had that $S_{\rho\mu\nu} = -S_{\rho\nu\mu}$ which implies that $PS_\gamma = S_\gamma$ or $g(S(V, \dot\gamma(\tau)), \dot\gamma(\tau)) = 0$ for all vectors $V \in T_{\gamma(\tau)}M$. More generally, we have that $g(S(V, W), W) = 0$ for all vectors $V, W \in T_{\gamma(\tau)}M$. Therefore

$$\partial_\tau^2 g(J, \dot\gamma) = g(D_\tau^2 J, \dot\gamma) = -g(D_\tau S(J, \dot\gamma), \dot\gamma) - g(R(J, \dot\gamma)\dot\gamma, \dot\gamma) = -\partial_\tau g(S(J, \dot\gamma), \dot\gamma) = 0, \tag{2.41}$$

where we have used the Jacobi Equation. That implies that if $g(J, \dot\gamma) = 0$ for two points on γ, it vanishes for all points on γ. This happens in particular for the Jacobi field related to conjugate points, so those Jacobi fields have to live in $N(\gamma)$.

For a totally anti-symmetric torsion we also have that

$$g(S_\gamma V, W) = g_{\alpha\beta}\left(S_\gamma\right)^\alpha{}_\mu V^\mu W^\beta = -\left(S_\gamma\right)_{\mu\beta} V^\mu W^\beta = g(V, -S_\gamma W) \tag{2.42}$$

for arbitrary vectors V, W, which proves that

$$S_\gamma^\dagger = -S_\gamma. \tag{2.43}$$

Now let X be a tensor field along γ, then

$$\begin{aligned} g\left[(D_\tau X)V, W\right] &= g\left[D_\tau(XV) - X(D_\tau V), W\right] \\ &= \partial_\tau g(V, X^\dagger W) - g(XV, D_\tau W) - g(D_\tau V, X^\dagger W) \\ &= -g(XV, D_\tau W) + g\left[V, D_\tau\left(X^\dagger W\right)\right] \\ &= -g\left[V, X^\dagger(D_\tau W)\right] + g\left[V, \left(D_\tau X^\dagger\right)W\right] + g\left[V, X^\dagger(D_\tau W)\right] \\ &= g\left[V, \left(D_\tau X^\dagger\right)W\right], \end{aligned} \tag{2.44}$$

which implies that

$$(D_\tau X)^\dagger = \left(D_\tau X^\dagger\right). \tag{2.45}$$

This also implies that $D_\tau S_\gamma$ is anti-symmetric. Using that and the anti-symmetry of S_γ, we find that S_γ, $D_\tau S_\gamma$ and $S_\gamma\sigma$ are traceless such that Eq. (2.32) reduces to

$$\dot\theta_A = -\mathrm{Ric}(\dot\gamma(\tau), \dot\gamma(\tau)) - \mathrm{tr}(S_\gamma\omega_A) - \mathrm{tr}(\omega_A^2) - \mathrm{tr}(\sigma_A^2) - \frac{\theta_A^2}{3}. \tag{2.46}$$

Lemma 2.1.3. *The vorticity tensor of a Jacobi tensor field A that vanishes at $\gamma(\tau_\mathrm{i})$ is*

$$\omega_A = -\frac{1}{2}S_\gamma. \tag{2.47}$$

Proof. Define the Wronskian of A by

$$W(A,A) = (D_\tau A)^\dagger A - A^\dagger D_\tau A. \tag{2.48}$$

The time derivative of the Wronskian along γ is given by

$$D_\tau W(A,A) = D_\tau\left[(D_\tau A)^\dagger\right] A - A^\dagger D_\tau^2 A. \tag{2.49}$$

Using property Eq. (2.45) for $X = D_\tau A$, Eq. (2.49) reduces to

$$\begin{aligned} D_\tau W(A,A) &= \left(D_\tau^2 A\right)^\dagger A - A^\dagger D_\tau^2 A \\ &= (-D_\tau(S_\gamma A) - R_\gamma A)^\dagger A + A^\dagger(D_\tau(S_\gamma A) + R_\gamma A), \end{aligned} \tag{2.50}$$

where we have used Eq. (2.17) ($PS_\gamma = S_\gamma$). Again with Eq. (2.45) we find that

$$\begin{aligned} D_\tau W(A,A) &= -D_\tau(S_\gamma A)^\dagger A - A^\dagger R_\gamma^\dagger A + A^\dagger D_\tau(S_\gamma A) + A^\dagger R_\gamma A \\ &= D_\tau(A^\dagger S_\gamma)A + A^\dagger D_\tau(S_\gamma A) + A^\dagger\left(R_\gamma - R_\gamma^\dagger\right)A \\ &= D_\tau(A^\dagger)S_\gamma A + A^\dagger S_\gamma D_\tau A + 2A^\dagger D_\tau(S_\gamma)A + A^\dagger\left(R_\gamma - R_\gamma^\dagger\right)A. \end{aligned} \tag{2.51}$$

We will now examine $R_\gamma - R_\gamma^\dagger$. We have that

$$(R_\gamma)^\rho{}_\mu = R^\rho{}_{\lambda\mu\nu}\dot\gamma^\lambda\dot\gamma^\nu. \tag{2.52}$$

Writing

$$\Gamma^\rho_{\mu\nu} = \{{}^{\rho}_{\mu\nu}\} + {}_S\Gamma^\rho_{\mu\nu} \tag{2.53}$$

(${}_S\Gamma^\rho_{\mu\nu}$ is a tensor) and denoting the components of the curvature tensor that correspond to the vanishing torsion case by $\breve{R}^\rho{}_{\lambda\mu\nu}$, we find that

$$\begin{aligned} R^\rho{}_{\lambda\mu\nu} &= \partial_\mu\left(\{{}^{\rho}_{\nu\lambda}\} + {}_S\Gamma^\rho_{\nu\lambda}\right) + \left(\{{}^{\beta}_{\nu\lambda}\} + {}_S\Gamma^\beta_{\nu\lambda}\right)\left(\{{}^{\rho}_{\mu\beta}\} + {}_S\Gamma^\rho_{\mu\beta}\right) - \partial_\nu\left(\{{}^{\rho}_{\mu\lambda}\} + {}_S\Gamma^\rho_{\mu\lambda}\right) \\ &\quad - \left(\{{}^{\beta}_{\mu\lambda}\} + {}_S\Gamma^\beta_{\mu\lambda}\right)\left(\{{}^{\rho}_{\nu\beta}\} + {}_S\Gamma^\rho_{\nu\beta}\right) \\ &= \breve{R}^\rho{}_{\lambda\mu\nu} + \left(\partial_\mu {}_S\Gamma^\rho_{\nu\lambda} + \{{}^{\rho}_{\mu\beta}\}{}_S\Gamma^\beta_{\nu\lambda} - \{{}^{\beta}_{\mu\lambda}\}{}_S\Gamma^\rho_{\nu\beta}\right) - \left(\partial_\nu {}_S\Gamma^\rho_{\mu\lambda} + \{{}^{\rho}_{\nu\beta}\}{}_S\Gamma^\beta_{\mu\lambda} - \{{}^{\beta}_{\nu\lambda}\}{}_S\Gamma^\rho_{\mu\beta}\right) \\ &\quad + {}_S\Gamma^\beta_{\nu\lambda}{}_S\Gamma^\rho_{\mu\beta} - {}_S\Gamma^\beta_{\mu\lambda}{}_S\Gamma^\rho_{\nu\beta} \\ &= \breve{R}^\rho{}_{\lambda\mu\nu} + \nabla_\mu {}_S\Gamma^\rho_{\nu\lambda} + \{{}^{\beta}_{\mu\nu}\}{}_S\Gamma^\rho_{\beta\lambda} - \nabla_\nu {}_S\Gamma^\rho_{\mu\lambda} - \{{}^{\beta}_{\nu\mu}\}{}_S\Gamma^\rho_{\beta\lambda} + {}_S\Gamma^\beta_{\nu\lambda}{}_S\Gamma^\rho_{\mu\beta} - {}_S\Gamma^\beta_{\mu\lambda}{}_S\Gamma^\rho_{\nu\beta} \\ &= \breve{R}^\rho{}_{\lambda\mu\nu} + \nabla_\mu {}_S\Gamma^\rho_{\nu\lambda} - \nabla_\nu {}_S\Gamma^\rho_{\mu\lambda} + {}_S\Gamma^\beta_{\nu\lambda}{}_S\Gamma^\rho_{\mu\beta} - {}_S\Gamma^\beta_{\mu\lambda}{}_S\Gamma^\rho_{\nu\beta}. \end{aligned} \tag{2.54}$$

Using that for totally anti-symmetric torsion (Eq. (1.15))

$${}_S\Gamma^\rho_{\mu\nu} = \frac{1}{2}S^\rho{}_{\mu\nu}, \tag{2.55}$$

we find

$$\begin{aligned} (R_\gamma)^\rho{}_\mu &= R^\rho{}_{\lambda\mu\nu}\dot\gamma^\lambda\dot\gamma^\nu \\ &= \left[\breve{R}^\rho{}_{\lambda\mu\nu} - \frac{1}{2}\nabla_\nu S^\rho{}_{\mu\lambda} + \frac{1}{4}S^\beta{}_{\mu\lambda}S^\rho{}_{\beta\nu}\right]\dot\gamma^\lambda\dot\gamma^\nu. \end{aligned} \tag{2.56}$$

This implies that

$$R_\gamma = \breve{R}_\gamma - \frac{1}{2}\left(D_\tau S\right)_\gamma + \frac{1}{4}S_\gamma S_\gamma. \tag{2.57}$$

We have that

$$\begin{aligned}(D_\tau S)_\gamma(V) &= (D_\tau S)(V,\dot\gamma) = D_\tau[S(V,\dot\gamma)] - S(D_\tau V,\dot\gamma) - S(V, D_\tau\dot\gamma)\\ &= D_\tau[S_\gamma(V)] - S_\gamma(D_\tau V)\\ &= (D_\tau S_\gamma)(V),\end{aligned} \tag{2.58}$$

where we have used that γ is a geodesic. So

$$R_\gamma = \breve{R}_\gamma - \frac{1}{2}D_\tau S_\gamma + \frac{1}{4}S_\gamma S_\gamma \tag{2.59}$$

and

$$\begin{aligned}R_\gamma - R_\gamma^\dagger &= \breve{R}_\gamma - \breve{R}_\gamma^\dagger - \frac{1}{2}D_\tau S_\gamma + \frac{1}{2}(D_\tau S_\gamma)^\dagger + \frac{1}{4}S_\gamma S_\gamma - \frac{1}{4}S_\gamma^\dagger S_\gamma^\dagger\\ &= -D_\tau S_\gamma,\end{aligned} \tag{2.60}$$

where we used that $(D_\tau S_\gamma)^\dagger = D_\tau S_\gamma^\dagger = -D_\tau S_\gamma$. Substitution in Eq. (2.51) yields

$$\begin{aligned}D_\tau W(A,A) &= D_\tau(A^\dagger)S_\gamma A + A^\dagger S_\gamma D_\tau A + 2A^\dagger D_\tau(S_\gamma)A - A^\dagger D_\tau(S_\gamma)A\\ &= D_\tau(A^\dagger)S_\gamma A + A^\dagger S_\gamma D_\tau A + A^\dagger D_\tau(S_\gamma)A\\ &= D_\tau(A^\dagger S_\gamma A).\end{aligned} \tag{2.61}$$

Considering this as a matrix equation (working in components), we have that

$$\partial_\tau\left(W(A,A) - A^\dagger S_\gamma A\right)^\mu{}_\nu = -\dot\gamma^\rho\Gamma^\mu_{\rho\lambda}\left(W(A,A) - A^\dagger S_\gamma A\right)^\lambda{}_\nu + \dot\gamma^\rho\Gamma^\lambda_{\rho\nu}\left(W(A,A) - A^\dagger S_\gamma A\right)^\mu{}_\lambda \tag{2.62}$$

and at τ_i, $\left(W(A,A) - A^\dagger S_\gamma A\right)^\mu{}_\nu = 0$ since $A(\tau_\mathrm{i}) = 0$. This is a linear system of first-order ODEs for the functions $\left(W(A,A) - A^\dagger S_\gamma A\right)^\mu{}_\nu$. Such a system has a unique solution and that implies that Eq. (2.62) is uniquely solved by

$$\left(W(A,A) - A^\dagger S_\gamma A\right)^\mu{}_\nu = 0 \tag{2.63}$$

and if this holds for one set of components, it should hold for the tensor. Hence

$$A^\dagger D_\tau A = (D_\tau A)^\dagger A - A^\dagger S_\gamma A \tag{2.64}$$

or

$$\begin{aligned}B_A &= (D_\tau A)A^{-1}\\ &= \left(A^\dagger\right)^{-1}(D_\tau A)^\dagger - S_\gamma\\ &= \left(A^{-1}\right)^\dagger(D_\tau A)^\dagger - S_\gamma\\ &= B_A^\dagger - S_\gamma,\end{aligned} \tag{2.65}$$

where we have used that $\left(A^{-1}\right)^\dagger A^\dagger = \left(AA^{-1}\right)^\dagger = I$, so $\left(A^\dagger\right)^{-1} = \left(A^{-1}\right)^\dagger$. Hence

$$\omega_A = \frac{1}{2}(B_A - B_A^\dagger) = -\frac{1}{2}S_\gamma. \tag{2.66}$$

□

This implies that Eq. (2.46) for the Jacobi tensor field A constructed in Eq. 2.39 reduces to

$$\dot\theta_A = -\mathrm{Ric}(\dot\gamma(\tau),\dot\gamma(\tau)) + \frac{1}{4}\mathrm{tr}(S_\gamma^2) - \mathrm{tr}(\sigma_A^2) - \frac{\theta_A^2}{3}. \tag{2.67}$$

Proposition 2.1.2. *Let $\gamma : \mathbb{R} \to M$ be a timelike geodesic and let $p = \gamma(\tau_0)$ for a $\tau_0 \in \mathbb{R}$, let A be the Jacobi tensor field, constructed as in Eq. (2.39) such that $A(\tau_0) = 0$. If for some $\tau_1 > \tau_0$ the expansion $\theta_A(\tau_1) < 0$ and if $\mathrm{Ric}(\dot\gamma,\dot\gamma) - \frac{1}{4}\mathrm{tr}(S_\gamma^2) \geq 0$ for all $\tau \geq \tau_1$, there will be a point conjugate to p along γ between $\gamma(\tau_1)$ and $\gamma(\tau_1 - \frac{3}{\theta_A(\tau_1)})$.*

Proof. Since σ_A is symmetric, $\mathrm{tr}(\sigma_A^2) \geq 0$. So we find that all the terms at the right-hand side of Eq. (2.67) are smaller or equal to zero such that we have

$$\dot{\theta}_A \leq -\frac{\theta_A^2}{3}. \tag{2.68}$$

Integration of this inequality yields that for $\tau > \tau_1$:

$$\theta_A \leq \frac{3}{\tau - \left(\tau_1 - \frac{3}{\theta_A(\tau_1)}\right)}, \tag{2.69}$$

hence, θ_A will become infinite for a $\tau_1 < \tau_0 \leq \tau_1 - \frac{3}{\theta_A(\tau_1)}$ and using lemma 2.1.2, this implies that there is a conjugate point to p at $\gamma(\tau_0)$. □

Proposition 2.1.3. *Let $\gamma : \mathbb{R} \to M$ be a timelike geodesic. If $\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) - \frac{1}{4}\mathrm{tr}(S_\gamma^2) \geq 0$ for all τ and if at some point $r = \gamma(\tau_1)$, $\breve{R}_\gamma \neq 0$, there will be points $p = \gamma(\tau_0)$ and $q = \gamma(\tau_2)$, $\tau_0 \neq \tau_2$, conjugate along γ.*

Proof. A Jacobi tensor field A is uniquely defined once we know $A(\tau_1)$ and $D_\tau A(\tau_1)$. Consider the set

$$P = \{A \mid A \text{ Jacobi tensor field with } A(\tau_1) = I,\ D_\tau A(\tau_1) - (D_\tau A)^\dagger(\tau_1) = -S_\gamma(\tau_1) \text{ and } \theta_A(\tau_1) \leq 0\}. \tag{2.70}$$

Suppose $A \in P$, at $\gamma(\tau_1)$:

$$W(A, A) - A^\dagger S_\gamma A = (D_\tau A)^\dagger - D_\tau A - S_\gamma = 0. \tag{2.71}$$

Then it follows in the same way as in lemma 2.1.3 that

$$\omega_A = -\frac{1}{2}S_\gamma. \tag{2.72}$$

This implies that θ_A obeys Eq. (2.67). If $\theta_A(\tau_1) < 0$, it follows in the same way as in proposition 2.1.2 that $\theta_A \to -\infty$ for some $\tau_1 < \tau < \tau_1 - \frac{3}{\theta_A(\tau_1)}$. If $\theta_A(\tau_1) = 0$, we see from Eq. (2.67) that $\theta_A(\tau_1) \leq 0$ for all $\tau > \tau_1$. When $\theta_A(\tau_3) < 0$ for some $\tau_3 > \tau_1$, then again it can be proven that $\theta_A \to -\infty$ for some $\tau_3 < \tau < \tau_3 - \frac{3}{\theta_A(\tau_3)}$. Suppose now that $\theta_A(\tau) = 0$ for all $\tau \geq \tau_1$. Then $\dot{\theta}_A(\tau) = 0$ for all $\tau \geq \tau_1$. From Eq. (2.67) it follows that $\mathrm{tr}(\sigma_A^2) = 0$ for all $\tau \geq \tau_1$. Since σ_A is symmetric, this implies that $\sigma_A = 0$ for all $\tau \geq \tau_1$. Hence $B_A = \omega_A = -\frac{1}{2}S_\gamma$ for all $\tau \geq \tau_1$. With Eq. (2.34) we then have that

$$\begin{aligned} R_\gamma &= -D_\tau B_A - D_\tau S_\gamma - S_\gamma B_A - B_A B_A = 0 \\ &= -\frac{1}{2}D_\tau S_\gamma + \frac{1}{4}S_\gamma^2 \end{aligned} \tag{2.73}$$

for all $\tau \geq \tau_1$ contradicting $\breve{R}_\gamma \neq 0$ (see Eq. 2.57). This proves that to every $A \in P$ we can associate the first point on γ where it becomes singular. If we start at a point $\tau_2 > \tau_1$ the Jacobi tensor field A as constructed in Eq. (2.39) such that $A(\tau_2) = 0$ will have vorticity $\omega_A = -\frac{1}{2}S_\gamma$. The Jacobi tensor $C(\tau) = A(\tau)A^{-1}(\tau_1)$ will then have $C(\tau_1) = I$ and at τ_1

$$\begin{aligned} D_\tau C - (D_\tau C)^\dagger &= (D_\tau A)\, A^{-1} - \left(A^{-1}\right)^\dagger (D_\tau A)^\dagger \\ &= B_A - B_A^\dagger \\ &= 2\omega_A \\ &= -S_\gamma. \end{aligned} \tag{2.74}$$

The tensor C is in P, but obviously C and A are singular at the same points. We can now finish the proof by following the reasoning in proposition 4.4.2 of [3] working with the space of matrices $D_\tau A(\tau_1)$ such that $D_\tau A(\tau_1) - (D_\tau A)^\dagger(\tau_1) = -S_\gamma(\tau_1)$. □

2.2 Spacelike Surface

Let H be a smooth submanifold of M and $i : H \to M$ be the inclusion map. We can consider TH as subspace of TM by identifying TH with i_*TH. We can also identify i^*g and g restricted to i_*TH.

Definition 2.2.1. A submanifold H of M is said to be *nondegenerate* if for each $p \in H$ and nonzero $v \in T_pH$, there exists a $w \in T_pH$ such that $g(v, w) \neq 0$.

Definition 2.2.2. A nondegenerate submanifold H of M is said to be *spacelike* if $g|\ T_pH \times T_pH$ is positive definite for each $p \in H$.

Nondegeneracy of a manifold implies that we can define

$$T_p^{\perp}H = \{v \in T_pM \mid g(v, w) = 0 \text{ for all } w \in T_pH\} \tag{2.75}$$

such that $T_p^{\perp}H \cap T_pH = \{0\}$. We may define the second fundamental form as follows.

Definition 2.2.3. Let H be a nondegenerate submanifold of M. Then the *second fundamental form*

$$\chi :\ T_p^{\perp}H \times T_pH \times T_pH \to \mathbb{R} \tag{2.76}$$

is defined by

$$\chi(n, x, y) = g(\nabla_X Y|_p, n), \tag{2.77}$$

where X, Y are local extensions of x, y.

Note that $\nabla_X Y|_p$ only depends on value $X(p) = x$, so the value of the second fundamental form is independent of the local extension X of x. Furthermore, we have that when $X, Y \in T_pH$ then $[X, Y] \in T_pH$ which implies that

$$\chi(n, x, y) = g(\nabla_X Y|_p, n) = g(S(x, y), n) + \chi(n, y, x). \tag{2.78}$$

Hence, the value of the second fundamental form is also independent of the local extension Y of y.

Definition 2.2.4. Let H be a nondegenerate submanifold of M. Then the *second fundamental form operator* $L_n :\ T_pH \to T_pH$ is defined by $g(L_n x, y) = \chi(n, x, y)$ for all $x, y \in T_pH$.

The second fundamental form operator may be constructed using a basis of T_pH and the linearity of χ.

Consider a spacelike hypersurface H and let n denote its unit timelike normal vector field, i.e. $g(n, n) = -1$. n defines a tensor field L_n and it can be proven [22] that

$$L_n(x) = -\nabla_x n \tag{2.79}$$

(the only property of the connection used in the proof is metric compatibility). The collection of unit speed geodesics such that $\dot{\gamma}(0) = n$ (in this section $\tau = 0$ corresponds to the geodesic at the spacelike hypersurface) determines a congruence of timelike geodesics normal to H. Let γ be a geodesic of this congruence and let Γ be a one-parameter family of this congruence such that Γ is a variation of γ. Let J denote the variation vector field of this variation along γ. Then J is a Jacobi field and $J(0)$ is a vector tangent to H which implies that J satisfies the initial condition

$$D_\tau J(0) = S(\dot{\gamma}, J) + (\nabla_J n)\,|_{\gamma(0)} = S(\dot{\gamma}, J) - L_n J. \tag{2.80}$$

This implies that we can define a conjugate point to a spacelike surface as follows.

Definition 2.2.5. Let γ be a timelike geodesic which is orthogonal to a spacelike three-surface H. A point q is said to be *conjugate* to H along γ if there exists a non-vanishing Jacobi vector field J along γ such that J is orthogonal to $\dot{\gamma}$, J vanishes at q and satisfies $D_\tau J = S(\dot{\gamma}, J) - L_n J$ at H.

From the definition of conjugate points to a surface it follows that to study these kind of conjugate points we can, similar to the case of a conjugate point to a point, work with a parallel orthonormal frame E_μ along γ such that $E_0 = \dot{\gamma}$. Let now J_i be the Jacobi field such that $J_i = E_i$ at H and that satisfies Eq. (2.80) at H. Let A be the tensor field that in this basis has the components

$$\begin{aligned} {A^k}_l(\tau) &= (J_l(\tau))^k; \\ {A^0}_0 = {A^k}_0 = {A^0}_l &= 0, \end{aligned} \tag{2.81}$$

for $k, l = 1, 2, 3$.

Lemma 2.2.1. *The tensor field A as constructed in Eq. (2.81) is a Jacobi tensor field and a point $\gamma(\tau_0)$ is conjugate to H if and only if $|\theta_A| \to \infty$ for $\tau \to \tau_0$.*

Proof. It follows in the same way as in lemma 2.1.2 that A is a Jacobi tensor field. We also have that $[D_\tau A(0)]\,(E_i) = D_\tau J_i(0) = S(\dot{\gamma}, J_i) - L_n J_i = -S_\gamma A E_i - L_n A E_i$, hence $D_\tau A(0) = -S_\gamma - L_n$. Every Jacobi field J which is orthogonal to $\dot{\gamma}$ and obeys Eq. (2.80) can then be expressed as AV where V is a parallel vector field along γ such that V is orthogonal to $\dot{\gamma}$. Therefore it follows that the points on γ conjugate to H will be given by the singular points of A. As before A will be singular if and only if $|\theta_A|$ becomes infinite. □

Lemma 2.2.2. *The vorticity tensor of the Jacobi tensor field A constructed in Eq. (2.81) is*

$$\omega_A = -\frac{1}{2} S_\gamma. \tag{2.82}$$

Proof. Using Eq. (2.78) we find that:

$$\begin{aligned} g(L_n x, y) &= \chi(n, x, y) \\ &= g(S(x, y), \dot{\gamma}) + g(L_n y, x) \\ &= S_{\mu\nu\lambda} x^\nu y^\lambda \dot{\gamma}^\mu + g(L_n y, x) \\ &= S_{\nu\lambda\mu} x^\nu y^\lambda \dot{\gamma}^\mu + g(L_n y, x) \\ &= g(S_\gamma y, x) + g(L_n y, x). \end{aligned} \tag{2.83}$$

Thus $L_n^\dagger = S_\gamma + L_n$ which implies that at $\tau = 0$ (the spacelike hypersurface)

$$\begin{aligned} W(A, A) - A^\dagger S_\gamma A &= (D_\tau A)^\dagger A - A^\dagger D_\tau A - A^\dagger S_\gamma A \\ &= (-S_\gamma - L_n)^\dagger + S_\gamma + L_n - S_\gamma \\ &= S_\gamma - L_n^\dagger + L_n \\ &= 0 \end{aligned} \tag{2.84}$$

such that in the same way as in lemma 2.1.3 we can derive that

$$\omega_A = -\frac{1}{2} S_\gamma. \tag{2.85}$$

□

By the previous two lemmas it follows that the derivative of the expansion related to the Jacobi tensor field (2.81) also obeys Eq. (2.67).

Proposition 2.2.1. *Let H be a spacelike hypersurface and let $\gamma : \mathbb{R} \to M$ be a timelike geodesic, such that $\gamma(0) \in H$ and γ is orthogonal to H. If $-\mathrm{tr}(L_{\dot{\gamma}(0)}) < 0$ and $\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) - \frac{1}{4}\mathrm{tr}(S_\gamma^2) \geq 0$ for all $\tau \geq 0$ there will be a point conjugate to H along γ between $\gamma(0)$ and $\gamma(-\frac{3}{\theta_1})$.*

Proof. This can be proven as in proposition 2.1.2 using the Jacobi tensor field A defined in (2.81), realizing that at $\gamma(0)$, $\theta_A = -\mathrm{tr}(L_{\dot{\gamma}(0)}) < 0$ and using lemma 2.2.1. □

2.3 Null Geodesic

We would like to also define conjugate points for a null geodesic $\gamma : [\tau_\mathrm{i}, \tau_\mathrm{f}] \to M$. Since we consider totally anti-symmetric torsion it can be seen from Eq. (2.41) that Jacobi fields that vanish at two points along γ have to live in $N(\gamma)$. Notice that for a null geodesic we also have that $\dot{\gamma} \in N(\gamma)$, so since we will be interested in the convergence of geodesics, it makes more sense to look at the projection of Jacobi fields to a quotient space formed by identifying vectors that differ by a multiple of $\dot{\gamma}$. This idea is implicitly used in [3] and further developed in [25].

We start defining a pseudo-orthonormal basis for the tangent space at every point. Let $E_0(\tau_\mathrm{i}) = \dot{\gamma}$ and $E_1(\tau_\mathrm{i}) = n \in T_{\gamma(\tau_\mathrm{i})}M$ be a null tangent vector such that $g(n, \dot{\gamma}(\tau_\mathrm{i})) = -1$. Choose spacelike tangent vectors $E_2, E_3 \in T_{\gamma(\tau_\mathrm{i})}M$ such that $g(n, E_j) = g(\dot{\gamma}(\tau_\mathrm{i}), E_j) = 0$ for $j = 1, 2$ and $g(E_i, E_j) = \delta_{ij}$. Extend these vectors to a parallel transported vector field along γ and define

$$N_\perp(\gamma(\tau)) = \{\lambda^2 E_2(\tau) + \lambda^3 E_3(\tau) | \lambda^j \in \mathbb{R}\}. \tag{2.86}$$

Then $N_\perp(\gamma(\tau)) \subset N(\gamma(\tau))$ and consists of spacelike vectors. We have the direct sum decomposition

$$N(\gamma(\tau)) = N_\perp(\gamma(\tau)) \oplus [\dot{\gamma}(\tau)] \tag{2.87}$$

for all $\tau \in [\tau_\mathrm{i}, \tau_\mathrm{f}]$, where

$$[\dot{\gamma}(\tau)] = \{\lambda \dot{\gamma}(\tau) | \lambda \in \mathbb{R}\}. \tag{2.88}$$

Define now the bundle

$$N_\perp(\gamma) = \bigsqcup_{\tau_\mathrm{i} \le \tau \le \tau_\mathrm{f}} N_\perp(\gamma(\tau)). \tag{2.89}$$

Obviously $[\dot{\gamma}(\tau)]$ is a vector subspace of $N(\gamma(\tau))$, so we can define the quotient vector space

$$G(\gamma(\tau)) = N(\gamma(\tau))/[\dot{\gamma}(\tau)] \tag{2.90}$$

and the quotient bundle

$$G(\gamma) = N(\gamma)/[\dot{\gamma}] = \bigsqcup_{\tau_\mathrm{i} \le \tau \le \tau_\mathrm{f}} G(\gamma(\tau)). \tag{2.91}$$

Define the projection map

$$\pi : N(\gamma(\tau)) \to G(\gamma(\tau)) \quad v \longmapsto v + [\dot{\gamma}(\tau)]. \tag{2.92}$$

We can see $N_\perp(\gamma(\tau))$ as the geometric realization of $G(\gamma(\tau))$ via the isomorphism

$$\phi : \quad N_\perp(\gamma(\tau)) \to G(\gamma(\tau)) \qquad v \mapsto v + [\dot{\gamma}(\tau)]. \tag{2.93}$$

The inverse ϕ^{-1} of this map is constructed as follows. Given $v \in G(\gamma(\tau))$ choose an $x \in N(\gamma(\tau))$ such that $\pi(x) = v$. Decompose x uniquely as $x = x_1 + \lambda\dot{\gamma}(\tau)$, where $x_1 \in N_\perp(\gamma(\tau))$. We define $\phi^{-1}(v) = x_1$.

We will project the metric, covariant derivative, curvature tensor and torsion tensor to $G(\gamma(\tau))$. Given $v, w \in G(\gamma(\tau))$, let $x, y \in N(\gamma(\tau))$ be such that $\pi(x) = v$, $\pi(y) = w$. We define the projected metric by

$$\bar{g}(v, w) = g(x, y). \tag{2.94}$$

This is well-defined, because x, y are orthogonal to $\dot{\gamma}$.

If we let $\Omega(\gamma)$ denote the piecewise smooth sections of $G(\gamma)$ and $\mathcal{T}_\perp(\gamma)$ the piecewise smooth sections of $N(\gamma)$, we can given $\bar{V} \in \Omega(\gamma)$ find a $V \in \mathcal{T}_\perp(\gamma)$ such that $\bar{V} = \pi(V)$. Define

$$D_\tau \bar{V} = \pi(D_\tau V). \tag{2.95}$$

Let $V_1 \in \mathcal{T}_\perp(\gamma)$ be such that we also have that $\pi(V_1) = \bar{V}$, hence $V_1 = V + f\dot{\gamma}$ for some smooth function $f : [\tau_\mathrm{i}, \tau_\mathrm{f}] \to \mathbb{R}$ and we get that $D_\tau V_1 = D_\tau V + \dot{f}\dot{\gamma}$. This implies that $D_\tau \bar{V}$ is well defined. This covariant derivative on $\Omega(\gamma)$ is also compatible with the metric $\bar{g}$ and has all the usual properties of a covariant derivative. It can also be shown that for $\bar{V} \in \Omega(\gamma)$

$$D_\tau \phi^{-1}(\bar{V}) = \phi^{-1}(D_\tau \bar{V}), \tag{2.96}$$

hence, covariant differentiation in $G(\gamma)$ and $N_\perp(\gamma)$ are the same.

We define the curvature endomorphism $\bar{R}_\gamma : G(\gamma(\tau)) \to G(\gamma(\tau))$ by

$$v \mapsto \bar{R}_\gamma(v) = \pi\left[R_\gamma(x)\right] \tag{2.97}$$

where $x \in N(\gamma(\tau))$ such that $v = \pi(x)$. This does not depend on the particular x we choose.

If we let $v, w \in G(\gamma(\tau))$, $x, y \in N(\gamma(\tau))$ such that $v = \pi(x)$, $w = \pi(y)$. Then

$$\bar{g}(\bar{R}_\gamma(v), w) = g(R_\gamma(x), y). \tag{2.98}$$

We define the torsion tensor $\bar{S}_\gamma : G(\gamma(\tau)) \to G(\gamma(\tau))$ by

$$v \mapsto \bar{S}_\gamma(v) = \pi\left[S_\gamma(x)\right] \tag{2.99}$$

which also does not depend on the particular $x \in N(\gamma(\tau))$ we choose such that $v = \pi(x)$. Notice that

$$\begin{aligned} \bar{g}\left[\bar{S}_\gamma(v), w\right] &= \bar{g}\left[\pi\left[S_\gamma(x)\right], w\right] \\ &= g\left(S_\gamma(x), y\right) \\ &= -g\left(x, S_\gamma(y)\right) \\ &= -\bar{g}\left[v, \bar{S}_\gamma(w)\right], \end{aligned} \tag{2.100}$$

so $\left(\bar{S}_\gamma\right)^\dagger = -\bar{S}_\gamma$.

Definition 2.3.1. A smooth section $\bar{J} \in \Omega(\gamma)$ is a *Jacobi class* in $G(\gamma)$ if $\bar{J}$ satisfies the Jacobi equation

$$D_\tau^2 \bar{J} + D_\tau \bar{S}_\gamma(\bar{J}) + \bar{R}_\gamma(\bar{J}) = [\dot{\gamma}]. \tag{2.101}$$

It can be shown (i.e. [22]) that given a Jacobi class $\bar{J} \in \Omega(\gamma)$, there is a two-parameter class of Jacobi fields $J_{\lambda,\mu} = J + \lambda\dot{\gamma} + \mu\tau\dot{\gamma} \in \mathcal{T}_\perp(\gamma)$, $\lambda, \mu \in \mathbb{R}$, such that $\bar{J} = \pi(J_{\lambda,\mu})$. It is not always the case that there lives a Jacobi field J in $N_\perp(\gamma(\tau))$ such that $\bar{J} = \pi(J)$. We will prove the following useful lemma (generalizing the proof from [22]).

Lemma 2.3.1. *Let $\bar{J} \in \Omega(\gamma)$ be a Jacobi class such that $\bar{J}(\tau_\mathrm{i}) = [\dot{\gamma}(\tau_\mathrm{i})]$, $\bar{J}(\tau_\mathrm{f}) = [\dot{\gamma}(\tau_\mathrm{f})]$. Then there is a unique Jacobi field $J \in \mathcal{T}_\perp(\gamma)$ such that $\bar{J} = \pi(J)$ and $J(\tau_\mathrm{i}) = J(\tau_\mathrm{f}) = 0$.*

Proof. There exists a Jacobi field $Y \in \mathcal{T}_\perp(\gamma)$ such that $\bar{J} = \pi(Y)$. Then $Y(\tau_\mathrm{i}) = c_1\dot{\gamma}(\tau_\mathrm{i})$, $Y(\tau_\mathrm{f}) = c_2\dot{\gamma}(\tau_\mathrm{f})$ with constants c_1, c_2. Define $\lambda = (c_2\tau_\mathrm{i} - c_1\tau_\mathrm{f})(\tau_\mathrm{f} - \tau_\mathrm{i})^{-1}$ and $\mu = \tau_\mathrm{f}^{-1}\left[(c_1\tau_\mathrm{f} - c_2\tau_\mathrm{i})(\tau_\mathrm{f} - \tau_\mathrm{i})^{-1} - c_2\right]$. The vector field $J = Y + \lambda\dot{\gamma} + \mu\tau\dot{\gamma} \in \mathcal{T}_\perp(\gamma)$ is also a Jacobi field with $\pi(J) = \bar{J}$ and it has $J(\tau_\mathrm{i}) = J(\tau_\mathrm{f}) = 0$. Suppose now that we have two Jacobi fields $J_1, J_2 \in \mathcal{T}_\perp(\gamma)$ with $J_1(\tau_\mathrm{i}) = J_1(\tau_\mathrm{f}) = J_2(\tau_\mathrm{i}) = J_2(\tau_\mathrm{f}) = 0$ and such that $\pi(J_1) = \pi(J_2) = \bar{J}$. $X = J_1 - J_2$ is a Jacobi field with $\pi(X) = [\dot{\gamma}]$, which implies that $X = f(\tau)\dot{\gamma}$. We have that

$$0 = D_\tau^2 X + D_\tau S(X, \dot{\gamma}) + R(X, \dot{\gamma})\dot{\gamma} = \ddot{f}\dot{\gamma}, \tag{2.102}$$

hence, using $f(\tau_\mathrm{i}) = f(\tau_\mathrm{f}) = 0$, $f = 0$. Therefore $J_1 = J_2$. ☐

Using this lemma we can now make the following definition.

Definition 2.3.2. Let $\gamma : [\tau_\mathrm{i}, \tau_\mathrm{f}] \to M$ be a null geodesic. Let $p = \gamma(\tau_1)$, $q = \gamma(\tau_2)$ for $\tau_\mathrm{i} \le \tau_1 < \tau_2 \le \tau_\mathrm{f}$. p is said to be *conjugate* to q along γ if there exists a Jacobi class $\bar{J} \in \Omega(\gamma)$ such that $\bar{J} \ne [\dot{\gamma}]$ and $\bar{J}(\tau_1) = [\dot{\gamma}(\tau_1)]$, $\bar{J}(\tau_2) = [\dot{\gamma}(\tau_2)]$.

Definition 2.3.3. A smooth tensor field $\bar{A} : G(\gamma) \to G(\gamma)$ is said to be a Jacobi tensor field if

$$\begin{aligned} D_\tau^2 \bar{A} + D_\tau \bar{S}_\gamma \bar{A} + \bar{R}\bar{A} &= 0 \\ \mathrm{Ker}(\bar{A}(\tau)) \cap \mathrm{Ker}(D_\tau \bar{A}(\tau)) &= \{0\} \end{aligned} \tag{2.103}$$

for all τ.

We can define $\bar{B}_{\bar{A}} = \left(D_\tau \bar{A}\right)\bar{A}^{-1}$ and define the expansion, vorticity tensor and shear tensor:

$$\begin{aligned} \bar{\theta}_{\bar{A}} &= \operatorname{tr}(\bar{B}_{\bar{A}}); \\ \bar{\omega}_{\bar{A}} &= \frac{1}{2}(\bar{B}_{\bar{A}} - \bar{B}^\dagger_{\bar{A}}); \\ \bar{\sigma}_{\bar{A}} &= \frac{1}{2}(\bar{B}_{\bar{A}} + \bar{B}^\dagger_{\bar{A}}) - \frac{\bar{\theta}_{\bar{A}}}{2}, \end{aligned} \tag{2.104}$$

where there is a 2 in the definition of the shear tensor because $G(\gamma(\tau))$ is two dimensional for all τ.

