UNIVERSITY OF UTRECHT

MASTER THESIS

Supersymmetric Sigma Models and Generalized Complex Geometry

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Chapter 1

Introduction

Supersymmetry has become an important concept in modern theoretical physics, especially in the search of unified theories. Its most important aspect is the symmetry between bosons and fermions, physical particles with integer and half–integer spin. One of the most notable occurrences of supersymmetry is in particle physics, where the supersymmetric standard model predicts the existence of a superpartner for every particle in the standard model. These superpartners would, among others, provide theoretical arguments for the qualitative differences between different kind of forces. Also very important, but perhaps more distant from the experimental world, is the role supersymmetry plays in string theory. The superstring, which is the model for string theory incorporating supersymmetry, has the advantage that it does not predict the existence of a bad behaving particle called the Tachyon. Beside these features, there is also a mathematical motivation to consider supersymmetry. It has been proved by Coleman and Mandula [12] in 1967 that under some technical conditions, there are no nontrivial extensions of the Poincaré algebra, which is the Lie algebra of the group of orthogonal symmetries and translations of space–time. However, Haag, Lopuszanski and Sohnius [20] discovered around 1975 that this theorem can be circumvented via the theory of superalgebras. They proved that the most general superalgebras extending the Poincaré algebra are given by the supersymmetry algebras which are, roughly speaking, classified by an integer N. In this sense, supersymmetry is the obvious next step after considering the usual space–time symmetries.

Beside their physical relevance, supersymmetric field theories turned out to be intimately related to geometry. One of the first who realized this was Zumino in 1979, who described in [38] non–linear sigma models on Kähler manifolds. Two years later, Alvarez–Gaume and Freedman [1] gave a classification of supersymmetric sigma models in terms of geometric structures. These sigma models consist of maps between a two–dimensional space called the *world–sheet* and some target space, taken to be a manifold in this setting. They established a connection between the amount of supersymmetry on this model and the type of geometry on the target space, which turned out to belong to the area of complex geometry. As most important case, the (2,2)–supersymmetric sigma model is associated to Kähler geometry. Somewhat later in 1985, Gates, Hull and Roček investigated in [16] a wider class of sigma models admitting the so–called B–fields, and they determined the associated geometries.

The most notable of these are the bi–hermitian structures, associated to (2,2)–supersymmetry. In the following years, little progress was made in finding examples of bi–hermitian manifolds that are not Kähler, until around 20 years later in 2003 when the field of generalized complex geometry was invented by Hitchin and Gualtieri [18]. They generalized the concept of Kähler geometry to generalized complex geometry, which turned out to be equivalent to bi–hermitian geometry. Using this generalized language, interesting new examples were produced and new insights into their general properties were obtained.

The link between supersymmetry and geometry became even stronger after Witten [36] invented the so–called topological twist in 1988. Twisting the $(2, 2)$–supersymmetric sigma model produces a topological field theory, a theory whose physical outcome is independent of the metric on the world–sheet of the model. Witten performed this twist for Kähler manifolds in the absence of B–fields and discovered two different models, called the A and the B model, where the B twist can only be performed if the Kähler structure is in fact Calabi–Yau. The motivation behind the twist is that in a topological field theory one can compute certain physical quantities more easily than in the original theory, where we sometimes lack the tools to compute them exactly. On the topological side the physical observables have the structure of a cohomology ring whose correlators, which are the physical quantities mentioned earlier, are related to geometric data such as intersection numbers on the manifold. In 2004, the topological twist for models with B–field was performed by Kapustin and Li [26], who made explicit use of the generalized complex point of view. They discovered that the twist can only be performed on generalized Calabi–Yau spaces, which include symplectic and ordinary Calabi–Yau structures (hence the A and the B model) as special cases. The physical cohomology ring is given by the Lie algebroid cohomology of the generalized complex structures, which on generalized Calabi–Yau spaces can be seen as an interpolation between ordinary de Rham cohomology and Dolbeault cohomology. In this sense, the idea of generalized geometry as a unification of complex and symplectic structures is the mathematical analogue of the development of sigma models with flux.