Proposition 2.3.1. *The derivative of the expansion of a Jacobi tensor field A is given by*

$$\partial_\tau \bar{\theta}_{\bar{A}} = -\operatorname{Ric}(\dot{\gamma},\dot{\gamma}) - \operatorname{tr}(\bar{S}_\gamma \bar{\omega}_{\bar{A}}) - \operatorname{tr}(\bar{\omega}^2_{\bar{A}}) - \operatorname{tr}(\bar{\sigma}^2_{\bar{A}}) - \frac{\bar{\theta}^2_{\bar{A}}}{2}. \tag{2.105}$$

Proof. Just as in the timelike case, we obtain $D_\tau \bar{B}_{\bar{A}} = -\bar{R}_\gamma - D_t \bar{S}_\gamma - \bar{S}_\gamma \bar{B} - \bar{B}\bar{B}$ and this leads to

$$\partial_\tau \bar{\theta}_A = -\operatorname{tr}(\bar{R}_\gamma) - \operatorname{tr}(D_\tau \bar{S}_\gamma) - \operatorname{tr}(\bar{S}_\gamma \bar{\omega}_{\bar{A}}) - \operatorname{tr}(\bar{S}_\gamma \bar{\sigma}_{\bar{A}}) - \frac{\bar{\theta}_{\bar{A}}}{3}\operatorname{tr}(\bar{S}_\gamma) - \operatorname{tr}(\bar{\omega}^2_{\bar{A}}) - \operatorname{tr}(\bar{\sigma}^2_{\bar{A}}) - \frac{\bar{\theta}^2_{\bar{A}}}{2}, \tag{2.106}$$

which gives, using that $\bar{S}_\gamma$ is anti-symmetric

$$\partial_\tau \bar{\theta}_{\bar{A}} = -\operatorname{tr}(\bar{R}_\gamma) - \operatorname{tr}(\bar{S}_\gamma \bar{\omega}_{\bar{A}}) - \operatorname{tr}(\bar{\omega}^2_{\bar{A}}) - \operatorname{tr}(\bar{\sigma}^2_{\bar{A}}) - \frac{\bar{\theta}^2_{\bar{A}}}{2}. \tag{2.107}$$

Let E_i be an orthonormal basis for $N_\perp(\gamma)$ at every point of γ, $i \in \{2,3\}$. Expand this basis with E_0 and E_1 to an orthonormal basis along γ, where E_0 is a timelike vector and $\dot{\gamma} = (E_0 + E_1)/\sqrt{2}$. Then

$$\begin{aligned} g\left(R_\gamma(E_1), E_1\right) - g\left(R_\gamma(E_0), E_0\right) &= g\left(R(E_1,\dot{\gamma})\dot{\gamma}, E_1\right) - g\left(R(E_0,\dot{\gamma})\dot{\gamma}, E_0\right) \\ &= \frac{1}{2} g\left(R(E_1,E_0)E_0, E_1\right) - \frac{1}{2} g\left(R(E_0,E_1)E_1, E_0\right) \\ &= 0 \end{aligned} \tag{2.108}$$

where we used the anti-symmetry of the curvature tensor (to prove this property one only needs metric compatibility). Hence

$$\begin{aligned} \operatorname{tr}(\bar{R}_\gamma) &= \sum_{i=2}^{3} \bar{g}\left(\bar{R}_\gamma(\bar{E}_i), \bar{E}_i\right) \\ &= \sum_{i=2}^{3} g\left(R_\gamma(E_i), E_i\right) \\ &= \sum_{i,j=0}^{3} g(E_i, E_j) g\left(R_\gamma(E_i), E_i\right) \\ &= \eta^{ij} E_i^\mu E_j^\nu g\left(R(\partial_\mu, \dot{\gamma})\dot{\gamma}, \partial_\nu\right) \\ &= \operatorname{Ric}(\dot{\gamma},\dot{\gamma}), \end{aligned} \tag{2.109}$$

where the last step follows as in the timelike case. Substitution in Eq. (2.107) yields

$$\partial_\tau \bar{\theta}_{\bar{A}} = -\operatorname{Ric}(\dot{\gamma},\dot{\gamma}) - \operatorname{tr}(\bar{S}_\gamma \bar{\omega}_{\bar{A}}) - \operatorname{tr}(\bar{\omega}^2_{\bar{A}}) - \operatorname{tr}(\bar{\sigma}^2_{\bar{A}}) - \frac{\bar{\theta}^2_{\bar{A}}}{2}. \tag{2.110}$$

This is the generalized Raychaudhuri equation for null geodesics and totally anti-symmetric torsion. □

We will now construct a Jacobi tensor field $\bar{A}$ just as we did in the timelike case. Let $\bar{E}_i = \pi(E_i)$ for $i = 2,3$ be the projected basis vectors of the geometric realization of $G(\gamma)$. Let $\bar{J}_i$ for $i = 2,3$ be the Jacobi class in $G(\gamma)$ with $\bar{J}_i(\tau_\mathrm{i}) = [\dot{\gamma}(\tau_\mathrm{i})]$ and $D_\tau \bar{J}_i(\tau_\mathrm{i}) = \bar{E}_i(\tau_\mathrm{i})$. Then the tensor that in this basis has components

$$\bar{A}^k{}_l = \left(\bar{J}_l\right)^k \tag{2.111}$$

for $l, k = 2, 3$ is a Jacobi tensor field along γ. A point along γ will be conjugate to $\gamma(\tau_1)$ along γ if and only if $\bar{A}$ is singular there which happens precisely when $|\bar{\theta}| \to \infty$. In the same way as for the timelike case, we can show that

$$\bar{\omega}_{\bar{A}} = -\frac{1}{2}\bar{S}_\gamma, \tag{2.112}$$

such that Eq. (2.110) reduces to

$$\partial_\tau \bar{\theta}_{\bar{A}} = -\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) + \frac{1}{4}\mathrm{tr}(\bar{S}^2_\gamma) - \mathrm{tr}(\bar{\sigma}^2_{\bar{A}}) - \frac{\bar{\theta}^2_{\bar{A}}}{2}. \tag{2.113}$$

Expanding now the orthonormal basis E_2, E_3 of $N_\perp(\gamma)$ with E_0 and E_1 to an orthonormal basis along γ, where E_0 is a timelike vector and $\dot{\gamma} = (E_0 + E_1)/\sqrt{2}$. Then

$$\begin{aligned}
-g\left[S^2_\gamma(E_0), E_0\right] + g\left[S^2_\gamma(E_1), E_1\right] &= -\frac{1}{\sqrt{2}} g\left[S_\gamma(S(E_0, E_1)), E_0\right] + \frac{1}{\sqrt{2}} g\left[S_\gamma(S(E_1, E_0)), E_1\right] \\
&= -g\left[S_\gamma(S(E_0, E_1)), \dot{\gamma}\right] \\
&= 0.
\end{aligned} \tag{2.114}$$

Hence

$$\begin{aligned}
\mathrm{tr}(\bar{S}^2_\gamma) &= \sum_{i=2}^{3} \bar{g}(\bar{S}^2_\gamma(\bar{E}_i), \bar{E}_i) \\
&= \sum_{i=2}^{3} g(S^2_\gamma(E_i), E_i) \\
&= \sum_{i,j=0}^{3} g(E_i, E_j) g(S^2_\gamma(E_i), E_i) \\
&= \eta^{ij} E^\mu_i E^\mu_j g(S^2_\gamma(\partial_\mu), \partial_\nu) \\
&= g^{\mu\nu} \left(S^2_\gamma\right)_{\nu\mu} \\
&= \mathrm{tr}(S^2_\gamma).
\end{aligned} \tag{2.115}$$

Thus we find that Eq. (2.113) becomes

$$\partial_\tau \bar{\theta}_{\bar{A}} = -\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) + \frac{1}{4}\mathrm{tr}(S^2_\gamma) - \mathrm{tr}(\bar{\sigma}^2_{\bar{A}}) - \frac{\bar{\theta}^2_{\bar{A}}}{2}. \tag{2.116}$$

Proposition 2.3.2. *Let $\gamma : \mathbb{R} \to M$ be a null geodesic, and let $p = \gamma(\tau_0)$. Let $\bar{A}$ the Jacobi tensor field constructed as in Eq. (2.111) such that $\bar{A}(\tau_0) = 0$. If for some $\tau_1 > \tau_0$ the expansion $\bar{\theta}_{\bar{A}}(\tau_1) = \bar{\theta}_1 < 0$ and if $\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) - \frac{1}{4}\mathrm{tr}(S^2_\gamma) \geq 0$ for all $\tau \geq \tau_1$, there will be a point conjugate to p along γ between $\gamma(\tau_1)$ and $\gamma(\tau_1 - \frac{2}{\theta_1})$.*

Proof. This can be proven as in proposition 2.1.2 using Eq. (2.116). □

Proposition 2.3.3. *Let $\gamma : \mathbb{R} \to M$ be a null geodesic. If $\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) - \frac{1}{4}\mathrm{tr}(S^2_\gamma) \geq 0$ for all τ and if at some point $r = \gamma(\tau_1)$ $\breve{\bar{R}}_\gamma \neq 0$, there will be points $p = \gamma(\tau_0)$, $q = \gamma(\tau_2)$ $\tau_0 \neq \tau_2$ conjugate along γ.*

Proof. This can be proven as in proposition 2.1.3. □

2.4 Spacelike Two-Surface

We also like to consider conjugate points to a spacelike two-surface P along a null geodesic that is orthogonal to the surface. For a point $p \in P$ g restricted to $T^\perp_p P$ is a 2-dimensional Lorentz metric. Hence, there are two null vector fields n_1, n_2 orthogonal to P. These two fields define tensor fields L_{n_i} on P. Locally we can then define a pseudo-orthonormal basis on P: $E_0 = n_1$, $E_1 = \beta n_2$ and E_2, E_3 chosen such that $g(E_0, E_1) = -1$, $g(E_0, E_i) = g(E_1, E_i) = 0$ and $g(E_i, E_j) = \delta_{ij}$ for $i, j = 2, 3$. The null vector fields E_0 and E_1 give rise to second fundamental form operators L_{E_0} and L_{E_1} which are $(1,1)$ tensors that are locally defined on P. If $\gamma : [\tau_\mathrm{i}, \tau_\mathrm{f}] \to M$ is a null geodesic such that $\dot\gamma(\tau_\mathrm{i}) = E_0(\gamma(\tau_\mathrm{i}))$ we can parallel transport the pseudo-orthonormal basis to get such a frame along γ. The space normal to $\dot\gamma$ will be spanned by E_0, E_2 and E_3. Just as in the previous section we can introduce the quotient space $G(\gamma(\tau)) = N(\gamma(\tau))/[\dot\gamma(\tau)]$ with corresponding quotient bundle $G(\gamma)$. If $\pi : N(\gamma) \to G(\gamma)$ denotes the projection then $\pi|_{T_pH} : T_pH \to G(\gamma(\tau_\mathrm{i}))$ where $p = \gamma(\tau_\mathrm{i})$ is a vector space isomorphism. Therefore we can project the second fundamental form operators to operators $\bar L_{E_i}$ by

$$\bar L_{E_i} = \pi \circ L_{E_i} \circ \left[\pi|_{T_pH}\right]^{-1}. \tag{2.117}$$

Let now Γ be a one-parameter family of the null geodesics defined by the null vector field $E_0 = n_1$ such that Γ is a variation of γ. Let J be the variation vector field of this variation, then J is a Jacobi field.

Lemma 2.4.1. *Let J be the variation vector field of the variation defined above. Then $D_\tau J(\tau_\mathrm{i}) = -L_{\dot\gamma(\tau_\mathrm{i})}(J(\tau_\mathrm{i})) - S(J(\tau_\mathrm{i}), \dot\gamma(\tau_\mathrm{i})) + \lambda\dot\gamma(\tau_\mathrm{i})$ for a $\lambda \in \mathbb{R}$.*

Proof. Since $g(T,T) = 0$ we have that

$$0 = W(g(T,T)) = 2g(D_wT, T) = 2g(S(W,T),T) + 2g(D_\tau W, T) = 2g(D_\tau W, T). \tag{2.118}$$

Hence, $g(D_\tau J(\tau_\mathrm{i}), \dot\gamma(\tau_\mathrm{i})) = 0$ and this implies that $D_\tau J(\tau_\mathrm{i}) \in N(\gamma)$. Extend $v \in T_pH$ to a vector field $V \in TH$ along the curve $w \mapsto \Gamma(w, \tau_\mathrm{i})$. Then

$$\begin{aligned} g(L_{\dot\gamma(\tau_\mathrm{i})}(J(\tau_\mathrm{i})), v) &= g\left(\nabla_J V|_p, T(0,\tau_\mathrm{i})\right) \\ &= \partial_w g\left(V, T\right) - g(v, D_wT|_{(0,\tau_\mathrm{i})}) \\ &= -g(v, S(J,\dot\gamma)) - g(v, D_\tau J(\tau_\mathrm{i})). \end{aligned} \tag{2.119}$$

This holds for every $v \in T_pH$ so we must have that $D_\tau J(\tau_\mathrm{i})|_{T_pH} = -L_{\dot\gamma(\tau_\mathrm{i})}(J(\tau_\mathrm{i})) - S(J, \dot\gamma)$ and that proves the lemma. □

This implies that the Jacobi class $\bar J = \pi(J)$ satisfies the initial condition

$$D_\tau \bar J(\tau_\mathrm{i}) = -\bar L_{\dot\gamma(\tau_\mathrm{i})}(\bar J(\tau_\mathrm{i})) - \bar S_\gamma(J(\tau_\mathrm{i})) \tag{2.120}$$

and this motivates the following definition.

Definition 2.4.1. Let $\gamma : [\tau_\mathrm{i}, \tau_\mathrm{f}] \to M$ be a null geodesic which is orthogonal to a spacelike two-surface P and such that $\gamma(\tau_\mathrm{i}) \in P$. A point $q = \gamma(\tau_2)$ is said to be *conjugate* to P if there exists a non-vanishing Jacobi class $\bar J \in \Omega(\gamma)$ such that $\bar J$ vanishes at q and

$$D_\tau \bar J(\tau_\mathrm{i}) = -\bar L_{\dot\gamma(\tau_\mathrm{i})}(\bar J(\tau_\mathrm{i})) - \bar S_\gamma(J(\tau_\mathrm{i})). \tag{2.121}$$

Proposition 2.4.1. *Let P be a spacelike two-surface and let $\gamma : \mathbb{R} \to M$ be a null geodesic, such that $\gamma(0) \in P$ and γ is orthogonal to P. If $\theta_1 = -\mathrm{tr}(L_{\dot\gamma(0)}) < 0$ and $\mathrm{Ric}(\dot\gamma,\dot\gamma) - \frac{1}{4}\mathrm{tr}(S_\gamma^2) \geq 0$ for all $\tau \geq 0$, there will be a point conjugate to P along γ between $\gamma(0)$ and $\gamma(-\frac{2}{\theta_1})$.*

Proof. This can be proven using similar reasoning as in Section 2.2 that leads to proposition 2.2.1. □

Chapter 3

Variations of the Arc Length

In this chapter we will prove more propositions needed for the singularity theorems in Chapter 5. In particular we will examine the length of non-spacelike curves, Eq. (1.20), and prove propositions related to this length. Furthermore we will see under which conditions we have a timelike curve between points or between a surface and a point. The propositions in this section have been proven in [3] for the case of vanishing torsion. We will extend them to totally anti-symmetric torsion. However, in the propositions and proofs we will often not directly reduce to this kind of torsion such that in Section 5.2 we can easily discuss the case of torsion that is not totally anti-symmetric.

In this chapter we consider curves γ that are piecewise smooth but may have a finite number of singular points; we only require that the two tangent vectors $\partial_\tau^- = \lim_{\tau\uparrow\tau_1}\dot\gamma$ and $\partial_\tau^+ = \lim_{\tau\downarrow\tau_1}\dot\gamma$ at a singular point τ_1 satisfy $g(\partial_\tau^-, \partial_\tau^+) = -1$. From now on we also assume that whenever we have a convex normal neighborhood U of a point $p \in M$, the map $\exp_p$ will be restricted to the neighborhood of the origin in T_pM that is diffeomorphic to U.

Lemma 3.0.1. *Let U be a convex normal coordinate neighborhood of $p \in M$. Then the timelike geodesics through p are orthogonal to the three-surfaces of constant $\sigma(q) = g(\exp_p^{-1}q, \exp_p^{-1}q)$ when $\sigma < 0$.*

Proof. Notice that a curve that is given by constant σ is given by $\lambda(w) = \exp_p(\tau_0 X(w))$, where $X(w)$ is a curve in T_pM such that $g(X(w), X(w)) = -1$ and τ_0 is a constant. So we have to show that timelike geodesics through p are orthogonal to these curves, where these are defined. Geodesics through p are given by $\gamma(\tau) = \exp_p(\tau X(w_0))$ where $X(w_0)$ is timelike. In terms of the surface $\alpha(w,\tau) = \exp_p(\tau X(w))$ we need to prove that

$$g\left(\partial_w, \partial_\tau\right) = 0. \tag{3.1}$$

Now

$$\begin{aligned}
\partial_\tau g\left(\partial_\tau, \partial_w\right) &= g\left(D_\tau\partial_\tau, \partial_w\right) + g\left(\partial_\tau, D_\tau\partial_w\right) \\
&= g\left(\partial_\tau, S\left(\partial_\tau, \partial_w\right)\right) + g\left(\partial_\tau, D_w\frac{\partial}{\partial\tau}\right) \\
&= \frac{1}{2}\partial_w g\left(\partial_\tau, \partial_\tau\right) = 0.
\end{aligned} \tag{3.2}$$

So $g(\partial_\tau, \partial_w)$ does not depend on τ. At $\tau = 0$, $\partial_w = 0$ and that proves this lemma. □

Since no arguments need to be used that include torsion, we will not give the proof of the following proposition [3].

Proposition 3.0.1. *Let U be a convex normal coordinate neighborhood of $p \in M$. The points that are connected to p by timelike (non-spacelike) curves in U are those of the form $\exp_p(X)$, $X \in T_pM$ where $g(X,X) < 0$ (≤ 0)*

Corollary 3.0.1. *Let U be a convex normal coordinate neighborhood of $p \in M$. If $q \in U$ can be reached from p by a non-spacelike curve, but not by a timelike curve, then q lies on a null geodesic from p.*

In analogy with Eq. (1.20) the length of a non-spacelike curve $\gamma : [\tau_{\rm i}, \tau_{\rm f}] \to M$ such that $\gamma(\tau_{\rm i}) = p$, $\gamma(\tau_{\rm f}) = q$ between points can be defined as

$$L(\gamma, p, q) = \int_{\tau_{\rm i}}^{\tau_{\rm f}} \left[-g(\dot\gamma, \dot\gamma)\right]^{1/2} d\tau, \tag{3.3}$$

where the integral is taken over the differentiable sections of the curve. Again, since we do not have to consider torsion, we will state the following proposition (proposition 4.5.3 from [3]) without proof:

Proposition 3.0.2. *Let $q,\ p \in U \subset M$ where U is a convex normal neighborhood. If q and p can be joined by a non-spacelike curve in U, the longest such curve is the unique non-spacelike geodesic in U between q and p. Furthermore, defining $\rho(p, q)$ as the length of this curve if it exists and as zero otherwise, $\rho(p, q)$ is a continuous function on $U \times U$.*

So far we have only been looking at curves between points that lie in the same convex normal neighborhood. We now also want to look at more general curves and see what conditions need to be satisfied by a longest curve. We will do this using proper variations of a curve between two points.

Definition 3.0.1. Let $\gamma : [\tau_{\rm i}, \tau_{\rm f}] \to M$ be a timelike curve such that $\gamma(\tau_{\rm i}) = p$, $\gamma(\tau_{\rm f}) = q$. A *proper variation* of this curve is a smooth map $\Gamma : (-\epsilon, \epsilon) \times [\tau_{\rm i}, \tau_{\rm f}] \to M$ such that:

- $\Gamma(0, \tau) = \gamma(\tau)$;
- there is a subdivision $\tau_{\rm i} = \tau_1 < ... < \tau_n = \tau_{\rm f}$ of $[\tau_{\rm i}, \tau_{\rm f}]$ such that Γ is smooth on $(-\epsilon, \epsilon) \times [\tau_i, \tau_{i+1}]$ for all $1 \le i \le n-1$;
- $\Gamma(w, \tau_{\rm i}) = p$ and $\Gamma(w, \tau_{\rm f}) = q$
- $\Gamma(w_0, \tau)$ is a timelike curve for each $w_0 \in (-\epsilon, \epsilon)$.

As before, we define $W = \partial_w \Gamma$, $T = \partial_\tau \Gamma$ and $V = W(0, \tau)$ is the variation vector field of Γ.

Given a vector field V along γ such that $V(\tau_{\rm i}) = V(\tau_{\rm f}) = 0$, $\Gamma(w, \tau) = \exp_{\gamma(\tau)}(wV(\tau))$ for $w \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$ defines a proper variation of γ.

Lemma 3.0.2. *Under a variation of a curve $\gamma : [\tau_{\rm i}, \tau_{\rm f}] \to M$ between $p, q \in M$, the derivative of the length of the curve is equal to:*

$$\begin{aligned} \frac{\partial L}{\partial w}|_{w=0} &= \sum_{i=1}^{n-1} \int_{\tau_i}^{\tau_{i+1}} \left\{ \left[-g(S(V, \dot\gamma), \dot\gamma) + g(V, D_\tau \dot\gamma)\right] f^{-1} - f^{-2} \frac{\partial f}{\partial \tau} g(V, \dot\gamma) \right\} d\tau \\ &\quad + \sum_{i=2}^{n-1} g\left(V, \left[f^{-1}\dot\gamma\right]_i\right), \end{aligned} \tag{3.4}$$

where $f(\tau) = \left[-g(\dot\gamma(\tau), \dot\gamma(\tau))\right]^{1/2}$ and $g\left(V, \left[f^{-1}\dot\gamma\right]_i\right)$ is the discontinuity at τ_i.

Proof. Define

$$f(w, \tau) = \left[-g(T(w, \tau), T(w, \tau))\right]^{1/2}. \tag{3.5}$$

We get:

$$\begin{aligned} \frac{\partial L}{\partial w} &= \sum_{i=1}^{n-1} \frac{\partial}{\partial w} \int_{\tau_i}^{\tau_{i+1}} \left[-g(T, T)\right]^{1/2} d\tau \\ &= -\sum_{i=1}^{n-1} \int_{\tau_i}^{\tau_{i+1}} g(D_w T, T) f^{-1} d\tau \\ &= -\sum_{i=1}^{n-1} \int_{\tau_i}^{\tau_{i+1}} \left[g(S(W, T), T) + g(D_\tau W, T)\right] f^{-1} d\tau \\ &= -\sum_{i=1}^{n-1} \int_{\tau_i}^{\tau_{i+1}} \left[g(S(W, T), T) + \frac{\partial}{\partial \tau} g(W, T) - g(W, D_\tau T)\right] f^{-1} d\tau. \end{aligned} \tag{3.6}$$

Evaluating at $w=0$ we get:

$$\begin{aligned}\frac{\partial L}{\partial w}|_{w=0} &= -\sum_{i=1}^{n-1}\int_{\tau_i}^{\tau_{i+1}}\left[g(S(V,\dot{\gamma}),\dot{\gamma})+\frac{\partial}{\partial\tau}g(V,\dot{\gamma})-g(V,D_\tau\gamma)\right]f^{-1}d\tau \\ &= \sum_{i=1}^{n-1}\int_{\tau_i}^{\tau_{i+1}}\left\{\left[-g(S(V,\dot{\gamma}),\dot{\gamma})+g(V,D_\tau\dot{\gamma})\right]f^{-1}-f^{-2}\frac{\partial f}{\partial\tau}g(V,\dot{\gamma})\right\}d\tau+\sum_{i=2}^{n-1}g\left(V,\left[f^{-1}\dot{\gamma}\right]_i\right).\end{aligned} \tag{3.7}$$

□

We can choose the parameter τ such that $g(\dot{\gamma},\dot{\gamma})=-1$ and Eq. (3.4) reduces to

$$\frac{\partial L}{\partial w}|_{w=0}=\sum_{i=1}^{n-1}\int_{\tau_i}^{\tau_{i+1}}\left[-g(S(V,\dot{\gamma}),\dot{\gamma})+g(V,D_\tau\dot{\gamma})\right]d\tau+\sum_{i=2}^{n-1}g\left(V,[\dot{\gamma}]_i\right). \tag{3.8}$$

When the torsion is totally anti-symmetric, we see from Eq. (3.8) that the first derivative vanishes for an (unbroken) geodesic γ, for all other curves there exists a variation such that the derivative is larger than zero. Hence, a longest curve necessarily needs to be a geodesic. We need to consider the second derivative to be able to say more.

Definition 3.0.2. We define a two-parameter proper variation of γ by a map $\Gamma:(-\epsilon_1,\epsilon_1)\times(-\epsilon_2,\epsilon_2)\times[\tau_{\rm i},\tau_{\rm f}]\to M$ such that:

- $\Gamma(0,0,\tau)=\gamma(\tau)$;
- there is a subdivision $\tau_{\rm i}=\tau_1<...<\tau_n=\tau_{\rm f}$ of $[\tau_{\rm i},\tau_{\rm f}]$ such that Γ is smooth on $(-\epsilon_1,\epsilon_1)\times(-\epsilon_2,\epsilon_2)\times[\tau_i,\tau_{i+1}]$ for all $1\le i\le n-1$;
- $\Gamma(w_1,w_2,\tau_{\rm i})=p$ and $\Gamma(w_1,w_2,\tau_{\rm f})=q$;
- $\Gamma(w_1,w_2,\tau)$ is a timelike curve for each $w_1\in(-\epsilon_1,\epsilon_1)$, $w_2\in(-\epsilon_2,\epsilon_2)$.

We define $W_1=\partial_{w_1}\Gamma$, $W_2=\partial_{w_2}\Gamma$ and $T=\partial_\tau\Gamma$ and let $V_1=W_1(0,0,\tau)$, $V_2=W_2(0,0,\tau)$ be the variation vector fields of Γ along γ.

Given two vector fields V_1,V_2 along γ such that $V_1(\tau_{\rm i})=V_1(\tau_{\rm f})=V_2(\tau_{\rm i})=V_2(\tau_{\rm f})=0$, $\Gamma=\exp_{\gamma(\tau)}(w_1V_1(\tau)+w_2V_2(\tau))$ is a proper variation of γ such that the variation vectors fields are V_1 and V_2.

Lemma 3.0.3. *Under a two-parameter variation of the geodesic timelike curve $\gamma:[\tau_{\rm i},\tau_{\rm f}]\to M$ between points $p,q\in M$, the second derivative of the length is equal to*

$$\begin{aligned}L(V_1,V_2)\equiv\frac{\partial^2 L}{\partial w_1\partial w_2}|_{w_i=0} &= \sum_{i=1}^{n-1}\int_{\tau_i}^{\tau_{i+1}}\Big\{g(V_2,D_\tau^2\left[V_1+g(V_1,\dot{\gamma})\dot{\gamma}\right])+g(V_2,R(V_1,\dot{\gamma})\dot{\gamma} \\ &\quad +D_\tau S(V_1,\dot{\gamma}))\Big\}d\tau+\sum_{i=2}^{n-1}g\left(V_2,\left[D_\tau\left(V_1+g(V_1,\dot{\gamma})\dot{\gamma}\right)\right]_i\right).\end{aligned} \tag{3.9}$$

Proof. Starting from Eq. (3.6) we find that

$$\begin{aligned}
\frac{\partial^2 L}{\partial w_1 \partial w_2} &= -\sum_{i=1}^{n-1} \frac{\partial}{\partial w_1} \int_{\tau_i}^{\tau_{i+1}} \left[g(S(W_2,T),T) + \frac{\partial}{\partial \tau} g(W_2,T) - g(W_2, D_\tau T) \right] f^{-1} d\tau \\
&= \sum_{i=1}^{n-1} \int_{\tau_i}^{\tau_{i+1}} \left\{ \left[-\frac{\partial}{\partial w_1} g(S(W_2,T),T) - \frac{\partial}{\partial \tau} g(D_{w_1} W_2, T) - \frac{\partial}{\partial \tau} g(W_2, D_{w_1} T) + g(D_{w_1} W_2, D_\tau T) \right. \right. \\
&\quad \left. + g(W_2, D_{w_1} D_\tau T) \right] f^{-1} + f^{-2} \frac{\partial f}{\partial w_1} \left. \left[g(S(W_2,T),T) + \frac{\partial}{\partial \tau} g(W_2,T) - g(W_2, D_\tau T) \right] \right\} d\tau \\
&= \sum_{i=1}^{n-1} \int_{\tau_i}^{\tau_{i+1}} \left\{ \left[-\frac{\partial}{\partial w_1} g(S(W_2,T),T) + g(D_{w_1} W_2, D_\tau T) + g(W_2, D_{w_1} D_\tau T) \right] f^{-1} \right. \\
&\quad - \left[g(D_{w_1} W_2, T) + g(W_2, D_{w_1} T) \right] f^{-2} \frac{\partial f}{\partial \tau} + f^{-2} \frac{\partial f}{\partial w_1} \left[g(S(W_2,T),T) - g(W_2, D_\tau T) \right] \\
&\quad \left. - f^{-2} \frac{\partial^2 f}{\partial \tau \partial w_1} g(W_2,T) + 2 f^{-3} \frac{\partial f}{\partial \tau} \frac{\partial f}{\partial w_1} g(W_2,T) \right\} d\tau \\
&\quad + \sum_{i=2}^{n-2} \left\{ g\left(D_{w_1} W_2, \left[f^{-1} T\right]_i\right) + g\left(W_2, \left[f^{-1} D_{w_1} T\right]_i\right) - g\left(W_2, \left[f^{-2} \frac{\partial f}{\partial w_1} T\right]_i\right) \right\} \qquad (3.10)
\end{aligned}$$

In the end we will look at $w_1 = w_2 = 0$, so we already know that some terms will go to zero, using that γ is a geodesic parametrized such that $f = 1$. Though it is mathematically not correct, we will obtain the same result if we already remove these terms:

$$\begin{aligned}
\frac{\partial^2 L}{\partial w_1 \partial w_2} &= \sum_{i=1}^{n-1} \int_{\tau_i}^{\tau_{i+1}} \left\{ -\frac{\partial}{\partial w_1} g(S(W_2,T),T) + g(W_2, D_{w_1} D_\tau T) + \frac{\partial f}{\partial w_1} g(S(W_2,T),T) \right. \\
&\quad \left. - \frac{\partial^2 f}{\partial w_1 \partial \tau} g(W_2,T) \right\} d\tau + \sum_{i=2}^{n-1} \left\{ g\left(W_2, [D_{w_1} T]_i\right) - g\left(W_2, \left[\frac{\partial f}{\partial w_1} T\right]_i\right) \right\}. \qquad (3.11)
\end{aligned}$$

Calculating the derivative of f with respect to w_1, we find that:

$$\frac{\partial}{\partial w_1} f = \frac{\partial}{\partial w_1} \sqrt{-g(T,T)} = \frac{-1}{\sqrt{-g(T,T)}} g(D_{w_1} T, T) = -f^{-1} \left[g(S(W_1,T),T) + g(D_\tau W_1, T) \right]. \qquad (3.12)$$

We also have that

$$g\left(W_2, D_{w_1} D_\tau T\right) = g\left(W_2, R(W_1,T)T + D_\tau D_{w_1} T\right) = g\left(W_2, R(W_1,T)T + D_\tau S(W_1,T) + D_\tau^2 W_1\right). \qquad (3.13)$$

Using these results and using that the torsion is totally anti-symmetric, we find that

$$\begin{aligned}
\frac{\partial^2 L}{\partial w_1 \partial w_2} &= \sum_{i=1}^{n-1} \int_{\tau_i}^{\tau_{i+1}} \left\{ g\left(W_2, R(W_1,T)T + D_\tau S(W_1,T) + D_\tau^2 W_1\right) + g(W_2,T) \frac{\partial}{\partial \tau} g(D_\tau W_1, T) \right\} d\tau \\
&\quad + \sum_{i=2}^{n-1} \left\{ g\left(W_2, [D_\tau W_1]_i\right) + g\left(W_2, [g(D_\tau W_1, T) T]_i\right) \right\} \\
&= \sum_{i=1}^{n-1} \int_{\tau_i}^{\tau_{i+1}} \left\{ g\left(W_2, D_\tau^2 \left[W_1 + g(W_1,T)T\right]\right) + g\left(W_2, R(W_1,T)T + D_\tau S(W_1,T)\right) \right\} d\tau \\
&\quad + \sum_{i=2}^{n-1} g\left(W_2, [D_\tau W_1 + g(W_1,T)T]_i\right). \qquad (3.14)
\end{aligned}$$

Evaluating at $w_1 = w_2 = 0$ yields:

$$\begin{aligned}\frac{\partial^2 L}{\partial w_1 \partial w_2}|_{w_i=0} &= \sum_{i=1}^{n-1} \int_{\tau_i}^{\tau_{i+1}} \left\{ g\left(V_2, D_\tau^2 \left[V_1 + g(V_1,\dot\gamma)\dot\gamma\right]\right) + g\left(V_2, R(V_1,\dot\gamma)\dot\gamma + D_\tau S(V_1,\dot\gamma)\right) \right\} d\tau \\ &+ \sum_{i=2}^{n-1} g\left(V_2, \left[D_\tau\left(V_1 + g(V_1,\dot\gamma)\dot\gamma\right)\right]_i\right). \end{aligned} \quad (3.15)$$

□

We see that this formula only depends on the components of the variation vector fields perpendicular to $\dot\gamma$. So this means we only have to consider variations with perpendicular variation vectors. Notice also that by construction we must have that $L(V_1, V_2) = L(V_2, V_1)$ and that L is linear in both arguments.

Proposition 3.0.3. *Let $\gamma : [\tau_\mathrm{i}, \tau_\mathrm{f}] \to M$ be a timelike geodesic such that $\gamma(\tau_\mathrm{i}) = p$, $\gamma(\tau_\mathrm{f}) = q$. When there is a conjugate point $r = \gamma(\tau_1)$ ($\tau_\mathrm{i} < \tau_1 < \tau_\mathrm{f}$) to p along γ there is a timelike curve between p and q that has a larger length than γ.*

Proof. Let J be the Jacobi field that vanishes at p and r. Extend it to $\tau_1 \le \tau \le \tau_\mathrm{f}$ by putting it to 0. Define the vector field K such that

$$g(K(\tau_1), D_\tau J(\tau_1)) = -1 \quad (3.16)$$

and K is orthogonal to $\dot\gamma$. Let

$$V = \epsilon K + \frac{1}{\epsilon} J, \quad (3.17)$$

where ϵ is a constant. Notice that V is orthogonal to $\dot\gamma$. With Eq. (3.9) we find that

$$L(V,V) = \epsilon^2 L(K,K) + 2L(K,J) + \frac{1}{\epsilon^2} L(J,J). \quad (3.18)$$

For the Jacobi field J we find that

$$\begin{aligned} L(J,J) &= \int_0^{\tau_1} \left\{ g(J, D_\tau^2 J + D_\tau S(J,\dot\gamma) + R(J,\dot\gamma)\dot\gamma) \right\} d\tau + g\left(J, [D_\tau J]\right)_{\tau=\tau_1} \\ &= 0. \end{aligned} \quad (3.19)$$

because of the Jacobi equation and $J(\tau_1) = 0$. In the same way we find that

$$\begin{aligned} L(K,J) &= \int_0^{\tau_1} \left\{ g(K, D_\tau^2 J + D_\tau S(J,\dot\gamma) + R(J,\dot\gamma)\dot\gamma) \right\} d\tau + g\left(K, [D_\tau J]\right)_{\tau=\tau_1} \\ &= g\left(K(\tau_1), -D_\tau J(\tau_1)\right) = 1. \end{aligned} \quad (3.20)$$

Hence

$$L(V,V) = \epsilon^2 L(K,K) + 2, \quad (3.21)$$

and by taking ϵ small enough, $L(V,V)$ will be positive, such that there exists a longer curve than γ between p and q. □

We would like to prove similar results for a curve γ between a spacelike three-surface H and a point $q \in M$. For that we can use a one-parameter proper variation of γ that is defined as before, except that instead of $\Gamma(w_0, \tau_\mathrm{i}) = p$ we require $\Gamma(w_0, \tau_\mathrm{i}) \in H$. We can also use two-parameter proper variations of γ that are defined as before, except that instead of $\Gamma(w_1, w_2, \tau_\mathrm{i}) = p$, we require $\Gamma(w_1, w_2, \tau_\mathrm{i}) \in H$.

Lemma 3.0.4. *Under a one-parameter proper variation of a curve $\gamma : [\tau_\mathrm{i}, \tau_\mathrm{f}] \to M$ between a spacelike hypersurface H and a point $q \in M$ the derivative of the length is given by:*

$$\frac{\partial L}{\partial w}|_{w=0} = \sum_{i=1}^{n-1} \int_{\tau_i}^{\tau_{i+1}} \left\{ \left[-g(S(V,\dot\gamma),\dot\gamma) + g(V, D_\tau\dot\gamma)\right] f^{-1} - f^{-2} \frac{\partial f}{\partial \tau} g(V,\dot\gamma) \right\} d\tau + g\left(V, f^{-1}\dot\gamma\right)|_{\tau=\tau_\mathrm{i}} + \sum_{i=2}^{n-1} g\left(V, \left[f^{-1}\dot\gamma\right]_i\right), \quad (3.22)$$

where $f(\tau) = \left[-g(\dot\gamma(\tau), \dot\gamma(\tau))\right]^{1/2}$.

Proof. This goes in a similar way as in lemma 3.0.2. □

Using a parametrization such that $f = 1$ and using that we consider totally anti-symmetric torsion, we obtain:

$$\frac{\partial L}{\partial w}|_{w=0} = \sum_{i=1}^{n-1} \int_{\tau_i}^{\tau_{i+1}} g(V, D_\tau \dot{\gamma}) d\tau + g(V, \dot{\gamma})|_{\tau=\tau_{\rm i}} + \sum_{i=2}^{n-1} g(V, [\dot{\gamma}]_i). \tag{3.23}$$

From this expression we can also conclude that a longest curve should necessarily be an unbroken geodesic curve orthogonal to H.

Lemma 3.0.5. *Under a two-parameter proper variation of a geodesic $\gamma : [\tau_{\rm i}, \tau_{\rm f}] \to M$ between a spacelike hypersurface H and a point $q \in M$, where γ and the other geodesics of the variation are orthogonal to H, the second derivative of the length is given by:*

$$\begin{aligned}\frac{\partial^2 L}{\partial w_1 \partial w_2}|_{w_i=0} &= \sum_{i=1}^{n-1} \int_{\tau_i}^{\tau_{i+1}} \left\{ g\left[V_2, D_\tau^2 \left(V_1 + g(V_1, \dot{\gamma})\dot{\gamma}\right)\right] + g\left(V_2, D_\tau S(V_1, \dot{\gamma}) + R(V_1, \dot{\gamma})\dot{\gamma}\right) \right\} d\tau \\ &\quad + \sum_{i=2}^{n-1} g\left(V_2, \left[D_\tau \left(V_1 + g(V_1, \dot{\gamma})\dot{\gamma}\right)\right]_i\right). \end{aligned} \tag{3.24}$$

Proof. Starting from the result of lemma 3.0.4 we see that with respect to the result of lemma 3.0.3 we get the extra terms

$$\begin{aligned} g\left(D_{w_1} W_2, f^{-1} T\right)|_{\tau=\tau_{\rm i}} + g\left(W_2, f^{-1} D_{w_1} T\right)|_{\tau=\tau_{\rm i}} - g\left(W_2, f^{-2} \frac{\partial f}{\partial w_1} T\right)|_{\tau=\tau_{\rm i}} &= \\ g\left(D_{w_1} W_2, f^{-1} T\right)|_{\tau=\tau_{\rm i}} + g\left(W_2, f^{-1} D_{w_1} T\right)|_{\tau=\tau_{\rm i}} + f^{-3} g\left(D_{w_1} T, T\right) g\left(W_2, T\right)|_{\tau=\tau_{\rm i}} &= \\ f^{-1} D_{w_1} g\left(W_2, T\right)|_{\tau=\tau_{\rm i}} + f^{-3} g\left(D_{w_1} T, T\right) g\left(W_2, T\right)|_{\tau=\tau_{\rm i}} & \end{aligned} \tag{3.25}$$

Now the variation is such that at $\tau = \tau_{\rm i}$ W_2 lies in H, so $g(W_2, T) = 0$ and that implies that these terms are zero. Hence:

$$\begin{aligned}\frac{\partial^2 L}{\partial w_1 \partial w_2}|_{w_i=0} &= \sum_{i=1}^{n-1} \int_{\tau_i}^{\tau_{i+1}} \left\{ g\left[V_2, D_\tau^2 \left(V_1 + g(V_1, \dot{\gamma})\dot{\gamma}\right)\right] + g\left(V_2, D_\tau S(V_1, \dot{\gamma}) + R(V_1, \dot{\gamma})\dot{\gamma}\right) \right\} d\tau \\ &\quad + \sum_{i=2}^{n-1} g\left(V_2, \left[D_\tau \left(V_1 + g(V_1, \dot{\gamma})\dot{\gamma}\right)\right]_i\right). \end{aligned} \tag{3.26}$$

□

Proposition 3.0.4. *Let $\gamma : [\tau_{\rm i}, \tau_{\rm f}] \to M$ be a timelike geodesic from a spacelike hypersurface H to a point q (i.e. $\gamma(\tau_{\rm i}) \in H$, $\gamma(\tau_{\rm f}) = q$, $\dot{\gamma}(\tau_{\rm i})$ orthogonal to H). When there is a conjugate point $r = \gamma(\tau_1)$ $(\tau_{\rm i} < \tau_1 < \tau_{\rm f})$ to H along γ, there is a timelike curve between H and q that has a larger length than γ.*

Proof. This follows in the same way as in proposition 3.0.3, using Eq. (3.24). □

We are now interested in what conditions need to be satisfied in order to have a timelike curve between points in spacetime. So suppose that $p, q \in M$ and that we have a non-spacelike curve $\gamma : [\tau_{\rm i}, \tau_{\rm f}] \to M$ such that $\gamma(\tau_{\rm i}) = p$ and $\gamma(\tau_{\rm f}) = q$. We want to know when it is possible to find a one-parameter proper variation $\Gamma(w, \tau)$ of γ such that $g(T, T)$ becomes negative everywhere, hence yields a timelike curve from p to q. With a variation Γ we get that

$$\begin{aligned} \partial_w g(T, T) &= 2g(D_w T, T) \\ &= 2g(S(W, T), T) + 2g(D_\tau W, T) \\ &= 2g(S(W, T), T) + 2\partial_\tau g(W, T) - 2g(W, D_\tau T). \end{aligned} \tag{3.27}$$

Evaluating in $w = 0$ gives

$$\partial_w g(T, T)|_{w=0} = 2g(S(V, \dot{\gamma}), \dot{\gamma}) + 2\partial_\tau g(V, \dot{\gamma}) - 2g(V, D_\tau \dot{\gamma}). \tag{3.28}$$

Proposition 3.0.5. *Let $p, q \in M$ and let $\gamma : [\tau_\mathrm{i}, \tau_\mathrm{f}] \to M$ be a non-spacelike curve which is not a null geodesic curve such that $\gamma(\tau_\mathrm{i}) = p$, $\gamma(\tau_\mathrm{f}) = q$, then p and q can also be joined by a timelike curve.*

Proof. If γ is not a timelike curve, it must have some parts where it is a null curve. The set $U \subset [\tau_\mathrm{i}, \tau_\mathrm{f}]$ where γ is null is a closed set, so without loss of generality we can assume γ to be a null curve. If γ is not a null geodesic curve, it either has a singular point or there must be an open interval where $D_\tau\dot{\gamma}$ is non-zero and not parallel to $\dot{\gamma}$. Consider first the second case. We then have that

$$g(D_\tau\dot{\gamma}, \dot{\gamma}) = \frac{1}{2}\partial_\tau \left(g(\dot{\gamma}, \dot{\gamma})\right) = 0. \tag{3.29}$$

This implies that $D_\tau\dot{\gamma}$ is a spacelike vector field at points where it is non-zero and not parallel to $\dot{\gamma}$. We can also define a vector field $X \in \mathcal{T}(\gamma)$ such that $g(X, \dot{\gamma}) < 0$. Define now the following functions along γ:

$$\begin{aligned}
h_1(\tau) &= g(D_\tau\dot{\gamma}, D_\tau\dot{\gamma}); && (3.30)\\
h_2(\tau) &= -g(X, \dot{\gamma}); && (3.31)\\
h_3(\tau) &= g(X, D_\tau\dot{\gamma}); && (3.32)\\
h_4(\tau) &= g(S(D_\tau\dot{\gamma}, \dot{\gamma}), \dot{\gamma}); && (3.33)\\
h_5(\tau) &= g(S(X, \dot{\gamma}), \dot{\gamma}); && (3.34)\\
h_6(\tau) &= \int_{\tau_\mathrm{i}}^{\tau} \frac{1}{h_2(\tau')} \left(h_5(\tau') - h_3(\tau')\right) d\tau'. && (3.35)
\end{aligned}$$

Let Γ be the variation of γ with variation vector

$$V = f_1 X + f_2 D_\tau\dot{\gamma}, \tag{3.36}$$

where

$$f_1(\tau) = \frac{1}{h_2(\tau)} e^{h_6(\tau)} \int_{\tau_\mathrm{i}}^{\tau} e^{-h_6(\tau')} \left(1 + f_2(\tau')h_4(\tau') - f_2(\tau')h_1(\tau')\right) d\tau' \tag{3.37}$$

and f_2 is an arbitrary function on $[\tau_\mathrm{i}, \tau_\mathrm{f}]$ with $f_2(\tau_\mathrm{i}) = f_2(\tau_\mathrm{f}) = 0$ and such that

$$f_1(\tau_\mathrm{f}) = \frac{1}{h_2(\tau_\mathrm{f})} e^{h_6(\tau_\mathrm{f})} \int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}} e^{-h_6(\tau')} \left(1 + f_2(\tau')h_4(\tau') - f_2(\tau')h_1(\tau')\right) d\tau' = 0. \tag{3.38}$$

This only does not work when $h_4(\tau) - h_1(\tau) = 0$ for all $\tau \in [\tau_\mathrm{i}, \tau_\mathrm{f}]$. For a totally anti-symmetric torsion $h_4(\tau) - h_1(\tau) = -h_1(\tau) \leq 0$. Because of condition Eq. (3.38), $f_1(\tau_\mathrm{i}) = f_1(\tau_\mathrm{f}) = 0$. We now have that:

$$\begin{aligned}
\partial_\tau g(V, \dot{\gamma}) &= \dot{f}_1 g(X, \dot{\gamma}) + f_1 \partial_\tau g(X, \dot{\gamma}) + \dot{f}_2 g(D_\tau\dot{\gamma}, \dot{\gamma}) + f_2 \partial_\tau g(D_\tau\dot{\gamma}, \dot{\gamma})\\
&= -\dot{f}_1 h_2 - f_1 \partial_\tau h_2\\
&= -\dot{h}_6 f_1 h_2 - \frac{1}{h_2} e^{h_6} e^{-h_6} \left(1 + f_2 h_4 - f_2 h_1\right) h_2\\
&= -\dot{h}_6 f_1 h_2 - 1 - f_2 h_4 + f_2 h_1\\
&= h_3 f_1 - h_5 f_1 - 1 - f_2 h_4 + f_2 h_1. && (3.39)
\end{aligned}$$

So using Eq. (3.28) we get:

$$\begin{aligned}
\frac{1}{2}\partial_w g(T, T)|_{w=0} &= g(S(V, \dot{\gamma}), \dot{\gamma}) + \partial_\tau g(V, \dot{\gamma}) - g(V, D_\tau\dot{\gamma})\\
&= f_1 h_5 + f_2 h_4 + \partial_\tau g(V, \dot{\gamma}) - f_1 h_3 - f_2 h_1\\
&= -1, && (3.40)
\end{aligned}$$

which is exactly what we wanted.

Suppose now that $\dot{\gamma}$ is continuous on segments $[\tau_i, \tau_{i+1}]$ where $1 \leq i \leq n-1$ and $\tau_1 = \tau_\mathrm{i}$, $\tau_n = \tau_\mathrm{f}$. If a segment $[\tau_i, \tau_{i+1}]$ is not a null geodesic curve, it can be varied to give a timelike curve between the points $\gamma(\tau_i)$ and $\gamma(\tau_{i+1})$. So we only have left to show that if γ is a curve that is geodesic on segments $[\tau_i, \tau_{i+1}]$, we can obtain a timelike curve between p and q. The parameter τ can be taken such that $D_\tau\dot{\gamma} = 0$ on each segment $[\tau_i, \tau_{i+1}]$. Since the discontinuity $\partial_\tau^+ - \partial_\tau^-$ at singular points is a spacelike vector, one can find a vector field X along $[\tau_{i-1}, \tau_{i+1}]$ such that $g(X, \dot{\gamma}) < 0$ on $[\tau_{i-1}, \tau_i]$ and $g(X, \dot{\gamma}) > 0$ on $[\tau_i, \tau_{i+1}]$. Define

$$h_{7,i}(\tau) = \int_{\tau_{i-1}}^{\tau} \frac{1}{h_2(\tau')} h_5(\tau')d\tau' \quad \tau_{i-1} \leq \tau \leq \tau_{i+1} \tag{3.41}$$

Let $g^- = g(X, \partial_{\tau_i}^-) < 0$ and $g^+ = g(X, \partial_{\tau_i}^+) > 0$. Let then Γ denote the variation with variation vector $V = f_1 X$ where

$$f_1 = \begin{cases} \frac{1}{h_2}(\tau_{i+1} - \tau_i)(\tau - \tau_{i-1})e^{h_{7,i}(\tau)} & \tau_{i-1} \leq \tau \leq \tau_i; \\ \frac{g^+}{g^-}\frac{1}{h_2}(\tau_i - \tau_{i-1})(\tau_{i+1} - \tau)e^{h_{7,i}(\tau)} & \tau_i \leq \tau \leq \tau_{i+1}, \end{cases} \tag{3.42}$$

such that V is continuous and smooth on the two segments. Using Eq. (3.28) we find for $\tau \in [\tau_{i-1}, \tau_i]$

$$\begin{aligned} \frac{1}{2}\partial_w g(T,T)|_{w=0} &= f_1 h_5 - (\partial_\tau f_1)\, h_2 - f_1 \partial_\tau h_2 \\ &= f_1 h_5 + h_2 \frac{1}{h_2} f_1 \partial_\tau h_2 - \frac{1}{h_2}(\tau_{i+1} - \tau_i)e^{h_{7,i}} h_2 - f_1 h_2 \partial_\tau h_{7,i} - f_1 \partial_\tau h_2 \\ &= -(\tau_{i+1} - \tau_i)e^{h_{7,i}} < 0. \end{aligned} \tag{3.43}$$

In the same way we find for $\tau \in [\tau_i, \tau_{i+1}]$ that

$$\frac{1}{2}\partial_w g(T,T)|_{w=0} = \frac{g^+}{g^-}(\tau_{i+1} - \tau_i)e^{h_{7,i}} < 0. \tag{3.44}$$

□

We would now like to do the same for a null geodesic $\gamma : [0,1] \to M$, but this is a bit trickier. In case of totally anti-symmetric torsion Eq. (3.28) reduces to

$$\partial_w g(T,T)|_{w=0} = 2\partial_\tau g(V, \dot{\gamma}). \tag{3.45}$$

Since for a proper variation we need to have $V(0) = V(1) = 0$, the variation vector should be orthogonal to $\dot{\gamma}$ to yield a timelike curve (if $g(V, \dot{\gamma})$ becomes non-zero, the derivative will be positive somewhere on $[0,1]$). So we should consider the second derivative of $g(T,T)$:

$$\frac{1}{2}\partial_w^2 g(T,T) = \partial_w g(S(W,T),T) + \partial_w \partial_\tau g(W,T) - \partial_w g(W, D_\tau T). \tag{3.46}$$

Using that $g(S(W,T),T) = 0$ and evaluation in $w = 0$ yields, using $D_\tau\dot{\gamma} = 0$:

$$\frac{1}{2}\partial_w^2 g(T,T)|_{w=0} = \partial_\tau \left[\partial_w g(W,T)\right]_{w=0} - g(V, D_\tau^2 V + D_\tau S(V, \dot{\gamma}) + R(V, \dot{\gamma})\dot{\gamma}). \tag{3.47}$$

We would now like to generalize proposition 4.5.12 from [3] to the case of non-vanishing (totally anti-symmetric) torsion. We will basically give the proof from [22] with a small adaptation because of the torsion.

Proposition 3.0.6. *Let $p, q \in M$ and $\gamma : [0,1] \to M$ be a null geodesic such that $\gamma(0) = p$, $\gamma(1) = q$. Suppose there is a point $\gamma(\tau_0) = r$, $0 < \tau_0 < 1$, conjugate to p along γ. Then there is a variation of γ which will give a timelike curve from p to q.*

Proof. We will suppose that r is the first conjugate point to p along γ. Notice that using proposition 3.0.5, it is sufficient to find a variation that will give a timelike curve between p and $\gamma(\tau)$ for a $\tau_0 < \tau < 1$. Since r is a conjugate point to q along γ, there exists a non-trivial Jacobi class $\bar{J}$ along γ such that $\bar{J}(0) = [\dot{\gamma}(0)]$, $\bar{J}(\tau_0) = [\dot{\gamma}(\tau_0)]$. We can write

$$\bar{J}(t) = f(t)\widehat{J}(t), \tag{3.48}$$

where $\widehat{J}$ is a smooth vector class along γ such that $\bar{g}(\widehat{J},\widehat{J})=1$ and $f:[0,1]\to\mathbb{R}$ is a smooth function. Since r is the first conjugate point to p we have that $f(0)=f(\tau_0)=0$ and we can assume that $f(\tau)>0$ for all $0<\tau<\tau_0$. $\bar{J}$ is non-trivial and $\bar{J}(\tau_0)=[\dot{\gamma}(\tau_0)]$, so $D_\tau\bar{J}(\tau_0)\neq[\dot{\gamma}(\tau_0)]$. Using $D_\tau\bar{J}(\tau)=\dot{f}(\tau)\widehat{J}(\tau)+f(\tau)D_\tau\widehat{J}(\tau)$ it follows that $\dot{f}(\tau_0)\neq 0$, hence $f(\tau)<0$ for $\tau\in(\tau_0,\tau_1]$ where $\tau_0<\tau_1<1$. Define now the continuous function

$$h(\tau)=\bar{g}\left(D_\tau^2\widehat{J}+D_\tau\bar{S}_\gamma(\widehat{J})+\bar{R}_\gamma(\widehat{J}),\widehat{J}\right). \tag{3.49}$$

On $[0,\tau_1]$ h has a minimum value h_0 such that we have a constant $a\in\mathbb{R}$, $a>0$ and

$$a^2+h_0>0. \tag{3.50}$$

Define now the vector class

$$\bar{V}=[b(e^{a\tau}-1)+f]\,\widehat{J}\equiv r(\tau)\widehat{J}, \tag{3.51}$$

where

$$b=-\frac{f(\tau_1)}{e^{a\tau_1}-1} \tag{3.52}$$

to make sure we have $\bar{V}(0)=[\dot{\gamma}(0)]$, $\bar{V}(\tau_1)=[\dot{\gamma}(\tau_1)]$. $\bar{J}$ is a Jacobi class, hence

$$\begin{aligned}
0 &= \bar{g}\left(D_\tau^2\bar{J}+D_\tau\bar{S}_\gamma(\bar{J})+\bar{R}_\gamma(\bar{J}),\widehat{J}\right)\\
&= \ddot{f}\bar{g}\left(\widehat{J},\widehat{J}\right)+2\dot{f}\bar{g}\left(D_\tau\widehat{J},\widehat{J}\right)+f\bar{g}\left(D_\tau^2\widehat{J},\widehat{J}\right)+\dot{f}\bar{g}\left(\bar{S}_\gamma(\widehat{J}),\widehat{J}\right)+f\bar{g}\left(D_\tau\bar{S}_\gamma(\widehat{J}),\widehat{J}\right)+f\bar{g}\left(\bar{R}_\gamma(\widehat{J}),\widehat{J}\right)\\
&= \ddot{f}+2\dot{f}\bar{g}\left(D_\tau\widehat{J},\widehat{J}\right)+fh+\dot{f}\bar{g}\left(\bar{S}_\gamma(\widehat{J}),\widehat{J}\right)\\
&= \ddot{f}+2\dot{f}\bar{g}\left(D_\tau\widehat{J},\widehat{J}\right)+fh.
\end{aligned} \tag{3.53}$$

We have that $\bar{g}\left(D_\tau\widehat{J},\widehat{J}\right)=\frac{1}{2}\partial_\tau\bar{g}\left(\widehat{J},\widehat{J}\right)=0$, hence

$$\ddot{f}=-fh. \tag{3.54}$$

This gives

$$\begin{aligned}
\bar{g}\left(\bar{V},D_\tau^2\bar{V}+\bar{R}_\gamma(\bar{V})+D_\tau\bar{S}_\gamma(\bar{V})\right) &= r\bar{g}\left(\widehat{J},\ddot{r}\widehat{J}+2\dot{r}D_\tau\widehat{J}+rD_\tau^2\widehat{J}+r\bar{R}_\gamma(\widehat{J})+\dot{r}\bar{S}_\gamma(\widehat{J})+rD_\tau\bar{S}_\gamma(\widehat{J})\right)\\
&= r\ddot{r}+r^2h\\
&= r\left[ba^2e^{a\tau}+\ddot{f}+be^{a\tau}h-bh+fh\right]\\
&= r\left[be^{a\tau}\left(a^2+h\right)-bh\right].
\end{aligned} \tag{3.55}$$

Notice that $b,a^2+h>0$, so $be^{a\tau}\left(a^2+h\right)-bh>b\left(a^2+h\right)-bh=ba^2>0$. So expression (3.55) is larger than zero precisely when r is. Since $f(\tau)>0$ for $\tau\in(0,\tau_0)$ we have that $r(\tau)>0$ for $\tau\in(0,\tau_2)$ and $r(\tau_2)=0$, where we can choose $\tau_0<\tau_2\leq\tau_1$. Let now $\tilde{V}\in\mathcal{T}_\perp(\gamma)$ such that $\pi(\tilde{V})=\bar{V}$. Since $\bar{V}(0)=[\dot{\gamma}(0)]$ and $\bar{V}(\tau_2)=[\dot{\gamma}(\tau_2)]$ we must have that $\tilde{V}(0)=\mu\dot{\gamma}(0)$, $\tilde{V}(\tau_2)=\lambda\dot{\gamma}(\tau_2)$ for $\mu,\lambda\in\mathbb{R}$. Define

$$V=\tilde{V}-\mu\dot{\gamma}+\frac{\mu-\lambda}{\tau_2}\tau\dot{\gamma}, \tag{3.56}$$

then $V(0)=V(\tau_2)=0$ and $\pi(V)=\bar{V}$, hence $g\left[V,D_\tau^2V+R(V,\dot{\gamma})\dot{\gamma}+D_\tau S(V,\dot{\gamma})\right]>0$ for all $0<\tau<\tau_2$. Hence we have $\delta\in\mathbb{R}$ such that

$$0<\delta<\min\left\{g\left(V,D_\tau^2V+R(V,\dot{\gamma})\dot{\gamma}+D_\tau S(V,\dot{\gamma})\right)\mid\tau\in\left[\frac{1}{4}\tau_2,\frac{3}{4}\tau_2\right]\right\}. \tag{3.57}$$

Define the function

$$\rho(\tau)=\begin{cases}-\delta\tau & 0\leq\tau\leq\frac{1}{4}\tau_2,\\ \delta(\tau-\frac{1}{2}\tau_2) & \frac{1}{4}\tau_2\leq\tau\leq\frac{3}{4}\tau_2,\\ \delta(\tau_2-\tau) & \frac{3}{4}\tau_2\leq\tau\leq\tau_2.\end{cases} \tag{3.58}$$

Using the pseudo-orthonormal frame E_0, E_1, E_3, E_4 that we saw before, we can now define the proper variation $\Gamma :\ (-\epsilon, \epsilon) \times [0, \tau_2] \to M$ of $\gamma|_{[0,\tau_2]}$ by

$$\begin{aligned} W(0,\tau) &= V(\tau) \\ D_w W(0,\tau) &= [g(V, D_\tau V) - \rho(\tau)]\, E_1. \end{aligned} \tag{3.59}$$

We then have that

$$\begin{aligned} \partial_\tau \left[\partial_w g(W,T)\right]_{w=0} &= \partial_\tau \left[g(D_w W, T) + g(W, D_w T)\right]_{w=0} \\ &= \partial_\tau \left[g(D_w W, T) + g(W, D_\tau W)\right]_{w=0} \\ &= \partial_\tau \left[g\left([g(V, D_\tau V) - \rho(\tau)]\, E_1, \dot{\gamma}\right) + g(V, D_\tau V)\right] \\ &= \partial_\tau \left[-g(V, D_\tau V) + \rho(\tau) + g(V, D_\tau V)\right] \\ &= \begin{cases} -\delta & 0 \le \tau \le \frac{1}{4}\tau_2, \\ \delta & \frac{1}{4}\tau_2 \le \tau \le \frac{3}{4}\tau_2, \\ -\delta & \frac{3}{4}\tau_2 \le \tau \le \tau_2. \end{cases} \end{aligned} \tag{3.60}$$

With Eq. (3.47) we then find that $\partial_w^2 g(T,T)|_{w=0} < 0$ such that this variation gives a timelike curve between p and $\gamma(\tau_2)$. □

Proposition 3.0.7. *Let $\gamma :\ [0,1] \to M$ be a null geodesic between a spacelike two-surface P and a point q (i.e. $\gamma(0) \in P$ and $\gamma(1) = q$), such that γ is orthogonal to P. If there is a point $r = \gamma(\tau_1)$, $0 < \tau_1 < 1$, conjugate to P along γ, then there is a variation of γ that gives a timelike curve from P to q.*

Proof. This follows in a similar way as in proposition 3.0.6. □

Chapter 4

Equations of Motion and Generalized Identities

In Chapters 2 and 3 we have completely focused on the geometric side of Einstein-Cartan theory. However, when treating the singularity theorems in Chapter 5 we need to translate geometric assumptions to the matter content of the universe. In general relativity we have the Einstein equation to do this, so to do the same in a theory with torsion, we need to derive the equations of motion of Einstein-Cartan theory. That is why in this chapter we derive the equations of motion that follow from the Einstein-Hilbert action plus a matter action. In Section 4.1 we do this in the metric formalism, where metric compatibility is assumed and the metric and torsion are taken as dynamical variables. In Section 4.2 we derive the equations in the metric-affine formalism, in which metric compatibility is not assumed and the metric and connection are taken as dynamical variables. Notice that in the latter formalism, the connection has no a priori dependence on the metric. After deriving the equations of motion in both formalisms we will show in Section 4.3 that they are equivalent for the Lagrangian of the Standard model and that the matter in this model induces totally anti-symmetric torsion. This equivalence was already known in the literature (e.g. [26, 27]), but what is done in this chapter can be seen as a nice review. When that is done, a generalized Bianchi identity and generalized conservation of energy-momentum is derived in Section 4.4.

Before doing all of this, we need to derive some geometric identities, in particular an application of Stokes theorem. This will be done for a manifold with dimension n.

Definition 4.0.1. The *divergence* of a vector field $V \in \mathcal{T}(M)$ is defined by

$$d(i_V dV_g) = (\mathrm{div}(V))dV_g, \tag{4.1}$$

where d is exterior differentiation, i is interior multiplication and dV_g is in coordinates equal to

$$\sqrt{-g}d^n x. \tag{4.2}$$

Lemma 4.0.1. *In coordinates:*

$$\mathrm{div}(V) = \partial_\mu V^\mu - \frac{1}{2} g_{\rho\nu}\partial_\mu(g^{\rho\nu})V^\mu. \tag{4.3}$$

Proof. The left-hand side of Eq. (4.1) is in coordinates equal to

$$\begin{aligned} d(i_V dV_g) &= d\left(\sqrt{-\det g}\sum_{\mu=1}^{n}(-1)^{\mu-1}dx^\mu(V)dx^1\wedge ...\wedge dx^{\mu-1}\wedge dx^{\mu+1}\wedge ...\wedge dx^n\right) \\ &= d\left(\sqrt{-\det g}\sum_{\mu=1}^{n}(-1)^{\mu-1}V^\mu dx^1\wedge ...\wedge dx^{\mu-1}\wedge dx^{\mu+1}\wedge ...\wedge dx^n\right) \\ &= \sum_{\nu=1}^{n}\sum_{\mu=1}^{n}(-1)^{\mu-1}\frac{\partial\sqrt{-\det g}V^\mu}{\partial x^\nu}d^\nu x\wedge dx^1\wedge ...\wedge dx^{\mu-1}\wedge dx^{\mu+1}\wedge ...\wedge dx^n \\ &= \sum_{\mu=1}^{n}\frac{\partial\sqrt{-\det g}V^\mu}{\partial x^\mu}d^n x \\ &= \sum_{\mu=1}^{n}\left[\frac{-1}{2\sqrt{-\det g}}\partial_\mu\left(\det g\right)V^\mu+\sqrt{-\det g}\partial_\mu V^\mu\right]d^n x. \end{aligned} \tag{4.4}$$

The derivative of the determinant of the metric can be calculated by using the identity

$$\det g = \exp(\mathrm{trace}(\log(g))). \tag{4.5}$$

We find

$$\begin{aligned} \partial_\mu\left(\det g\right) &= \det g\partial_\mu \mathrm{trace}(\log(g)) \\ &= \det g \mathrm{trace}(g^{-1}\partial_\mu g) \\ &= \left(\det g\right)g^{\rho\lambda}\partial_\mu g_{\rho\lambda}. \end{aligned} \tag{4.6}$$

Hence:

$$\begin{aligned} d(i_V dV_g) &= \sum_{\mu=1}^{n}\left[\frac{1}{2}\sqrt{-\det g}g^{\rho\nu}\partial_\mu\left(g_{\rho\nu}\right)V^\mu+\sqrt{-\det g}\partial_\mu V^\mu\right]d^n x \\ &= \sum_{\mu=1}^{n}\left[-\frac{1}{2}\sqrt{-\det g}g_{\rho\nu}\partial_\mu\left(g^{\rho\nu}\right)V^\mu+\sqrt{-\det g}\partial_\mu V^\mu\right]d^n x, \end{aligned} \tag{4.7}$$

and this implies, using the definition of divergence 4.1, that in coordinates

$$\mathrm{div}(V) = \partial_\mu V^\mu - \frac{1}{2}g_{\rho\nu}\partial_\mu(g^{\rho\nu})V^\mu. \tag{4.8}$$

□

Normal coordinates Normally normal coordinates are defined for a covariant derivative that is metric compatible and has vanishing torsion, but we will consider the general case. In this general case, geodesics γ are still defined by

$$\nabla_{\dot\gamma}\dot\gamma = 0, \tag{4.9}$$

such that we have an exponential map $\exp_p$ at a point $p \in M$ defined by

$$\exp_p: \ T_pM \to M \quad V \mapsto \gamma_V(1), \tag{4.10}$$

where γ_V is the geodesic such that $\gamma_V(0) = p$ and $\dot\gamma_V(0) = V$. This map is a diffeomorphism when restricted to a small enough neighborhood U of $0 \in T_pM$. Using that, we can define normal coordinates in a neighborhood of p by choosing an orthonormal basis of T_pM and letting $\exp_p(V)$ for $V \in U$ having coordinates of V in this orthonormal basis. This immediately implies that in these coordinates

$$g_{\mu\nu}(p) = \eta_{\mu\nu}. \tag{4.11}$$

Furthermore, geodesics γ_V for $V \in U$ (in normal coordinates) are given by

$$\gamma_V(\tau) = (\tau V^1, ..., \tau V^n) \tag{4.12}$$

because $\gamma_V(0) = p$ and $\dot{\gamma}_V(0) = V$ and $\gamma_V(\tau) = \exp_p(\tau V)$, which has the coordinates given above. Using the defining formula for a geodesic, we find that in normal coordinates at p

$$0 = \partial_\tau V^\rho + \dot{\gamma}^\mu_V \Gamma^\rho_{\mu\nu} \dot{\gamma}^\nu_V = \Gamma^\rho_{(\mu\nu)} V^\mu V^\nu \tag{4.13}$$

for all $V \in U$, which implies that

$$\Gamma^\rho_{(\mu\nu)}(p) = 0. \tag{4.14}$$

Now we have enough tools to prove the following lemma.

Lemma 4.0.2. *Let V be a vectorfield defined on a manifold (M,g) with boundary ∂M such that $V|_{\partial M} = 0$. Then*

$$\int_M d^n x \sqrt{-\det g} \nabla_\mu V^\mu = \int_M d^n x \sqrt{-\det g} \left(\frac{1}{2} g_{\rho\nu} \nabla_\mu (g^{\rho\nu}) + S_\mu \right) V^\mu, \tag{4.15}$$

where $S_\mu = S^\nu{}_{\nu\mu}$.

Proof. Choosing normal coordinates based on $p \in M$ as our particular coordinate system gives $\Gamma^\rho_{(\mu\nu)} = 0$ at p. Therefore, using Eq. (4.3), it follows that at p

$$\begin{aligned}
\nabla_\mu V^\mu &= \partial_\mu V^\mu + \Gamma^\mu_{[\mu\nu]} V^\nu \\
&= \mathrm{div}(V) + \frac{1}{2} g_{\rho\nu} \partial_\mu (g^{\rho\nu}) V^\mu + \frac{1}{2} S_\mu V^\mu \\
&= \mathrm{div}(V) + \frac{1}{2} g_{\rho\nu} \nabla_\mu (g^{\rho\nu}) V^\mu - \frac{1}{2} g_{\rho\nu} \Gamma^\rho_{[\mu\lambda]} g^{\lambda\nu} V^\mu - \frac{1}{2} g_{\rho\nu} \Gamma^\nu_{[\mu\lambda]} g^{\rho\lambda} V^\mu + \frac{1}{2} S_\mu V^\mu \\
&= \mathrm{div}(V) + \frac{1}{2} g_{\rho\nu} \nabla_\mu (g^{\rho\nu}) V^\mu - \frac{1}{2} S^\lambda{}_{\mu\lambda} V^\mu + \frac{1}{2} S_\mu V^\mu \\
&= \mathrm{div}(V) + \frac{1}{2} g_{\rho\nu} \nabla_\mu (g^{\rho\nu}) V^\mu + S_\mu V^\mu.
\end{aligned} \tag{4.16}$$

It follows from Stokes theorem that

$$\int_M d^n x \sqrt{-\det g} \, \mathrm{div}(V) = \int_M \mathrm{div}(V) dV_g = \int_M d(i_V dV_g) = \int_{\partial M} i_V dV_g = 0 \tag{4.17}$$

because $V = 0$ on ∂M. Using Eq. (4.16) we find that

$$\begin{aligned}
\int_M d^n x \sqrt{-\det g} \nabla_\mu V^\mu &= \int_M d^n x \sqrt{-\det g} \left(\mathrm{div}(V) + \frac{1}{2} g_{\rho\nu} \nabla_\mu (g^{\rho\nu}) V^\mu + S_\mu V^\mu \right) \\
&= \int_M d^n x \sqrt{-\det g} \left(\frac{1}{2} g_{\rho\nu} \nabla_\mu (g^{\rho\nu}) + S_\mu \right) V^\mu.
\end{aligned} \tag{4.18}$$

□

Corollary 4.0.1. *In case of metric compatibility:*

$$\int_M d^n x \sqrt{-\det g} \nabla_\mu V^\mu = \int_M d^n x \sqrt{-\det g} S_\mu V^\mu. \tag{4.19}$$

4.1 Metric Formalism

We derive the equations of motion for an action

$$S = \frac{1}{16\pi G_\mathrm{N}} S_\mathrm{H} + S_\mathrm{m}, \tag{4.20}$$

where $S_{\rm H}$ is the Hilbert action, defined by

$$S_{\rm H} = \int d^4x\sqrt{-{\rm det}g}R, \tag{4.21}$$

and $S_{\rm m}$ is the matter action. We will take the variation with respect to the metric $g^{\mu\nu}$ and the torsion $S^{\rho}{}_{\mu\nu}$. We first define some tensors for the matter part of the action:

Definition 4.1.1. The *energy-momentum tensor* of matter is given by

$$T_{\mu\nu} = \frac{-2}{\sqrt{-\det g}}\frac{\delta S_{\rm m}}{\delta g^{\mu\nu}}, \tag{4.22}$$

and the *spin tensor* of matter is given by

$$\Pi_{\rho}{}^{\mu\nu} = \frac{-4}{\sqrt{-{\rm det}g}}\frac{\delta S_{\rm m}}{\delta S^{\rho}{}_{\mu\nu}}. \tag{4.23}$$

Proposition 4.1.1. *The variation of the action S (Eq. (4.20)) with respect to the metric $g^{\mu\nu}$ leads to the following equation of motion (e.g. [6]):*

$$G_{(\mu\nu)} + S^{\rho}\left(-S_{(\mu\nu)\rho} + S_{\rho}g_{\mu\nu} - S_{(\mu}g_{\nu)\rho}\right) - \nabla^{\rho}\left(-S_{(\mu\nu)\rho} + S_{\rho}g_{\mu\nu} - S_{(\mu}g_{\nu)\rho}\right) = 8\pi G_{\rm N}T_{\mu\nu}. \tag{4.24}$$

Proof. Varying $S_{\rm H}$ with respect to the metric $g^{\mu\nu}$ yields

$$\delta_g S_{\rm H} = \int d^4x\left[\delta_g\left(\sqrt{-{\rm det}g}\right)R + \sqrt{-{\rm det}g}R_{\mu\nu}\delta_g g^{\mu\nu} + \sqrt{-{\rm det}g}g^{\rho\lambda}\delta_g R_{\rho\lambda}\right]. \tag{4.25}$$

Using that

$$\delta_g\left(\sqrt{-{\rm det}g}\right) = -\frac{1}{2}\sqrt{-{\rm det}g}g_{\mu\nu}\delta_g g^{\mu\nu} \tag{4.26}$$

(this follows in the same way as in Eq. (4.6)) reduces Eq. (4.25) to

$$\delta_g S_{\rm H} = \int d^nx\sqrt{-{\rm det}g}\left[\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta_g g^{\mu\nu} + g^{\rho\lambda}\delta_g R_{\rho\lambda}\right]. \tag{4.27}$$

With Eq. (1.10) it follows that

$$\begin{aligned} \delta_g R_{\rho\lambda} &= \partial_\alpha\delta_g\Gamma^{\alpha}_{\lambda\rho} + \left(\delta_g\Gamma^{\beta}_{\lambda\rho}\right)\Gamma^{\alpha}_{\alpha\beta} + \Gamma^{\beta}_{\lambda\rho}\left(\delta_g\Gamma^{\alpha}_{\alpha\beta}\right) - \partial_\lambda\delta_g\Gamma^{\alpha}_{\alpha\rho} - \left(\delta_g\Gamma^{\beta}_{\alpha\rho}\right)\Gamma^{\alpha}_{\lambda\beta} - \Gamma^{\beta}_{\alpha\rho}\left(\delta_g\Gamma^{\alpha}_{\lambda\beta}\right) \\ &= \nabla_\alpha\delta_g\Gamma^{\alpha}_{\lambda\rho} + \Gamma^{\beta}_{\alpha\lambda}\left(\delta_g\Gamma^{\alpha}_{\beta\rho}\right) - \nabla_\lambda\delta_g\Gamma^{\alpha}_{\alpha\rho} - \Gamma^{\beta}_{\lambda\alpha}\left(\delta_g\Gamma^{\alpha}_{\beta\rho}\right) \\ &= \nabla_\alpha\delta_g\Gamma^{\alpha}_{\lambda\rho} - \nabla_\lambda\delta_g\Gamma^{\alpha}_{\alpha\rho} + S^{\beta}{}_{\alpha\lambda}\left(\delta_g\Gamma^{\alpha}_{\beta\rho}\right). \end{aligned} \tag{4.28}$$

Then by metric compatibility and Corollary 4.0.1 (the variation is as always assumed to be 0 at the boundary):

$$\begin{aligned} \int d^4x\sqrt{-{\rm det}g}g^{\rho\lambda}\delta_g R_{\rho\lambda} &= \int d^4x\sqrt{-{\rm det}g}g^{\rho\lambda}\left[\nabla_\alpha\delta_g\Gamma^{\alpha}_{\lambda\rho} - \nabla_\lambda\delta_g\Gamma^{\alpha}_{\alpha\rho} + S^{\beta}{}_{\alpha\lambda}\delta_g\Gamma^{\alpha}_{\beta\rho}\right] \\ &= \int d^4x\sqrt{-{\rm det}g}\left[\nabla_\alpha\left(g^{\lambda\rho}\delta_g\Gamma^{\alpha}_{\lambda\rho} - g^{\rho\alpha}\delta_g\Gamma^{\beta}_{\beta\rho}\right) + g^{\lambda\rho}S^{\beta}{}_{\alpha\lambda}\delta_g\Gamma^{\alpha}_{\beta\rho}\right] \\ &= \int d^4x\sqrt{-{\rm det}g}\left[S_\alpha\left(g^{\lambda\rho}\delta_g\Gamma^{\alpha}_{\lambda\rho} - g^{\rho\alpha}\delta_g\Gamma^{\beta}_{\beta\rho}\right) - S^{\beta\rho}{}_{\alpha}\delta_g\Gamma^{\alpha}_{\beta\rho}\right] \\ &= \int d^4x\sqrt{-{\rm det}g}\left(-S^{\mu\nu}{}_{\alpha} + S_\alpha g^{\mu\nu} - S_\beta\delta^{\mu}_{\alpha}g^{\nu\beta}\right)\delta_g\Gamma^{\alpha}_{\mu\nu} \\ &= \int d^4x\sqrt{-{\rm det}g}\tilde{S}^{\mu\nu}{}_{\alpha}\delta_g\Gamma^{\alpha}_{\mu\nu}. \end{aligned} \tag{4.29}$$

where we defined

$$\tilde{S}^{\mu\nu}{}_{\alpha} \equiv -S^{\mu\nu}{}_{\alpha} + S_\alpha g^{\mu\nu} - S_\beta\delta^{\mu}_{\alpha}g^{\nu\beta}. \tag{4.30}$$

Notice that $\tilde{S}_{\mu\nu\alpha}$ is anti-symmetric in the last two indices. We calculate $\delta_g\Gamma^{\alpha}_{\mu\nu}$ using metric compatibility: $0 = \overset{\delta}{\nabla}_{\rho}\left(g_{\mu\nu} + \delta g_{\mu\nu}\right)$, where $\overset{\delta}{\nabla}$ is the covariant derivative defined by the varied connection.

$$\begin{aligned}
0 &= \overset{\delta}{\nabla}_{\rho}(g_{\mu\nu} + \delta_g g_{\mu\nu}) \\
&= \nabla_{\rho}(g_{\mu\nu} + \delta_g g_{\mu\nu}) - \delta_g\Gamma^{\lambda}_{\rho\mu}(g_{\lambda\nu} + \delta_g g_{\lambda\nu}) - \delta_g\Gamma^{\lambda}_{\rho\nu}(g_{\mu\lambda} + \delta_g g_{\mu\lambda}) \\
&= \nabla_{\rho}(g_{\mu\nu} + \delta_g g_{\mu\nu}) - g_{\lambda\nu}\delta_g\Gamma^{\lambda}_{\rho\mu} - g_{\mu\lambda}\delta_g\Gamma^{\lambda}_{\rho\nu} \\
&= \nabla_{\rho}\delta_g g_{\mu\nu} - g_{\lambda\nu}\delta_g\Gamma^{\lambda}_{\rho\mu} - g_{\mu\lambda}\delta_g\Gamma^{\lambda}_{\rho\nu}.
\end{aligned} \tag{4.31}$$

Hence

$$\begin{aligned}
\nabla_{\mu}\delta_g g_{\nu\rho} + \nabla_{\nu}\delta_g g_{\mu\rho} - \nabla_{\rho}\delta_g g_{\mu\nu} &= g_{\lambda\nu}\delta_g\Gamma^{\lambda}_{\mu\rho} + g_{\lambda\rho}\delta_g\Gamma^{\lambda}_{\mu\nu} + g_{\lambda\mu}\delta_g\Gamma^{\lambda}_{\nu\rho} + g_{\lambda\rho}\delta_g\Gamma^{\lambda}_{\nu\mu} - g_{\lambda\nu}\delta_g\Gamma^{\lambda}_{\rho\mu} - g_{\lambda\mu}\delta_g\Gamma^{\lambda}_{\rho\nu} \\
&= g_{\lambda\nu}\delta_g S^{\lambda}{}_{\mu\rho} + g_{\lambda\rho}\delta_g\Gamma^{\lambda}_{\mu\nu} + g_{\lambda\mu}\delta_g S^{\lambda}{}_{\nu\rho} + g_{\lambda\rho}\delta_g\Gamma^{\lambda}_{\mu\nu} - g_{\lambda\rho}\delta_g S^{\lambda}{}_{\mu\nu} \\
&= 2g_{\lambda\rho}\delta_g\Gamma^{\lambda}_{\mu\nu} + g_{\lambda\nu}\delta_g S^{\lambda}{}_{\mu\rho} + g_{\lambda\mu}\delta_g S^{\lambda}{}_{\nu\rho} - g_{\lambda\rho}\delta_g S^{\lambda}{}_{\mu\nu}.
\end{aligned} \tag{4.32}$$

This implies

$$\delta_g\Gamma^{\alpha}_{\mu\nu} = g^{\alpha\rho}g_{\rho\lambda}\delta_g\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\alpha\rho}\left(\nabla_{\mu}\delta_g g_{\nu\rho} + \nabla_{\nu}\delta_g g_{\mu\rho} - \nabla_{\rho}\delta_g g_{\mu\nu} - g_{\lambda\nu}\delta_g S^{\lambda}{}_{\mu\rho} - g_{\lambda\mu}\delta_g S^{\lambda}{}_{\nu\rho} + g_{\lambda\rho}\delta_g S^{\lambda}{}_{\mu\nu}\right). \tag{4.33}$$

Since we consider the variation with respect to the metric:

$$\delta_g\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\nabla_{\mu}\delta_g g_{\nu\beta} + \nabla_{\nu}\delta_g g_{\mu\beta} - \nabla_{\beta}\delta_g g_{\mu\nu}\right). \tag{4.34}$$

Substitution in Eq. (4.29) and application of Corollary 4.0.1 yields:

$$\begin{aligned}
\int d^4x\sqrt{-\mathrm{det}g}\,g^{\rho\lambda}\delta_g R_{\rho\lambda} &= \frac{1}{2}\int d^4x\sqrt{-\mathrm{det}g}\,g^{\alpha\beta}\tilde{S}^{\mu\nu}{}_{\alpha}\left(\nabla_{\mu}\delta_g g_{\nu\beta} + \nabla_{\nu}\delta_g g_{\mu\beta} - \nabla_{\beta}\delta_g g_{\mu\nu}\right) \\
&= \frac{1}{2}\int d^4x\sqrt{-\mathrm{det}g}\left(\tilde{S}^{\mu\nu\beta}\nabla_{\mu}\delta_g g_{\nu\beta} - 2\tilde{S}^{\mu\nu\beta}\nabla_{\beta}\delta_g g_{\mu\nu}\right) \\
&= -\int d^4x\sqrt{-\mathrm{det}g}\left(S_{\beta}\tilde{S}^{\mu\nu\beta} + \nabla_{\beta}\tilde{S}^{\mu\nu\beta}\right)\delta_g g_{\mu\nu} \\
&= \int d^4x\sqrt{-\mathrm{det}g}\left(S_{\alpha}\tilde{S}^{\rho\lambda\alpha} + \nabla_{\alpha}\tilde{S}^{\rho\lambda\alpha}\right)g_{\rho\mu}g_{\lambda\nu}\delta_g g^{\mu\nu}.
\end{aligned} \tag{4.35}$$

We have used that

$$\delta_g g_{\rho\lambda} = -g_{\rho\mu}g_{\lambda\nu}\delta_g g^{\mu\nu}. \tag{4.36}$$

Substitution in Eq. (4.27) yields

$$\delta_g S_{\mathrm{H}} = \int d^4x\sqrt{-\mathrm{det}g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + S_{\alpha}\tilde{S}_{\mu\nu}{}^{\alpha} - \nabla_{\alpha}\tilde{S}_{\mu\nu}{}^{\alpha}\right)\delta_g g^{\mu\nu}. \tag{4.37}$$

Hence

$$\begin{aligned}
\frac{1}{\sqrt{-\mathrm{det}g}}\frac{\delta S_{\mathrm{H}}}{\delta g^{\mu\nu}} &= R_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}R + S_{\alpha}\tilde{S}_{(\mu\nu)}{}^{\alpha} - \nabla_{\alpha}\tilde{S}_{(\mu\nu)}{}^{\alpha} \qquad (4.38)\\
&= R_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}R + S^{\rho}\left(-S_{(\mu\nu)\rho} + S_{\rho}g_{\mu\nu} - S_{(\mu}g_{\nu)\rho}\right) - \nabla^{\rho}\left(-S_{(\mu\nu)\rho} + S_{\rho}g_{\mu\nu} - S_{(\mu}g_{\nu)\rho}\right).
\end{aligned}$$

So for the total action

$$S = \frac{1}{16\pi G_{\mathrm{N}}}S_{\mathrm{H}} + S_{\mathrm{m}}, \tag{4.39}$$

we obtain

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g^{\mu\nu}} \\ &= \frac{1}{16\pi G_{\mathrm{N}}}\frac{1}{\sqrt{-\det g}}\frac{\delta S_{\mathrm{H}}}{\delta g^{\mu\nu}} + \frac{1}{\sqrt{-\det g}}\frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}} \\ &= \frac{1}{16\pi G_{\mathrm{N}}}\frac{1}{\sqrt{-\det g}}\frac{\delta S_{\mathrm{H}}}{\delta g^{\mu\nu}} - \frac{1}{2}T_{\mu\nu}. \end{aligned} \tag{4.40}$$

Thus the equation that follows from variation with respect to the metric is given by:

$$G_{(\mu\nu)} + S^{\rho}\left(-S_{(\mu\nu)\rho} + S_{\rho}g_{\mu\nu} - S_{(\mu}g_{\nu)\rho}\right) - \nabla^{\rho}\left(-S_{(\mu\nu)\rho} + S_{\rho}g_{\mu\nu} - S_{(\mu}g_{\nu)\rho}\right) = 8\pi G_{\mathrm{N}}T_{\mu\nu}. \tag{4.41}$$

□

Proposition 4.1.2. *The variation of the action S (Eq. (4.20)) with respect to the torsion $S^{\rho}{}_{\mu\nu}$ leads to the Cartan equation (e.g. [6]):*

$$S^{\nu\mu}{}_{\rho} - S^{\mu\nu}{}_{\rho} + S_{\rho}{}^{\mu\nu} - 2S^{\nu}\delta^{\mu}_{\rho} + 2S^{\mu}\delta^{\nu}_{\rho} = 8\pi G_{\mathrm{N}}\Pi_{\rho}{}^{\mu\nu}. \tag{4.42}$$

Proof. To obtain the Cartan equation, we vary with respect to $S^{\rho}{}_{\mu\nu}$. In a similar way as in proposition 4.1.1, using Eqs. (4.27), (4.29) and (4.33), we find

$$\begin{aligned} \delta_S S_{\mathrm{H}} &= \int d^4x\sqrt{-\det g}\, g^{\rho\lambda}\delta_S R_{\rho\lambda} \\ &= \int d^4x\sqrt{-\det g}\,\tilde{S}^{\mu\nu}{}_{\alpha}\delta_S\Gamma^{\alpha}_{\mu\nu} \\ &= \frac{1}{2}\int d^4x\sqrt{-\det g}\,\tilde{S}^{\mu\nu\rho}\left(-g_{\lambda\nu}\delta S^{\lambda}{}_{\mu\rho} - g_{\lambda\mu}\delta S^{\lambda}{}_{\nu\rho} + g_{\lambda\rho}\delta S^{\lambda}{}_{\mu\nu}\right) \\ &= \frac{1}{2}\int d^4x\sqrt{-\det g}\left(2\tilde{S}^{\mu\nu}{}_{\rho} + \tilde{S}_{\rho}{}^{\nu\mu}\right)\delta S^{\rho}{}_{\mu\nu}. \end{aligned} \tag{4.43}$$

Hence, with Eq. (4.30):

$$\frac{1}{\sqrt{-\det g}}\frac{\delta S_{\mathrm{H}}}{\delta S^{\rho}{}_{\mu\nu}} = \frac{1}{2}\left(S^{\nu\mu}{}_{\rho} - S^{\mu\nu}{}_{\rho} + S_{\rho}{}^{\mu\nu} - 2S^{\nu}\delta^{\mu}_{\rho} + 2S^{\mu}\delta^{\nu}_{\rho}\right). \tag{4.44}$$

Again considering the total action

$$S = \frac{1}{16\pi G_{\mathrm{N}}}S_{\mathrm{H}} + S_{\mathrm{m}}, \tag{4.45}$$

we find

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta S^{\rho}{}_{\mu\nu}} \\ &= \frac{1}{16\pi G_{\mathrm{N}}}\frac{1}{\sqrt{-\det g}}\frac{\delta S_{\mathrm{H}}}{\delta S^{\rho}{}_{\mu\nu}} + \frac{1}{\sqrt{-\det g}}\frac{\delta S_{\mathrm{m}}}{\delta S^{\rho}{}_{\mu\nu}} \\ &= \frac{1}{16\pi G_{\mathrm{N}}}\frac{1}{\sqrt{-\det g}}\frac{\delta S_{\mathrm{H}}}{\delta S^{\rho}{}_{\mu\nu}} - \frac{1}{4}\Pi_{\rho}{}^{\mu\nu}. \end{aligned} \tag{4.46}$$

Thus the Cartan equation is given by

$$S^{\nu\mu}{}_{\rho} - S^{\mu\nu}{}_{\rho} + S_{\rho}{}^{\mu\nu} - 2S^{\nu}\delta^{\mu}_{\rho} + 2S^{\mu}\delta^{\nu}_{\rho} = 8\pi G_{\mathrm{N}}\Pi_{\rho}{}^{\mu\nu}. \tag{4.47}$$

□

Corollary 4.1.1. *The Cartan equation can also be written as*

$$S_{\nu\mu\rho} = 8\pi G_{\mathrm{N}}\left(\Pi_{[\rho\mu]\nu} - \frac{1}{2}g_{\nu[\rho}\Pi^{\lambda}{}_{\mu]\lambda}\right). \tag{4.48}$$

Proof. Eq. (4.42) can be written as:

$$-S_{\nu\rho\mu}+S_{\mu\rho\nu}+S_{\rho\mu\nu}-2S_\nu g_{\mu\rho}+2S_\mu g_{\nu\rho}=8\pi G_{\rm N}\Pi_{\rho\mu\nu}. \tag{4.49}$$

Contracting this equation with $g^{\nu\rho}$ yields

$$S_\mu = 2\pi G_{\rm N}\Pi^\nu{}_{\mu\nu}. \tag{4.50}$$

Substitution of this result in Eq. (4.49) gives:

$$-S_{\nu\rho\mu}+2S_{(\mu\rho)\nu}=8\pi G_{\rm N}\left(\Pi_{\rho\mu\nu}+\frac{1}{2}g_{\rho\mu}\Pi^\lambda{}_{\nu\lambda}-\frac{1}{2}g_{\rho\nu}\Pi^\lambda{}_{\mu\lambda}\right). \tag{4.51}$$

Taking the anti-symmetric part in μ,ρ gives

$$-S_{\nu\rho\mu}=-S_{\nu[\rho\mu]}=8\pi G_{\rm N}\left(\Pi_{[\rho\mu]\nu}-\frac{1}{2}g_{\nu[\rho}\Pi^\lambda{}_{\mu]\lambda}\right). \tag{4.52}$$

Hence

$$S_{\nu\mu\rho}=8\pi G_{\rm N}\left(\Pi_{[\rho\mu]\nu}-\frac{1}{2}g_{\nu[\rho}\Pi^\lambda{}_{\mu]\lambda}\right). \tag{4.53}$$

□

4.2 Metric-Affine Formalism

Within the metric-affine formalism we take the variation with respect to $g^{\mu\nu}$ and $\Gamma^\rho_{\mu\nu}$, where the connection is independent from the metric.