This thesis is organized as follows. We start in Chapter 2 with a general introduction to sigma models, and specify to the model we will be interested in. To this end we describe some basic ingredients from string theory and explain in some detail what kind of model comes out. Following up, an introduction to the concept of supersymmetry is given, focusing mainly on the relevant case of two dimensions. Representations of the so–called (p, q)–supersymmetry algebra are constructed using a concept called *superspace*. As mentioned above, incorporating these representations on our model enforces some geometrical constraints on the target space, and this relation between the type of supersymmetric representations and the types of geometries of the target space is the central theme of this thesis. The focus in this chapter is on the conceptual, and the computations will have to wait for Chapter 4. In this chapter there is not yet a B–field, and the main reference is [1].

In Chapter 3 we introduce the relevant mathematical theory, called *generalized complex geometry*, whose main contributors are Hitchin, Gualtieri and Cavalcanti. To make a clear distinction between the algebraic and differential structures we start with a linear algebraic discussion, and introduce the

Clifford algebra and the space of spinors for vector spaces of the form $V \oplus V^*$, where V is a real finite dimensional vector space. We then proceed to a manifold M, and introduce the Courant bracket on $T \oplus T^*$, the tangent plus cotangent bundle of M. After the necessary ingredients are in place, we introduce generalized complex structures, and following up, generalized Kähler structures of which we also give an explicit example. In the end of this chapter we introduce the notion of a gerbe, which is the mathematical theory behind the B–field. Most of the relevant material can be found in [18].

The technical computations that explicitly relate supersymmetry and geometry are the subject of Chapter 4. In the presence of B–fields, these have first been done by Gates, Hull and Roček in [16]. Care has been taken to present the calculations in such a way that a minimal effort is needed to follow all the steps, without exaggerating on the amount of tedious intermediate steps. Readers not interested in these detailed calculations can safely skip this part, as only the final outcome is relevant to the other chapters.

Finally, Chapter 5 is centered on the *topological twist*. We begin with a discussion about quantization, in order to explain what it means for a theory to be topological. After this physical intermezzo, we come back to the sigma model and discuss the problems that arise when the world–sheet is generalized to an arbitrary Riemann surface, and how the topological twist resolves these issues. The part of the $(2, 2)$–algebra that survives the twist gives rise to a nilpotent operator Q, and we end the chapter with a calculation of its associated cohomology ring. The main references for this chapter are [36] and [26].

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I would like to thank my two supervisors Gil Cavalcanti[1] and Stefan Vandoren[2] for their supervision during this project. In the combined area of physics and mathematics, it is often difficult to make the translation between both areas, especially because of the great differences in literature. Both Gil and Stefan were of great help in overcoming this issue, and I especially enjoyed our joint meetings, in which the three of us together faced this 'cultural barrier'. The common interest they have in both areas made them well suited for supervising this project.

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Chapter 2

Sigma models

A sigma model is a theory studying maps from one geometric space to another, with the purpose of describing some physical process. These spaces usually come equipped with extra geometric structures, depending on the particular type of model. For us these geometric spaces will always be manifolds, perhaps with a smooth boundary. Phrased in such a general language, a lot of physical processes are described by sigma models. Think of classical mechanics for instance, which can be regarded as the study of maps from an interval to a symplectic manifold, or, perhaps, a field theory where one considers the sections of some vector bundle over space–time.

As diverse as these models can be, one feature that they share is the existence of an action[1]. If we let X be the space of maps for the sigma–model, the action is a map $S : X \to \mathbb{C}$ that describes the physical behavior of the system. At the classical level, these laws follow from the stationary points of S, in the sense that the physical solutions are those for which the action has an extremal value. It is often the case that the maps of the sigma model, which from now on we refer to as *fields*, are all defined on a 'source' manifold that we appropriately denote by Σ, while the targets are perhaps varying for the different kinds of fields. In these cases the action can be written in terms of a more local quantity L, called the Lagrangian density, which is a function from the space of fields to the space of densities on Σ. This Lagrangian usually looks like a partial differential operator, i.e. a polynomial in the fields and their derivatives. The action can then be written as an integral of L over Σ.

As an easy example, consider a set of n functions $\varphi^1, \ldots, \varphi^n$ on $\mathbb{R}^m$, acting for the moment as Σ. The action is given by $\int_{\mathbb{R}^m} L(x, \varphi^i, \partial_\mu \varphi^j, \ldots) dx$, where the Lagrangian L in general depends on the space–time point x, and on φ^i and all its derivatives, although in most cases it does not depend on third derivatives or higher. For $(\varphi^1, \ldots, \varphi^n)$ to form a stationary point, we insist that $\frac{d}{dt} S(\varphi + t\gamma)|_{t=0} = 0$ with γ^i arbitrary functions with compact support. This equation translates in

$$0 = \frac{d}{dt} S(\varphi + t\gamma)|_{t=0} = \int_{\mathbb{R}^m} \sum_{i,\alpha} \frac{\delta L}{\delta(\partial_\alpha \varphi^i)} \partial_\alpha \gamma^i = \int_{\mathbb{R}^m} \sum_{i,\alpha} (-1)^{|\alpha|} \partial_\alpha \left(\frac{\delta L}{\delta(\partial_\alpha \varphi^i)} \right) \gamma^i, \qquad (2.0.1)$$

[1] The word 'action' is appropriate as its dimension equals energy times time, and has nothing to do with group actions.

where the α summation is over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $|\alpha| := \alpha_1 + \ldots + \alpha_m$, and we use the short-hand notation $\partial_\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_m}^{\alpha_m}$. So e.g., $\partial_{(0,\ldots,0)}\varphi^i = \varphi^i$. In the third equality we performed a partial integration, which is allowed as the γ^i have compact support. Note that $\frac{\delta L}{\delta(\partial_\alpha \varphi^i)}$ stands for the partial differentiation to the variable $\partial_\alpha \varphi^i$. As the γ^i are arbitrary, this equation implies the Euler–Lagrange equations

$$\sum_\alpha (-1)^{|\alpha|} \partial_\alpha \left(\frac{\delta L}{\delta(\partial_\alpha \varphi^i)} \right) = 0. \tag{2.0.2}$$

It is remarkable that all known physical theories allow for a description in terms of an action, and perhaps more peculiar is that in the quantum theory not only the critical points, but the entire behavior of S becomes important (cf. Chapter 5).

2.1 String theory as a two–dimensional sigma model

Although there are many examples of sigma models appearing everywhere in physics, we will be focusing on one particular type. Let Σ be a two–dimensional manifold, possibly with boundary, and let M be a compact manifold equipped with a Riemannian metric. The maps of this sigma model will be smooth maps $\varphi : \Sigma \to M$, later to be extended with fields of a different type to create a super–symmetric theory (cf. Section (2.3)). The main reason for studying this particular model comes from string theory, so we first give a short introduction to string theory and explain how it gives rise to this model.

String theory is the generalization of point–particle physics to extended one–dimensional objects called strings. String theory was invented to provide a unifying theory of quantum mechanics and general relativity, two theories that seemingly cannot be merged together within the framework of point particles. Strings can be either closed or open, as a one–dimensional compact manifold is either a circle or a closed interval. If we denote by M our space–time, a pseudo-Riemannian manifold of signature $(1, n-1)$ (1 being the 'negative direction'), the configuration space of a string propagating in M is given by $LM := \{\gamma : S^1 \to M\}$ (the loop space of M) for closed strings and $PM := \{\gamma : [0,1] \to M\}$ (the path space of M) for open strings.

As a string propagates through M, it sweeps out a two–dimensional region, which is referred to as the world–sheet of the string. To be more precise, we define the world sheet for the non–interacting closed string to be a cylinder $S^1 \times [\tau_0, \tau_1]$. The coordinate $\tau \in [\tau_0, \tau_1]$ should be thought of as the eigentime of the string, and is not directly related to time of an external observer. We use the coordinate σ for the space-like direction, i.e. the S^1 part of the world–sheet. The propagation of the string is now described by a map $\varphi : \Sigma \to M$, and as such we recognize in string theory a two–dimensional sigma model, but note that M above is not yet compact nor Riemannian. Analogously there is a sigma model for the open string, but for this entire thesis we will only be interested in closed strings.