Definition 4.2.1.

$$\Delta_\rho{}^{\mu\nu}=\frac{-2}{\sqrt{-\text{det}g}}\frac{\delta S_{\rm m}}{\delta\Gamma^\rho_{\mu\nu}}$$

Proposition 4.2.1. *Variation of the action (Eq. (4.20)) with respect to the metric $g^{\mu\nu}$ yields the equation*

$$G_{(\mu\nu)} \;=\; 8\pi G_{\rm N}T_{\mu\nu}. \tag{4.54}$$

Proof. Following the derivation in proposition 4.1.1, replacing every δ_g by δ_Γ, this can be seen from Eq. (4.27). □

Proposition 4.2.2. *Variation of the action (Eq. (4.20)) with respect to the connection gives the equation*

$$\begin{aligned}-\nabla_\rho\left(g^{\mu\nu}\right)+\frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\nabla_\rho(g^{\alpha\beta})+\delta^\mu_\rho\left[\nabla_\alpha\left(g^{\alpha\nu}\right)-\frac{1}{2}g^{\nu\alpha}g_{\beta\lambda}\nabla_\alpha(g^{\beta\lambda})\right]&\\ +g^{\mu\nu}S_\rho-\delta^\mu_\rho g^{\nu\alpha}S_\alpha-S^{\mu\nu}{}_\rho \;=\; 8\pi G_{\rm N}\Delta_\rho{}^{\mu\nu}.&\end{aligned} \tag{4.55}$$

Proof. When we vary the Hilbert action with respect to $\Gamma^\rho_{\mu\nu}$ we find using Eqs. (4.27) and (4.28):

$$\begin{aligned}\delta_\Gamma S_{\rm H} &= \int d^4x\sqrt{-\text{det}g}g^{\rho\lambda}\delta_\Gamma R_{\rho\lambda}\\ &= \int d^4x\sqrt{-\text{det}g}g^{\rho\lambda}\left[\nabla_\alpha\delta_\Gamma\Gamma^\alpha_{\lambda\rho}-\nabla_\lambda\delta_\Gamma\Gamma^\alpha_{\alpha\rho}+S^\beta{}_{\alpha\lambda}\left(\delta_\Gamma\Gamma^\alpha_{\beta\rho}\right)\right]\\ &= \int d^4x\sqrt{-\text{det}g}\left[\nabla_\alpha\left(g^{\rho\lambda}\delta_\Gamma\Gamma^\alpha_{\lambda\rho}-g^{\rho\alpha}\delta_\Gamma\Gamma^\beta_{\beta\rho}\right)-\nabla_\alpha\left(g^{\rho\lambda}\right)\delta_\Gamma\Gamma^\alpha_{\lambda\rho}+\nabla_\lambda\left(g^{\rho\lambda}\right)\delta_\Gamma\Gamma^\alpha_{\alpha\rho}+g^{\rho\lambda}S^\beta{}_{\alpha\lambda}\left(\delta_\Gamma\Gamma^\alpha_{\beta\rho}\right)\right].\end{aligned} \tag{4.56}$$

Using lemma 4.0.2, we find

$$\begin{aligned}
\delta_\Gamma S_{\mathrm{H}} &= \int d^4x\sqrt{-\mathrm{det}g}\Bigg[\left(\frac{1}{2}g_{\mu\nu}\nabla_\alpha(g^{\mu\nu})+S_\alpha\right)\left(g^{\rho\lambda}\delta_\Gamma\Gamma^\alpha_{\lambda\rho}-g^{\rho\alpha}\delta_\Gamma\Gamma^\beta_{\beta\rho}\right) \\
&\quad -\nabla_\alpha\left(g^{\rho\lambda}\right)\delta_\Gamma\Gamma^\alpha_{\lambda\rho}+\nabla_\lambda\left(g^{\rho\lambda}\right)\delta_\Gamma\Gamma^\alpha_{\alpha\rho}+g^{\rho\lambda}S^\beta{}_{\alpha\lambda}\left(\delta_\Gamma\Gamma^\alpha_{\beta\rho}\right)\Bigg] \\
&= \int d^4x\sqrt{-\mathrm{det}g}\Bigg[\frac{1}{2}g_{\mu\nu}\nabla_\alpha(g^{\mu\nu})g^{\rho\beta}+g^{\rho\beta}S_\alpha-\frac{1}{2}g_{\mu\nu}\nabla_\lambda(g^{\mu\nu})g^{\rho\lambda}\delta^\beta_\alpha \\
&\quad -\delta^\beta_\alpha g^{\rho\lambda}S_\lambda-\nabla_\alpha\left(g^{\rho\beta}\right)+\nabla_\lambda\left(g^{\rho\lambda}\right)\delta^\beta_\alpha-S^{\beta\rho}{}_\alpha\Bigg]\delta_\Gamma\Gamma^\alpha_{\beta\rho}. \qquad (4.57)
\end{aligned}$$

Thus when we vary the total action, Eq. (4.20), with respect to the connection $\Gamma^\rho_{\mu\nu}$, we get the equation:

$$\begin{aligned}
-\nabla_\rho\left(g^{\mu\nu}\right)+\frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\nabla_\rho(g^{\alpha\beta})+\delta^\mu_\rho\left[\nabla_\alpha\left(g^{\alpha\nu}\right)-\frac{1}{2}g^{\nu\alpha}g_{\beta\lambda}\nabla_\alpha(g^{\beta\lambda})\right]& \\
+g^{\mu\nu}S_\rho-\delta^\mu_\rho g^{\nu\alpha}S_\alpha-S^{\mu\nu}{}_\rho &= 8\pi G_{\mathrm{N}}\Delta_\rho{}^{\mu\nu}. \qquad (4.58)
\end{aligned}$$

□

We would like now to derive an expression for the covariant derivative of the metric.

Proposition 4.2.3.

$$\nabla_\rho\left(g^{\mu\nu}\right)=-\frac{1}{3}g^{\mu\nu}S_\rho-\frac{1}{3}\delta^\mu_\rho S^\nu-S^{\mu\nu}{}_\rho+\frac{8}{3}\pi G_{\mathrm{N}}\delta^\mu_\rho\Delta_\lambda{}^{\lambda\nu}+4\pi G_{\mathrm{N}}g^{\mu\nu}\Delta_{\rho\lambda}{}^\lambda-\frac{4}{3}\pi G_{\mathrm{N}}g^{\mu\nu}\Delta^\lambda{}_{\lambda\rho}-8\pi G_{\mathrm{N}}\Delta_\rho{}^{\mu\nu}. \qquad (4.59)$$

Proof. Contracting Eq. (4.55) with $g_{\mu\nu}$ gives

$$\frac{1}{2}g_{\mu\nu}\nabla_\rho\left(g^{\mu\nu}\right)+g_{\mu\rho}\nabla_\alpha\left(g^{\alpha\mu}\right) = -2S_\rho+8\pi G_{\mathrm{N}}\Delta_{\rho\mu}{}^\mu. \qquad (4.60)$$

Contraction of Eq. (4.55) with δ^ρ_μ gives

$$3\nabla_\mu\left(g^{\mu\nu}\right)-\frac{3}{2}g^{\rho\nu}g_{\alpha\beta}\nabla_\rho(g^{\alpha\beta})=2S^\nu+8\pi G_{\mathrm{N}}\Delta_\mu{}^{\mu\nu}, \qquad (4.61)$$

and contracting Eq. (4.61) with $g_{\rho\nu}$ yields

$$3g_{\rho\nu}\nabla_\mu\left(g^{\mu\nu}\right)-\frac{3}{2}g_{\alpha\beta}\nabla_\rho(g^{\alpha\beta})=2S_\rho+8\pi G_{\mathrm{N}}\Delta^\mu{}_{\mu\rho}. \qquad (4.62)$$

Using Eqs. (4.60) and (4.62), we get

$$\begin{aligned}
\frac{3}{2}g_{\alpha\beta}\nabla_\rho(g^{\alpha\beta}) &= 3g_{\rho\nu}\nabla_\mu\left(g^{\mu\nu}\right)-2S_\rho-8\pi G_{\mathrm{N}}\Delta^\mu{}_{\mu\rho} \\
&= 3\left(-2S_\rho+8\pi G_{\mathrm{N}}\Delta_{\rho\mu}{}^\mu-\frac{1}{2}g_{\alpha\beta}\nabla_\rho\left(g^{\alpha\beta}\right)\right)-2S_\rho-8\pi G_{\mathrm{N}}\Delta^\mu{}_{\mu\rho} \qquad (4.63)
\end{aligned}$$

or

$$g_{\alpha\beta}\nabla_\rho(g^{\alpha\beta})=-\frac{8}{3}S_\rho+8\pi G_{\mathrm{N}}\Delta_{\rho\lambda}{}^\lambda-\frac{8}{3}\pi G_{\mathrm{N}}\Delta^\lambda{}_{\lambda\rho}. \qquad (4.64)$$

In the same way we find from Eqs. (4.61) and (4.64) that

$$\begin{aligned}
3\nabla_\mu\left(g^{\mu\nu}\right) &= \frac{3}{2}g^{\rho\nu}g_{\alpha\beta}\nabla_\rho(g^{\alpha\beta})+2S^\nu+8\pi G_{\mathrm{N}}\Delta_\mu{}^{\mu\nu} \\
&= \frac{3}{2}g^{\nu\rho}\left(-\frac{8}{3}S_\rho+8\pi G_{\mathrm{N}}\Delta_{\rho\lambda}{}^\lambda-\frac{8}{3}\pi G_{\mathrm{N}}\Delta^\lambda{}_{\lambda\rho}\right)+2S^\nu+8\pi G_{\mathrm{N}}\Delta_\lambda{}^{\lambda\nu} \\
&= -2S^\nu+12\pi G_{\mathrm{N}}\Delta^{\nu\lambda}{}_\lambda+4\pi G_{\mathrm{N}}\Delta_\lambda{}^{\lambda\nu}. \qquad (4.65)
\end{aligned}$$

Or

$$\nabla_\alpha \left(g^{\alpha\nu}\right) = -\frac{2}{3}S^\nu + 4\pi G_{\mathrm{N}}\Delta^{\nu\lambda}{}_\lambda + \frac{4}{3}\pi G_{\mathrm{N}}\Delta_\lambda{}^{\lambda\nu}. \tag{4.66}$$

Substitution of Eqs. (4.64) and (4.66) in Eq. (4.55) yields

$$\begin{aligned}
\nabla_\rho \left(g^{\mu\nu}\right) &= \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\nabla_\rho(g^{\alpha\beta}) + \delta^\mu_\rho\left[\nabla_\alpha\left(g^{\alpha\nu}\right) - \frac{1}{2}g^{\nu\alpha}g_{\beta\lambda}\nabla_\alpha(g^{\beta\lambda})\right] + g^{\mu\nu}S_\rho - \delta^\mu_\rho g^{\nu\alpha}S_\alpha - S^{\mu\nu}{}_\rho - 8\pi G_{\mathrm{N}}\Delta_\rho{}^{\mu\nu} \\
&= \frac{1}{2}g^{\mu\nu}\left(-\frac{8}{3}S_\rho + 8\pi G_{\mathrm{N}}\Delta_{\rho\lambda}{}^\lambda - \frac{8}{3}\pi G_{\mathrm{N}}\Delta^\lambda{}_{\lambda\rho}\right) + \delta^\mu_\rho\left[-\frac{2}{3}S^\nu + 4\pi G_{\mathrm{N}}\Delta^{\nu\lambda}{}_\lambda + \frac{4}{3}\pi G_{\mathrm{N}}\Delta_\lambda{}^{\lambda\nu}\right. \\
&\quad \left. -\frac{1}{2}g^{\nu\alpha}\left(-\frac{8}{3}S_\alpha + 8\pi G_{\mathrm{N}}\Delta_{\alpha\lambda}{}^\lambda - \frac{8}{3}\pi G_{\mathrm{N}}\Delta^\lambda{}_{\lambda\alpha}\right)\right] + g^{\mu\nu}S_\rho - \delta^\mu_\rho g^{\nu\alpha}S_\alpha - S^{\mu\nu}{}_\rho - 8\pi G_{\mathrm{N}}\Delta_\rho{}^{\mu\nu} \\
&= -\frac{1}{3}g^{\mu\nu}S_\rho - \frac{1}{3}\delta^\mu_\rho S^\nu - S^{\mu\nu}{}_\rho + \frac{8}{3}\pi G_{\mathrm{N}}\delta^\mu_\rho\Delta_\lambda{}^{\lambda\nu} + 4\pi G_{\mathrm{N}}g^{\mu\nu}\Delta_{\rho\lambda}{}^\lambda - \frac{4}{3}\pi G_{\mathrm{N}}g^{\mu\nu}\Delta^\lambda{}_{\lambda\rho} - 8\pi G_{\mathrm{N}}\Delta_\rho{}^{\mu\nu}.
\end{aligned} \tag{4.67}$$

□

4.3 Equivalence Metric and Metric-affine Formalism

In this section we will show that for the Standard model Lagrangian (not including any renormalization terms), the metric-affine formalism yields the same equations of motion as the metric formalism does. In particular, in the metric-affine formalism metric compatibility follows from the field equations instead of assuming it as we did in the metric formalism. We will do this separately for matter actions independent of the connection and the Dirac action before coming to the full Standard model.

The Hilbert action is invariant under the projective transformation (e.g. [28])

$$\Gamma^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu} + \delta^\rho_\mu\xi_\nu. \tag{4.68}$$

This is because the Ricci tensor transforms as

$$R_{\mu\nu} \to R_{\mu\nu} - 2\partial_{[\mu}\xi_{\nu]}, \tag{4.69}$$

which implies that

$$R \to R. \tag{4.70}$$

If the matter action is also invariant under this transformation, it can be seen as a gauge transformation and it can be used to fix 4 degrees of freedom of the connection. A very convenient gauge choice is

$$S_\mu = 0. \tag{4.71}$$

This will be enough to derive equivalence of the two formalisms for the matter in the Standard model. The observation of this equivalence is not new, it has been partly done in e.g. [26] and [27].

A problem of Eq. (4.55) is that when contracted with δ^ρ_ν, it yields

$$0 \;=\; 8\pi G_{\mathrm{N}}\Delta_\lambda{}^{\mu\lambda}. \tag{4.72}$$

This can be seen as an inconsistency of Eq. (4.55), because matter for which the right-hand side of Eq. (4.72) does not vanish cannot be described using these equations (for the matter in the Standard model the right-hand side already vanishes). The inconsistency can be resolved by adding a term

$$\int d^4x\sqrt{-\det g}B^\mu S_\mu, \tag{4.73}$$

to the total action, Eq. (4.20). Here, B^μ is a Lagrange multiplier. This basically results in $S_\mu = 0$ and an extra term in Eq. (4.55). With this extra term the Hilbert action is not invariant under projective transformations anymore. See e.g. [26] for a discussion of the resulting action.

4.3.1 Matter Action Independent of Connection

In this subsection we will consider actions that are independent of the connection. Two very familiar examples are the action of a scalar field

$$S_\varphi = \frac{1}{2}\int d^4x\sqrt{-\det g}\left(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right) \tag{4.74}$$

and the (kinetic part of the) action of the electromagnetic field

$$S_A = \frac{1}{4}\int d^4x\sqrt{-\det g}F^{\mu\nu}F_{\mu\nu}, \tag{4.75}$$

where

$$F = dA \tag{4.76}$$

for a vector field A, such that in coordinates

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.77}$$

We start with the metric-affine formalism. For a matter action that does not depend on the connection

$$\Delta_\rho{}^{\mu\nu} = 0. \tag{4.78}$$

Independence of the connection also implies that the total action, Eq. (4.20), is invariant under the projective transformation (4.68). Therefore we can choose the gauge

$$S_\mu = 0, \tag{4.79}$$

such that Eq. (4.59) reduces to

$$\nabla_\rho\left(g^{\mu\nu}\right) = -S^{\mu\nu}{}_\rho. \tag{4.80}$$

The right-hand side is symmetric under interchanging μ and ν and anti-symmetric under interchanging ν and ρ and this implies that it vanishes because

$$S^{\mu\nu\rho} = S^{\nu\mu\rho} = -S^{\nu\rho\mu} = -S^{\rho\nu\mu} = S^{\rho\mu\nu} = S^{\mu\rho\nu} = -S^{\mu\nu\rho}. \tag{4.81}$$

Therefore we have metric compatibility and vanishing torsion.

Vanishing torsion is also what you find in the metric formalism from Eq. (4.48). Since the matter action does not depend on the connection, the energy-momentum tensor in both formalisms is the same, which implies that the equations of motion that follow from varying with respect to the metric (Eqs. (4.24) and (4.54) are the same). Hence, for matter actions that do not depend on the connection, the two formalisms are equivalent. This became clear after choosing a gauge.

4.3.2 Dirac Action

Before we can explain what the Dirac action looks like in curved spacetime, we need to introduce some more geometric tools. Normally vectors are always expressed in a basis ∂_μ that is defined by a choice of coordinates x^μ. Of course at every point $p \in M$ one can choose arbitrary frames of T_pM to express vectors in. A property of a vectorbundle is that locally one can choose an orthonormal frame $e_i = e_i^\mu\partial_\mu$ such that $g(e_i, e_j) = \eta_{ij}$. Such a frame does not have to be related to a choice of coordinates. In this new frame, the metric has components η_{ij} such that we can consider it as flat spacetime. To transfer back to the coordinate frame ∂_μ we need the inverse of the matrix defined by e_i^μ. Let $e_\nu^j = \eta^{ji}g_{\mu\nu}e_i^\mu$, then $e_\nu^j e_i^\nu = \eta^{jk}g_{\mu\nu}e_k^\mu e_i^\nu = \eta^{jk}\eta_{ki} = \delta_i^j$ and we have found the inverse matrix. The matrix defined by e_i^μ is also called a vierbein or tetrad. Using the vierbein every tensor can now be expressed in terms of its components in this orthonormal basis. We will use Latin indices for the orthonormal basis and Greek indices for a basis induced by a coordinate system. The vierbein enables us to switch back and forth between Latin and Greek indices. We can even consider mixed tensors where some of the indices are Latin and some of them Greek.

Notice that the vierbein can be chosen in many different ways, let t_i denote another orthonormal frame, then $e_i^\mu = \Lambda^\mu{}_\nu t_i^\nu$ for some matrix Λ and $e_\mu^i = \Lambda_\mu{}^\nu t_\nu^i$. This implies that the metric in the new coordinate system has components

$$g_{ab} = \Lambda_a{}^i \Lambda_b{}^j \eta_{ij}. \tag{4.82}$$

Requiring that the new frame is also orthonormal implies that the transformations Λ we consider are local Lorentz transformations.

There are now two choices: a choice of coordinate system and a choice of orthonormal frame. Since $\partial_\mu V^a$ does not transform as vector under a transformation to another orthonormal frame, we need to define

$$\nabla_\mu V^a = \partial_\mu V^a + \omega^a_{\mu b} V^b, \tag{4.83}$$

where $\omega^a_{\mu b}$ is called the spin connection. We will now derive a relation between $\omega^a_{\mu b}$ and $\Gamma^\rho_{\mu\nu}$.

Lemma 4.3.1.

$$\omega^a_{\mu b} = e^\nu_b e^a_\rho \Gamma^\rho_{\mu\nu} - e^\nu_b \partial_\mu \left(e^a_\nu\right). \tag{4.84}$$

Proof. Using $V = V^\mu \partial_\mu = V^a e_a$ we find

$$\nabla V = \nabla_\mu \left(V^\rho \partial_\rho\right) dx^\mu = \left(\partial_\mu V^\rho + \Gamma^\rho_{\mu\nu} V^\nu\right) dx^\mu \otimes \partial_\rho \tag{4.85}$$

and

$$\begin{aligned}
\nabla V &= \nabla_\mu \left(V^a e_a\right) dx^\mu \\
&= \left(\partial_\mu V^a + \omega^a_{\mu b} V^b\right) dx^\mu \otimes e_a \\
&= \left(\partial_\mu \left(e^a_\nu V^\nu\right) + \omega^a_{\mu b} e^b_\nu V^\nu\right) dx^\mu \otimes \left(e^\rho_a \partial_\rho\right) \\
&= e^\rho_a \left(e^a_\nu \partial_\mu V^\nu + \partial_\mu \left(e^a_\nu\right) V^\nu + \omega^a_{\mu b} e^b_\nu V^\nu\right) dx^\mu \otimes \partial_\rho \\
&= \left(\partial_\mu V^\rho + e^\rho_a \partial_\mu \left(e^a_\nu\right) V^\nu + \omega^a_{\mu b} e^\rho_a e^b_\nu V^\nu\right) dx^\mu \otimes \partial_\rho.
\end{aligned} \tag{4.86}$$

Comparing Eqs. (4.85) and (4.86) yields:

$$\Gamma^\rho_{\mu\nu} = e^\rho_a \partial_\mu \left(e^a_\nu\right) + \omega^a_{\mu b} e^\rho_a e^b_\nu \tag{4.87}$$

or

$$\omega^a_{\mu b} = e^\nu_b e^a_\rho \Gamma^\rho_{\mu\nu} - e^\nu_b \partial_\mu \left(e^a_\nu\right). \tag{4.88}$$

□

Lemma 4.3.1 implies

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho + \omega^a_{\mu b} e^b_\nu = \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho + e^b_\nu \left(e^\lambda_b e^a_\rho \Gamma^\rho_{\mu\lambda} - e^\lambda_b \partial_\mu \left(e^a_\lambda\right)\right) = 0. \tag{4.89}$$

This result is sometimes mentioned as the tetrad postulate, but it is always true.

In the same way as we have a covariant derivative for tensors, we also have a covariant derivative for a spinor ψ:

$$\nabla_\mu \psi = \partial_\mu \psi + \Gamma_\mu \psi. \tag{4.90}$$

Notice that since $\nabla_\mu \psi$ has to transform as a spinor: under a local Lorentz transformation $\Lambda(x)$

$$\psi \to S(x)\psi = \exp(-\frac{i}{4}\omega^{ab}(x)\sigma_{ab})\psi, \tag{4.91}$$

where $\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b]$, ω^{ab} constants such that $\omega^{ab} = -\omega^{ba}$ and the gamma matrices γ^a form a representation of the Clifford algebra that is defined by

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} I. \tag{4.92}$$

When

$$\Gamma_\mu \to S \Gamma_\mu S^{-1} - \left(\partial_\mu S\right) S^{-1} \tag{4.93}$$

then

$$\nabla_\mu \psi \to (\partial_\mu S)\,\psi + S\partial_\mu \psi + \left[S\Gamma_\mu S^{-1} - (\partial_\mu S)\,S^{-1}\right] S\psi = S\nabla_\mu \psi, \tag{4.94}$$

exactly what we wanted. We also have that

$$\bar{\psi} \to \bar{\psi} S^{-1}, \tag{4.95}$$

where $\bar{\psi} = \psi^\dagger \gamma^0$. It immediately follows that with

$$\nabla_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \bar{\psi}\Gamma_\mu \tag{4.96}$$

$$\begin{aligned}\nabla_\mu \bar{\psi} \;\to\; & \left(\partial_\mu \bar{\psi}\right) S^{-1} + \bar{\psi}\partial_\mu S^{-1} - \bar{\psi} S^{-1}\left(S\Gamma_\mu S^{-1} - (\partial_\mu S)\,S^{-1}\right) \\ & = \left(\partial_\mu \bar{\psi}\right) S^{-1} + \bar{\psi}\partial_\mu S^{-1} - \bar{\psi} S^{-1}\left(S\Gamma_\mu S^{-1} + S\partial_\mu S^{-1}\right) \\ & = \left(\nabla_\mu \bar{\psi}\right) S^{-1}. \end{aligned} \tag{4.97}$$

To derive an expression for Γ_μ in terms of the affine connection, we will roughly follow [29].

Lemma 4.3.2.

$$\Gamma_\mu = \frac{1}{8}\omega^a_{\mu b}[\gamma_a, \gamma^b] - A_\mu, \tag{4.98}$$

where A_μ is a multiple of the unit matrix.

Proof. Notice that

$$\partial_\mu\left(\bar{\psi}\gamma^a\psi\right) = \nabla_\mu\left(\bar{\psi}\gamma^a\psi\right) - \omega^a_{\mu b}\bar{\psi}\gamma^b\psi = \partial_\mu\left(\bar{\psi}\gamma^a\psi\right) - \bar{\psi}\Gamma_\mu\gamma^a\psi + \bar{\psi}\left(\nabla_\mu\gamma^a\right)\psi + \bar{\psi}\gamma^a\Gamma_\mu\psi - \omega^a_{\mu b}\bar{\psi}\gamma^b\psi, \tag{4.99}$$

hence

$$\nabla_\mu \gamma^a = \omega^a_{\mu b}\gamma^b + \Gamma_\mu\gamma^a - \gamma^a\Gamma_\mu = \omega^a_{\mu b}\gamma^b + [\Gamma_\mu, \gamma^a]. \tag{4.100}$$

Now

$$\nabla_\mu \gamma^a = \frac{1}{2}\left(\nabla_\mu \eta^{ab}\right)\gamma_b \tag{4.101}$$

[30] such that

$$\omega^a_{\mu b}\gamma_a\gamma^b + \gamma_a\Gamma_\mu\gamma^a - 4\Gamma_\mu = \frac{1}{2}\gamma_a\left(\nabla_\mu\eta^{ab}\right)\gamma_b. \tag{4.102}$$

We solve Eq. (4.102) by a solution of the form

$$\Gamma_\mu = \frac{1}{8}\omega^a_{\mu b}\gamma_a\gamma^b - \frac{1}{8}\omega^a_{\mu b}\gamma^b\gamma_a - A_\mu, \tag{4.103}$$

where A_μ has two spinor indices and one vector index. Substitution of Eq. (4.103) in Eq. (4.102) and

$$\gamma_a\gamma^b\gamma^c\gamma^a = 2\delta^b_a\gamma^c\gamma^a - \gamma^b\gamma_a\gamma^c\gamma^a = 2\gamma^c\gamma^b - 2\gamma^b\gamma^c + 4\gamma^b\gamma^c = 4\eta^{bc} \tag{4.104}$$

yields

$$\frac{1}{2}\omega^a_{\mu b}\gamma_a\gamma^b + \frac{1}{2}\omega^a_{\mu b}\gamma^b\gamma_a - \gamma_a A_\mu\gamma^a + 4A_\mu = \frac{1}{2}\gamma_a\left(\nabla_\mu\eta^{ab}\right)\gamma_b = \frac{1}{2}\gamma_a\left(\omega^a_{\mu c}\eta^{cb} + \omega^b_{\mu c}\eta^{ac}\right)\gamma_b = \frac{1}{2}\omega^a_{\mu c}\gamma_a\gamma^c + \frac{1}{2}\omega^a_{\mu c}\gamma^c\gamma_a \tag{4.105}$$

such that

$$-\gamma_a A_\mu \gamma^a + 4A_\mu = 0. \tag{4.106}$$

This implies that A_μ should be a multiple of the unit matrix. □

In the literature (e.g. [29]) this A_μ is usually set to zero by viewing it as an extra non-gravitational field. This implies that we have

$$\Gamma_\mu = \frac{1}{8}\omega^a_{\mu b}[\gamma_a, \gamma^b]. \tag{4.107}$$

Now we have all the tools to derive what the Dirac action looks like in curved spacetime. The Dirac action in flat spacetime is given by

$$\begin{aligned} S_{\mathrm{D}} &= \int d^4x \left(\frac{i}{2}\bar{\psi}\gamma^a\partial_a\psi - \frac{i}{2}\partial_a\left(\bar{\psi}\right)\gamma^a\psi + m\bar{\psi}\psi \right) \\ &= \int d^4x\sqrt{-\det g}\left(\frac{i}{2}\bar{\psi}\gamma^a\nabla_a\psi - \frac{i}{2}\nabla_a\left(\bar{\psi}\right)\gamma^a\psi + m\bar{\psi}\psi \right), \end{aligned} \tag{4.108}$$

where $g_{\mu\nu} = \eta_{\mu\nu}$ and $\nabla_a = \partial_a$. Take this as the form of the Dirac action when in curved spacetime we work in flat coordinates. We can then transform to obtain the Dirac action in curved coordinates:

$$S_{\mathrm{D}} = \int d^4x\sqrt{-\det g}\left(\frac{i}{2}\bar{\psi}\gamma^\mu\nabla_\mu\psi - \frac{i}{2}\nabla_\mu\left(\bar{\psi}\right)\gamma^\mu\psi + m\bar{\psi}\psi \right), \tag{4.109}$$

where

$$\gamma^\mu = e_a^\mu\gamma^a \tag{4.110}$$

and

$$\begin{aligned} \nabla_\mu\psi &= \partial_\mu\psi + \Gamma_\mu\psi = \partial_\mu\psi + \frac{1}{8}\omega^a_{\mu b}[\gamma_a,\gamma^b]\psi = \partial_\mu\psi + \frac{1}{8}\left(e_b^\nu e_\rho^a\Gamma^\rho_{\mu\nu} - e_b^\nu\partial_\mu\left(e_\nu^a\right)\right)[\gamma_a,\gamma^b]\psi; \\ \nabla_\mu\bar{\psi} &= \left(\partial_\mu\bar{\psi} - \bar{\psi}\Gamma_\mu\right) = \partial_\mu\bar{\psi} - \frac{1}{8}\omega^a_{\mu b}\bar{\psi}[\gamma_a,\gamma^b] = \partial_\mu\bar{\psi} - \frac{1}{8}\left(e_b^\nu e_\rho^a\Gamma^\rho_{\mu\nu} - e_b^\nu\partial_\mu\left(e_\nu^a\right)\right)\bar{\psi}[\gamma_a,\gamma^b]. \end{aligned} \tag{4.111}$$

Lemma 4.3.3. *For the Dirac action*

$$\Delta_\rho{}^{\mu\nu} = -\frac{i}{2}e_\rho^d e_b^\nu e_c^\mu\eta_{ad}\bar{\psi}\gamma^{[c}\gamma^a\gamma^{b]}\psi. \tag{4.112}$$

Proof. From Eq. (4.109)

$$\begin{aligned} \Delta_\rho{}^{\mu\nu} &= -\frac{2}{\sqrt{-\det g}}\frac{\delta S_{\mathrm{D}}}{\delta\Gamma^\rho_{\mu\nu}} \\ &= -2\left[\frac{i}{16}\bar{\psi}\left(\gamma^\mu[\gamma_\rho,\gamma^\nu] + [\gamma_\rho,\gamma^\nu]\gamma^\mu\right)\psi\right] \\ &= -\frac{i}{8}\bar{\psi}\{\gamma^\mu,[\gamma_\rho,\gamma^\nu]\}\psi. \end{aligned} \tag{4.113}$$

Hence:

$$\Delta_\rho{}^{\mu\nu} = -\frac{i}{8}e_\rho^d e_b^\nu e_c^\mu\eta_{ad}\bar{\psi}\left\{\gamma^c,[\gamma^a,\gamma^b]\right\}\psi. \tag{4.114}$$

Consider the following equality:

$$\{\gamma^a,[\gamma^b,\gamma^c]\} = \frac{1}{3}\left(\gamma^a[\gamma^b,\gamma^c] + \gamma^b[\gamma^c,\gamma^a] + \gamma^c[\gamma^a,\gamma^b]\right) + \frac{1}{3}\left([\gamma^b,\gamma^c]\gamma^a + [\gamma^c,\gamma^a]\gamma^b + [\gamma^a,\gamma^b]\gamma^c\right). \tag{4.115}$$

The right-hand side is clearly anti-symmetric under interchanging a and b. Now using

$$\begin{aligned} \gamma^{[a}\gamma^b\gamma^{c]} &= \frac{1}{6}\left(\gamma^a[\gamma^b,\gamma^c] + \gamma^b[\gamma^c,\gamma^a] + \gamma^c[\gamma^a,\gamma^b]\right) \\ &= \frac{1}{6}\left([\gamma^b,\gamma^c]\gamma^a + [\gamma^c,\gamma^a]\gamma^b + [\gamma^a,\gamma^b]\gamma^c\right) \end{aligned} \tag{4.116}$$

we find that

$$\{\gamma^a,[\gamma^b,\gamma^c]\} = 2\gamma^{[a}\gamma^b\gamma^{c]} + 2\gamma^{[a}\gamma^b\gamma^{c]} = 4\gamma^{[a}\gamma^b\gamma^{c]}. \tag{4.117}$$

Substitution of this result in Eq. (4.114) yields

$$\Delta_\rho{}^{\mu\nu} = -\frac{i}{2}e_\rho^d e_b^\nu e_c^\mu\eta_{ad}\bar{\psi}\gamma^{[c}\gamma^a\gamma^{b]}\psi. \tag{4.118}$$

□

This implies that $\Delta^{\rho\mu\nu}$ is totally anti-symmetric such that also the Dirac action is invariant under the projective transformation (4.68). Hence, we can fix the gauge by setting S_μ to zero. From Eq. (4.59) we then find:

$$\nabla_\rho \left(g^{\mu\nu}\right) = -S^{\mu\nu}{}_\rho - 8\pi G_{\mathrm{N}} \Delta_\rho{}^{\mu\nu}. \tag{4.119}$$

Taking the (anti-)symmetric part in μ and ν, we obtain

$$\begin{aligned} \nabla_\rho \left(g^{\mu\nu}\right) &= -S^{(\mu\nu)}{}_\rho; \\ S^{[\mu\nu]}{}_\rho &= -8\pi G_{\mathrm{N}} \Delta_\rho{}^{\mu\nu}. \end{aligned} \tag{4.120}$$

But even more, Eq. (4.119) implies that

$$\nabla_{(\rho} g_{\nu)\mu} = 0, \tag{4.121}$$

such that $\nabla_\rho g_{\nu\mu}$ is anti-symmetric in its first 2 indices and symmetric in its last 2 indices, which implies:

$$-S^{(\mu\nu)}{}_\rho = \nabla_\rho \left(g^{\mu\nu}\right) = 0. \tag{4.122}$$

So we have metric compatibility, the torsion is totally anti-symmetric and:

$$S^{\mu\nu}{}_\rho = -8\pi G_{\mathrm{N}} \Delta_\rho{}^{\mu\nu}. \tag{4.123}$$

In the metric formalism, the only part of the connection that couples is the torsion part (since the tensor that is coupled to the connection is totally anti-symmetric), this implies that:

$$\Delta_\rho{}^{\mu\nu} = \Pi_\rho{}^{\mu\nu} \tag{4.124}$$

and from that and Eq. (4.48) it follows that the Cartan equations in both formalisms are the same. In the same way it turns out that the Einstein equation in both formalisms gives the same. Hence, the two formalisms are equivalent for fermions.

4.3.3 Standard Model

The analysis of the previous two sections shows that the metric and metric-affine formalism are equivalent for the Standard model, because the Lagrangian of the Standard model is build from the three specific actions we considered. All extra terms in this Lagrangian that cannot be rewritten as one of the three actions do not contain covariant derivatives and it follows from Section 4.3.1 that the two formalisms are also equivalent for these terms. When terms from renormalization are included, the projective symmetry is possibly broken and the two formalisms are not equivalent anymore.