What action should be written down for such strings? When describing a theory extending some

older theory, an obvious requirement is that in the limit of the old theory, one should obtain the same results. Hence, we should look for inspiration at the action for a point–particle, whose action is proportional to the length of the worldline γ (the trajectory the particle follows):

$$S(\gamma) = -m \int_{[\tau_0,\tau_1]} \sqrt{-\langle \gamma'(\tau), \gamma'(\tau) \rangle} d\tau, \tag{2.1.1}$$

where we assume that the particle follows a time-like path $\langle \gamma'(\tau), \gamma'(\tau) \rangle \leq 0$, which means that it is moving slower than the speed of light. To see that this is the correct action, take $M = \mathbb{R}^{1,3}$ and parameterize the path via $\gamma(t) = (t, \overrightarrow{x}(t))$, so that the above formula reduces to

$$S(\gamma) = -m \int_{[t_0,t_1]} \sqrt{1 - (\dot{\overrightarrow{x}}(t))^2} dt. \tag{2.1.2}$$

From this we can read off the momentum p_i, defined by $p_i := \frac{\delta L}{\delta \dot{x}_i}$, and the Hamiltonian (the energy) which is defined via $H := p_i \dot{x}_i - L$, and they are given by the well known expressions

$$\overrightarrow{p} = \frac{m \dot{\overrightarrow{x}}}{\sqrt{1 - (\dot{\overrightarrow{x}})^2}}, \; H = \sqrt{m^2 + \overrightarrow{p}^2}. \tag{2.1.3}$$

The obvious analogue for a string is the area of its embedded worldsheet, i.e. the area of $\varphi(\Sigma)$, which can be written as

$$S(\varphi) = -T \int_\Sigma d\sigma d\tau \sqrt{-\det \varphi^* g}, \tag{2.1.4}$$

where g is the metric on M and T is a constant related to the tension of the string, and the pullback of the metric is defined via $\varphi^* g_p(v, w) = g_{\varphi(p)}(d_p\varphi(v), d_p\varphi(w))$. This action is called the Nambu-Goto action ([19]).

The square root in the Nambu-Goto action makes it complicated to quantize[2], as it prevents us from using the standard path–integral techniques for perturbation series (Feynman diagrams), a problem which can be solved as follows. Instead of pulling back the metric g, we consider an independent metric h on Σ, and define the Polyakov action ([19]):

$$S(\varphi) = -\frac{1}{4\pi\alpha'} \int_\Sigma \sqrt{-h} h^{\alpha\beta} g_{ij} \partial_\alpha \varphi^i \partial_\beta \varphi^j d\tau d\sigma, \tag{2.1.5}$$

where α, β denote the coordinates τ, σ on Σ, $i, j, \ldots$ are coordinates on M, and h is the determinant of $h_{\alpha\beta}$. If we put h on–shell, meaning that we use the equation of motion as imposed by the action, the Polyakov action reduces to the Nambu-Goto action, and in this sense both actions are classically equivalent. In the quantum theory, this is not necessarily the case. For aesthetic reasons we drop the proportionality constants, since they play no important role here, and the action (2.1.5) can then be written mathematically in the form

$$S(\varphi) = \int_\Sigma g(d\varphi \,\hat{,}\, \star\, d\varphi), \tag{2.1.6}$$

[2]Quantization stands for the process of passing from a classical theory to a quantum theory.

where the 2–form $g(d\varphi \,\hat{,}\, \star\, d\varphi)$ is defined by

$$g(d\varphi \,\hat{,}\, \star\, d\varphi)(v, w) = g(d\varphi(v), \star d\varphi(w)) - g(d\varphi(w), \star d\varphi(v)),$$

with the orientation on Σ given by the ordered pair τ, σ, and $\star$ denoting the Hodge star induced by the metric h and the specified orientation.