4.4 Generalized Identities

In this section we generalize the Bianchi identity $\nabla_\mu G^{\mu\nu}$ and conservation of the energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$ that we know from general relativity. To do this, we use diffeomorphism invariance, i.e. that

$$\int \phi^* \omega = \int \omega, \tag{4.125}$$

where ω is a 4-form and ϕ is a diffeomorphism. A globally defined vector field V on M defines a flow $\phi(\tau, m):\ D \to M$ where $D \subset \mathbb{R} \times M$ such that $(0, m) \in D$ for all $m \in M$ and $V(m) = \partial_\tau \phi(\tau, m)|_{\tau=0}$. For a fixed τ, all $m \in M$ such that $(\tau, m) \in D$ form a set M' and $\phi_\tau:\ M' \to M\ m \mapsto \phi(\tau, m)$ is a diffeomorphism to its image. The idea is now that if we take τ infinitesimally small, we get a diffeomorphism $\phi:\ M \to M$ such that we can calculate the difference of the two integrals in Eq. (4.125) which should be equal to 0. Taking an action we obtain

$$0 = \delta_\phi S = \int \left(\frac{\delta S}{\delta g_{\mu\nu}} \delta_\phi g_{\mu\nu} + \frac{\delta S}{\delta \Gamma^\rho_{\mu\nu}} \delta_\phi \Gamma^\rho_{\mu\nu} + \frac{\delta S}{\delta \varphi_i} \delta_\phi \varphi_i \right), \tag{4.126}$$

where φ_i are the matter fields in the action. When the equations of motion are obeyed:

$$\frac{\delta S}{\delta \varphi_i} = 0. \tag{4.127}$$

Since we consider τ infinitesimally small, we must have that

$$\delta_\phi g|_p = \lim_{\tau \to 0} \frac{\phi_\tau^* g_{\phi_\tau(p)} - g|_p}{\tau} \equiv \left(\mathcal{L}_V g\right)|_p, \tag{4.128}$$

where $\mathcal{L}$ is the Lie derivative and $p \in M$. In components this is equal to (e.g. [31]):

$$\begin{aligned}
\delta_\phi g_{\mu\nu} &= \left(\mathcal{L}_V g\right)_{\mu\nu} \\
&= V(g_{\mu\nu}) + g_{\lambda\nu}\partial_\mu V^\lambda + g_{\mu\lambda}\partial_\nu V^\lambda \\
&= V^\lambda \nabla_\lambda g_{\mu\nu} + V^\lambda \Gamma^\alpha_{\lambda\mu} g_{\alpha\nu} + V^\lambda \Gamma^\alpha_{\lambda\nu} g_{\mu\alpha} + g_{\lambda\nu}\nabla_\mu V^\lambda - g_{\lambda\nu}\Gamma^\lambda_{\mu\alpha}V^\alpha + g_{\mu\lambda}\nabla_\nu V^\lambda - g_{\mu\lambda}\Gamma^\lambda_{\nu\alpha}V^\alpha \\
&= V^\lambda \nabla_\lambda g_{\mu\nu} + V^\lambda S^\alpha{}_{\lambda\mu} g_{\alpha\nu} + V^\lambda S^\alpha{}_{\lambda\nu} g_{\mu\alpha} + g_{\lambda\nu}\nabla_\mu V^\lambda + g_{\mu\lambda}\nabla_\nu V^\lambda \\
&= V^\lambda \nabla_\lambda g_{\mu\nu} + V^\lambda S_{\nu\lambda\mu} + V^\lambda S_{\mu\lambda\nu} + g_{\lambda\nu}\nabla_\mu V^\lambda + g_{\mu\lambda}\nabla_\nu V^\lambda.
\end{aligned} \tag{4.129}$$

We now derive what the Lie derivative of the connection looks like

Lemma 4.4.1.

$$\delta_\phi \Gamma^\rho_{\mu\nu} = \left(\mathcal{L}_V \Gamma\right)^\rho_{\mu\nu} = \nabla_\mu \nabla_\nu V^\rho - S^\rho{}_{\nu\lambda}\nabla_\mu V^\lambda - \nabla_\mu\left(S^\rho{}_{\nu\lambda}\right)V^\lambda + R^\rho{}_{\nu\lambda\mu}V^\lambda. \tag{4.130}$$

Proof. In the same way as we had for the metric we have that:

$$\delta_\phi \Gamma|_p = \lim_{\tau \to 0} \frac{\phi_\tau^* \Gamma_{\phi_\tau(p)} - \Gamma|_p}{\tau} \equiv \left(\mathcal{L}_V \Gamma\right)|_p. \tag{4.131}$$

The pullback of the covariant derivative is defined by

$$\left(\phi^* \nabla\right)_X Y = \phi_*^{-1}\left(\nabla_{\phi_* X}\phi_* Y\right) \tag{4.132}$$

for arbitrary vector fields X, $Y \in \mathcal{T}(M)$ [23]. In components this is equal to

$$\begin{aligned}
X^\mu \partial_\mu Y^\rho + X^\mu \left(\phi^*\Gamma\right)^\rho_{\mu\nu} Y^\nu &= \frac{\partial\left(\phi^{-1}\right)^\rho}{\partial x'^\lambda}\frac{\partial \phi^\beta}{\partial x^\mu} X^\mu \left[\partial'_\beta\left(\frac{\partial\phi^\lambda}{\partial x^\nu}(\phi^{-1}(x'))Y^\nu(\phi^{-1}(x'))\right) + \Gamma^\lambda_{\beta\alpha}\frac{\partial\phi^\alpha}{\partial x^\nu}Y^\nu\right] \\
&= \frac{\partial\left(\phi^{-1}\right)^\rho}{\partial x'^\lambda}\frac{\partial \phi^\beta}{\partial x^\mu} X^\mu \left[\frac{\partial^2\phi^\lambda}{\partial x^\alpha \partial x^\nu}\frac{\partial\left(\phi^{-1}\right)^\alpha}{\partial x'^\beta}Y^\nu + \frac{\partial\phi^\lambda}{\partial x^\nu}\frac{\partial\left(\phi^{-1}\right)^\alpha}{\partial x'^\beta}\partial_\alpha Y^\nu + \Gamma^\lambda_{\beta\alpha}\frac{\partial\phi^\alpha}{\partial x^\nu}Y^\nu\right] \\
&= \delta^\alpha_\mu X^\mu \frac{\partial\left(\phi^{-1}\right)^\rho}{\partial x'^\lambda}\frac{\partial^2\phi^\lambda}{\partial x^\alpha\partial x^\nu}Y^\nu + \delta^\rho_\nu\delta^\alpha_\mu X^\mu\partial_\alpha Y^\nu + \frac{\partial\left(\phi^{-1}\right)^\rho}{\partial x'^\lambda}\frac{\partial\phi^\beta}{\partial x^\mu}\frac{\partial\phi^\alpha}{\partial x^\nu}\Gamma^\lambda_{\beta\alpha}X^\mu Y^\nu \\
&= \frac{\partial\left(\phi^{-1}\right)^\rho}{\partial x'^\lambda}\frac{\partial^2\phi^\lambda}{\partial x^\mu\partial x^\nu}X^\mu Y^\nu + X^\mu\partial_\mu Y^\rho + \frac{\partial\left(\phi^{-1}\right)^\rho}{\partial x'^\lambda}\frac{\partial\phi^\beta}{\partial x^\mu}\frac{\partial\phi^\alpha}{\partial x^\nu}\Gamma^\lambda_{\beta\alpha}X^\mu Y^\nu.
\end{aligned} \tag{4.133}$$

This implies that

$$\left(\phi^*\Gamma\right)^\rho_{\mu\nu} X^\mu Y^\nu = \frac{\partial\left(\phi^{-1}\right)^\rho}{\partial x'^\lambda}\left[\frac{\partial^2\phi^\lambda}{\partial x^\mu\partial x^\nu} + \frac{\partial\phi^\beta}{\partial x^\mu}\frac{\partial\phi^\alpha}{\partial x^\nu}\Gamma^\lambda_{\beta\alpha}\right]X^\mu Y^\nu \tag{4.134}$$

for arbitrary X, Y. Hence:

$$\left(\phi^*\Gamma\right)^\rho_{\mu\nu} = \frac{\partial\left(\phi^{-1}\right)^\rho}{\partial x'^\lambda}\left(\frac{\partial^2\phi^\lambda}{\partial x^\mu\partial x^\nu} + \frac{\partial\phi^\beta}{\partial x^\mu}\frac{\partial\phi^\alpha}{\partial x^\nu}\Gamma^\lambda_{\beta\alpha}\right). \tag{4.135}$$

For the infinitesimal diffeomorphism ϕ we can write

$$\phi(x) = \left(x^0 + \tau V^0(x), ..., x^3 + \tau V^3(x)\right) \tag{4.136}$$

such that

$$\phi^{-1}(x') = \left(x'^0 - \tau V^0(x'), ..., x'^3 - \tau V^3(x')\right). \tag{4.137}$$

Therefore to first order

$$\begin{aligned}
(\phi^*\Gamma)^\rho_{\mu\nu}(x) &= \left(\delta^\rho_\lambda - \tau\partial'_\lambda V^\rho\right)\left[\tau\partial_\mu\partial_\nu V^\lambda + \left(\delta^\beta_\mu + \tau\partial_\mu V^\beta\right)\left(\delta^\alpha_\nu + \tau\partial_\nu V^\alpha\right)\left(\Gamma^\lambda_{\beta\alpha}(x) + \tau V^\sigma\partial_\sigma\Gamma^\lambda_{\beta\alpha}(x)\right)\right] \\
&= \left(\delta^\rho_\lambda - \tau\partial_\lambda V^\rho\right)\left[\tau\partial_\mu\partial_\nu V^\lambda + \Gamma^\lambda_{\mu\nu}(x) + \tau V^\sigma\partial_\sigma\Gamma^\lambda_{\mu\nu}(x) + \tau\Gamma^\lambda_{\beta\nu}(x)\partial_\mu V^\beta + \tau\Gamma^\lambda_{\mu\alpha}(x)\partial_\nu V^\alpha\right] \\
&= \tau\partial_\mu\partial_\nu V^\rho + \Gamma^\rho_{\mu\nu}(x) - \tau\Gamma^\lambda_{\mu\nu}(x)\partial_\lambda V^\rho + \tau V^\lambda\partial_\lambda\Gamma^\rho_{\mu\nu}(x) + \tau\Gamma^\rho_{\lambda\nu}(x)\partial_\mu V^\lambda + \tau\Gamma^\rho_{\mu\lambda}(x)\partial_\nu V^\lambda.
\end{aligned} \tag{4.138}$$

So

$$\begin{aligned}
(\mathcal{L}_V\Gamma)^\rho_{\mu\nu} &= \partial_\mu\partial_\nu V^\rho - \Gamma^\lambda_{\mu\nu}\partial_\lambda V^\rho + V^\lambda\partial_\lambda\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\lambda\nu}\partial_\mu V^\lambda + \Gamma^\rho_{\mu\lambda}\partial_\nu V^\lambda \\
&= \partial_\mu\left(\nabla_\nu V^\rho - \Gamma^\rho_{\nu\lambda}V^\lambda\right) - \Gamma^\lambda_{\mu\nu}\partial_\lambda V^\rho + V^\lambda\partial_\lambda\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\lambda\nu}\partial_\mu V^\lambda + \Gamma^\rho_{\mu\lambda}\partial_\nu V^\lambda \\
&= \nabla_\mu\nabla_\nu V^\rho + \Gamma^\lambda_{\mu\nu}\nabla_\lambda V^\rho - \Gamma^\rho_{\mu\lambda}\nabla_\nu V^\lambda - \partial_\mu\left(\Gamma^\rho_{\nu\lambda}\right)V^\lambda - \Gamma^\rho_{\nu\lambda}\partial_\mu V^\lambda - \Gamma^\lambda_{\mu\nu}\partial_\lambda V^\rho \\
&\quad + V^\lambda\partial_\lambda\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\lambda\nu}\partial_\mu V^\lambda + \Gamma^\rho_{\mu\lambda}\partial_\nu V^\lambda \\
&= \nabla_\mu\nabla_\nu V^\rho + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\alpha}V^\alpha - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\alpha}V^\alpha - \partial_\mu\left(\Gamma^\rho_{\nu\lambda}\right)V^\lambda - S^\rho{}_{\nu\lambda}\partial_\mu V^\lambda + V^\lambda\partial_\lambda\Gamma^\rho_{\mu\nu} \\
&= \nabla_\mu\nabla_\nu V^\rho - S^\rho{}_{\nu\lambda}\partial_\mu V^\lambda + \Bigg(\partial_\lambda\Gamma^\rho_{\mu\nu} - \partial_\mu\left(S^\rho{}_{\nu\lambda}\right) - \partial_\mu\Gamma^\rho_{\lambda\nu} + \Gamma^\alpha_{\mu\nu}S^\rho{}_{\alpha\lambda} \\
&\quad + \Gamma^\alpha_{\mu\nu}\Gamma^\rho_{\lambda\alpha} - S^\alpha{}_{\nu\lambda}\Gamma^\rho_{\mu\alpha} - \Gamma^\alpha_{\lambda\nu}\Gamma^\rho_{\mu\alpha}\Bigg)V^\lambda \\
&= \nabla_\mu\nabla_\nu V^\rho - S^\rho{}_{\nu\lambda}\partial_\mu V^\lambda - \nabla_\mu\left(S^\rho{}_{\nu\lambda}\right)V^\lambda - \Gamma^\alpha_{\mu\lambda}S^\rho{}_{\nu\alpha}V^\lambda + R^\rho{}_{\nu\lambda\mu}V^\lambda \\
&= \nabla_\mu\nabla_\nu V^\rho - S^\rho{}_{\nu\lambda}\nabla_\mu V^\lambda - \nabla_\mu\left(S^\rho{}_{\nu\lambda}\right)V^\lambda + R^\rho{}_{\nu\lambda\mu}V^\lambda.
\end{aligned} \tag{4.139}$$

Since $\mathcal{L}_V\Gamma$ is a tensor, this is its expression in coordinates. □

Expression (4.130) agrees with the one in [32], but notice that they use slightly different conventions. With Eqs. (4.126), (4.127), (4.129) and (4.130), we obtain:

$$\begin{aligned}
0 &= \delta_\phi S \\
&= \int d^4x\sqrt{-\det g}\left[\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}\delta_\phi g_{\mu\nu} + \frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}\delta_\phi\Gamma^\rho_{\mu\nu}\right] \\
&= \int d^4x\sqrt{-\det g}\Bigg[\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}\left(V^\lambda\nabla_\lambda g_{\mu\nu} + g_{\lambda\nu}\nabla_\mu V^\lambda + g_{\mu\lambda}\nabla_\nu V^\lambda + V^\lambda S_{\nu\lambda\mu} + V^\lambda S_{\mu\lambda\nu}\right) \\
&\quad + \frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}\left(\nabla_\mu\nabla_\nu V^\rho + S^\rho{}_{\lambda\nu}\nabla_\mu V^\lambda + V^\lambda\nabla_\mu S^\rho{}_{\lambda\nu} + R^\rho{}_{\nu\lambda\mu}V^\lambda\right)\Bigg].
\end{aligned} \tag{4.140}$$

Using lemma 4.0.2 we find that

$$\begin{aligned}
0 = \int d^4x\sqrt{-\det g}\Bigg[\Bigg(&\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}\nabla_\lambda g_{\mu\nu} - \nabla_\mu\left(g_{\lambda\nu}\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}\right) + \left[\frac{1}{2}g_{\alpha\beta}\nabla_\mu(g^{\alpha\beta}) + S_\mu\right]g_{\lambda\nu} \\
&\cdot\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}} - \nabla_\nu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}g_{\mu\lambda}\right) + \left[\frac{1}{2}g_{\alpha\beta}\nabla_\nu(g^{\alpha\beta}) + S_\nu\right]\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}g_{\mu\lambda} \\
&+\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}S_{\nu\lambda\mu} + \frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}S_{\mu\lambda\nu}\Bigg)V^\lambda + \Bigg(-\nabla_\mu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}\right)\nabla_\nu V^\rho \\
&+\left[\frac{1}{2}g_{\alpha\beta}\nabla_\mu(g^{\alpha\beta}) + S_\mu\right]\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}\nabla_\nu V^\rho - \nabla_\mu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}S^\rho{}_{\lambda\nu}\right)V^\lambda + \left[\frac{1}{2}g_{\alpha\beta}\nabla_\mu(g^{\alpha\beta}) + S_\mu\right] \\
&\cdot\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}S^\rho{}_{\lambda\nu}V^\lambda + \frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}V^\lambda\nabla_\mu S^\rho{}_{\lambda\nu} + \frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}R^\rho{}_{\nu\lambda\mu}V^\lambda\Bigg)\Bigg].
\end{aligned} \tag{4.141}$$

Again with lemma 4.0.2:

$$
\begin{aligned}
0 \;=\; & \int d^4x\sqrt{-\det g}\Bigg[\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}\nabla_\lambda g_{\mu\nu}-\nabla_\mu\left(g_{\lambda\nu}\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}\right)+\left[\frac{1}{2}g_{\alpha\beta}\nabla_\mu(g^{\alpha\beta})+S_\mu\right] \\
& \cdot g_{\lambda\nu}\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}-\nabla_\nu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}g_{\mu\lambda}\right)+\left[\frac{1}{2}g_{\alpha\beta}\nabla_\nu(g^{\alpha\beta})+S_\nu\right]\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}g_{\mu\lambda} \\
& +\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}S_{\nu\lambda\mu}+\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}S_{\mu\lambda\nu}+\nabla_\nu\nabla_\mu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\lambda_{\mu\nu}}\right) \\
& -\left[\frac{1}{2}g_{\alpha\beta}\nabla_\nu(g^{\alpha\beta})+S_\nu\right]\nabla_\mu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\lambda_{\mu\nu}}\right)-\nabla_\nu\left[\left[\frac{1}{2}g_{\alpha\beta}\nabla_\mu(g^{\alpha\beta})+S_\nu\right]\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\lambda_{\mu\nu}}\right] \\
& +\left[\frac{1}{2}g_{\sigma\gamma}\nabla_\nu(g^{\sigma\gamma})+S_\nu\right]\left[\frac{1}{2}g_{\alpha\beta}\nabla_\mu(g^{\alpha\beta})+S_\mu\right]\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\lambda_{\mu\nu}}-\nabla_\mu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}S^\rho{}_{\lambda\nu}\right) \\
& +\left[\frac{1}{2}g_{\alpha\beta}\nabla_\mu(g^{\alpha\beta})+S_\mu\right]\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}S^\rho{}_{\lambda\nu}+\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}\nabla_\mu S^\rho{}_{\lambda\nu}+\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}R^\rho{}_{\nu\lambda\mu}\Bigg]V^\lambda.
\end{aligned}
\tag{4.142}
$$

We then find that

$$
\begin{aligned}
0 \;=\; & \frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}\nabla_\lambda g_{\mu\nu}-\nabla_\mu\left(g_{\lambda\nu}\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}\right)+\left[\frac{1}{2}g_{\alpha\beta}\nabla_\mu(g^{\alpha\beta})+S_\mu\right]g_{\lambda\nu}\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}} \\
& -\nabla_\nu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}g_{\mu\lambda}\right)+\left[\frac{1}{2}g_{\alpha\beta}\nabla_\nu(g^{\alpha\beta})+S_\nu\right]\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}g_{\mu\lambda}+\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}S_{\nu\lambda\mu} \\
& +\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}S_{\mu\lambda\nu}+\nabla_\nu\nabla_\mu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\lambda_{\mu\nu}}\right)-\left[\frac{1}{2}g_{\alpha\beta}\nabla_\nu(g^{\alpha\beta})+S_\nu\right]\nabla_\mu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\lambda_{\mu\nu}}\right) \\
& -\nabla_\nu\left[\left[\frac{1}{2}g_{\alpha\beta}\nabla_\mu(g^{\alpha\beta})+S_\mu\right]\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\lambda_{\mu\nu}}\right] \\
& +\left[\frac{1}{2}g_{\sigma\gamma}\nabla_\nu(g^{\sigma\gamma})+S_\nu\right]\left[\frac{1}{2}g_{\alpha\beta}\nabla_\mu(g^{\alpha\beta})+S_\mu\right]\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\lambda_{\mu\nu}}-\nabla_\mu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}S^\rho{}_{\lambda\nu}\right) \\
& +\left[\frac{1}{2}g_{\alpha\beta}\nabla_\mu(g^{\alpha\beta})+S_\mu\right]\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}S^\rho{}_{\lambda\nu}+\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}\nabla_\mu S^\rho{}_{\lambda\nu}+\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}R^\rho{}_{\nu\lambda\mu}.
\end{aligned}
\tag{4.143}
$$

This can be rewritten as

$$
\begin{aligned}
0 \;=\; & \frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}\nabla_\lambda g_{\mu\nu}-2\nabla_\mu\left(g_{\lambda\nu}\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}\right)+\left[g_{\alpha\beta}\nabla_\mu(g^{\alpha\beta})+2S_\mu\right]g_{\lambda\nu}\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}} \\
& -2\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}S_{\nu\mu\lambda}+\nabla_\nu\nabla_\mu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\lambda_{\mu\nu}}\right)-\left[\frac{1}{2}g_{\alpha\beta}\nabla_\nu(g^{\alpha\beta})+S_\nu\right]\nabla_\mu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\lambda_{\mu\nu}}\right) \\
& -\nabla_\nu\left[\left[\frac{1}{2}g_{\alpha\beta}\nabla_\mu(g^{\alpha\beta})+S_\mu\right]\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\lambda_{\mu\nu}}\right] \\
& +\left[\frac{1}{2}g_{\sigma\gamma}\nabla_\nu(g^{\sigma\gamma})+S_\nu\right]\left[\frac{1}{2}g_{\alpha\beta}\nabla_\mu(g^{\alpha\beta})+S_\mu\right]\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\lambda_{\mu\nu}}-\nabla_\mu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}\right)S^\rho{}_{\lambda\nu} \\
& +\left[\frac{1}{2}g_{\alpha\beta}\nabla_\mu(g^{\alpha\beta})+S_\mu\right]\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}S^\rho{}_{\lambda\nu}+\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}R^\rho{}_{\nu\lambda\mu},
\end{aligned}
\tag{4.144}
$$

where we used the symmetry of $g_{\mu\nu}$. Using now that for the Standard model we can work in a gauge such that $S_\mu = 0$ and we have metric compatibility, Eq. (4.144) reduces to:

$$
\begin{aligned}
0 \;=\; & -2\nabla_\mu\left(g_{\lambda\nu}\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}\right)-2\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}S_{\nu\mu\lambda}+\nabla_\nu\nabla_\mu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\lambda_{\mu\nu}}\right) \\
& -\nabla_\mu\left(\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}\right)S^\rho{}_{\lambda\nu}+\frac{1}{\sqrt{-\det g}}\frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}}R^\rho{}_{\nu\lambda\mu}.
\end{aligned}
\tag{4.145}
$$

We now apply this identity to different actions.

Hilbert action Using

$$\delta g_{\rho\lambda} = -g_{\rho\mu} g_{\lambda\nu} \delta g^{\mu\nu} \tag{4.146}$$

we find

$$\frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} = \frac{\delta S}{\delta g^{\rho\lambda}} \delta g^{\rho\lambda} = -\frac{\delta S}{\delta g^{\rho\lambda}} g^{\rho\mu} g^{\lambda\nu} \delta g_{\mu\nu}. \tag{4.147}$$

Hence

$$\frac{\delta S}{\delta g_{\mu\nu}} = -\frac{\delta S}{\delta g^{\rho\lambda}} g^{\rho\mu} g^{\lambda\nu}. \tag{4.148}$$

For the Hilbert action we find from Eqs. (4.27) and (4.57) that in the gauge $S_\mu = 0$

$$\begin{aligned} \frac{1}{\sqrt{-\det g}} \frac{\delta S}{\delta g^{\rho\lambda}} &= G_{(\rho\lambda)}; \\ \frac{1}{\sqrt{-\det g}} \frac{\delta S}{\delta \Gamma^{\rho}_{\mu\nu}} &= -S^{\mu\nu}{}_{\rho}. \end{aligned} \tag{4.149}$$

Substitution of these results in Eq. (4.145) yields the generalized Bianchi identity for the case $S_\mu = \nabla_\rho g_{\mu\nu} = 0$ (so what we have when the matter Lagrangian is the one from the Standard model):

$$\begin{aligned} 0 &= 2g_{\lambda\nu} \nabla_\mu G^{(\mu\nu)} + 2G^{(\mu\nu)} S_{(\nu\mu)\lambda} - \nabla_\nu \nabla_\mu \left(S^{\mu\nu}{}_{\lambda}\right) \\ &\quad + \nabla_\mu \left(S^{\mu\nu}{}_{\rho}\right) S^{\rho}{}_{\lambda\nu} - S^{\mu\nu}{}_{\rho} R^{\rho}{}_{\nu\lambda\mu}. \end{aligned} \tag{4.150}$$

The second term in the right-hand side of Eq. (4.150) is zero for totally anti-symmetric torsion.

Matter action independent of connection In this case we have seen that we obtain metric compatibility and vanishing torsion. Also

$$\begin{aligned} \frac{1}{\sqrt{-\det g}} \frac{\delta S_{\mathrm{m}}}{\delta g_{\mu\nu}} &= \frac{1}{2} T^{\mu\nu}; \\ \frac{1}{\sqrt{-\det g}} \frac{\delta S_{\mathrm{m}}}{\delta \Gamma^{\rho}_{\mu\nu}} &= 0. \end{aligned} \tag{4.151}$$

The identity we obtain is thus

$$0 = \nabla_\mu T^{\mu\nu}. \tag{4.152}$$

Dirac Action Working in the gauge $S_\mu = 0$, we found metric compatibility. Also:

$$\begin{aligned} \frac{1}{\sqrt{-\det g}} \frac{\delta S_{\mathrm{m}}}{\delta g_{\mu\nu}} &= \frac{1}{2} T^{\mu\nu}; \\ \frac{1}{\sqrt{-\det g}} \frac{\delta S_{\mathrm{m}}}{\delta \Gamma^{\rho}_{\mu\nu}} &= -\frac{1}{2} \Delta_{\rho}{}^{\mu\nu}, \end{aligned} \tag{4.153}$$

which results in the following identity:

$$\begin{aligned} 0 &= -g_{\lambda\nu} \nabla_\mu \left(T^{\mu\nu}\right) - T^{\mu\nu} S_{(\nu\mu)\lambda} - \frac{1}{2} \nabla_\nu \nabla_\mu \left(\Delta_{\lambda}{}^{\mu\nu}\right) + \frac{1}{2} \nabla_\mu \left(\Delta_{\rho}{}^{\mu\nu}\right) S^{\rho}{}_{\lambda\nu} - \frac{1}{2} \Delta_{\rho}{}^{\mu\nu} R^{\rho}{}_{\nu\lambda\mu} \\ &= -g_{\lambda\nu} \nabla_\mu \left(T^{\mu\nu}\right) - \frac{1}{2} \nabla_\nu \nabla_\mu \left(\Delta_{\lambda}{}^{\mu\nu}\right) + 4\pi G_{\mathrm{N}} \nabla_\mu \left(\Delta^{\rho\mu\nu}\right) \Delta_{\nu\lambda\rho} + \frac{1}{2} \Delta^{\rho\mu\nu} R_{\rho\nu\mu\lambda}. \end{aligned} \tag{4.154}$$

This means that the energy-momentum tensor corresponding to the Dirac field is possibly not conserved, which implies that energy and momentum is locally not conserved.

Chapter 5

Singularity Theorems in Einstein-Cartan Theory

In this chapter we will finally generalize the singularity theorems of Hawking and Penrose [1, 2, 3] to spacetimes with totally anti-symmetric torsion. In Chapter 2 and 3 we extended all the propositions that are needed that involve torsion. The other propositions that are used in [3] only use arguments not dependent on torsion so do not have to be generalized, they can be found in Chapter 6 of [3]. In Chapter 4 we derived the equations of motion for the theory that can be used to translate geometric conditions to conditions on the matter content of the universe.

This chapter is organized as follows. In Section 5.1 we state the singularity theorems that we obtain for totally anti-symmetric torsion and discuss their assumptions. We will also compare with a different method to prove these theorems. In Section 5.2 we discuss whether the singularity theorems can also be generalized directly to non totally anti-symmetric torsion. In particular we give a way to construct a null geodesically incomplete spacetime with vectorial torsion.

5.1 Totally Anti-Symmetric Torsion

For the discussion of singularities it will be useful to have a certain arrow of time. In our daily life this seems to be well-defined. This makes it reasonable to believe that there is a local arrow of time defined continuously at every point in spacetime. In the language of manifolds this is called an orientation and we require that it should be possible to define continuously a division of non-spacelike vectors in two classes: future- and past-oriented. We call such a spacetime time-orientable. There are spacetimes (M, g) that are not time-orientable but they always have a double cover that has a time orientation. From now on we will assume that the spacetime is time-orientable or that we are dealing with its time-oriented covering space. If there is a singularity in the cover, there should be one in the space itself. Using the time-orientability we can say that curves are future or past directed.

To formulate the generalized singularity theorems, we need a couple of definitions:

Definition 5.1.1. The *chronological future* $I^+(P)$ of a set $P \subset M$ is the set of all points $q \in M$ such that there is a $p \in P$ and a future-directed timelike curve $\gamma : [\tau_\mathrm{i}, \tau_\mathrm{f}] \to M$ such that $\gamma(\tau_\mathrm{i}) = p$ and $\gamma(\tau_\mathrm{f}) = q$.

Definition 5.1.2. The *causal future* $J^+(P)$ of a set $P \subset M$ is the set of all points $q \in M$ such that there is a $p \in P$ and a future-directed non-spacelike curve $\gamma : [\tau_\mathrm{i}, \tau_\mathrm{f}] \to M$ such that $\gamma(\tau_\mathrm{i}) = p$ and $\gamma(\tau_\mathrm{f}) = q$.

Definition 5.1.3. A set $P \subset M$ is said to be *achronal* if $I^+(P) \cap P = \emptyset$, so there exists no two points in P with timelike separation.

Definition 5.1.4. The *future (past) horismos* of a set $P \subset M$ is $E^+(P) = J^+(P) - I^+(P)$ ($E^-(P) = J^-(P) - I^-(P)$).

Definition 5.1.5. The *future (past) Cauchy development* $D^+(P)$ ($D^-(P)$) of a set $P \subset M$, is the set of all points $q \in M$ such that every past (future)-inextendible non-spacelike curve through q intersects P.

Definition 5.1.6. A *Cauchy surface* is a spacelike hypersurface P which every non-spacelike curve intersects exactly once and such that

$$D(P) \equiv D^+(P) \cup D^-(P) = M. \tag{5.1}$$

Definition 5.1.7. A *closed trapped surface* P is a C^2 closed (i.e. compact without boundary) spacelike two-surface such that the two families n_i of null geodesics orthogonal to P are converging at P, i.e. $-\mathrm{tr}(L_{n_i})$ is negative at the surface.

Definition 5.1.8. The *strong causality condition* holds at a point $p \in M$ if for every neighborhood U of p there exists a neighborhood V such that $p \in V \subset U$ and such that no non-spacelike curve intersects V more than once.

The singularity theorems in the case of totally anti-symmetric torsion can then be stated as follows (proofs are the same as the proofs of theorem 1, 2, 3 and 4 in Chapter 8 in [3]):

Theorem 5.1.1. *A spacetime (M, g) cannot be null geodesically complete if all of the following conditions are met:*

1. $\mathrm{Ric}(K, K) - \frac{1}{4}\mathrm{tr}(S_K^2) \geq 0$ *for all null vectors K;*
2. *There is a non-compact Cauchy surface in M;*
3. *There is a closed trapped surface in M.*

Theorem 5.1.2. *Spacetime (M, g) is not timelike and null geodesically complete if the following conditions all hold:*

1. $\mathrm{Ric}(K, K) - \frac{1}{4}\mathrm{tr}(S_K^2) \geq 0$ *for all non-spacelike vectors K;*
2. *The generic condition is satisfied, i.e. every timelike geodesic γ contains a point at which $\breve{R}_\gamma \neq 0$ and every null geodesic γ contains a point at which $\breve{\breve{R}}_\gamma \neq 0$;*
3. *The chronology condition holds on M (i.e. there are no closed timelike curves);*
4. *There exists at least one of the following:*
    (a) *A compact achronal set without edge,*
    (b) *A closed trapped surface,*
    (c) *A point p such that on every past (or every future) null geodesic from p the expansion $\bar{\theta}_{\bar{A}}$ of the null geodesics from p becomes negative (i.e. the null geodesics from p are focused by the matter or curvature and start to reconverge).*

Theorem 5.1.3. *The following conditions cannot all hold:*

1. *Every inextendible non-spacelike geodesic contains a pair of conjugate points;*
2. *The chronology condition holds on M;*
3. *There is an achronal set P such that $E^+(P)$ or $E^-(P)$ is compact.*

Theorem 5.1.4. *If all of the following holds:*

1. $\mathrm{Ric}(K,K) - \frac{1}{4}\mathrm{tr}(S_K^2) \geq 0$ *for all non-spacelike vectors K;*
2. *The strong causality condition holds on (M,g);*
3. *There is some past-directed unit timelike vector W at a point p and a positive constant b such that if V is the unit tangent vector to the past-directed timelike geodesics through p, then on each such geodesic the expansion θ_A of these geodesics becomes less than $-3c/b$ within a distance b/c from p, where $c = -g(W,V)$,*

then there is a past incomplete non-spacelike geodesic through p.

Notice that one can indeed use the proof of theorem 3 in Chapter 8 of [3] since for totally anti-symmetric torsion we find from Eq. (2.22) that for a timelike geodesic

$$\theta_A = \mathrm{tr}\left(-S_\gamma + \nabla_\mu T^\rho|_{w=0}\right) = \mathrm{tr}\left(\nabla_\mu T^\rho|_{w=0}\right), \tag{5.2}$$

and similarly for a null geodesic, which is what is used in the proof (notice that this is mathematically not well-defined and ideally we would like a proof that uses the tensor A).

Theorem 5.1.5. *Spacetime is not timelike geodesically complete if the following two conditions hold:*

1. $\mathrm{Ric}(K,K) - \frac{1}{4}\mathrm{tr}(S_K^2) \geq 0$ *for all non-spacelike vectors K;*
2. *There exists a compact spacelike three-surface P (without edge) such that the unit normals to P are everywhere converging, i.e. $-\mathrm{tr}(L_n) < 0$ (or everywhere diverging, i.e. $-\mathrm{tr}(L_n) > 0$) on P.*

Theorems 5.1.2 and 5.1.3 are equivalent [3]. We will now discuss some of the assumptions of the theorems. The geometric condition

$$\mathrm{Ric}(K,K) - \frac{1}{4}\mathrm{tr}(S_K^2) \geq 0 \tag{5.3}$$

that appears as assumption in every theorem has to hold for all null and/or timelike geodesics because every non-spacelike vector corresponds to such a geodesic. Via the equations of motion the Ricci and torsion tensors can be coupled to the matter in the universe. For the matter in the Standard model the equations derived from the metric formalism and metric-affine formalism are equivalent (classically). Using that torsion is totally anti-symmetric, Eqs. (4.24) and (4.48) can be rewritten as

$$\begin{aligned} R_{(\mu\nu)} &= 8\pi G_\mathrm{N}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \\ S_{\mu\nu\rho} &= -8\pi G_\mathrm{N}\Pi_{\mu\nu\rho}, \end{aligned} \tag{5.4}$$

where $T = g^{\mu\nu}T_{\mu\nu}$. Using the Einstein and Cartan equations, condition (5.3) becomes

$$\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T - 2\pi G_\mathrm{N}\Pi^\alpha{}_{\beta\mu}\Pi^\beta{}_{\alpha\nu}\right)K^\mu K^\nu \geq 0. \tag{5.5}$$

Since the term $-2\pi G_\mathrm{N}\Pi^\alpha{}_{\beta\mu}\Pi^\beta{}_{\alpha\nu}K^\mu K^\nu$ is positive it seems that this condition is easier satisfied than the one for vanishing torsion $\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)K^\mu K^\nu \geq 0$, the strong energy condition. This seems to imply that singularities should form 'easier' in spacetimes with totally anti-symmetric torsion. Quantum corrections can of course cause violation of condition (5.5).

The condition

$$\breve{R}_\gamma \neq 0 \tag{5.6}$$

for timelike geodesics γ can be understood for vanishing torsion using Eq. (2.1), the Jacobi equation

$$D_\tau^2 J = -D_\tau S_\gamma(J) - R_\gamma(J) = -\breve{R}_\gamma(J). \tag{5.7}$$

In Newtonian theory, the acceleration of each particle is given by the gradient of the potential Φ and the relative acceleration of two particles with separation J^μ is given by $J^\mu \partial_\mu (\nabla\Phi)$. For vanishing torsion $\breve{R}_\gamma$ is analogous to $\partial_\mu (\nabla\Phi)$. The effect of this term can be seen as follows. If one considers a sphere of particles falling towards the earth, this sphere will deform because particles closer to the earth fall faster than particles further away from the earth. Hence, we can say that $\breve{R}_\gamma \neq 0$ if the test particle moving on γ will encounter some matter or radiation on its way. In a physical universe one can expect this to be the case. If torsion is not vanishing, it seems to be less clear if condition (5.6) is satisfied in a physical universe. If we want to do a similar analysis as for the vanishing torsion case, we have that

$$D_\tau S_\gamma + R_\gamma \neq 0 \tag{5.8}$$

at some point on γ if the test particle moving on γ will encounter some matter or radiation on its way. However from Eq. (2.57) it follows that this is not equivalent to condition (5.6). Of course, when the test particle encounters matter or radiation that does not induce torsion (i.e. when it encounters bosons), it is equivalent to condition (5.6). For the condition $\breve{\tilde{R}}_\gamma \neq 0$ for null geodesics γ we can have a similar discussion.

So far we have treated torsion on the same footing as the metric. One can actually also use Eq. (2.54) to expand Eq. (4.24) in terms corresponding to the Levi-Civita connection and torsion terms (e.g. [6, 13]). Then one can use Eq. (4.48) to substitute for torsion. This way one finds the Einstein equation of general relativity with an adapted energy-momentum tensor (this is called 'integration out torsion'). Hence, we have general relativity with a changed Einstein equation. If one does that, the assumption $\breve{R}_\gamma \neq 0$ is expected to be satisfied in a physical universe, while in our approach this is not immediately clear. So to improve upon what we have done, one can prove propositions 2.1.3 and 2.3.3 under conditions for which it is more clear that they are satisfied in a physical universe. Below, we will come back to integrating out torsion since it gives an easy and elegant way of extending the singularity theorems.

That there must be a closed trapped surface somewhere in a physical spacetime is proved in Chapter 9 of [3]. It seems that some and probably most stars with masses $M > 1.5\mathrm{M}_{\mathrm{Sun}}$ will collapse when they are gone through their stages of fusion. For spherical symmetric collapse the solution outside of the star will be described by the Schwarzschild solution (when other matter is sufficiently far away), so when the star is collapsed to within its event horizon we automatically have a closed trapped surface. If the collapse is not spherically symmetric (this will happen when the star is rotating or has a non-vanishing magnetic field) it can be shown [3] that if the deviations from spherical symmetry are small, one still obtains a closed trapped surface. There are at least 10^9 stars in our galaxy that have masses larger than $1.5\mathrm{M}_{\mathrm{Sun}}$ and the number of black holes in our galaxy formed by the collapse of a star is estimated to be between 10^7 and 10^9. That means that there are a large number of trapped surfaces.

As discussed in [3], the existence of a Cauchy surface is a weakness of theorem 5.1.1. An example by Bardeen shows that it is a necessary condition. In that example one takes the same global structure as the Reissner-Nordström solution except that the singularity is smoothed out such that they are just the origin of polar coordinates. The spacetime obeys all conditions of theorem 5.1.1 except for the Cauchy surface. The existence of a Cauchy surface is a weakness of theorem 5.1.1 and that is why theorem 5.1.2 is more convenient. In that respect is also theorem 5.1.4 very useful, since also the assumptions of that theorem are satisfied in a number of physical situations. However it is possible that violation of causality (e.g. in theorem 5.1.2 one can have a closed timelike curve) prevents the occurrence of a singularity. Theorem 5.1.5 shows that not in all cases singularities can be prevented by violation of causality.

In this thesis we gave very direct generalizations of the proofs given in [3] for totally anti-symmetric torsion, but as for instance observed in [6, 13] one can also generalize the proofs by integrating out torsion. Using Eq. (2.54) one can write

$$\mathrm{Ric}(K,K) \;=\; \breve{\mathrm{Ric}}(K,K) + \left(\nabla_{\mu S}\Gamma^{\mu}_{\nu\lambda} - \nabla_{\nu S}\Gamma^{\mu}_{\mu\lambda} + {}_S\Gamma^{\beta}_{\nu\lambda}{}_S\Gamma^{\mu}_{\mu\beta} - {}_S\Gamma^{\beta}_{\mu\lambda}{}_S\Gamma^{\mu}_{\nu\beta}\right) K^\lambda K^\nu. \tag{5.9}$$

For totally anti-symmetric torsion we have

$$ {}_S\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} S^{\rho}{}_{\mu\nu} \tag{5.10} $$

and we find that

$$ \begin{aligned} \mathrm{Ric}(K,K) &= \breve{\mathrm{Ric}}(K,K) + \frac{1}{2}\left(\nabla_\mu S^{\mu}{}_{\nu\lambda} - \frac{1}{2} S^{\beta}{}_{\mu\lambda} S^{\mu}{}_{\nu\beta}\right) K^\lambda K^\nu \\ &= \breve{\mathrm{Ric}}(K,K) + \frac{1}{4}\mathrm{tr}(S_K^2). \end{aligned} \tag{5.11} $$

Realizing now that the geodesics for the Levi-Civita connection are the same as for totally anti-symmetric torsion, we can also use the theorems from Hawking and Penrose to say something about their completeness. In their theorems one has the assumption

$$ \breve{\mathrm{Ric}}(K,K) \geq 0 \tag{5.12} $$

which is equivalent (at the geometric side) to the one we found, (5.3), which can be seen using Eq. (5.11). Notice that $\mathrm{tr}(S_K^2) \leq 0$. When using the Einstein equation to translate the left-hand side of Eq. (5.11) to the matter content of the universe, one finds that after integrating out torsion one has general relativity with an adapted energy-momentum tensor (substitute Eq. (4.48) for the torsion).

[6, 13] used their method of integrating out torsion because they claim that test particles move on curves with maximal length, hence the curves that are geodesics with respect to the Levi-Civita connection. Using expansion (5.9) with

$$ {}_S\Gamma^{\rho}_{\mu\nu} = -S_{(\mu\nu)}{}^{\rho} + \frac{1}{2} S^{\rho}{}_{\mu\nu} \tag{5.13} $$

one can express the Einstein equation in terms of the metric, torsion and the energy-momentum tensor. Then one can substitute Eq. (4.48) and one has the Einstein equation of general relativity with an adapted energy-momentum tensor. Since the curves considered are geodesics with respect to the Levi-Civita connection, one can apply the singularity theorems of Hawking and Penrose with an adapted energy-momentum tensor.

When one assumes that test particles also for general torsion have geodesics as their trajectories, it becomes more complicated. That is why we generalized the proofs in the way we did it. This way it is easier to see where things go wrong when one considers general torsion. This is what we will do in the next section.

5.2 General Torsion

For general torsion it seems hard to give a direct generalization of the proofs of the theorems in [3]. Since for general torsion in general a geodesic between two points is not the curve between those points that has maximal length, the proofs of theorems 5.1.2, 5.1.3, 5.1.4 and 5.1.5 directly fail because they use that fact. For the proof of theorem 5.1.1 one needs the proof of proposition 3.0.6 (also needed in the proofs of theorems 5.1.2 and 5.1.3). This proof becomes very hard to generalize since one cannot use that $g(S(W,T),T) = 0$ which alters expression (3.47). The proof of proposition 3.0.5 that is needed in the proof of theorem 5.1.5 only does not hold when $g(S(D_\tau\dot\gamma,\dot\gamma),\dot\gamma) = g(D_\tau\dot\gamma, D_\tau\dot\gamma)$ for all points on γ between the points p and q.

All the theorems also need the propositions about the existence of conjugate points. They correspond to conditions (5.3) and (5.6). From the proof of proposition 2.1.2 and Eq. (2.32) it is clear that condition (5.3) needs to be replaced by

$$ \mathrm{Ric}(\dot\gamma(\tau),\dot\gamma(\tau)) + \mathrm{tr}(D_\tau(PS_\gamma)) + \mathrm{tr}(PS_\gamma\omega_A) + \mathrm{tr}(PS_\gamma\sigma_A) + \frac{\theta_A}{3}\mathrm{tr}(PS_\gamma) + \mathrm{tr}(\omega_A^2) \geq 0 \tag{5.14} $$

for all timelike geodesics γ and all starting points $\gamma(\tau)$ (the matrix A is related to this starting point). For totally anti-symmetric torsion we could calculate the vorticity tensor ω_A, but for general torsion this seems to be quite impossible which is why condition (5.14) is more complicated. We would also need a condition for null geodesics, but we did not derive a Raychaudhuri equation for general torsion in that case since it was not clear to us how to do that. Condition (5.14) seems like a rather hard condition to check for an arbitrary spacetime and even harder to motivate from a physics point of view. There has been some work on singularity theorems in spacetimes with non totally anti-symmetric torsion [11], [33] and they indeed assumed something similar to condition (5.14). They also had to assume that the length of the geodesics is maximal. Both assumptions seem to be way too restrictive and one can doubt about how much sense it makes to do that.

Vectorial Torsion

Despite the increasing difficulty for general torsion, we succeeded in giving a construction of a null geodesically incomplete spacetime with a specific kind of torsion, namely vectorial torsion (see the decomposition (1.7)):

$$S^{\rho}{}_{\mu\nu} = g^{\rho}_{\mu}A_{\nu} - g^{\rho}_{\nu}A_{\mu}, \tag{5.15}$$

where A^{μ} is a vectorfield. This is possible because under sufficient assumptions on A^{μ}, an incomplete null geodesic with respect to $\breve{\nabla}$ (the connection with respect to the Levi-Civita connection) can be mapped to an incomplete null geodesic with respect to ∇ (the connection with torsion). First of all it is interesting to see from what sort of matter action we obtain vectorial torsion. According to the equation of motion for torsion, Eq. (4.42) in the metric formalism, we need a matter action such that

$$\Pi_{\rho}{}^{\mu\nu} = -\frac{4}{\sqrt{-\det g}}\frac{\delta S_{\mathrm{m}}}{\delta S^{\rho}{}_{\mu\nu}} \tag{5.16}$$

satisfies

$$\begin{aligned} 8\pi G_{\mathrm{N}}\Pi_{\rho}{}^{\mu\nu} &= S^{\nu\mu}{}_{\rho} - S^{\mu\nu}{}_{\rho} + S_{\rho}{}^{\mu\nu} - 2S^{\nu}\delta^{\mu}_{\rho} + 2S^{\mu}\delta^{\nu}_{\rho} \\ &= 8g^{[\nu}_{\rho}A^{\mu]}. \end{aligned} \tag{5.17}$$

One way to obtain this torsion is by a matter action that is non-minimally coupled to gravity:

$$S_{\mathrm{m}} = \int d^4x\sqrt{-\det g}\left(F_{\mu\nu}F^{\mu\nu} + \frac{1}{4\pi G_{\mathrm{N}}}S_{\mu}A^{\mu}\right) \tag{5.18}$$

where $F_{\mu\nu} = \partial_{\mu}A_{v} - \partial_{\nu}A_{\mu}$. When one integrates out torsion, using $S_{\mu} = 3A_{\mu}$, one finds the Proca action (in curved spacetime) that describes a massive vector field.

Lemma 5.2.1. *Let γ_1 be a null geodesic with respect to $\breve{\nabla}$, then we can reparametrize γ_1 such that locally it is a null geodesic with respect to ∇. Let γ_1 be a null geodesic with respect to ∇, then we can reparametrize γ_1 such that it locally is a geodesic with respect to $\breve{\nabla}$.*

Proof. For a null curve γ, we have that

$$\begin{aligned} (\nabla_{\dot\gamma}\dot\gamma)^{\rho} &= \dot\gamma^{\mu}\partial_{\mu}\dot\gamma^{\rho} + \Gamma^{\rho}_{\mu\nu}\dot\gamma^{\mu}\dot\gamma^{\nu} \\ &= \left(\breve{\nabla}_{\dot\gamma}\dot\gamma\right)^{\rho} - S_{(\mu\nu)}{}^{\rho}\dot\gamma^{\mu}\dot\gamma^{\nu} \\ &= \left(\breve{\nabla}_{\dot\gamma}\dot\gamma\right)^{\rho} + \left(g^{\rho}_{\mu}A_{\nu} - g_{\mu\nu}A^{\rho}\right)\dot\gamma^{\mu}\dot\gamma^{\nu} \\ &= \left(\breve{\nabla}_{\dot\gamma}\dot\gamma\right)^{\rho} + g(A,\dot\gamma)\dot\gamma^{\rho}. \end{aligned} \tag{5.19}$$

Therefore, when γ_1 is a null geodesic with respect to $\breve{\nabla}$ we have that

$$(\nabla_{\dot\gamma_1}\dot\gamma_1)^{\rho} = g(A,\dot\gamma_1)\dot\gamma_1^{\rho}. \tag{5.20}$$

Let $p = \gamma_1(\tau_0)$ be a point on γ_1 and define

$$f(\tau) = \tau_0 + \int_{\tau_0}^{\tau}\frac{1}{C_{\gamma_1} + \int_{\tau_0}^{\tau'}g(A,\dot\gamma_1)d\tau''}d\tau', \tag{5.21}$$

where the constant $C_{\gamma_1} \neq 0$. This function will be well-defined and monotonously increasing or decreasing for $\tau \in I_1 = [\tau_0 - \epsilon, \tau_0 + \epsilon]$ for ϵ small enough. Let $\tau \in I_2 = (\tau_0 - \delta, \tau_0 + \delta)$ where δ is chosen such that $f(I_2)$ is a subset of the domain of definition of γ_1 and $I_2 \subset I_1$. Defining

$$\gamma(\tau) = \gamma_1(f(\tau)), \tag{5.22}$$

we find that

$$\begin{aligned}
(\nabla_{\dot\gamma}\dot\gamma)^\rho &= \frac{\partial^2\gamma^\rho}{\partial\tau^2} + \dot\gamma^\mu \Gamma^\rho_{\mu\nu}\dot\gamma^\nu \\
&= \frac{\partial}{\partial\tau}\left(\dot f \dot\gamma_1^\rho\right) + \dot f^2 \Gamma^\rho_{\mu\nu}\dot\gamma_1^\nu\dot\gamma_1^\mu \\
&= \left[\ddot f + \dot f^2 g(A,\dot\gamma_1)\right]\dot\gamma_1^\rho \\
&= \left[-\frac{g(A,\dot\gamma_1)}{\left(C_{\gamma_1} + \int_{\tau_0}^{\tau'} g(A,\dot\gamma_1)d\tau''\right)^2} + \left(\frac{1}{C_{\gamma_1} + \int_{\tau_0}^{\tau'} g(A,\dot\gamma_1)d\tau''}\right)^2 g(A,\dot\gamma_1)\right]\dot\gamma_1^\rho \\
&= 0.
\end{aligned} \tag{5.23}$$

Hence around $\gamma_1(\tau_0)$ is γ_1 a null geodesic with respect to ∇.

When γ_1 is a null geodesic curve with respect to ∇ we find that

$$\left(\breve\nabla_{\dot\gamma_1}\dot\gamma_1\right)^\rho = -g(A,\dot\gamma_1)\dot\gamma_1^\rho. \tag{5.24}$$

With a similar argument as before we find that locally γ_1 is a geodesic with respect to $\breve\nabla$. □

Notice that using this lemma one can construct spacetimes with vectorial torsion that are null geodesically incomplete. To do this one starts with a spacetime that is null geodesically incomplete if we consider null geodesics with respect to $\breve\nabla$ (e.g. choose a spacetime that satisfies the conditions of theorem 5.1.1). Therefore, there is a maximal null geodesic $\gamma : I \to M$ such that $I \neq \mathbb{R}$. We can assume I to be bounded from above because if it is not, we can do an affine transformation $\tau' = -\tau + \alpha$, where α is a constant. Choose a vector field A, such that there are a $\tau_0 \in I$ and constant C_γ such that the function

$$f(\tau) = \tau_0 + \int_{\tau_0}^{\tau} \frac{1}{C_\gamma + \int_{\tau_0}^{\tau'} g(A,\dot\gamma)d\tau''} d\tau' \; \tau \in [\tau_0, \infty) \cap I \tag{5.25}$$

is well-defined, strictly increasing and $f(\tau) \geq \tau$. It follows from the proof of lemma 5.2.1 that $\gamma_1 : [\tau_0, \tau_1) \to M$, $\gamma_1(\tau) = \gamma(f(\tau))$ (where τ_1 is such that $f([\tau_0, \tau_1)) = [\tau_0, \infty) \cap I$) is a null geodesic with respect to ∇ for $\tau \geq \tau_0$. The geodesic γ_1 cannot be expanded to $\tau \geq \tau_1$ because with lemma 5.2.1 one can expand γ outside I, which is impossible. Hence, we have an incomplete null geodesic with respect to ∇.

The function (5.25) is strictly increasing when

$$\frac{1}{C_\gamma + \int_{\tau_0}^{\tau'} g(A,\dot\gamma)d\tau''} > 0 \tag{5.26}$$

is positive, hence

$$C_\gamma + \int_{\tau_0}^{\tau} g(A,\dot\gamma)d\tau' \tag{5.27}$$

should be positive. Notice that when one chooses a vectorfield A such that $\int_{\tau_0}^{\tau} g(A,\dot\gamma)d\tau'$ is positive and $\int_{I\cap[\tau_0,\infty)} g(A,\dot\gamma)d\tau'$ is well-defined (and not ∞), we can consider αA for a constant α and get

$$\frac{1}{.1 + \alpha\int_{\tau_0}^{\tau'} g(A,\dot\gamma)d\tau''} \geq 1 \tag{5.28}$$

for all $\tau \in [\tau_0, \infty) \cap I$ by choosing α small enough. Then

$$f(\tau) \geq \tau. \tag{5.29}$$

To give an example where we can apply this construction, we take the metric

$$ds^2 = -dt^2 + t \sum_{i=1}^{3} \left(dx^i\right)^2 \tag{5.30}$$

for $t \geq 0$. This is the FLRW metric with $\kappa = 0$ and $a(t) = \sqrt{t}$. We treat spacetimes with this metric extensively in Chapter 6, but for now the only thing that is really important is that we have a singularity when $t = 0$. Let a geodesic be given by $\gamma(\tau) = \big(t(\tau), x^i(\tau)\big)$ and let $u^\mu = d\gamma^\mu / d\tau$. All geodesics will be incomplete (when $t \to 0$). The geodesic equations (with respect to $\breve{\nabla}$) are given by

$$\begin{aligned} \frac{du^0}{d\tau} + \frac{1}{2}\left(u^i\right)^2 &= 0 \\ \frac{du^i}{d\tau} + \frac{1}{t} u^0 u^i &= 0. \end{aligned} \tag{5.31}$$

The second equation can be rewritten as

$$\frac{d}{d\tau}\left[t u^i\right] = 0 \tag{5.32}$$

with solution

$$u^i = \frac{C_i}{t}, \tag{5.33}$$

where C_i are constants. The constraint equation for null geodesics is

$$0 = -\left(u^0\right)^2 + t\left(u^i\right)^2, \tag{5.34}$$

such that

$$u^0 = -\sqrt{\frac{C}{t}}, \tag{5.35}$$

where $C = \sum_i C_i^2$. This can be solved by

$$t = \left(1 - \frac{3}{2}\sqrt{C}\tau\right)^{2/3}, \tag{5.36}$$

choosing $\tau = 0$ to correspond to $t = 1$. For an arbitrary vector A we then get

$$g(A, \dot{\gamma}) = A^0 \sqrt{\frac{C}{t}} + A^i C_i \tag{5.37}$$

Integration yields

$$\int_{\tau_0}^{\tau'} g(A, \dot{\gamma}) d\tau'' = -\int_{t_0}^{t} (A^0 + \frac{\sqrt{t}}{\sqrt{C}} A^i C_i) dt' = -A^0(t - t_0) - \frac{2}{3}\frac{A^i C_i}{\sqrt{C}}\left(t^{3/2} - t_0^{3/2}\right) \tag{5.38}$$

which is well-defined when $t \to 0$ and A and t_0 can be chosen such that this is positive for all $t < t_0$ (e.g. by choosing $A^0 > 0$ and A^i such that $A^i C_i > 0$). Hence, we have seen that we can construct a spacetime with vectorial torsion that is null geodesically incomplete.

Chapter 6

FLRW Spacetime

In this chapter we will study singularities and torsion in spacetimes that are spatially homogeneous and isotropic, but evolving in time. The most general metric that corresponds to such a spacetime is given by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-\kappa r^2} + r^2 \left(d\theta^2 + \sin^2(\theta) d\varphi^2 \right) \right], \tag{6.1}$$

where we use spherical coordinates, κ represents the (constant) curvature of spacelike 3-surfaces and $a : \mathbb{R} \supset I \to \mathbb{R}_{\geq 0}$ (I connected) is the scale factor. The interval I has the form $[t_0, t_1]$, $t_0, t_1 \in \mathbb{R}$, when $a(t_0) = a(t_1) = 0$ and $a(t) > 0$ on (t_0, t_1). It has the form $[t_0, \infty)$, $t_0 \in \mathbb{R}$, when $a(t_0) = 0$ and $a(t) > 0$ for $t > t_0$ and it has the form $(-\infty, t_0]$ when $a(t_0) = 0$ and $a(t) > 0$ for $t < t_0$. This metric is a good description of our universe, since from experiments as WMAP and Planck, it follows that our universe is spatially homogeneous and isotropic when averaged over large scales.

This chapter is organized as follows. In Section 6.1 we describe the geodesics in an FLRW spacetime using its symmetries. Then we argue in Section 6.2 that such a spacetime contains an initial singularity if and only if $a(t)$ vanishes for a certain $t \in I$ (one can do the same for a future singularity, but we restrict our attention to singularities in the past). Then in Section 6.3 we examine the relation between the existence of conjugate points and singularities in this spacetime. One of the conditions of the singularity theorems in Chapter 5 implies that if a geodesic is complete, it contains conjugate points. We will show that for certain FLRW spacetimes with $\kappa \leq 0$ the existence of a singularity implies that all points on a geodesic are conjugate to the point of the geodesic at the singularity. To do this we slightly adapt the definition of conjugate points. All the points on a geodesic conjugate to a certain point also implies a singularity. Lastly we give a condition for a geodesic in a spacetime with $\kappa = 0$ to have conjugate points. In the first sections we will put torsion to zero, however in Section 6.4 we will model torsion as an energy density and combine it first with a perfect radiation fluid and then with a perfect matter fluid. We will see that in this case a singularity at the beginning of the universe is avoided. As far as we know, the work done in this chapter is new.

6.1 Geodesics

In this section we will study the geodesics in an FLRW spacetime using the symmetry of this spacetime. This spacetime has 6 Killing vectors ξ, given by

$$\begin{aligned} \xi^t &= 0 \\ \xi^r &= \sqrt{1-\kappa r^2} \left\{ \sin(\theta) \left[\cos(\varphi) a_1 + \sin(\varphi) a_2 \right] + \cos(\theta) a_3 \right\} \\ \xi^\theta &= \frac{\sqrt{1-\kappa r^2}}{r} \left\{ \cos(\theta) \left[\cos(\varphi) a_1 + \sin(\varphi) a_2 \right] - \sin(\theta) a_3 \right\} + \sin(\varphi) b_1 - \cos(\varphi) b_2 \\ \xi^\varphi &= \frac{\sqrt{1-\kappa r^2}}{r} \frac{1}{\sin(\theta)} \left[\cos(\varphi) a_2 - \sin(\varphi) a_1 \right] + \cot(\theta) \left[\cos(\varphi) b_1 + \sin(\varphi) b_2 \right] - b_3, \end{aligned} \tag{6.2}$$

where a_i and b_i are six constants [34]. Let γ be a geodesic, then we can use 2 of the 3 rotation symmetries to choose coordinates such that $\theta = \pi/2$ for γ. The quantity $\xi_\mu \dot{\gamma}^\mu$ is conserved along γ when ξ is a Killing vector. The Killing vectors that give non-vanishing $\xi_\mu \dot{\gamma}^\mu$ are

$$\begin{aligned} \xi^t &= 0 \\ \xi^r &= \sqrt{1-\kappa r^2}\,[\cos(\varphi) a_1 + \sin(\varphi) a_2] \\ \xi^\theta &= 0 \\ \xi^\varphi &= \frac{\sqrt{1-\kappa r^2}}{r}\,[\cos(\varphi) a_2 - \sin(\varphi) a_1] - b_3. \end{aligned} \tag{6.3}$$

With ∂_φ as Killing vector we find the conserved angular momentum

$$L = a^2 r^2 \dot{\gamma}^\varphi. \tag{6.4}$$

Choosing $a_1 = 1$ and $a_2 = b_3 = 0$ in (6.3) we find the conserved quantity

$$\begin{aligned} K_1 &= \frac{a^2 \cos(\varphi)}{\sqrt{1-\kappa r^2}} \dot{\gamma}^r - r a^2 \sqrt{1-\kappa r^2} \sin(\varphi) \dot{\gamma}^\varphi \\ &= \frac{a^2 \cos(\varphi)}{\sqrt{1-\kappa r^2}} \dot{\gamma}^r - L \frac{\sqrt{1-\kappa r^2}}{r} \sin(\varphi) \end{aligned} \tag{6.5}$$

and choosing $a_2 = 1$ and $a_1 = b_3 = 0$ in (6.3) we find the quantity

$$\begin{aligned} K_2 &= \frac{a^2 \sin(\varphi)}{\sqrt{1-\kappa r^2}} \dot{\gamma}^r + r a^2 \sqrt{1-\kappa r^2} \cos(\varphi) \dot{\gamma}^\varphi \\ &= \frac{a^2 \sin(\varphi)}{\sqrt{1-\kappa r^2}} \dot{\gamma}^r + L \frac{\sqrt{1-\kappa r^2}}{r} \cos(\varphi). \end{aligned} \tag{6.6}$$

Vanishing angular momentum $L = 0$ corresponds either to a radial geodesic or to a geodesic that has constant r, θ, φ (a comoving geodesic). When the geodesic passes the point that corresponds to $r = 0$, we can choose different coordinates from the start such that γ is not radial. Hence for a geodesic that is not comoving, we can always choose coordinates such that $L > 0$ for γ.

With Eqs. (6.5) and (6.6) we find that

$$\dot{\gamma}^r = \frac{\sqrt{1-\kappa r^2} K_1 r + L(1-\kappa r^2)\sin(\varphi)}{a^2 \cos(\varphi) r} = \frac{\sqrt{1-\kappa r^2} K_2 r - L(1-\kappa r^2)\cos(\varphi)}{a^2 \sin(\varphi) r}. \tag{6.7}$$

Using the normalization of a geodesic ϵ, where $\epsilon = 0$ for null geodesics and $\epsilon = -1$ for timelike geodesics, and Eqs. (6.4) and (6.7), we find that

$$\begin{aligned} \epsilon &= -\left(\dot{\gamma}^t\right)^2 + \frac{a^2}{1-\kappa r^2}\left(\dot{\gamma}^r\right)^2 + a^2 r^2 \left(\dot{\gamma}^\varphi\right)^2 \\ &= -\left(\dot{\gamma}^t\right)^2 + \frac{1}{a^2}\left(\frac{K_1^2 r^2 + L^2(1-\kappa r^2)\sin^2(\varphi) + 2K_1 r L \sin(\varphi)\sqrt{1-\kappa r^2}}{r^2\cos^2(\varphi)} + \frac{L^2}{r^2}\right) \\ &= -\left(\dot{\gamma}^t\right)^2 + \frac{1}{a^2}\left(K_1^2 + K_2^2 + \kappa L^2\right). \end{aligned} \tag{6.8}$$

Hence

$$\dot{\gamma}^t = \pm \frac{\sqrt{K_1^2 + K_2^2 + \kappa L^2 - \epsilon a^2}}{a}. \tag{6.9}$$

Using Eq. (6.7) we find that

$$K_1 r \sin(\varphi) + L\sqrt{1-\kappa r^2}\sin^2(\varphi) = K_2 r \cos(\varphi) - L\sqrt{1-\kappa r^2}\cos^2(\varphi), \tag{6.10}$$

which leads to

$$[K_1 \sin(\varphi) - K_2 \cos(\varphi)]\, r = -L\sqrt{1-\kappa r^2}. \tag{6.11}$$

When we take the square of both sides and rewrite it, we find that (when $L \neq 0$)

$$r = \frac{L}{\sqrt{\left(K_2\cos(\varphi) - K_1\sin(\varphi)\right)^2 + \kappa L^2}}, \tag{6.12}$$

(when L is negative, one gets an extra minus sign) which leads, using Eqs. (6.4) and (6.9), to

$$\begin{aligned} \frac{d\varphi}{dt} &= \frac{d\tau}{dt}\frac{d\varphi}{d\tau} \\ &= \frac{1}{\sqrt{K_1^2 + K_2^2 + \kappa L^2 - \epsilon a^2}}\frac{L}{ar^2} \\ &= \frac{\left(K_2\cos(\varphi) - K_1\sin(\varphi)\right)^2 + \kappa L^2}{aL\sqrt{K_1^2 + K_2^2 + \kappa L^2 - \epsilon a^2}}. \end{aligned} \tag{6.13}$$

Hence

$$\int_{\varphi_1}^{\varphi} \frac{L}{\left(K_2\cos(\varphi') - K_1\sin(\varphi')\right)^2 + \kappa L^2} d\varphi' = \int_{t_1}^{t} \frac{1}{a\sqrt{K_1^2 + K_2^2 + \kappa L^2 - \epsilon a^2}} dt'. \tag{6.14}$$

A primitive of the integrand on the left-hand side is given by

$$\begin{cases} \kappa > 0: & -\frac{1}{\sqrt{\kappa K_1^2 + \kappa K_2^2 + \kappa^2 L^2}} \arctan\left(\frac{K_1K_2 - \left(K_1^2 + \kappa L^2\right)\tan(\varphi)}{L\sqrt{\kappa K_1^2 + \kappa K_2^2 + \kappa^2 L^2}}\right) \\ \kappa < 0: & \frac{1}{\sqrt{-\kappa K_1^2 - \kappa K_2^2 - \kappa^2 L^2}} \tanh^{-1}\left(\frac{K_1K_2 - \left(K_1^2 + \kappa L^2\right)\tan(\varphi)}{L\sqrt{-\kappa K_1^2 - \kappa K_2^2 - \kappa^2 L^2}}\right) \\ \kappa = 0,\ K_2 \neq 0: & \frac{L\sin(\varphi)}{K_2(K_2\cos(\varphi) - K_1\sin(\varphi))} \\ \kappa = 0,\ K_2 = 0: & -\frac{L}{K_1^2\tan(\varphi)}. \end{cases} \tag{6.15}$$

The last one follows because it is impossible that $K_2 = K_1 = 0$.

6.2 Singularities[1]

Hubble's law, the observed abundance of elements, the cosmic background radiation and the large scale structure formation in the universe are strong evidence that the universe expanded from an initial very high dense state to how we observe it now. However, what happened exactly during this hot density state is still an open problem. One of the questions that needs to be answered is whether there was a singularity at the beginning of spacetime. Such a singularity is defined as a non-spacelike geodesic that is incomplete in the past. We would like to stress that the motivation behind this definition is that test particles move on these trajectories and thus have only traveled for a finite proper time.

The flatness, horizon and magnetic monopole problem can be solved with a period of exponential expansion in the very early universe [35], [36]. To avoid a singularity before that period, it was suggested that one can have past-eternal inflation in which the universe starts from an almost static universe and flows towards a period of exponential expansion. This way the universe would not have a beginning. One of the characteristics of inflationary models is that the Hubble parameter $H = \dot{a}/a$ is positive. In [37] it was shown that when the average Hubble parameter along a geodesic H_{av} is positive, the geodesic is past-incomplete such that we would have a singularity. This is also applicable to models of eternal inflation in which the average Hubble parameter along geodesics does not go to zero sufficiently fast (i.e. such that we do not have that H_{av} is zero). In [38], a model of eternal inflation was given with all non-spacelike geodesics complete, but in [39] these kind of models were shown to be quantum mechanically unstable. Hence, this would imply that also models of eternal inflation start from a singularity.

In [40] it was pointed out that in De Sitter space the test particles that follow those past-incomplete trajectories and have a non-vanishing velocity, will have an energy that becomes arbitrarily large when going back in the past. In this section we generalize this to general Friedmann-Lemaître-Robertson-Walker (FLRW)

[1] The material of this section can also be found in [15]. We only added a few extra comments.

spacetime. It means that the energy of such a test particle can become super-Planckian at some initial time such that their description breaks down. This is the reason one should not consider those trajectories when defining a singularity. When one only considers the trajectories of test particles that do not have a breakdown of the description of their trajectory, one finds that the only FLRW spacetimes that start from a singularity are the ones with a scale factor that vanishes at some initial time. This implies that models of eternal inflation or bouncing models are singularity free provided one requires sub-Planckian test particles at all times.

In this section we first consider the past-(in)completeness of geodesics in spacetimes with an FLRW metric. We review the general singularity theorems of [3] (that can be found in Chapter 5) applied to these models and we review the more general (in the context of cosmology) argument of [37]. After that we consider how the energies of test particles change in time.

6.2.1 Past-(in)completeness of Geodesics

Normalize the scale factor a, such that $a(t_1) = 1$ for a certain time t_1 and let γ be a future-directed geodesic. From Eq. (6.9) we find that

$$\dot{\gamma}^t = \frac{\sqrt{K_1^2 + K_2^2 + \kappa L^2 - \epsilon a^2}}{a} = \frac{\sqrt{|\vec{V}(t_1)|^2 - \epsilon a^2}}{a}, \tag{6.16}$$

where $|\vec{V}|^2 = g_{ij}\dot{\gamma}^i\dot{\gamma}^j$. We thus have a past-incomplete geodesic when

$$\int d\tau = \int_{t_0}^{t} \frac{a}{\sqrt{|\vec{V}(t_1)|^2 - \epsilon a^2}} dt \tag{6.17}$$

for an initial velocity $|\vec{V}(t_1)|$ is finite. Here t_0 is $-\infty$ if $a(t) > 0$ for all t, otherwise $t_0 \in \mathbb{R}$ is taken such that $a(t_0) = 0$ (one can also do this when a blows up at t_0).

Consider first $t_0 \in \mathbb{R}$, then using normalization $\epsilon = -1$

$$\int_{t_0}^{t} \frac{a}{\sqrt{|\vec{V}(t_1)|^2 + a^2}} dt' \leq \int_{t_0}^{t} \frac{a}{\sqrt{a^2}} dt' = t - t_0, \tag{6.18}$$

which is finite. This means these models are always singular, but notice that in this case we can have that all the null geodesics are complete. The null geodesics are also incomplete in case $a(t_0) = 0$. An example when the null geodesics are complete is given by $a(t) = 1/t$ for $t > 0$.

When $t_0 = -\infty$ and the integral (6.17) is converging, we cannot immediately conclude that geodesics are past-incomplete. It is possible that we only consider a part of the actual spacetime manifold. An example is given by $\kappa = 0$, and the Hubble parameter $H = \dot{a}/a$ satisfying $\dot{H}/H^2 = 0$, in which case $a(t) = e^{Ht}$ with H constant. If the whole manifold would be covered by these coordinates, it would result in past-incomplete geodesics. However, this model only describes one half of the larger De Sitter space; the whole space is described by choosing $\kappa = 1$, $a(t) = \cosh(Ht)/H$ which yields complete geodesics. See also [41] and [42].When the integral (6.17) is diverging one can conclude that geodesics in that specific coordinate patch are past-complete. Of course, one can also assume that a certain model with $t_0 = -\infty$ covers the whole spacetime. Then the past-(in)completeness of a geodesic is determined by the integral (6.17). If $a(t) \geq A \in \mathbb{R}_{>0}$ for all $t \leq t_1$ we find that since $a/\sqrt{|\vec{V}(t_1)|^2 + a^2}$ is increasing as function of a, that

$$\frac{a}{\sqrt{|\vec{V}(t_1)|^2 + a^2}} \geq \frac{A}{\sqrt{|\vec{V}(t_1)|^2 + A^2}} \tag{6.19}$$

such that

$$\int_{-\infty}^{t} \frac{a}{\sqrt{|\vec{V}(t_1)|^2 + a^2}} dt' \geq \int_{-\infty}^{t} \frac{A}{\sqrt{|\vec{V}(t_1)|^2 + A^2}} dt' = \infty \tag{6.20}$$

and in the same way null geodesics are complete. This implies that such spacetimes are non-singular. Hence for a spacetime to have a non-spacelike geodesic that is past-incomplete, $a(t)$ needs to become arbitrarily small.

There are a few theorems that prove that an FLRW spacetime contains a (past-)incomplete geodesic. We considered the theorems of Hawking and Penrose ([2], [3]) in Chapter 5 (here we put torsion to 0). These theorems state that when

$$R_{\mu\nu}K^{\mu}K^{\nu} \geq 0 \tag{6.21}$$

for all non-spacelike vectors K and the spacetime obeys a few other conditions such as containing a trapped surface, there is a non-spacelike geodesic that is incomplete. Using Eqs. (1.10) and (1.16), one finds that the non-vanishing components of the Ricci tensor for the metric (6.1) are given by

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a}; \\ R_{11} &= \frac{a\ddot{a}+2\dot{a}^2+2\kappa}{1-\kappa r^2}; \\ R_{22} &= r^2(a\ddot{a}+2\dot{a}^2+2\kappa); \\ R_{33} &= r^2(a\ddot{a}+2\dot{a}^2+2\kappa)\sin^2(\theta). \end{aligned} \tag{6.22}$$

Using these components we find that condition (6.21) yields:

$$\begin{aligned} 0 \leq R_{\lambda\nu}K^{\lambda}K^{\nu} &= -3\frac{\ddot{a}}{a}\left(K^0\right)^2 + \left(a\ddot{a}+2\dot{a}^2+2\kappa\right)\left(\frac{\left(K^1\right)^2}{1-\kappa r^2} + r^2\left(K^2\right)^2 + r^2\sin^2(\theta)\left(K^3\right)^2\right) \\ &= -3\frac{\ddot{a}}{a}\left(K^0\right)^2 + \frac{1}{a^2}\left(a\ddot{a}+2\dot{a}^2+2\kappa\right)\left(K^2 + \left(K^0\right)^2\right) \\ &= \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{\kappa}{a^2}\right)K^2 - 2\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2}\right]\left(K^0\right)^2. \end{aligned} \tag{6.23}$$

We can restrict to K being the tangent vector of null and timelike geodesics. Substitution of Eq. (6.9) in Eq. (6.23) yields

$$\begin{aligned} 0 \leq R_{\lambda\nu}K^{\lambda}K^{\nu} &= \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{\kappa}{a^2}\right)\epsilon - 2\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2}\right]\frac{|\vec{V}(t_1)|^2 - \epsilon a^2}{a^2} \\ &= 3\frac{\ddot{a}}{a}\epsilon - 2\frac{|\vec{V}(t_1)|^2}{a^2}\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2}\right]. \end{aligned} \tag{6.24}$$

This should hold for all timelike and null geodesics, so for all values of $|\vec{V}(t_1)|$ and for $\epsilon \in \{0, -1\}$. For $\epsilon = 0$ it yields

$$\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2} \leq 0 \tag{6.25}$$

and for $\epsilon = -1$ it yields that

$$3\frac{\ddot{a}}{a} \leq -2\frac{|\vec{V}(t_1)|^2}{a^2}\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2}\right]. \tag{6.26}$$

The right-hand side is positive by condition (6.25) and this inequality should hold for all $|\vec{V}(t_1)| \geq 0$ which implies that we have the following two conditions:

$$\begin{aligned} \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2} &\leq 0; \\ \ddot{a} &\leq 0. \end{aligned} \tag{6.27}$$

Notice that for $\kappa \geq 0$ the second condition implies the first, but that this does not happen for $\kappa < 0$. Concluding, the geometric condition of the singularity theorems for this metric reads:

$$\begin{aligned} \kappa \geq 0 &: \ \ddot{a} \leq 0; \\ \kappa < 0 &: \ \begin{cases} \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2} & \leq 0; \\ \ddot{a} & \leq 0. \end{cases} \end{aligned} \tag{6.28}$$

In particular $\ddot{a} \leq 0$ at all time, or that the spacetime is non-accelerating. Notice that when $\ddot{a} \leq 0$, a will always be zero at some time t_0 (this might be in the future), unless a is a positive constant ($H = 0$) in which case we do not have past-incomplete geodesics. Hence, when we want to use the theorems of Hawking and Penrose to say something about an initial singularity in an FLRW spacetime, we need a metric that has a scale parameter a that becomes zero at some time in the past.

Describing the matter content of the universe by a perfect fluid

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}, \tag{6.29}$$

where p is the pressure, ρ the energy density and $U^\mu = (1, 0, 0, 0)$, the condition (6.28) translates via the Friedmann equations (Einstein equation for this metric),

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G_\mathrm{N}}{3}\rho - \frac{\kappa}{a^2}; \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G_\mathrm{N}}{3}(\rho + 3p), \end{aligned} \tag{6.30}$$

to

$$\begin{aligned} \kappa \geq 0 &: \ \rho + 3p \geq 0; \\ \kappa < 0 &: \ \begin{cases} \rho + p & \geq 0; \\ \rho + 3p & \geq 0. \end{cases} \end{aligned} \tag{6.31}$$

Although it seems that for $\kappa \geq 0$ we have less restrictions, it is actually impossible to have an FLRW spacetime with non-negative spatial curvature and $\rho + 3p \geq 0$ and $\rho + p < 0$, because from Eq. (6.30) we see that we need $\rho \geq 0$ for $\kappa \geq 0$. In Fig. 6.1 one finds an illustration of condition (6.31).

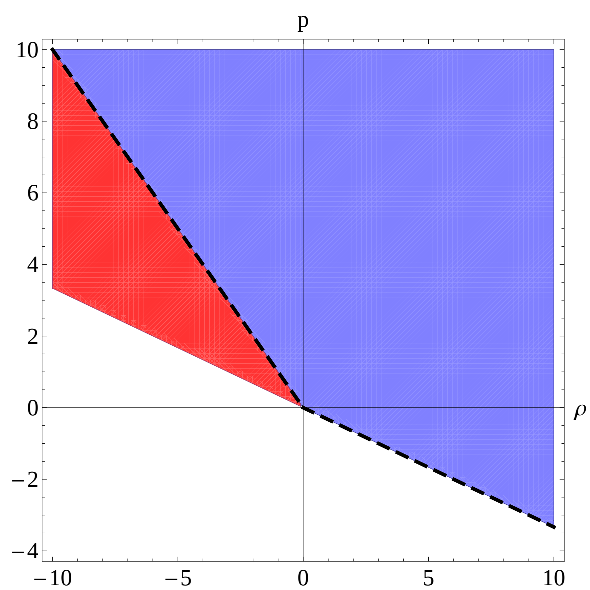

Figure 6.1: Illustration of condition (6.31). For $\kappa < 0$ one needs (ρ, p) in the blue area above the dashed line to apply the Hawking-Penrose singularity theorems. For $\kappa \geq 0$, we have less restrictions, the red shaded area below the dashed line is also included, but it is impossible for an FLRW spacetime with non-negative spatial curvature to be in that area.

Another theorem that proves that a geodesic is past-incomplete was published in [37] and is also applicable to spacetimes that have $a(t) > 0$ for all t. It says that when the average Hubble parameter $H = \dot{a}/a$ along a non-spacelike geodesic, H_av, satisfies $H_\mathrm{av} > 0$, the geodesic must be past-incomplete. For the metric (6.1), the argument is as follows. Consider a non-spacelike geodesic $\gamma(\tau)$ between an initial point $\gamma(\tau_\mathrm{i})$ and a final point $\gamma(\tau_\mathrm{f})$. We can integrate H along the geodesic, using Eq. (6.16):

$$
\begin{aligned}
\int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}} H d\tau &= \int_{t_\mathrm{i}}^{t_\mathrm{f}} \frac{\dot{a}}{\sqrt{|\vec{V}(t_1)|^2 - \epsilon a^2}} dt \\
&= \int_{a(t_\mathrm{i})}^{a(t_\mathrm{f})} \frac{da}{\sqrt{|\vec{V}(t_1)|^2 - \epsilon a^2}} \\
&= \begin{cases} \frac{1}{|\vec{V}(t_1)|}\left[a(t_\mathrm{f}) - a(t_\mathrm{i})\right], & \epsilon = 0 \\ \log\left(\frac{a(t_\mathrm{f}) + \sqrt{|\vec{V}(t_1)|^2 + a(t_\mathrm{f})^2}}{a(t_\mathrm{i}) + \sqrt{|\vec{V}(t_1)|^2 + a(t_\mathrm{i})^2}}\right) & \epsilon = -1 \end{cases} \\
&\leq \begin{cases} \frac{a(t_\mathrm{f})}{|\vec{V}(t_1)|}, & \epsilon = 0 \\ \log\left(\frac{a(t_\mathrm{f}) + \sqrt{|\vec{V}(t_1)|^2 + a(t_\mathrm{f})^2}}{|\vec{V}(t_1)|}\right) & \epsilon = -1. \end{cases}
\end{aligned}
\tag{6.32}
$$

Notice that for the second equality sign, one should break up the integration domain into parts where $a = a(t)$ is injective, but that one will end up with the same result. Hence, this integral as function of the initial affine parameter τ_i is restricted by some fixed final τ_f. This means that when

$$
H_\mathrm{av} = \frac{1}{\tau_\mathrm{f} - \tau_\mathrm{i}} \int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}} H d\tau > 0 \tag{6.33}
$$

τ_i has to be some finite value such that the geodesic is past-incomplete. Notice, that it is still possible to construct an FLRW spacetime that has $H > 0$ at all times and complete geodesics. For this we need that H_av must become zero when $\tau_\mathrm{i} \to -\infty$. Examples are for instance given by spacetimes with $H > 0$ and $a \to a_0 > 0$ for $t \to -\infty$ (in this case we will have that $H \to 0$ as $t \to -\infty$).

6.2.2 Energy of Test Particles

As stated before, the definition of a singularity is based on the trajectories of massive test particles and massless particles. For cosmological spacetimes with an FLRW metric, we would like to study the energies of test particles over time. We will generalize the argument given in [40] for De Sitter space to a general FLRW spacetime.

Using Eq. (6.16) we find that for massive test particles

$$
|\vec{V}|^2 \equiv g_{ij}\dot{\gamma}^i\dot{\gamma}^j = \epsilon + \left(\dot{\gamma}^0\right)^2 = \frac{|\vec{V}(t_1)|^2}{a^2}. \tag{6.34}
$$

We already saw that in order for a spacetime to have a past-incomplete non-spacelike geodesic, the scale parameter a needs to become arbitrarily small. With Eq. (6.34) this then implies that when the particle has a velocity $|\vec{V}(t_1)|$ at time t_1, the velocity and hence the energy $E^2 = m^2\left(1 + \frac{|\vec{V}(t_1)|^2}{a^2}\right)$ of a test particle with mass m becomes arbitrarily large when moving back to the past.

The statement above for massive test particles carries over to photons. In this case the angular frequency as observed by a comoving observer goes as

$$
\omega = \dot{\gamma}^0 = \frac{\omega(t_1)}{a}. \tag{6.35}
$$

Thus also the energy of photons $E = \hbar\omega$ will become arbitrarily large when moving back to the past.

In [40] it was noted that one cannot have particles with arbitrarily high energies because if such a particle has a nonvanishing interaction cross section with any particle with a non-zero physical number density, then the particle will interact with an infinite number of them, breaking the Cosmological principle. However, the particle's energy cannot become arbitrarily high because it will reach the Planck energy $E_\mathrm{P} = \sqrt{\frac{\hbar}{G_\mathrm{N}}} \approx$

$1.22 \cdot 10^{19}$ GeV at some time t. With this energy, the particle's Compton wavelength is approximately equal to its Schwarzschild radius such that it will form a black hole. Therefore, the description of the particle's trajectory will break down. Scattering processes involving vacuum fluctuations may cause the test particle's energy to never reach the Planck energy. If these processes are significant the particle's trajectory is not a geodesic anymore. Near the Planck energy scattering processes are dominated by processes that involve the exchange of a graviton [43]. To estimate this effect we consider photon-photon scattering with the exchange of a graviton. We model the loss of energy of the photon when going back in time as

$$\frac{d}{dt}E = (-H - \sigma n)\, E, \tag{6.36}$$

where n is the number density of virtual photons and σ is the cross section of the scattering process. The particle gains energy from the expansion of the universe because $-H$ is positive (when going back in time) and it looses energy from the scattering with virtual photons. We estimate the density of virtual photons as one per Hubble volume:

$$n = \frac{1}{V_{\mathrm{H}}} = -\frac{3H^3}{4\pi}. \tag{6.37}$$

The differential cross section for photon-photon scattering with the exchange of a graviton for unpolarized photons is [44]

$$\frac{d\sigma}{d\Omega} = \frac{(16\pi G_{\mathrm{N}})^2}{8\pi^2}\frac{k^2}{\sin^2(\theta)}\left[1 + \cos^{16}\left(\frac{1}{2}\theta\right) + \sin^{16}\left(\frac{1}{2}\theta\right)\right] \tag{6.38}$$

where k is the momentum of the photon and θ is the scattering angle. Since we are primarily interested in large momentum exchange, we neglect small angle scatterings when calculating the total cross section of this process:

$$\begin{aligned}
\sigma &= \int \frac{d\sigma}{d\Omega} d\Omega \\
&= \frac{(16\pi G_{\mathrm{N}})^2}{\pi}\frac{k^2}{4}\int_{-1+\xi}^{1-\xi} \frac{1 + \frac{1}{256}(1+x)^8 + \frac{1}{256}(1-x)^8}{1-x^2} dx \\
&= \frac{(16\pi G_{\mathrm{N}})^2}{\pi}\frac{k^2}{2}\int_{\xi}^{1} \frac{1 + \frac{1}{256}(2-y)^8 + \frac{1}{256}y^8}{y(2-y)} dy \\
&= \frac{(16\pi G_{\mathrm{N}})^2}{\pi}\frac{k^2}{4}\left[2\log\frac{1}{\xi} - \frac{363}{140} + \log(4) + \mathcal{O}(\xi)\right],
\end{aligned} \tag{6.39}$$

where we have the relation $\sin(\theta/2) = \sqrt{\xi/2}$. Taking only angles $.26\pi < \theta < .74\pi$ into account for the scattering, we have that $2\log\frac{1}{\xi} - \frac{363}{140} + \log(4) \approx 1$. With Eqs. (6.36), (6.37) and (6.39) we find that the energy of the test photon does not increase when

$$H \sim \sigma n = 48 G_{\mathrm{N}}^2 E^2 H^3, \tag{6.40}$$

where $E = k$ is the photon energy. Using the Hubble parameter of cosmic inflation which typically is about $-\hbar H \approx 10^{13}$ GeV, we find from (6.40) that the scattering process becomes significant when

$$\left(\frac{E}{E_{\mathrm{P}}}\right)^2 \sim \frac{E_{\mathrm{P}}^2}{48\hbar^2 H^2} \approx 10^{10}. \tag{6.41}$$

Hence, processes involving gravitons will not cause the particle's energy to stay smaller than the Planck energy and a black hole will form. This implies that the description of the particle's trajectory (as a geodesic) breaks down, either because of interaction processes or by the formation of a black hole. The latter definitely happens when the initial energy is near the Planck energy.

Up to now, the maximum energy of a single particle that has been measured is of the order of 10^{20} eV [45] which is eight orders of magnitude smaller than the Planck scale. These particles were all cosmic ray particles, so their probable origin is a supernova, an active galactic nucleus, a quasar or a gamma ray-burst.

Even when using this energy as an upper bound for the energy of test particles, we have that the description of the trajectories of non-comoving test particles breaks down at times that are certainly later than the Planck era, the period where we have to take quantum gravitational effects into account. In [40] the arbitrarily high energies of test particles were used to argue that these particles should be forbidden in De Sitter space. This can be done by using a different time arrow in the two patches of De Sitter space that one has in the flat slicing. That way the two coordinate patches become non-communicating and describe eternally inflating spacetimes. We will not look into these kind of constructions for general FLRW spacetimes but we want to use the arbitrarily high energies of test particles to give a consistent definition of a singularity. When the particle's description breaks down before it reaches the beginning of its trajectory, it is not very useful to use that particle as an indication for an initial singularity. That is the reason why we suggest to define a singularity in spacetimes with an FLRW metric that has a parameter a that becomes arbitrarily small, as a timelike geodesic with $|\vec{V}(t_1)| = 0$ that is past-incomplete. For such trajectories, we have that $dt = d\tau$ which means that a spacetime has no initial singularity when $a(t) > 0$ for all $t \in \mathbb{R}$. Hence, an FLRW spacetime starts from a singularity precisely when $a(t_0) = 0$ at some initial finite time t_0.

6.3 Relation between Conjugate Points and Singularities

Since the relation between the occurrence of conjugate points and singularities is evident in the singularity theorems, we would like to study these conjugate points in FLRW spacetime. It is sometimes stated that in this spacetime the singularity is equivalent to the occurrence of conjugate points. In this section we will first change the definition of conjugate points a bit motivated by an example. After that we will prove a couple of theorems about a singularity implying conjugate points and vice versa for certain FLRW models.