A well known feature of string theory is the existence of a critical dimension. If a classical theory has gauge symmetries, which for strings is the conformal symmetry of the metric h, these symmetries can be broken after quantization. In that case one says that the theory has obtained an anomaly. For strings this is the case, and the particular type of string theory we described above, called the bosonic string, has a conformal anomaly except in dimension 26. When we discuss supersymmetry, we will come to the definition of the superstring, whose critical dimension is known to be 10. This makes the latter a better candidate for a unifying theory, but in either case, there are some extra dimensions that somehow are hidden to us. One way to hide them is via compactification. One assumes that somehow these extra dimensions take the form of compact spaces, whose typical length scales are so small that they are almost invisible. It is this compact part that we are interested in, and we will assume that space–time is of the form $N \times M$, where N is a four–dimensional pseudo–Riemannian manifold describing our visible space–time, while M is a compact Riemannian manifold describing the compact, 'hidden' part. Furthermore, we assume that the propagation of a string can be described by studying separately its behavior in N and M. Effectively, if we write $\varphi = (\varphi^1, \varphi^2)$ with respect to the splitting $N \times M$, the action splits as

$$S(\varphi) = S_1(\varphi^1) + S_2(\varphi^2). \tag{2.1.7}$$

So the fields of the sigma model we get from bosonic string theory are maps $\varphi : \Sigma \to M$ from compact surfaces Σ to compact Riemannian manifolds (M, g), and metrics h on Σ. Finally, to simplify matters we also assume these surfaces to be oriented.

2.2 What is supersymmetry?

As already mentioned in the introduction, for certain physical reasons it is advantageous to impose supersymmetry on the sigma model. Before we show how to do this, we spend a few words on the general concept of supersymmetry.

If one would ask a physicist what supersymmetry is, he or she would most likely phrase the answer in terms of interchanging bosons or fermions. Although this is perfectly fine, let us phrase the answer in more mathematical terms. A relativistic theory on flat Minkowski space ($\mathbb{R}^{1,d-1}$) is invariant under the Poincaré group, which is defined as the group of isometries and translations of Minkowski space. The natural question to ask for any theory is then, are there more symmetries beside these? Of course there can be much more, but a 'no-go' theorem due to Coleman and Mandula [12] tells us that under

some mild conditions on the scattering matrix[3], and the existence of a mass gap[4], these symmetries must be Lorentz scalars. To explain what Lorentz scalar in this context means, let us look at the Lie algebra of the full symmetry group. The Poincaré algebra sits in there as a subalgebra, and the Coleman–Mandula Theorem states that the full Lie algebra must be the direct sum of the Poincaré algebra and the other symmetries under consideration. It thus tells us that any generator of the extra symmetries commutes with the Poincaré generators, and in this sense form Lorentz scalars: invariant under Lorentz transformations (perhaps 'Poincaré scalar' would be a more appropriate name).

Now that we know that we cannot, in a non–trivial way, extend the Poincaré algebra to a bigger Lie algebra, the next question is whether this is possible if we consider Lie superalgebras, and that turns out to be the case. Firstly, recall that a Lie superalgebra consists of a vector space V (for us over the field $\mathbb{C}$) with a decomposition into an even and an odd part;

$$V = V_0 \oplus V_1, \tag{2.2.1}$$

equipped with a graded Lie bracket, i.e. a bilinear map

$$[,] : V \times V \to V, \tag{2.2.2}$$

satisfying

$$\begin{aligned} &\text{Graded anti–symmetry :} \quad [X,Y] = -(-1)^{ab}[Y,X] \\ &\text{Graded Jacobi identity :} \quad (-1)^{ac}[X,[Y,Z]] + (-1)^{cb}[Z,[X,Y]] + (-1)^{ba}[Y,[Z,X]] = 0, \end{aligned} \tag{2.2.3}$$

for homogeneous elements $X \in V_a, Y \in V_b$ and $Z \in V_c$, with $a, b, c \in \{0, 1\}$.

The structure of such a Lie superalgebra containing the Poincaré algebra is restricted due to the Coleman–Mandula Theorem and the two constraints (2.2.3), and it turns out ([20]) that the most general form is given by the usual Poincaré generators, plus N odd spinorial generators Q^a ($a = 1, \ldots, N$), also referred to as supercharges, and at most $\frac{1}{2}N(N-1)$ even generators, referred to as the central charges because they lie in the center of the algebra. Spinorial means that each Q^a itself is a spinor, so it has $2^{\lfloor \frac{d}{2} \rfloor}$ components Q^a_α in d dimensions ($\lfloor \frac{d}{2} \rfloor$ denoting the largest integer smaller or equal than $\frac{d}{2}$), and it behaves as a spinor with respect to the Lorentz generators. Here N is a positive number, and we refer to the above algebra as the N extended Poincaré superalgebra, or N extended supersymmetry.