Let us first start to examine a specific model. We will study the FLRW metric with $\kappa = 0$ and $a(t) = \sqrt{t}$. This models a universe with a perfect radiation fluid for which $\rho \propto 1/a^4$. To derive the geodesics we use Cartesian coordinates for the metric (6.1)

$$ds^2 = -dt^2 + a(t)^2 \left(dx^i\right)^2 . \tag{6.42}$$

Let a geodesic be given by $\gamma(\tau) = \big(t(\tau), x^i(\tau)\big)$ and let $u^\mu = d\gamma^\mu/d\tau$. The geodesic equations are given by

$$\begin{aligned} \frac{du^0}{d\tau} + a\dot{a}\left(u^i\right)^2 &= 0 \\ \frac{du^i}{d\tau} + 2\frac{\dot{a}}{a}u^0u^i &= 0. \end{aligned} \tag{6.43}$$

The second equation can be rewritten as

$$\frac{d}{d\tau}\left[a^2u^i\right] = 0 \tag{6.44}$$

with solution

$$u^i = \frac{C_i}{a^2}, \tag{6.45}$$

where C_i are constants. The constraint equation is

$$\epsilon = -\left(u^0\right)^2 + a^2\left(u^i\right)^2, \tag{6.46}$$

so

$$u^0 = \sqrt{-\epsilon + a^2\left(u^i\right)^2} = \sqrt{-\epsilon + \frac{C}{a^2}} = \frac{\sqrt{C - \epsilon a^2}}{a}, \tag{6.47}$$

where $C = \sum_i C_i^2$ (this corresponds with expression (6.9)). Let us now consider timelike geodesics, $\epsilon = -1$ and choose $C = 1$. We can then solve Eq. (6.47) for $a(t) = \sqrt{t}$ by

$$\sqrt{t + t^2} - \sinh^{-1}\left(\sqrt{t}\right) = \tau, \tag{6.48}$$

where we chose τ such that $\tau = 0$ at the singularity. From Eq. (6.45) and Eq. (6.47) we find that

$$\frac{dx^i}{dt} = \frac{1}{\sqrt{1+a^2}} \frac{C_i}{a}, \tag{6.49}$$

which is solved by

$$x^i \;=\; 2C_i \sinh^{-1}\left(\sqrt{t}\right) + D_i, \tag{6.50}$$

where D_i are constants (notice that we have the restriction $1 = \sum_i C_i^2$). We consider the geodesic

$$\gamma = \left(t, 2\sinh^{-1}\left(\sqrt{t}\right), 0, 0\right) \tag{6.51}$$

and we want to examine conjugate points along this geodesic. In the spirit of Chapter 2 we construct the matrix A of Eq. (2.39) corresponding to the point $t_1 = 10$ for this geodesic. An orthonormal basis that is parallel transported along this geodesic is given by

$$\begin{aligned}
E_0 &= \left(\frac{\sqrt{1+a(t)^2}}{a(t)}, \frac{1}{a(t)^2}, 0, 0\right) = \left(\sqrt{\frac{1+t}{t}}, \frac{1}{t}, 0, 0\right) \\
E_1 &= \left(\frac{1}{a(t)}, \frac{\sqrt{1+a(t)^2}}{a(t)^2}, 0, 0\right) = \left(\frac{1}{\sqrt{t}}, \frac{\sqrt{1+t}}{t}, 0, 0\right) \\
E_2 &= \left(0, 0, \frac{1}{a(t)}, 0\right) = \left(0, 0, \frac{1}{\sqrt{t}}, 0\right) \\
E_3 &= \left(0, 0, 0, \frac{1}{a(t)}\right) = \left(0, 0, 0, \frac{1}{\sqrt{t}}\right).
\end{aligned} \tag{6.52}$$

We now need the Jacobi fields J_i for $i \in \{1, 2, 3\}$ such that $J_i(t_1) = 0$ and $D_\tau J_i(t_1) = E_i(t_1)$. The differential equations for the first 2 components of the Jacobi fields only depend on each other. The differential equation for J_i^k, $k \in \{2, 3\}$ is given by

$$\frac{(1+2t)\left(J_i^k\right)' + 2t(1+t)\left(J_i^k\right)''}{t} = 0. \tag{6.53}$$

This implies that

$$\begin{aligned}
J_1 &= (h_1(t), h_2(t), 0, 0) \\
J_2 &= (0, 0, h_3(t), 0) \\
J_3 &= (0, 0, 0, h_3(t)).
\end{aligned} \tag{6.54}$$

We can solve for h_3 explicitly and find

$$h_3(t) = -2\sqrt{10}\left(\sinh^{-1}\left(\sqrt{10}\right) - \sinh^{-1}\left(\sqrt{t}\right)\right). \tag{6.55}$$

We have to solve for h_1 and h_2 numerically. We can use the function $g(J_1, E_0)$ to see how good this numerical solution is since this function should be identically 0. The matrix A is then given by

$$A = \begin{pmatrix} -\frac{1}{\sqrt{t}}h_1 + \sqrt{1+t}h_2 & 0 & 0 \\ 0 & \sqrt{t}h_3 & 0 \\ 0 & 0 & \sqrt{t}h_3 \end{pmatrix}, \tag{6.56}$$

which has determinant

$$\det A = t\left(-\frac{1}{\sqrt{t}}h_1 + \sqrt{1+t}h_2\right)h_3^2. \tag{6.57}$$

Notice that at $t = 0$, $\sqrt{t}h_3(t) = 0$, which means that $\gamma(10)$ is conjugate to the point at the singularity. However, we do not have a Jacobi field that vanishes in the usual sense. Indeed from Eq. (6.55) we see that this Jacobi field does not vanish when we use another coordinate system. The norm of the Jacobi field however, is zero in both coordinate systems. This strange behavior is caused because the metric is not well defined at the singularity.

We use this example as a motivation to generalize the definition of conjugate points. From a physical point of view it is the norm of the Jacobi field that matters since this corresponds to the distance between particles moving on nearby geodesics. That is why we will also say that we have conjugate points on a timelike geodesic when the norm of the Jacobi field vanishes. Such a Jacobi field should in principle still be perpendicular to the geodesic (otherwise one could just get that it is a null vector). As long as the metric is well defined this definition is the same as our original definition. Notice that the vanishing of the determinant of the matrix A is equivalent to a Jacobi field J perpendicular to $\dot{\gamma}$ and such that $g(E_i, J) = 0$ for all i. From

$$g(J, J) = \sum_i g(E_i, J)^2 \tag{6.58}$$

we conclude that $g(E_i, J) = 0$ for all i is equivalent to $g(J, J) = 0$.

With this new definition two points on a geodesic can be conjugate in two different ways. The first one is that geodesics are indeed converging to one point (to first order), the second one is that that does not happen, but that the norm of the Jacobi field vanishes. We found that the vanishing of the determinant of A is equivalent to this new definition if the Jacobi field is perpendicular to the geodesic. In the same way one can give a generalized definition of conjugate points for null geodesics.

6.3.1 A Singularity Implies Conjugate Points

We will now first prove a theorem that states that under a certain condition on $a(t)$, every point on a geodesic is conjugate to the point of the geodesic at the singularity. Here we do not know whether geodesics actually converge to that point.

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Theorem 6.3.1. *Let $\gamma(\tau(t))$ be a non-comoving geodesic (such that $C \equiv K_1^2 + K_2^2 + \kappa L^2 \neq 0$) in a spacetime with FLRW metric such that $a(t_0) = 0$ for a certain t_0 and a is smooth for $t > t_0$. Let*

$$f(t) = 3\ddot{a} + 2\frac{C}{a}\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2}\right] \tag{6.59}$$

and define

$$f_+(t) = \begin{cases} f(t) & \text{for } t \text{ where } f(t) \geq 0 \\ 0 & \text{for } t \text{ where } f(t) < 0 \end{cases} \tag{6.60}$$

$$f_-(t) = \begin{cases} -f(t) & \text{for } t \text{ where } f(t) \leq 0 \\ 0 & \text{for } t \text{ where } f(t) > 0. \end{cases}$$

Every point $\gamma(\tau(t))$ for $t \neq t_0$ is conjugate to $\gamma(\tau(t_0))$ if the following conditions are satisfied:

$$\lim_{t \to t_0} \int_t^{t_1} a(t') \int_{t'}^{t_1} \frac{1}{a}\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2}\right] dt'' dt' = -\infty \tag{6.61}$$

$$\lim_{t \to t_0} \int_t^{t_1} f_+ dt' = \alpha \in \mathbb{R}_{\geq 0} \tag{6.62}$$

for a $t_1 > t_0$.

Proof. Let γ be a timelike geodesic. Let $\gamma(\tau(t_2))$ be a point on this geodesic and let A denote the corresponding matrix that describes the Jacobi fields that vanish at this point (Eq. (2.39)). To show that $\gamma(\tau(t_2))$ is conjugate to $\gamma(\tau(t_0))$, we will show that $\lim_{t \to t_0} \det A = 0$, or equivalently $\lim_{t \to t_0} \log(\det A) = -\infty$. Unfortunately, we cannot use Lemma 2.1.2, because the metric is not well-defined at t_0. However we can use that

$$\theta_A = \partial_\tau \log(\det A). \tag{6.63}$$

Consider the Raychaudhuri equation, Eq. (2.67) and use Eq. (6.24) to obtain

$$\begin{aligned} \frac{d\theta_A}{dt} = \frac{d\tau}{dt}\frac{d\theta_A}{d\tau} &= \frac{a}{\sqrt{C+a^2}}\left(3\frac{\ddot{a}}{a} + 2\frac{C}{a^2}\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2}\right] - \mathrm{tr}\left(\sigma_A^2\right) - \frac{\theta_A^2}{3}\right) \\ &\leq \frac{1}{\sqrt{C+a^2}}\left(3\ddot{a} + 2\frac{C}{a}\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2}\right]\right) \\ &= \frac{1}{\sqrt{C+a^2}}\left(f_+ - f_-\right). \end{aligned} \tag{6.64}$$

We find from condition (6.61) that

$$\begin{aligned} \lim_{t\to t_0}\int_t^{t_1} a\int_{t'}^{t_1}\left(3\ddot{a} + 2\frac{C}{a}\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2}\right]\right)dt''dt' &= 3\dot{a}(t_1)\int_{t_0}^{t_1} a dt - 3\lim_{t\to t_0}\int_t^{t_1} a\dot{a}dt' \\ &\quad +2C\lim_{t\to t_0}\int_t^{t_1} a\int_{t'}^{t_1}\frac{1}{a}\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2}\right]dt''dt' \\ &= 3\dot{a}(t_1)\int_{t_0}^{t_1} a dt - \frac{3}{2}a(t_1)^2 \\ &\quad +2C\lim_{t\to t_0}\int_t^{t_1} a\int_{t'}^{t_1}\frac{1}{a}\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2}\right]dt''dt' \\ &= -\infty. \end{aligned} \tag{6.65}$$

Hence:

$$-\infty = \lim_{t\to t_0}\int_t^{t_1} a\int_{t'}^{t_1}\left(f_+ - f_-\right)dt''dt' = \lim_{t\to t_0}\int_t^{t_1} a\int_{t'}^{t_1} f_+ dt''dt' - \lim_{t\to t_0}\int_t^{t_1} a\int_{t'}^{t_1} f_- dt''dt'. \tag{6.66}$$

From condition (6.62) it follows that

$$\int_t^{t_1} f_+ dt' \tag{6.67}$$

is a function that is α at t_0, 0 at t_1 and strictly decreasing. Hence

$$a(t)\int_t^{t_1} f_+ dt' \tag{6.68}$$

is vanishing at t_0 and t_1 and continuous and positive in between. This implies that

$$\lim_{t\to t_0}\int_t^{t_1} a\int_{t'}^{t_1} f_+ dt''dt' = \beta \in \mathbb{R}_{\geq 0} \tag{6.69}$$

and together with Eq. (6.66) this gives

$$\lim_{t\to t_0}\int_t^{t_1} a\int_{t'}^{t_1} f_- dt''dt' = \infty. \tag{6.70}$$

Choose $t_3 \leq t_1$ such that $\gamma(t_3)$ is not conjugate to $\gamma(t_2)$. Then for $t < t_3$ we find with Eq. (6.64) that

$$\begin{aligned} \theta_A(t) &= -\int_{\theta_A(t)}^{\theta_A(t_3)} d\theta_A + \theta_A(t_3) \\ &\geq -\int_t^{t_3}\frac{1}{\sqrt{C+a^2}}\left(f_+ - f_-\right)dt' + \theta_A(t_3) \\ &\geq -\frac{1}{\sqrt{C}}\int_t^{t_3} f_+ dt' + \frac{1}{\sqrt{C+a_{\max}^2}}\int_t^{t_3} f_- dt' + \theta_A(t_3), \end{aligned} \tag{6.71}$$

where $a_{\max} = \max\{a(t)|t_0 \leq t \leq t_3\}$. Then

$$\begin{aligned}
\log\left(\det A(t)\right) &= -\int_t^{t_3} \frac{a}{\sqrt{C+a^2}} \theta_A dt' + \log\left(\det A(t_3)\right) \\
&\leq \frac{1}{\sqrt{C}} \int_t^{t_3} \frac{a}{\sqrt{C+a^2}} \int_{t'}^{t_3} f_+ dt'' dt' - \frac{1}{\sqrt{C+a_{\max}^2}} \int_t^{t_3} \frac{a}{\sqrt{C+a^2}} \int_{t'}^{t_3} f_- dt'' dt' \\
&\quad -\theta_A(t_3) \int_t^{t_3} \frac{a}{\sqrt{C+a^2}} dt' + \log\left(\det A(t_3)\right) \\
&\leq \frac{1}{C} \int_t^{t_3} a \int_{t'}^{t_3} f_+ dt'' dt' - \frac{1}{C+a_{\max}^2} \int_t^{t_3} a \int_{t'}^{t_3} f_- dt'' dt' \\
&\quad -\theta_A(t_3) \int_t^{t_3} \frac{a}{\sqrt{C+a^2}} dt' + \log\left(\det A(t_3)\right). \qquad (6.72)
\end{aligned}$$

With Eqs. (6.69), (6.70) and (6.72) (notice that the integral of $f_\pm$ over the interval $[t_3, t_1]$ is finite) it then follows that

$$\begin{aligned}
\lim_{t\to t_0} \log\left(\det A(t)\right) &\leq \frac{1}{C} \lim_{t\to t_0} \int_t^{t_3} a \int_{t'}^{t_3} f_+ dt'' t'' - \frac{1}{C+a_{\max}^2} \lim_{t\to t_0} \int_t^{t_3} a \int_{t'}^{t_3} f_- dt'' dt' \\
&\quad -\theta_A(t_3) \int_{t_0}^{t_3} \frac{a}{\sqrt{C+a^2}} dt' + \log\left(\det A(t_3)\right) = -\infty. \qquad (6.73)
\end{aligned}$$

That means that $\gamma(\tau(t_0))$ is conjugate to $\gamma(\tau(t_2))$ along γ.

Consider now a null geodesic γ. The Raychaudhuri equation, Eq. (2.113), reads:

$$\begin{aligned}
\frac{d\bar{\theta}_{\bar{A}}}{dt} &= \frac{a}{\sqrt{C}} \left(2\frac{C}{a^2} \left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2} \right] - \operatorname{tr}\left(\bar{\sigma}_{\bar{A}}^2\right) - \frac{\bar{\theta}_{\bar{A}}^2}{2} \right) \\
&\leq 2\frac{\sqrt{C}}{a} \left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2} \right]. \qquad (6.74)
\end{aligned}$$

Choosing again a $t_3 \leq t_1$ such that $\gamma(t_3)$ is not conjugate to $\gamma(t_2)$, it follows that for $t < t_3$

$$\begin{aligned}
\bar{\theta}_{\bar{A}}(t) &= -\int_{\bar{\theta}_{\bar{A}}(t)}^{\bar{\theta}_{\bar{A}}(t_3)} d\theta_A + \bar{\theta}_{\bar{A}}(t_3) \\
&\geq -2\sqrt{C} \int_t^{t_3} \frac{1}{a} \left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2} \right] dt' + \bar{\theta}_{\bar{A}}(t_3). \qquad (6.75)
\end{aligned}$$

This then implies that

$$\begin{aligned}
\log\left(\det \bar{A}(t)\right) &= -\int_t^{t_3} \frac{a}{\sqrt{C}} \bar{\theta}_{\bar{A}} dt' + \log\left(\det \bar{A}(t_3)\right) \\
&\leq 2\int_t^{t_3} a \int_{t'}^{t_3} \frac{1}{a} \left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2} \right] dt'' dt' - \bar{\theta}_{\bar{A}}(t_3) \frac{1}{\sqrt{C}} \int_t^{t_3} a dt' + \log\left(\det \bar{A}(t_3)\right). \quad (6.76)
\end{aligned}$$

Condition (6.61) then implies

$$\lim_{t\to t_0} \log\left(\det \bar{A}(t)\right) \leq 2 \lim_{t\to t_0} \int_t^{t_3} a \int_{t'}^{t_3} \frac{1}{a} \left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2} \right] dt'' dt' - \bar{\theta}_{\bar{A}}(t_3) \frac{1}{\sqrt{C}} \int_{t_0}^{t_3} a dt' + \log\left(\det \bar{A}(t_3)\right) = -\infty. \qquad (6.77)$$

Hence $\lim_{t\to t_0} \log\left(\det \bar{A}(t)\right) = -\infty$ such that $\gamma(\tau(t_0))$ is conjugate to every point on γ along this geodesic. □

Notice that one can rewrite condition (6.61) by partially integrating the first term such that one obtains

$$\begin{aligned} -\infty &= \lim_{t\to t_0}\int_t^{t_1} a(t')\int_{t'}^{t_1}\frac{1}{a}\left[\frac{\ddot a}{a}-\frac{\dot a^2}{a^2}-\frac{\kappa}{a^2}\right]dt''dt' \\ &= \lim_{t\to t_0}\int_t^{t_1} a(t')\int_{t'}^{t_1}\frac{1}{a}\left[\frac{d}{dt}H-\frac{\kappa}{a^2}\right]dt''dt' \\ &= \frac{\dot a(t_1)}{a^2(t_1)}\int_{t_0}^{t_1} a dt' - \lim_{t\to t_0}\int_t^{t_1}\frac{\dot a}{a}dt' + \lim_{t\to t_0}\int_t^{t_1} a(t')\int_{t'}^{t_1}\frac{\dot a^2-\kappa}{a^3}dt''dt' \\ &= \frac{\dot a(t_1)}{a^2(t_1)}\int_{t_0}^{t_1} a dt' - \log(a(t_1)) + \lim_{t\to t_0}\log(a(t)) + \lim_{t\to t_0}\int_t^{t_1} a(t')\int_{t'}^{t_1}\frac{\dot a^2-\kappa}{a^3}dt''dt', \end{aligned} \tag{6.78}$$

such that condition (6.61) is definitely satisfied when

$$\lim_{t\to t_0}\int_t^{t_1} a\int_{t'}^{t_1}\frac{\dot a^2-\kappa}{a^3}dt''dt' \tag{6.79}$$

is not ∞.

Also condition (6.62) is satisfied as soon as f is negative for $t\in(t_0,t_0+\epsilon)$.

Theorem 6.3.1 can also be proven under different conditions. One set of such conditions would be for instance

$$\begin{aligned} \lim_{t\to t_0}\int_t^{t_1}\frac{a}{\sqrt{C+a^2}}\int_{t'}^{t_1}\frac{1}{\sqrt{C+a^2}}\left(3\ddot a+2\frac{C}{a}\left[\frac{\ddot a}{a}-\frac{\dot a^2}{a^2}-\frac{\kappa}{a^2}\right]\right)dt''dt' &= -\infty; \\ \lim_{t\to t_0}\int_t^{t_1} a\int_{t'}^{t_1}\frac{1}{a}\left[\frac{\ddot a}{a}-\frac{\dot a^2}{a^2}-\frac{\kappa}{a^2}\right]dt''dt' &= -\infty \end{aligned} \tag{6.80}$$

which would have made the proof really easy.

Theorem 6.3.1 is applicable to FLRW spacetimes with physical realistic scale factors. Take for instance $a(t)=t^{1/n}$ for $n>1$ ($n=2$ corresponds to the scale factor of an FLRW spacetime with $\kappa=0$ and a perfect homogeneous radiation fluid and $n=3/2$ to $\kappa=0$ and a perfect homogeneous matter fluid). We then find that for $n\neq 3$

$$\begin{aligned} \int_t^{t_1} a\int_{t'}^{t_1}\frac{1}{a}\left[\frac{\ddot a}{a}-\frac{\dot a^2}{a^2}-\frac{\kappa}{a^2}\right]dt''dt' &= \int_t^{t_1} t'^{1/n}\int_{t'}^{t_1}\left[-\frac{1}{n}\frac{1}{t^{1/n+2}}-\frac{\kappa}{t^{3/n}}\right]dt''dt' \\ &= \int_t^{t_1} t'^{1/n}\left[\frac{1}{1+n}\frac{1}{t^{1/n+1}}+\frac{n}{3-n}\frac{\kappa}{t^{3/n-1}}\right]_{t'}^{t_1}dt' \\ &= \alpha\int_t^{t_1} t'^{1/n}dt' - \int_t^{t_1}\left(\frac{1}{1+n}\frac{1}{t}+\frac{n}{3-n}\frac{\kappa}{t^{2/n-1}}\right)dt' \\ &= \alpha\frac{n}{1+n}\left(t_1^{1/n+1}-t^{1/n+1}\right)-\left[\frac{1}{1+n}\log t+\frac{n}{3-n}\frac{n}{2n-2}\frac{\kappa}{t^{2/n-2}}\right]_t^{t_1}, \end{aligned} \tag{6.81}$$

where

$$\alpha=\frac{1}{1+n}\frac{1}{t_1^{1/n+1}}+\frac{n}{3-n}\frac{\kappa}{t_1^{3/n-1}}. \tag{6.82}$$

We find that this scale factor obeys condition (6.61). Similarly, one can show that condition (6.61) is satisfied for $n=3$.

Also

$$3\ddot a+2\frac{C}{a}\left[\frac{\ddot a}{a}-\frac{\dot a^2}{a^2}-\frac{\kappa}{a^2}\right]=3\frac{1}{n}(\frac{1}{n}-1)\frac{1}{t^{-1/n+2}}-2C\left(\frac{1}{n}\frac{1}{t^{1/n+2}}+\frac{\kappa}{t^{3/n}}\right) \tag{6.83}$$

which goes to $-\infty$ when $t\to 0$ and that implies that condition (6.62) is satisfied.

Notice that when $a(t) = t$

$$\begin{aligned}\int_t^{t_1} a \int_{t'}^{t_1} \frac{1}{a}\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2}\right] dt'' dt' &= -\int_t^{t_1} t' \int_{t'}^{t_1} \frac{1+\kappa}{t^3} dt'' dt' = \frac{1}{2}\int_t^{t_1} t' \left[\frac{1+\kappa}{t^2}\right]_{t'}^{t_1} dt' \\ &= \frac{1}{4}\frac{1+\kappa}{t_1^2}\left(t_1^2 - t^2\right) - \frac{1}{2}\left(1+\kappa\right)\left(\log t_1 - \log(t)\right), \end{aligned} \tag{6.84}$$

such that condition (6.61) is not satisfied for $\kappa \leq -1$.

We will now give a more general proof when $\kappa \leq 0$. For that we first need to examine when a timelike non-comoving geodesic $\gamma(\tau)$ has a well defined point $\gamma(\tau_0)$ at the singularity. We observe from Eq. (6.12) that for $\kappa \leq 0$, $r \to \infty$ at a certain angle. This is because

$$\left(K_2 \cos(\varphi) - K_1 \sin(\varphi)\right)^2 = -\kappa L^2, \tag{6.85}$$

will be satisfied at two angles φ (when $-\kappa L^2 = 0$ only for one angle, but that does not matter for the argument), because the left-hand side takes values between 0 and $K_1^2 + K_2^2$, and

$$K_1^2 + K_2^2 = \frac{a^4}{1-\kappa r^2}\left(\dot{\gamma}^r\right)^2 + \frac{L^2}{r^2} - \kappa L^2 \tag{6.86}$$

which means that

$$0 \leq -\kappa L^2 \leq K_1^2 + K_2^2. \tag{6.87}$$

The divergence of r at a finite angle implies that $\lim_{\tau\to\tau_0} \varphi(\tau)$ is converging because from Eq. (6.4) we see that φ is monotonously increasing or decreasing and φ is bounded by the value where $r \to \infty$. This means that $\gamma(\tau_0)$ is a well defined point, precisely when r does not diverge for $\tau \to \tau_0$.

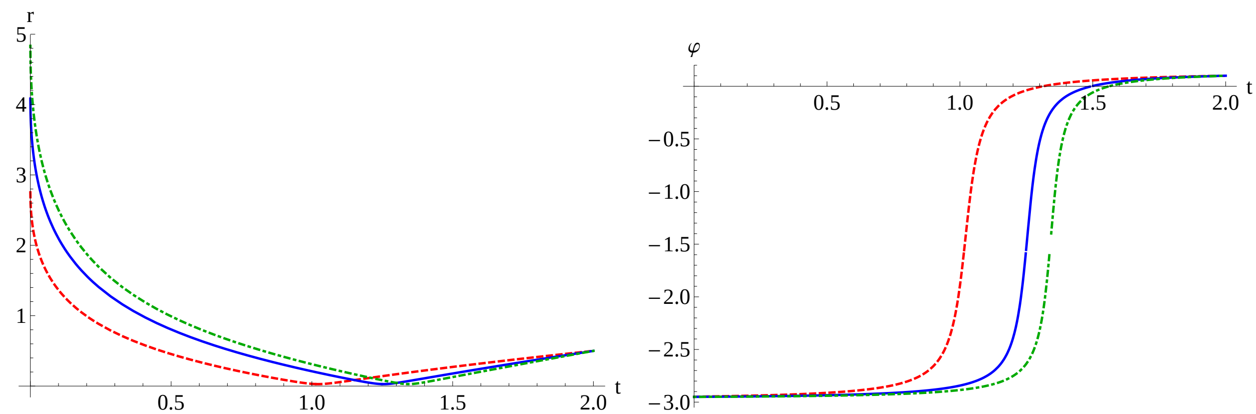
Theorem 6.3.2. *Let $\gamma(\tau)$ be a timelike non-comoving geodesic in a spacetime with FLRW metric with $\kappa \leq 0$ such that $a(t_0) = 0$ for a certain t_0. If $\gamma(\tau_0) = \gamma(\tau(t_0))$ is a well defined point in the manifold, then every point $\gamma(\tau(t))$ for $t \neq t_0$ is conjugate to $\gamma(\tau_0)$.*

Proof. Let $\gamma(\tau)$ be a timelike non-comoving geodesic with constants K_1, K_2 and L ((6.4), (6.5) and (6.6)), let $\gamma(\tau_1)$ be an initial point where τ_1 corresponds to a time t_1 and let τ_0 correspond to t_0:

$$\tau_0 = \tau_1 + \int_{t_1}^{t_0} \frac{a}{\sqrt{K_1^2 + K_2^2 + \kappa L^2 + a^2}} dt. \tag{6.88}$$

Let now $\varphi_0 = \gamma^\varphi(\tau_0)$.

To prove this theorem we will construct a one parameter family of geodesics in such a way that we can calculate the Jacobi field related to it. In Fig. 6.2 one can find an example where we used the FLRW metric with $a(t) = \sqrt{t}$, $\kappa = -1$ and showed the evolution of r and φ for geodesics with initial point $(t, r, \theta, \varphi) = (2, 0.5, \pi/2, 0.1)$ and initial velocity $\dot{\gamma}_w^r(\tau_1) = 1 + w$, $\dot{\gamma}_w^\theta(\tau_1) = 0$ and $\dot{\gamma}_w^\varphi(\tau_1) = 0.1\,(1 + w)$ for $w \in \{-0.5, 0, 1\}$. At the singularity $t = 0$ the Jacobi field that corresponds to this family of geodesics is well defined, such that its norm vanishes (assume that all geodesics in the family have equal length) and we have that $\gamma(0)$ is conjugate to $\gamma(\tau_1)$.



(a) **(b)**

Figure 6.2: Illustration for this proof. Choose initial parameters such that all geodesics have equal length and such that one can calculate φ and r at the singularity. That way one gets a Jacobi field such that its norm vanishes at the singularity.

We will now do this for a general geodesic in an FLRW spacetime with $\kappa \leq 0$.

We will construct a one-parameter family of geodesics $\gamma_w(\tau) = \Gamma(w, \tau)$, where $w \in (-\delta, \delta)$, (δ to be determined later) such that $\gamma_0(\tau) = \gamma(\tau)$ and $\gamma_w(\tau_1) = \gamma(\tau_1) = (t_1, r_1, \pi/2, \varphi_1)$. Since $L \neq 0$ we have that $\dot{\gamma}_0^{\varphi}(\tau_1) \neq 0$. We choose $\gamma_w^{\theta}(\tau) = \pi/2$ and as initial conditions

$$\begin{aligned} \dot{\gamma}_w^r(\tau_1) &= \dot{\gamma}^r(\tau_1) f(w), \\ \dot{\gamma}_w^{\varphi}(\tau_1) &= \dot{\gamma}^{\varphi}(\tau_1) f(w), \end{aligned} \tag{6.89}$$

where $f : (-\delta, \delta) \to \mathbb{R}$ and $f(0) = 1$. The only other parameter left to vary is the normalization $\epsilon(w)$ ($\epsilon(0) = -1$). Notice, that the constants (6.4), (6.5) and (6.6) that follow from the Killing vectors for γ_w are related to the constants of $\gamma = \gamma_0$ by

$$\begin{aligned} K_{1,w} &= K_1 f(w) \\ K_{2,w} &= K_2 f(w) \\ L_w &= L f(w). \end{aligned} \tag{6.90}$$

We now first require that $\gamma_w^t(\tau_0) = t_0$ to get $J^t(\tau_0) = 0$. We do this by considering the function

$$h_1(f, \epsilon) = \tau_1 + \int_{t_1}^{t_0} \frac{a}{\sqrt{(K_1^2 + K_2^2 + \kappa L^2) f^2 - \epsilon a^2}} dt'. \tag{6.91}$$

This function gives the value of the affine parameter corresponding to the geodesic with parameter f, ϵ at $t = t_0$. We want this parameter to be the same:

$$h_1(f, \epsilon) = \tau_0. \tag{6.92}$$

The integrand of h_1 is partial differentiable with respect to f and ϵ and the partial derivatives are continuous for $(f, \epsilon) \in V$ a neighborhood of $(f, \epsilon) = (1, -1)$. This means that h_1 is continuous differentiable for $(f, \epsilon) \in V$. The partial derivative with respect to ϵ is given by

$$\frac{\partial h_1}{\partial \epsilon} = \frac{1}{2} \int_{t_1}^{t_0} \frac{a^3}{\left((K_1^2 + K_2^2 + \kappa L^2) f^2 - \epsilon a^2\right)^{3/2}} dt', \tag{6.93}$$

which is certainly non-zero in $(1, -1)$ since the integrand is positive for all $t \in [t_0, t_1]$. By the implicit function theorem we then have a continuous differentiable function $\epsilon : (1 - \delta_1, 1 + \delta_1) \to W$ for a certain δ_1 (choose it such that $\left(K_1^2 + K_2^2 + \kappa L^2\right) f^2 > 0$), where W an open interval around -1. Thus all geodesics with parameters f, $\epsilon(f)$ have the same length.

We will now construct f in terms of w such that $\gamma_w^\varphi(t_0) = \varphi_0(w+1)$ and therefore we consider the function

$$h_2(f) = \int_{t_1}^{t_0} \frac{1}{a\sqrt{(K_1^2 + K_2^2 + \kappa L^2)\, f^2 - \epsilon(f) a^2}} dt'. \tag{6.94}$$

This is again a continuous differentiable function of f for $f \in (1-\delta_1, 1+\delta_1)$. Since $\gamma_w^t(\tau_0) = t_0$ we can use Eq. (6.14) to obtain:

$$w(f) = \begin{cases} \frac{1}{\varphi_0}\arctan\left[\frac{L\sqrt{-\kappa K_1^2 - \kappa K_2^2 - \kappa^2 L^2}\tanh\left(\sqrt{-\kappa K_1^2-\kappa K_2^2-\kappa^2 L^2}\, f h_2(f)\right) - K_1 K_2}{-\left(K_1^2 + \kappa L^2\right)}\right] - 1 & \kappa < 0 \\ \frac{1}{\varphi_0}\arctan\left[\frac{h_2(f) K_2^2 f}{L + h_2(f) K_2 K_1 f}\right] - 1 & \kappa = 0,\ K_2 \neq 0 \\ \frac{1}{\varphi_0}\arctan\left[-\frac{L}{K_1^2 f h_2(f)}\right] - 1 & \kappa = 0,\ K_2 = 0. \end{cases} \tag{6.95}$$

We want to use the inverse function theorem and for that we need to show that $\frac{dw}{df}(1) \neq 0$. Consider first the case $\kappa = 0$ and $K_2 \neq 0$. We have $\frac{dw}{df}(1) \neq 0$ precisely when

$$\frac{d}{df}\left[\frac{h_2(f) K_2^2 f}{L + h_2(f) K_2 K_1 f}\right]_{f=1} \neq 0. \tag{6.96}$$

Now

$$\frac{d}{df}\left[\frac{h_2(f) K_2^2 f}{L + h_2(f) K_2 K_1 f}\right]_{f=1} = \left[\frac{\frac{dh_2}{df} f + h_2(f)}{\left(L + h_2(f) K_2 K_1 f\right)^2} K_2^2 L\right]_{f=1} = \frac{\frac{dh_2}{df}(1) + h_2(1)}{\left(L + h_2(1) K_2 K_1\right)^2} K_2^2 L. \tag{6.97}$$

We have that

$$\frac{dh_2}{df}(1) + h_2(1) = \int_{t_1}^{t_0} \frac{-K_1^2 - K_2^2 + \frac{1}{2}\frac{d\epsilon}{df}(1) a^2 + K_1^2 + K_2^2 - \epsilon(1) a^2}{a\left(K_1^2 + K_2^2 - \epsilon(1) a^2\right)^{3/2}} dt = \int_{t_1}^{t_0} \frac{\frac{1}{2}\frac{d\epsilon}{df}(1) a^2 + a^2}{a\left(K_1^2 + K_2^2 + \kappa L^2 + a^2\right)^{3/2}} dt \tag{6.98}$$

and we must have that $\frac{d\epsilon}{df}(1) > 0$ because otherwise we can never have that $h_1(f, \epsilon(f))$ is independent of f. Since $K_2 \neq 0$ and $L \neq 0$ we have that w is a continuous differentiable function of f around 1 and its derivative is non-zero in $f = 1$. By the inverse function theorem, there exists a continuous function $f : (-\delta, \delta) \to (1-\delta_1, 1+\delta_1)$ and such that $\gamma_w^\varphi(\tau_0) = \varphi_0(w+1)$.

For $\kappa = 0$, $K_2 = 0$ we find in a similar way that

$$\frac{d}{df}\left[-\frac{L}{K_1^2 f h_2(f)}\right]_{f=1} = \left[\frac{L\left(h_2(f) + \frac{dh_2}{df}\right)}{K_1^2\left(f h_2(f)\right)^2}\right]_{f=1} = \frac{L\left(h_2(1) + \frac{dh_2}{df}(1)\right)}{K_1^2\left(h_2(1)\right)^2}, \tag{6.99}$$

so $w(f)$ can be inverted by similar reasoning as in the previous case.

For $\kappa < 0$, $\frac{dw}{df}(1) \neq 0$ precisely when

$$\frac{d}{df}\left[\sqrt{-\kappa K_1^2 - \kappa K_2^2 - \kappa^2 L^2}\, f h(f)\right]_{f=1} \neq 0. \tag{6.100}$$

Now

$$\frac{d}{df}\left[\sqrt{-\kappa K_1^2 - \kappa K_2^2 - \kappa^2 L^2}\, f h_2(f)\right]_{f=1} = \sqrt{-\kappa K_1^2 - \kappa K_2^2 - \kappa^2 L^2}\left[\frac{dh_2}{df}(1) + h_2(1)\right] \tag{6.101}$$

Hence, we have that $w(f)$ can be inverted by similar reasoning as for $\kappa = 0$.

Hence we have constructed the family of geodesics γ_w. This family corresponds to a Jacobi field J such that $J(\tau_1) = 0$. Furthermore using Eq. (6.12) we find that:

$$\gamma_w^r = \frac{L}{\sqrt{\left(K_2\cos(\varphi_w) - K_1\sin(\varphi_w)\right)^2 + \kappa L^2}}, \tag{6.102}$$

which we can use to calculate J at τ_0:

$$J(\tau_0) = \partial_w \gamma_w(t)|_{w=0,\tau=\tau_0} = \left(0, \frac{L\left(K_2\cos(\varphi_0) - K_1\sin(\varphi_0)\right)\left(K_2\sin(\varphi_0) + K_1\cos(\varphi_0)\right)}{\left[\left(K_2\cos(\varphi_0) - K_1\sin(\varphi_0)\right)^2 + \kappa L^2\right]^{3/2}}\varphi_0, 0, \varphi_0\right), \quad (6.103)$$

which is well-defined. This means that $g\left(J(\tau_0), J(\tau_0)\right) = 0$ which proves that $\gamma(\tau_1)$ is conjugate to $\gamma(\tau_0)$ along γ. Notice that we not necessarily have that $g(J, \dot{\gamma}) = 0$, but at least J is a spacelike vector at the singularity. □

We want to stress that theorem 6.3.2 is less satisfactory than theorem 6.3.1 because in this case the Jacobi field that has vanishing norm is not necessarily perpendicular to the geodesic. To improve on this theorem one should try to use a similar construction to find a Jacobi field with vanishing norm that is perpendicular to $\dot{\gamma}$.

Theorem 6.3.2 only holds for non-comoving timelike geodesics γ with a well defined point $\gamma(t_0)$ at the singularity. We would like to prove a similar theorem for timelike geodesics that do not satisfy this condition, hence have that $r \to \infty$ for $t \to t_0$. From Eq. (6.12) it follows that geodesics have this behavior if and only if $\left(K_2\cos(\varphi) - K_1\sin(\varphi)\right)^2 + \kappa L^2 \to 0$. Let φ be an angle such that $\left(K_2\cos(\varphi) - K_1\sin(\varphi)\right)^2 + \kappa L^2 = 0$, then

$$\begin{aligned} K_1^2\left[K_2^2\cos^2(\varphi) + K_1^2\sin^2(\varphi) - 2K_1K_2\sin(\varphi)\cos(\varphi) + \kappa L^2\right] &= \\ -\kappa L^2\left(K_2^2\cos^2(\varphi) + K_1^2\sin^2(\varphi) - 2K_1K_2\cos(\varphi)\sin(\varphi) + \kappa L^2\right). & \end{aligned} \quad (6.104)$$

This can be rewritten as

$$\begin{aligned} K_1^2K_2^2\cos^2(\varphi) + \left(K_1^2 + \kappa L^2\right)^2\left(1 - \cos^2(\varphi)\right) - 2K_1K_2\left(K_1^2 + \kappa L^2\right)\sin(\varphi)\cos(\varphi) &= \\ L^2\left(-\kappa K_1^2 - \kappa K_2^2 - \kappa^2 L^2\right)\cos^2(\varphi). & \end{aligned} \quad (6.105)$$

Dividing by $\cos^2(\varphi)$ yields

$$\left[K_1K_2 - \left(K_1^2 + \kappa L^2\right)\tan(\varphi)\right]^2 = L^2\left(-\kappa K_1^2 - \kappa K_2^2 - \kappa^2 L^2\right). \quad (6.106)$$

This implies that for $\kappa < 0$, $\lim_{t \to t_0} r = \infty$ is equivalent to

$$\lim_{t \to t_0} \frac{K_1K_2 - \left(K_1^2 + \kappa L^2\right)\tan(\varphi)}{L\sqrt{-\kappa K_1^2 - \kappa K_2^2 - \kappa^2 L^2}} = -1, \quad (6.107)$$

but then it follows from Eq. (6.15) and Eq. (6.14) that

$$\lim_{t \to t_0} \int_{t_1}^{t} \frac{1}{a\sqrt{K_1^2 + K_2^2 + \kappa L^2 + a^2}} dt' = -\infty. \quad (6.108)$$

When $\kappa = 0$, $r \to \infty$ is equivalent to $K_2\cos(\varphi) - K_1\sin(\varphi) \to 0$. Thus also in this case it follows from Eq. (6.15) and Eq. (6.14) that

$$\lim_{t \to t_0} \int_{t_1}^{t} \frac{1}{a\sqrt{K_1^2 + K_2^2 + \kappa L^2 + a^2}} dt' = -\infty. \quad (6.109)$$

In principle one can do a similar construction as in theorem (6.3.2) for a geodesic γ with property (6.109) and initial point $\gamma(\tau_1)$. Namely construct a family of geodesics γ_w such that $\gamma_0 = \gamma$, $\gamma_w(\tau_1) = \gamma(\tau_1)$ and

$$\begin{aligned} \dot{\gamma}_w^r(\tau_1) &= \dot{\gamma}^r(\tau_1)f(w), \\ \dot{\gamma}_w^\varphi(\tau_1) &= \dot{\gamma}^\varphi(\tau_1)f(w), \end{aligned} \quad (6.110)$$

where $f : (-\delta, \delta) \to \mathbb{R}_{>0}$ and $f(0) = 1$. The constants (6.4), (6.5) and (6.6) are related to the constants of $\gamma = \gamma_0$ by

$$\begin{aligned} K_{1,w} &= K_1 f(w) \\ K_{2,w} &= K_2 f(w) \\ L_w &= L f(w). \end{aligned} \quad (6.111)$$

The normalization $\epsilon(w)$ of γ_w can be fixed in the same way as in theorem 6.3.2 such that $\gamma_w^t(\tau_0) = t_0$.

From Eqs. (6.14), (6.15) and (6.109) it follows now that

$$\begin{cases} \kappa < 0: & \frac{1}{f(w)\sqrt{-\kappa K_1^2-\kappa K_2^2-\kappa^2 L^2}} \tanh^{-1}\left(\frac{K_1K_2-\left(K_1^2+\kappa L^2\right)\tan(\varphi_w)}{L\sqrt{-\kappa K_1^2-\kappa K_2^2-\kappa^2L^2}}\right) \to -\infty \\ \kappa = 0,\ K_2 \neq 0: & \frac{L\sin(\varphi_w)}{f(w)K_2(K_2\cos(\varphi_w)-K_1\sin(\varphi_w))} \to -\infty \\ \kappa = 0,\ K_2 = 0: & -\frac{L}{f(w)K_1^2\tan(\varphi_w)} \to -\infty. \end{cases} \tag{6.112}$$

This means that

$$\begin{cases} \kappa < 0: & \frac{K_1K_2-\left(K_1^2+\kappa L^2\right)\tan(\varphi_w)}{L\sqrt{-\kappa K_1^2-\kappa K_2^2-\kappa^2L^2}} \to -1 \\ \kappa = 0,\ K_2 \neq 0: & \frac{L\sin(\varphi_w)}{K_2(K_2\cos(\varphi_w)-K_1\sin(\varphi_w))} \to -\infty \\ \kappa = 0,\ K_2 = 0: & -\frac{L}{K_1^2\tan(\varphi_w)} \to -\infty \end{cases} \tag{6.113}$$

and this will give a unique solution for φ_w independent of w (and the function f). This means that $\gamma_w^{\varphi}(\tau) \to \varphi_0$ when $\tau \to \tau_0$. The radius, Eq. (6.12), does not depend on f, but it goes to ∞ for all w. Ideally one would like to prove that $\gamma_w^r/\gamma^r \to 1$, because in that case one would have a Jacobi field corresponding to γ_w that vanishes in the limit of $\tau \to \tau_0$. It seems hard to do this unfortunately.