For the precise general expression of all the generators and relations we refer to [17], Section 3.2c. We shall only be considering this algebra in two dimensions without central charges, in which case this algebra simplifies considerably. Also, in two dimensions the spin representation is reducible (see appendix (A)), which implies that each spinorial charge Q^a has two components $Q^a_\pm$, each component living in a one–dimensional irreducible representation of $Spin(1,1)$. Therefore, there is no need

[3] The matrix of correlation functions between initial and final states.

[4] A nonzero energy difference between the vacuum and the first excited energy state.

to consider the Q^a as a single, spinorial object, but we can consider its components separately. In particular, it is not necessary to take as many $+$ as $-$ components, so that in two dimensions there is the notion of (p,q) extended supersymmetry, p standing for the number of Q^a_+ charges and q for the number of Q^a_- charges. In summary, the algebra we will be interested in is the following (p,q) extended Poincaré superalgebra, where $p,q \in \mathbb{Z}_{\geq 0}$:

$$\begin{aligned} &\text{Even generators:} && L, P_\pm, \\ &\text{Odd generators:} && Q^1_+, \ldots, Q^p_+, Q^1_-, \ldots, Q^q_-, \\ &\text{Relations:} && [L, P_\pm] = \mp P_\pm, \;\; [L, Q^a_\pm] = \mp \frac{1}{2} Q^a_\pm, \;\; \{Q^a_\pm, Q^b_\pm\} = \delta^{ab} P_\pm. \end{aligned} \tag{2.2.4}$$

The generator L denotes the Lorentz generator, which is just a boost in the spacial direction[5], and $P_\pm$ denote the translation generators (more commonly referred to as the momentum generators) in the light–cone directions $\sigma^\pm = \sigma \pm \tau$. The $\pm$ indices for the supercharges Q have a different origin, as these denote the components of the spinor Q.

2.3 Superspace and (1,1)–supersymmetric sigma models

To define a supersymmetric field theory, we need to represent the Poincaré superalgebra as symmetries acting on the fields, and for us it will suffice to represent the algebra on–shell, i.e. for fields satisfying their equations of motion. An elegant approach to construct these representations was invented by Salam and Strathdee [31], using the formalism of superspace. To understand their construction, observe that the natural way to represent the ordinary Poincaré algebra is by its action on space time or, what is more convenient when considering fields on space time, by its induced action on the functions on space time. Recall that if a Lie group G acts on a space M, it inherits a group action on the set of functions on M, defined by $(g \cdot f)(x) := f(g^{-1} \cdot x)$ where $g \in G, x \in M, f : M \to \mathbb{C}$. Differentiating this action gives a representation of the Lie algebra of G on this space of functions, and we start with this representation on Σ with coordinates[6] $\sigma^\pm := \sigma \pm \tau$ and

$$L = \sigma^+ \partial_+ - \sigma^- \partial_-, \qquad\qquad P_\pm = -2i\partial_\pm, \tag{2.3.1}$$

where $\partial_\pm := \frac{\partial}{\partial \sigma^\pm}$. The particular factors and minus signs are conventional, but one quickly verifies that this defines a representation of the Poincaré algebra in 2 dimensions. Now we want to extend this to a representation of the Poincaré superalgebra, and the trick is to introduce extra coordinates. To also incorporate the fermionic character of the supercharges, we let those coordinates be anti–commuting Grassmann numbers. These will be denoted by $\theta^+_1, \ldots, \theta^+_p$ and $\theta^-_1, \ldots, \theta^-_q$. They transform under the action of L as the components of a spinor, i.e. $\theta^\pm_a \to e^{\pm\alpha/2}\theta^\pm_a$, see also appendix A. This immediately suggests the following correction to the Lorentz generator

$$L = \sigma^+ \partial_+ - \sigma^- \partial_- \longrightarrow L = \sigma^+ \partial_+ - \sigma^- \partial_- + \frac{1}{2}\theta^+_a \frac{\partial}{\partial \theta^+_a} - \frac{1}{2}\theta^-_a \frac{\partial}{\partial \theta^-_a}, \tag{2.3.2}$$

[5] Hyperbolic rotation in the (τ, σ)–plane.