6.3.2 Conjugate Points Imply a Singularity

In the previous section we proved two theorems that show that when $a(t_0) = 0$ for some t_0 it is possible that all points on a geodesic are conjugate to the point $\gamma(t_0)$. This seems like strange behavior and must come from the metric not being well-defined at t_0. Indeed, from Morse Index Theory it follows that there can only be a finite number of points of γ between some point $\gamma(t_1)$ and $\gamma(t_0)$ conjugate to $\gamma(t_0)$ when the metric is well-defined (see e.g. [22]). In the timelike version of this theory one considers a timelike geodesic $\gamma:\ [\tau_{\rm i}, \tau_{\rm f}] \to M$ and the index form $I:\ V^{\perp}(\gamma) \times V^{\perp}(\gamma) \to \mathbb{R}$

$$I(X,Y) = -\int_{\tau_{\rm i}}^{\tau_{\rm f}} \left[g\left(D_\tau X, D_\tau Y\right) - g\left(R\left(X,\dot\gamma\right)\dot\gamma, Y\right)\right] d\tau. \tag{6.114}$$

Let $V_0^{\perp}(\gamma)$ be the subspace of $V^{\perp}(\gamma)$ with elements X such that $X(\tau_{\rm i}) = X(\tau_{\rm f}) = 0$. The index of γ is then

$$\mathrm{Ind}\,(\gamma) = \limsup\left\{\dim A:\ A \text{ is a vector subspace of } V_0^{\perp}(\gamma) \text{ and } I|A\times A \text{ is positive definite}\right\}. \tag{6.115}$$

If $J_\tau(\gamma)$ denotes the $\mathbb{R}-$vector space of smooth Jacobi fields along γ such that $J(\tau_{\rm i}) = J(\tau) = 0$ for $\tau_{\rm i} < \tau \le \tau_{\rm f}$, then one can show that $\mathrm{Ind}\,(\gamma)$ is finite and

$$\mathrm{Ind}\,(\gamma) = \sum_{\tau_{\rm i}<\tau\le\tau_{\rm f}} \dim J_\tau(\gamma). \tag{6.116}$$

For a null geodesic γ one can use a similar argument using null Morse index theory. This proves that whenever we find that all the points of a geodesic γ are conjugate to a certain point $\gamma(t_0)$ we have that $a(t_0) = 0$ and hence a singularity.

We now prove a theorem that shows in which cases an FLRW spacetime with $\kappa = 0$ has conjugate points.

Theorem 6.3.3. *Let γ be a timelike geodesic with constant C in an FLRW spacetime with $\kappa = 0$, let $a(t) > 0$ for all t and let $\gamma(\tau_{\rm i})$ and $\gamma(\tau_{\rm f})$ be two points on this geodesic. These points are conjugate if and only if*

$$\int_{t_{\rm i}}^{t_{\rm f}} \frac{1}{a\sqrt{C+a^2}}dt = 2C\int_{t_{\rm i}}^{t_{\rm f}} \frac{\dot a}{a^3}\int_{t_{\rm i}}^{t} \frac{a\left(1-\alpha a^2\right)}{\left(C+a^2\right)^{3/2}}dt'dt, \tag{6.117}$$

where

$$\alpha = \frac{\int_{t_{\rm i}}^{t_{\rm f}} \frac{a}{(C+a^2)^{3/2}}dt}{\int_{t_{\rm i}}^{t_{\rm f}} \frac{a^3}{(C+a^2)^{3/2}}dt}. \tag{6.118}$$

Proof. We will use Cartesian coordinates (t, x_1, x_2, x_3). We can choose these coordinates such that for γ, $x_2 = x_3 = 0$. We furthermore have that

$$\begin{aligned} \dot{\gamma}^t &= \frac{\sqrt{C + a^2}}{a} \\ \dot{\gamma}^{x_1} &= \frac{\sqrt{C}}{a^2} \end{aligned} \tag{6.119}$$

(Eq. 6.45) and as usual we choose $g(\dot{\gamma}, \dot{\gamma}) = -1$. Let $\gamma(\tau_{\rm i}) = \left(t_{\rm i}, x^1_{\rm i}, 0, 0\right)$. A Jacobi field that vanishes at two points $\gamma(\tau_{\rm i})$, $\gamma(\tau_{\rm f})$ corresponds to a variation $\Gamma(w, \tau)$ such that $\Gamma(0, \tau) = \gamma(\tau)$, $\Gamma(w, \tau_{\rm i}) = \gamma(\tau_{\rm i})$ and $\partial_w \Gamma(0, \tau_{\rm f}) = 0$. We know that $\gamma_w(\tau) = \Gamma(w, \tau)$ satisfies

$$\begin{aligned} \dot{\gamma}^t_w &= \frac{\sqrt{C(w) - \epsilon(w) a^2}}{a}; \\ \dot{\gamma}^{x_1}_w &= \frac{C_1(w)}{a^2}; \\ \dot{\gamma}^{x_2}_w &= \frac{C_2(w)}{a^2}; \\ \dot{\gamma}^{x_3}_w &= \frac{C_3(w)}{a^2}, \end{aligned} \tag{6.120}$$

where $C(w) = C_1(w)^2 + C_2(w)^2 + C_3(w)^2$. Consider first the function

$$F(u, w) = \int_{t_{\rm i}}^{u} \frac{a}{\sqrt{C(w) - \epsilon(w) a^2}} dt. \tag{6.121}$$

We know that for $\tau_{\rm i} \leq \tau \leq \tau_{\rm f}$, $\tau - \tau_{\rm i} = F(t(w, \tau), w) \equiv G(w)$ for all w, hence

$$0 = \frac{dG}{dw} = \frac{\partial F}{\partial u}\frac{\partial t}{\partial w} + \frac{\partial F}{\partial w} = \frac{a}{\sqrt{C(w) - \epsilon(w) a^2}}\frac{\partial t}{\partial w} - \frac{1}{2}\int_{t_{\rm i}}^{t} \frac{a\left(\dot{C}(w) - \dot{\epsilon}(w) a^2\right)}{\left(C(w) - \epsilon(w) a^2\right)^{3/2}} dt. \tag{6.122}$$

Evaluation in $w = 0$ yields

$$\frac{a}{\sqrt{C + a^2}}\frac{\partial t}{\partial w}|_{w=0}(\tau) = \frac{1}{2}\int_{t_{\rm i}}^{t} \frac{a\left(\dot{C}(0) - \dot{\epsilon}(0) a^2\right)}{\left(C + a^2\right)^{3/2}} dt. \tag{6.123}$$

Hence

$$\frac{\partial t}{\partial w}|_{w=0}(\tau) = \frac{1}{2}\frac{\sqrt{C + a^2}}{a}\int_{t_{\rm i}}^{t} \frac{a\left(\dot{C}(0) - \dot{\epsilon}(0) a^2\right)}{\left(C + a^2\right)^{3/2}} dt \tag{6.124}$$

(here $\gamma^t(\tau) = t$). Evaluation in $\tau = \tau_{\rm f}$ $(t = t_{\rm f})$ yields

$$0 = \int_{t_{\rm i}}^{t_{\rm f}} \frac{a\left(\dot{C}(0) - \dot{\epsilon}(0) a^2\right)}{\left(C + a^2\right)^{3/2}} dt. \tag{6.125}$$

Therefore:

$$\dot{\epsilon}(0) = \alpha \dot{C}(0). \tag{6.126}$$

We can integrate Eq. (6.120):

$$\Gamma(w, \tau) = \left(t(w, \tau), \int_{\tau_{\rm i}}^{\tau} \frac{C_1(w)}{a^2(t(w, \tau))} d\tau' + x^1_0, \int_{\tau_{\rm i}}^{\tau} \frac{C_2(w)}{a^2(t(w, \tau))} d\tau', \int_{\tau_{\rm i}}^{\tau} \frac{C_3(w)}{a^2(t(w, \tau))} d\tau'\right). \tag{6.127}$$

It then follows that

$$\partial_w\Gamma(w,\tau) = \left(\partial_w t, \int_{\tau_i}^{\tau}\left(\frac{\dot{C}_1(w)}{a^2(t(w,\tau'))} - 2\frac{C_1(w)\dot{a}\partial_w t(w,\tau')}{a^3(t(w,\tau'))}\right)d\tau',\right.$$
$$\left.\int_{\tau_i}^{\tau}\left(\frac{\dot{C}_2(w)}{a^2(t(w,\tau'))} - 2\frac{C_2(w)\dot{a}\partial_w t(w,\tau')}{a^3(t(w,\tau'))}\right)d\tau', \int_{\tau_i}^{\tau}\left(\frac{\dot{C}_3(w)}{a^2(t(w,\tau'))} - 2\frac{C_3(w)\dot{a}\partial_w t(w,\tau')}{a^3(t(w,\tau'))}\right)d\tau'\right). \tag{6.128}$$

Evaluation in $(w,\tau) = (0,\tau_f)$ yields

$$\begin{aligned} 0 &= \partial_w\Gamma(0,\tau_f) \\ &= \left(0, \int_{\tau_i}^{\tau_f}\left(\frac{\dot{C}_1(0)}{a^2(\tau)} - 2\frac{C_1(0)\dot{a}\partial_w t(0,\tau)}{a^3(\tau)}\right)d\tau, \dot{C}_2(0)\int_{\tau_i}^{\tau_f}\frac{1}{a^2(\tau)}d\tau, \dot{C}_3(0)\int_{\tau_i}^{\tau_f}\frac{1}{a^2(\tau)}d\tau\right), \end{aligned} \tag{6.129}$$

which implies $\dot{C}_2(0) = \dot{C}_3(0) = 0$. Eq. (6.124) and the x^1 coordinate in Eq. (6.129) yield

$$\begin{aligned} \int_{t_i}^{t_f}\frac{\dot{C}_1(0)}{a\sqrt{C+a^2}}dt &= 2C_1(0)\int_{t_i}^{t_f}\frac{\dot{a}\partial_w t(0,\tau(t))}{a^2\sqrt{C+a^2}}dt \\ &= C_1(0)\int_{t_i}^{t_f}\frac{\dot{a}}{a^3}\int_{t_i}^{t}\frac{a\left(\dot{C}(0)-\dot{\epsilon}(0)a^2\right)}{(C+a^2)^{3/2}}dt'dt. \end{aligned} \tag{6.130}$$

Using Eq. (6.126) we find

$$\dot{C}_1(0)\int_{t_i}^{t_f}\frac{1}{a\sqrt{C+a^2}}dt = 2C_1^2(0)\dot{C}_1(0)\int_{t_i}^{t_f}\frac{\dot{a}}{a^3}\int_{t_i}^{t}\frac{a\left(1-\alpha a^2\right)}{(C+a^2)^{3/2}}dt'dt. \tag{6.131}$$

If $\dot{C}_1(0) = 0$, we have that $\dot{\epsilon}(0) = 0$ and we find the Jacobi field $J = 0$. Hence Eq. (6.131) yields condition (6.117) if $\gamma(\tau_i)$ and $\gamma(\tau_f)$ are conjugate. If condition (6.117) is satisfied, it follows with Eq. (6.131) that we can choose a variation with $\dot{C}_1(0) \neq 0$, $\dot{C}_2(0) = \dot{C}_3(0) = 0$ and $\dot{\epsilon}(0) = \alpha\dot{C}(0)$. From Eqs. (6.123), (6.128) and (6.131) we find that $\partial_w\Gamma(0,\tau_f) = 0$. □

Notice that when a is constant, the right-hand side of Eq. (6.117) vanishes, which implies there are no conjugate points in an FLRW spacetime with this scale factor. This makes sense, because this is Minkowski space. Also comoving geodesics (that have $C = 0$) do not have conjugate points. Notice that when one can choose times t_i, t_f and a constant $C \geq 0$ in such a way that condition (6.117) is satisfied, there are conjugate points in a spacetime with $a(t) > 0$ at all times. This means that conjugate points do not necessarily imply a singularity. At the other hand, when all points on a geodesic are conjugate to the point at the singularity, that does imply a singularity.

6.4 Including Torsion

We will now study torsion in a spatially homogeneous, isotropic universe in two settings. First together with a perfect radiation fluid and then with a perfect matter fluid. Torsion is induced by fermions and by integrating out torsion in the Einstein equation, and studying the Dirac equation in conformal spacetime it turns out that for torsion

$$\rho_s = -\frac{\rho_1}{a^6}, \tag{6.132}$$

where $\rho_1 \geq 0$.

6.4.1 Radiation and Torsion

Consider a spatially homogeneous, isotropic universe in which we have a perfect homogeneous fluid of radiation, which density goes as

$$\rho_r = \frac{\rho_0}{a^4}, \tag{6.133}$$

where $\rho_0 > 0$, and torsion, which density goes as Eq. (6.132). The Friedmann equation, Eq. (6.30) reads

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G_\mathrm{N}}{3}\left(\frac{\rho_0}{a^4} - \frac{\rho_1}{a^6}\right) - \frac{\kappa}{a^2}. \tag{6.134}$$

We will examine the behavior of a using the right-hand side of Eq. (6.134):

$$f(a) = -\kappa a^4 + \frac{8\pi G_\mathrm{N}}{3}\rho_0 a^2 - \frac{8\pi G_\mathrm{N}}{3}\rho_1. \tag{6.135}$$

We have $\dot{a} = 0$ exactly when

$$f(a) = 0, \tag{6.136}$$

which is solved for $\kappa = 0$ by

$$a = \pm\sqrt{\frac{\rho_1}{\rho_0}}. \tag{6.137}$$

Since $f \to \infty$ for $a \to \pm\infty$ and $f(0) \leq 0$, this always results in a bounce and therefore a non-singular spacetime. When $\kappa \neq 0$ Eq. (6.136) is solved by

$$a = \pm\sqrt{\frac{-\frac{8\pi G_\mathrm{N}}{3}\rho_0 \pm \sqrt{\left(\frac{8\pi G_\mathrm{N}}{3}\rho_0\right)^2 - \frac{32\pi G_\mathrm{N}}{3}\kappa\rho_1}}{-2\kappa}}. \tag{6.138}$$

For $\kappa > 0$, this has two positive and two negative real solutions in case that

$$\rho_1 \leq \frac{2\pi G_\mathrm{N}}{3\kappa}\rho_0^2 \tag{6.139}$$

and otherwise it has zero real solutions and is $f(a) \leq 0$. When Eq. (6.136) has solutions, we find that for $a \to \pm\infty$, $f(a) \to -\infty$ and $f(0) \leq 0$ so in this case we have that a solution $a(t)$ will be bounded by two values. This means that for non-vanishing torsion we do not have a singularity.

For $\kappa < 0$, $f(a) = 0$ always has two real solutions: one positive and one negative. Now $f(a) \to \infty$ for $a \to \pm\infty$ so we find that

$$a(t) \geq \sqrt{-\frac{4\pi G_\mathrm{N}}{3\kappa}\rho_0} \cdot \sqrt{-1 + \sqrt{1 - \frac{3\kappa}{2\pi G_\mathrm{N}}\frac{\rho_1}{\rho_0^2}}}, \tag{6.140}$$

which will result in complete geodesics and hence no singularity.

We will now expand on the example that we started with in Section 6.3. We will work in conformal time

$$g_{\mu\nu} = a(\eta)^2\eta_{\mu\nu}, \tag{6.141}$$

since in this way we can solve for the scale parameter explicitly. Notice that we can transform back and forth between time t and conformal time η using the relation

$$dt = a(\eta)d\eta. \tag{6.142}$$

The Friedmann equation, Eq. (6.134) (we take $\kappa = 0$), in conformal time reads

$$\frac{a'^2}{a^4} = \frac{8\pi G_\mathrm{N}}{3}\left(\frac{\rho_0}{a^4} - \frac{\rho_1}{a^6}\right), \tag{6.143}$$

where we denote

$$a' = \frac{da}{d\eta}. \tag{6.144}$$

Eq. (6.143) is solved by

$$a(\eta) = \sqrt{\frac{\rho_1}{\rho_0} + \frac{8\pi G_\mathrm{N}\rho_0}{3}\eta^2} \equiv \sqrt{\alpha + \beta\eta^2}. \tag{6.145}$$

We have chosen the integration constants such that the bounce or singularity takes place at $\eta = 0$. In Fig. 6.3 one can find this a for $\alpha = \beta = 1$. Taking $\alpha = 0$ and transforming to time t using Eq. (6.142), one indeed finds that $a(t) \propto \sqrt{t}$.

Using that $dt = a(\eta)d\eta$, we find that the geometric condition from the singularity theorems, Eq. (6.24) (which is used to predict conjugate points) in conformal time is given by:

$$-3\left(\frac{a''}{a}-\left(\frac{a'}{a}\right)^2\right)\frac{\epsilon}{a^2}+2\frac{C}{a^4}\left[\frac{a''}{a}-2\left(\frac{a'}{a}\right)^2\right]\leq 0. \tag{6.146}$$

For Eq. (6.145) this yields:

$$-3\left(\alpha-\beta\eta^2\right)\epsilon+2C\left[\frac{\alpha-2\beta\eta^2}{\alpha+\beta\eta^2}\right]\leq 0. \tag{6.147}$$

So for $\alpha = 0$, we see that we get that

$$3\beta\eta^2\epsilon-4C\leq 0, \tag{6.148}$$

which is clearly satisfied. Of course we also have a singularity here. For $\alpha > 0$ we find that the condition gets broken for some timelike geodesics for

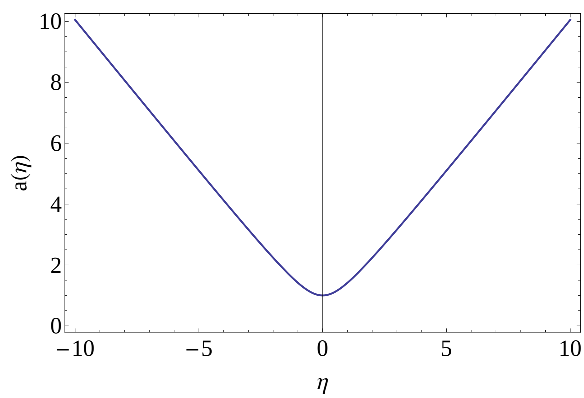


Figure 6.3: Scale parameter of a spatially homogeneous, isotropic universe with radiation and torsion.

$$\eta^2 < \frac{\alpha}{\beta} = \frac{3\rho_1}{8\pi G_{\mathrm{N}}\rho_0^2} \tag{6.149}$$

(take C small enough). It gets broken for all geodesics when

$$\eta^2 < \frac{\alpha}{2\beta} = \frac{3\rho_1}{16\pi G_{\mathrm{N}}\rho_0^2}. \tag{6.150}$$

This model has a bounce for $\rho_1 > 0$. Since it becomes singular for $\rho_1 = 0$ it is an excellent example to examine the relation between conjugate points and singular points. For that we first have to construct the geodesics. Let a geodesic be given by

$$\gamma = (\eta, x^i), \tag{6.151}$$

and let $u^\mu = \frac{d}{d\tau}\gamma^\mu$, where τ is an affine parameter. Then the geodesic equation gives rise to two equations:

$$\frac{du^0}{d\tau}+\frac{a'}{a}\left(\left(u^0\right)^2+\sum_i\left(u^i\right)^2\right) = 0 \tag{6.152}$$

$$\frac{du^i}{d\tau}+2\frac{a'}{a}u^0u^i = 0. \tag{6.153}$$

Using that $a'u^0 = \frac{d}{d\tau}a$ and that we can write $\left(u^i\right)^2 = \left(u^0\right)^2+\frac{\epsilon}{a^2}$, Eqs. (6.152) and (6.153) can be written as

$$\frac{d}{d\eta}\left[a^4\left(u^0\right)^2+a^2\epsilon\right] = 0; \tag{6.154}$$

$$\frac{d}{d\tau}\left[a^2u^i\right] = 0.$$

which leads to

$$u^0 = \frac{\sqrt{C-a^2\epsilon}}{a^2}; \tag{6.155}$$

$$u^i = \frac{C_i}{a^2}. \tag{6.156}$$

Normalization implies

$$C = \sum_i C_i^2.$$

Let us now focus on timelike geodesics: $\epsilon = -1$. Then Eqs. (6.155) and (6.156) can be written as

$$u^0 = \frac{\sqrt{C+a^2}}{a^2}; \tag{6.157}$$

$$\frac{dx^i}{d\eta} = \frac{d\tau}{d\eta}u^i = \frac{C_i}{\sqrt{C+a^2}}. \tag{6.158}$$

For the scale parameter that we are considering, Eq. (6.157) reads

$$\frac{d\eta}{d\tau} = \frac{\sqrt{C+\alpha+\beta\eta^2}}{\alpha+\beta\eta^2} \tag{6.159}$$

and Eq. (6.158) gives us

$$\frac{dx^i}{d\eta} = \frac{C_i}{\sqrt{C+\alpha+\beta\eta^2}}. \tag{6.160}$$

This is solved by

$$x^i = \frac{C_i}{\sqrt{\beta}} \log\left(\beta\eta + \sqrt{C\beta+\alpha\beta+\beta^2\eta^2}\right) + D_i. \tag{6.161}$$

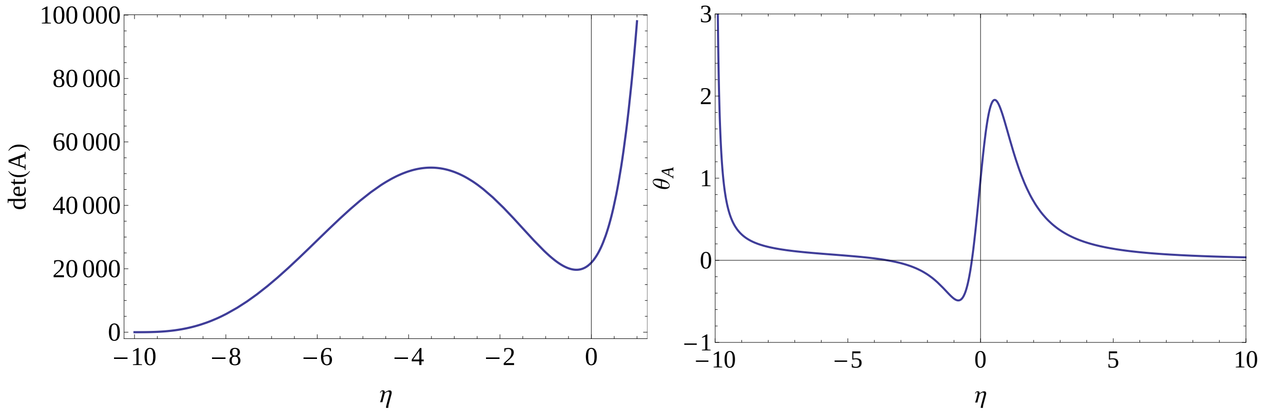
We consider the geodesic

$$\gamma = \left(\eta, \frac{1}{\sqrt{\beta}} \log\left(\beta\eta + \sqrt{\beta+\alpha\beta+\beta^2\eta^2}\right), 0, 0\right) \tag{6.162}$$

and construct the matrix A (Eq. 2.39) corresponding to the point $\gamma(-10)$ for this geodesic. An orthonormal, parallel transported basis along γ is given by

$$\begin{aligned} E_0 &= \left(\frac{\sqrt{1+a(\eta)^2}}{a(\eta)^2}, \frac{1}{a(\eta)^2}, 0, 0\right) \\ E_1 &= \left(\frac{1}{a(\eta)^2}, \frac{\sqrt{1+a(\eta)^2}}{a(\eta)^2}, 0, 0\right) \\ E_2 &= \left(0, 0, \frac{1}{a(\eta)}, 0\right) \\ E_3 &= \left(0, 0, 0, \frac{1}{a(\eta)}\right). \end{aligned} \tag{6.163}$$

We now need the Jacobi fields J_i for $i \in \{1,2,3\}$ such that $J_i(-10) = 0$ and $D_\tau J_i(-10) = E_i(0)$. We solve for the Jacobi fields numerically (although just as in Section 6.3, one can solve for 2 of the Jacobi fields analytically). When we look at the determinant of the matrix A (which is zero at a conjugate point to $\gamma(-10)$) for different values of α and β, we find that this function goes through a minimum. A plot of the determinant of A can be found in Fig. 6.4a. We see that the determinant of A is 0 at $\eta = -10$ and becomes bigger at first, then it decreases again and has a minimum a little bit before the bounce at $\eta = 0$ after which it keeps increasing. In Fig. 6.4b the more familiar expansion parameter θ_A is shown. Here we see that indeed θ_A is increasing for $-1 < \eta < 1$ where the singularity condition is broken. We observe that there is no conjugate point along this geodesic to $\gamma(-10)$.



(a) Determinant of the matrix A corresponding to $\gamma(-10)$. The determinant vanishes at the initial point and has a minimum just before $\eta = 0$.

(b) The expansion parameter corresponding to $\gamma(-10)$. Notice that θ_A is increasing for $-1 < \eta < 1$ where the singularity condition is broken.

Figure 6.4

This minimum in the determinant is a common feature for different values of α, β if the initial point $\gamma(\eta_1)$ is far enough from $\gamma(0)$. In Fig. 6.5 we show the dimensionless variable $\eta\sqrt{\beta}$ ($1/\sqrt{\beta}$ is a typical time for the bounce) at which the determinant of A has a minimum for different values of α, β. We see that this minimum is further away for larger α because the minimum value of a is $\sqrt{\alpha}$.

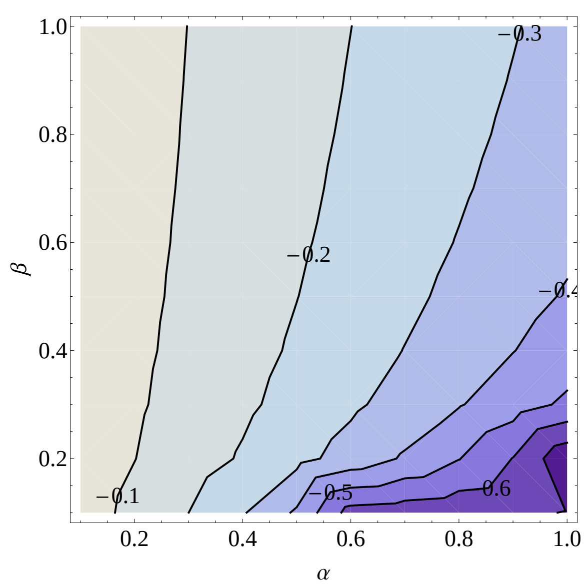


Figure 6.5: The value of $\eta\sqrt{\beta}$ at which the determinant of A has a minimum as function of α and β.

6.4.2 Matter and Torsion

Consider a spatially homogeneous, isotropic universe where we have a perfect homogeneous fluid of matter, which density goes as

$$\rho_{\rm m} = \frac{\rho_0}{a^3}, \tag{6.164}$$

and torsion, which density goes as Eq. (6.132). The Friedmann equation, Eq. (6.30) reads

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G_{\rm N}}{3}\left(\frac{\rho_0}{a^3} - \frac{\rho_1}{a^6}\right) - \frac{\kappa}{a^2}. \tag{6.165}$$

We will examine the behavior of a using the right-hand side of Eq. (6.165):

$$f(a) = \frac{8\pi G_{\rm N}}{3}\left(\rho_0 a^3 - \rho_1\right) - \kappa a^4. \tag{6.166}$$

We have that $\dot{a} = 0$ exactly when

$$f(a){=}0. \tag{6.167}$$

For $\kappa = 0$ this results in one solution which means we have a bounce and $a(t)^3 \geq \frac{\rho_1}{\rho_0}$. This immediately implies that all geodesics are complete as we have seen in Section 6.2. For $\kappa > 0$ we see that f has a maximum for $a_1 = \frac{2\pi G_{\rm N}\rho_0}{\kappa}$ and the only other extremum can be found at $a{=}0$, but this is a saddle point. $f(0) = -\frac{8\pi G_{\rm N}}{3}\rho_1 \leq 0$ and $f(a_1) = 5\frac{1}{3}\frac{\pi^4 G_{\rm N}^4\rho_0^4}{\kappa^3} - \frac{8\pi G_{\rm N}}{3}\rho_1$. Hence when

$$\rho_1 \leq 2\frac{\pi^3 G_{\rm N}^3\rho_0^4}{\kappa^3}, \tag{6.168}$$

f crosses zero twice for $0 \leq a_0 \leq a_1 \leq a_2$ such that a bounces back and forth between a_0 and a_2 and when

$$\rho_1 > 2\frac{\pi^3 G_{\rm N}^3\rho_0^4}{\kappa^3}, \tag{6.169}$$

f is negative everywhere. This implies that for non-vanishing torsion we always have complete geodesics.

For $\kappa < 0$ we see that the function f has a minimum at $a_1 = \frac{2\pi G_{\rm N}\rho_0}{\kappa} < 0$ and the only other extremum is at $a = 0$. The function f has the value$-\frac{8\pi G_{\rm N}}{3}\rho_1 \leq 0$ in 0 and $5\frac{1}{3}\frac{\pi^4 G_{\rm N}^4\rho_0^4}{\kappa^3} - \frac{8\pi G_{\rm N}}{3}\rho_1 < 0$ in a_1. This implies that f is zero twice for $a_0 \leq a_1 < 0 \leq a_2$. Hence we again have a bounce for non-vanishing torsion and therefore no singularities. Also all geodesics are complete.

The Friedmann equation corresponding to matter, torsion and $\kappa = 0$, Eq. (6.165), is solved by

$$a(t) = \left(\frac{\rho_1}{\rho_0} + \frac{9}{4}\frac{8\pi G_{\rm N}\rho_0}{3}t^2\right)^{1/3} \equiv \left(\alpha + \beta t^2\right)^{1/3} \tag{6.170}$$

where the bounce ($\alpha > 0$) or singularity ($\alpha = 0$) takes place at $t = 0$.

We can again look at the singularity condition (6.23) to see what we get in this case:

$$0 \;\geq\; 2C(\alpha - \beta t^2) + (-3\alpha + \beta t^2)(\alpha + \beta t^2)^{2/3}\epsilon. \tag{6.171}$$

This leads to a violation of the condition for certain timelike geodesics (C small enough) when

$$t^2 < 3\frac{\alpha}{\beta} = \frac{\rho_1}{2\pi G_{\rm N}\rho_0^2} \tag{6.172}$$

and for all geodesics when

$$t^2 < \frac{\alpha}{\beta} = \frac{\rho_1}{6\pi G_{\rm N}\rho_0^2}. \tag{6.173}$$

One finds similar behavior as in the radiation and torsion case when considering the matrix A (Eq. (2.39)) corresponding to an initial point $\gamma(t_1)$. The determinant of A can again have a minimum before the bounce at $t = 0$.

Conclusion

In this thesis we studied the singularity theorems of Hawking and Penrose in the context of spacetimes with non-vanishing torsion. Assuming that test particles also move on geodesics for general torsion, we used an incomplete non-spacelike geodesic as definition of a singularity. We started with a very general approach to see whether we could generalize the propositions proven by Hawking and Penrose and used to prove the singularity theorems in the context of general relativity to spacetimes with arbitrary torsion. To do this we needed to study the generalized Jacobi and Raychaudhuri equations. We found an extra term with respect to the literature in the Raychaudhuri equation for a timelike geodesic and arbitrary torsion. We succeeded in proving the propositions of Hawking and Penrose for the case of totally anti-symmetric torsion and found that the resulting singularity theorems agreed with the ones proven in a different way in [14]. An important step in doing that was to derive that the vorticity $\omega_A = -\frac{1}{2}S_\gamma$, where γ is a geodesic. Our way of proving the singularity theorems is more cumbersome than what has been done in [14], but we did it because we were hoping that we could generalize the theorems to spacetimes with non totally anti-symmetric torsion. In that case the proofs in [14] are not valid anymore because they assumed particles to move on curves of maximal length instead of on geodesics. Only for totally anti-symmetric torsion both sets of curves are equal. Unfortunately, we had to conclude that it is impossible to generalize the Hawking-Penrose theorems to non totally anti-symmetric torsion in a very direct way (using the theorems that already exist). One has to come up with completely new arguments to attack this problem. We only succeeded in giving a construction of null geodesically incomplete spacetimes with vectorial torsion, by observing that we can map null geodesics with respect to the Levi-Civita connection to null geodesics with respect to the connection with vectorial torsion. It would be very interesting if one could generalize the singularity theorems to spacetimes with non totally anti-symmetric torsion.

We also reviewed the ways to derive the equations of motion for Einstein-Cartan theory. We compared the metric formalism, in which the metric and torsion are taken as dynamical variables and metric compatibility is assumed, with the metric-affine formalism, in which the metric and connection are taken as dynamical variables and no metric compatibility is assumed. It turned out that for the case of totally anti-symmetric torsion (this is the torsion one gets for the matter in the Standard model) the both formalisms are actually equivalent. Metric compatibility follows from the equations of motion in the metric-affine formalism. To derive this equivalence one has to use a gauge symmetry of the action and since this symmetry is probably broken when adding renormalization corrections to the action, the equivalence does possibly not hold anymore when one introduces quantum corrections. This part was not new, although there are a lot of mistakes in the literature. It would be interesting to see what happens with the conditions of the singularity theorems when one includes quantum corrections, so this is something for future work. If the two formalisms are not equivalent anymore, one has to use the metric formalism to make the translation from geometry to matter in the singularity theorems since in the proofs one uses metric compatibility extensively.

In general relativity one has the Bianchi identity and conservation of the energy-momentum tensor that follow from diffeomorphism invariance of the theory. We derived a generalization of these equalities to spacetimes with torsion.

We concluded by studying the singularity theorems in FLRW spacetimes, mostly with respect to an initial singularity. To do this we put torsion to zero. We first argued that one should not consider all geodesics when looking for an incomplete one, but that one should only consider the comoving ones (this part can also be found in our paper [15]). This is because non-comoving particles that follow an incomplete geodesics have an energy that blows up when going back to the past. This means that (if they kept following that geodesic) their energy will be larger than the Planck energy at some initial time at which point they form a black hole. This means that those particles do not reach the beginning of their incomplete trajectory which is why they should not indicate a singularity. After defining what a singularity is in FLRW spacetimes, we examined the relation between conjugate points and singularities which plays an important role in many of the singularity theorems. We showed that for a class of singular FLRW spacetimes all points on certain geodesics are conjugate to the point of the geodesic at the singularity. We also argued that when all points on a geodesic are conjugate to a certain point one must have a singularity. Lastly we showed under which condition a geodesic in an FLRW spacetime with flat spacelike three-surfaces has conjugate points. We have to mention that we used a slightly different definition of conjugate points. Normally, this is defined as two points along a geodesic where a Jacobi field vanishes. Loosely speaking, one can see this as a one-parameter family of geodesics leaving from one point and coming back at another point. We extended this definition and also included cases where the norm of a Jacobi field vanishes at two points along the geodesic (this is not equivalent at the singularity) because it is the distance between geodesics that matters. As far as we know the part on the relation between conjugate points and singularities is completely new. There is still some work to do, since we did not have time to treat all kind of models extensively. One can try to prove similar theorems as we did here for other classes of FLRW spacetimes. For instance examine under what conditions one has conjugate points in spacetimes with positive or negative curved spatial three-surfaces. One can also examine what happens in spacetimes that are small perturbations of FLRW spacetime.

After this we introduced torsion in FLRW spacetime as an energy density, combined it with a perfect radiation fluid and a perfect matter fluid, and examined what happened with singularities and conjugate points. This part was mainly meant as example and also to see what happens with the initial singularity. We actually found that one gets a bounce, but this has been noticed before and more rigorously, even including quantum corrections, e.g. [16].

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Appendix

In this appendix we intend to give a very brief overview of tensors and differential forms. We will omit all proofs, see e.g. [46] for a very thorough introduction to manifolds.

Tensors

Definition. Let V be a vector space. A *covariant k-tensor* T on V is a multi-linear function

$$T:\ V \times ... \times V \to \mathbb{R}$$

where we have k copies of V. The *vector space of covariant k-tensors* on V is denoted by $T^k V$

Definition. The *tensor product* $\otimes$ of a k_1-tensor T_1 and a k_2-tensor T_2 is

$$\left(T_1 \otimes T_2\right)\left(X_1, ..., X_{k_1+k_2}\right) = T_1(X_1, ..., X_{k_1}) T_2(X_{k_1+1}, ..., X_{k_1+k_2}).$$

Proposition. *Let (E_i) be a basis for a vector space V and let $\left(\epsilon^i\right)$ be its dual basis (so $\epsilon^j(E_i) = \delta_i^j$). Then the set of all covariant k-tensors of the form*

$$\epsilon^{i_1} \otimes ... \otimes \epsilon^{i_k}$$

forms a basis of $T^k V$.

Since the dual space V^* is also a vector space, we can give the following definition.

Definition. Let V be a vector space. A *contravariant l-tensor* T on V is a multi-linear function

$$T:\ V^* \times ... \times V^* \to \mathbb{R}$$

where we have l copies of V^*. The *space of contravariant l-tensors* on V is denoted by $T_l V$.

Definition. Let V be a vector space. A *mixed tensor* T of type (k,l) on V is a multi-linear function

$$T:\ V \times ... \times V \times V^* \times ... \times V^* \to \mathbb{R}$$

where we have k copies of V and l copies of V^*. The *space of mixed (k,l)-tensors* on V is denoted by $T_l^k V$.

Let now M denote an n-dimensional smooth manifold. For each point $p \in M$, $T_p M$ is a vectorspace with a corresponding space of k-vectors.

Definition. The *bundle of covariant k-tensors* on M is

$$T^k M = \coprod_{p \in M} T^k(T_p M);$$

The *bundle of contravariant l-tensors* on M is

$$T_l M = \coprod_{p \in M} T_l(T_p M);$$

The *bundle of mixed (k,l)-tensors* on M is

$$T_l^k M = \coprod_{p \in M} T_l^k(T_p M).$$

Definition. A *covariant k-tensor field* is a map

$$T: \; M \to T^k M.$$

This map can locally be expressed as

$$T = T_{\mu_1 \ldots \mu_k} dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_k}.$$

A covariant k-tensor field is smooth when the components $T_{\mu_1 \ldots \mu_k}$ are smooth in every coordinate chart.

Smooth contravariant l-tensorfields and smooth mixed (k,l)-tensorfields are defined in a similar way.

Definition. The *space of smooth covariant k-tensor fields* is denoted by

$$\mathcal{T}^k(M).$$

The *space of smooth contravariant l-tensor fields* is denoted by

$$\mathcal{T}^l(M).$$

The *space of smooth mixed (k,l)-tensor fields* is denoted by

$$\mathcal{T}_l^k(M).$$

An example of a covariant 2-tensor that we use over and over in this thesis is of course the metric.

Differential Forms

Differential forms are tensors that are totally anti-symmetric. They are needed to define integration on manifolds. We will again first focus on a vector space V with dimension n.

Definition. An *alternating k-tensor* on a vector space V is a covariant k-tensor T such that

$$T(X_1, ..., X_i, ..., X_j, ..., X_k) = -T(X_1, ..., X_j, ..., X_i, ..., X_k)$$

Denote the *space of alternating k-tensors* by $\Lambda^k(V)$.

Notice that the space of these tensors is again a vector space. We will now define a basis for this vector space.

Proposition. *Let $I = (i_1, ..., i_k)$ such that $1 < i_1 < ... < i_k < n$ (this is called a multi-index). The covariant k-tensors*

$$\epsilon^I(X_1, ..., X_k) = \det \begin{pmatrix} \epsilon^{i_1}(X_1) & . & . & . & \epsilon^{i_1}(X_k) \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ \epsilon^{i_k}(X_1) & . & . & . & \epsilon^{i_k}(X_k) \end{pmatrix}$$

form a basis of $\Lambda^k(V)$.

We will now define a product on the space of alternating tensors.

Definition. Let S_k denote the symmetric group. The *alternating projection of a covariant k-tensor T* is given by

$$\text{Alt}(T)(X_1, ..., X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign}(\sigma) T(X_{\sigma(1)}, ..., X_{\sigma(k)}).$$

Definition. The *wedge product* of a tensor $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$ is

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!} \text{Alt}(\omega \otimes \eta).$$

Lemma. *For multi-indices I, J,*

$$\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}$$

where $IJ = (i_1, ..., i_k, j_1, ..., j_l)$.

Proposition. *The wedge product is bilinear, associative and anti-commutative: for $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$*

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega.$$

Furthermore for a multi-index I

$$\epsilon^{i_1} \wedge ... \wedge \epsilon^{i_k} = \epsilon^I,$$

and for any covectors ω^1, $\omega^k \in V^*$.

$$\omega^1 \wedge ... \wedge \omega^k (X_1, ..., X_k) = \det(\omega^i(X_j)).$$

We now turn to the definition of differential forms on a manifold M.

Definition. The *bundle of alternating k-tensors* on M is denoted by

$$\Lambda^k M = \coprod_{p \in M} \Lambda^k(T_p M)$$

and a *differential k-form* or a *k-form* is defined by a smooth tensor field

$$\omega : \; M \to \Lambda^k M.$$

The *space of k-forms* is denoted by

$$\mathcal{A}^k(M).$$

Definition. Let $V \in \mathcal{T}(M)$. *Interior multiplication* is the map

$$\begin{aligned} i_V : \; \mathcal{A}^k(M) \; &\to \; \mathcal{A}^{k-1}(M); \\ i_V \omega(Y_1, ..., Y_k) \; &= \; \omega(V, Y_2, ..., Y_k). \end{aligned} \tag{6.174}$$

Lemma. *Let $V \in \mathcal{T}(M)$ and $\omega^1, ..., \omega^k \in \mathcal{A}^1$. Then*

$$i_V \left(\omega^1 \wedge ... \wedge \omega^k\right) = \sum_i (-1)^{i-1} \omega^i(V) \omega^1 \wedge ... \wedge \omega^{i-1} \wedge \omega^{i+1} \wedge ... \wedge \omega^k.$$

Definition. *Exterior differentiation* is the map, locally expressed by

$$\begin{aligned} d : \; \mathcal{A}^k(M) \; &\to \; \mathcal{A}^{k+1}(M); \\ d\left(\sum_I \omega_I dx^{i_1} \wedge ... \wedge dx^{i_n}\right) \; &= \; \sum_I \sum_i \frac{\partial \omega_I}{\partial x^i} dx^i \wedge dx^{i_1} \wedge ... \wedge dx^{i_n}. \end{aligned} \tag{6.175}$$

Definition. Let $\phi : \; M \to N$ be a smooth map and ω a k-form on N. The *pullback* $\phi^* \omega$ of ω is

$$\phi^* \omega(X_1, ..., X_k) = \omega(\phi_* X_1, ..., \phi_* X_k),$$

where $(\phi_* X_1)(f) = X_1(f \circ \phi)$.

Integration on Manifolds

On an n-dimensional manifold integration of n-forms can be defined. The manifold should also have an orientation and the form should be compactly supported, but we will not go into the details of how exactly integration is defined. First of all, notice that an n-form ω always (locally) has the form

$$\omega = f dx^1 \wedge ... \wedge dx^n$$

where f is a smooth function. $dx^1 \wedge ... \wedge dx^n$ is often denoted as $d^n x$, such that integrals have the expression

$$\int_V \omega = \int_V f d^n x,$$

where $V \subset M$. We now state two propositions, which are very important.

Proposition. *Diffeomorphism invariance. Let $\phi :\ M \to N$ be a diffeomorphism. Then*

$$\int_N \omega = \int_M \phi^* \omega.$$

Theorem. *Stokes theorem. Let $\omega \in \mathcal{A}^{n-1}(M)$, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$