[6] Recall that until now Σ is either a cylinder or a strip, which are both flat and have global coordinates τ, σ.

where a summation over a is understood. Functions on both the even and odd coordinates are called *superfields* and we assume these to be analytic in the odd coordinates. As these square to zero, this means that every superfield is expressed through a finite power series in the θ's. Furthermore, we will only consider superfields of fixed statistics, meaning that all homogeneous terms in the θ's are all either bosonic or fermionic. We define odd derivatives in the obvious way:

$$\frac{\partial}{\partial\theta_a^\pm}\theta_b^\pm = \delta_{ab}, \qquad \frac{\partial}{\partial\theta_a^\pm}\theta_b^\mp = 0, \tag{2.3.3}$$

and we extend them as odd derivations to arbitrary polynomials in the θ's. Likewise we define an integration, which is ordinary integration for the even coordinates and equal to differentiation for the odd coordinates (so $\int d\theta_a^\pm\theta_b^\pm = \delta_{ab}$ etc.). With these coordinates and derivatives there are a number of operators we can pick to define the Q's, but looking at the relations (2.2.4) we see that such operators must have spin $\pm\frac{1}{2}$. This basically restricts us to the operators $\theta_a^\pm\partial_\pm$ and $\frac{\partial}{\partial\theta_a^\pm}$. Each of these has the right commutation with the Lorentz generator, but will not square to the momentum operators $P_\pm$. Therefore we decompose them into the following operators

$$Q_\pm^a = \frac{\partial}{\partial\theta_a^\pm} - i\theta_a^\pm\partial_\pm \quad \text{and} \quad D_\pm^a = \frac{\partial}{\partial\theta_a^\pm} + i\theta_a^\pm\partial_\pm. \tag{2.3.4}$$

We wrote $Q_\pm^a$ for the first operator, suggestively indicating that this will define the representation of the supercharges. Indeed, one quickly verifies that with these choices for L, $P_\pm$ and $Q_\pm^a$ we have a representation of the (p,q) Poincaré superalgebra. For later use we give the anti–commutator of the D's, which is opposite to that of the Q's:

$$\{D_\pm^a, D_\pm^b\} = -\delta^{ab}P_\pm. \tag{2.3.5}$$

So far we did not use the explicit form of our sigma model, and the representation theory above can be applied to more general two–dimensional field theories. Moreover, there are no constraints on such a field theory to have this supersymmetric representation. Let us now specialize to the sigma model of Section 2.1. In local coordinates $i, j, \ldots$ on M the map φ is described by n functions $\varphi^1, \ldots, \varphi^n$, and we can regard them as for instance the leading terms of bosonic superfields Φ^i. Imposing (p,q) supersymmetry on these Φ^i straightaway using the above construction leads to a large number of new fields, as the expansion of Φ^i in the odd variables in (p,q) superspace gives 2^{p+q} coefficients. Furthermore, it is a priori not clear what we must write down for the action, in order for the bosonic part to reduce to (2.1.5). For these reasons we first try to incorporate $(1,1)$ supersymmetry on our sigma model, where for the moment we gauge fix the metric h to be flat

$$h_{+-} = h_{-+} = 0, \qquad h_{++} = h_{--} = \frac{1}{2}, \tag{2.3.6}$$

and forget that it represents a dynamical field, so that our action reduces to

$$S(\varphi) = \int_\Sigma g_{ij}\partial_+\varphi^i\partial_-\varphi^j d\sigma^+ d\sigma^-, \tag{2.3.7}$$

in coordinates $\sigma^\pm = \sigma \pm \tau$. Now we extend the bosonic fields ϕ^i to bosonic superfields Φ^i, whose taylor expansions equal

$$\Phi^i = \varphi^i + \theta^+\psi_+^i + \theta^-\psi_-^i + \theta^-\theta^+F^i. \tag{2.3.8}$$

The new fields $\psi_\pm^i$ and F^i appear as a consequence of the superspace construction, and since Φ^i is bosonic, the $\psi_\pm^i$ are fermionic and the F^i are bosonic. We shall see later that the F^i do not give new physical degrees of freedom, and therefore are referred to as auxiliary fields. The most obvious candidate for the action in analogy to (2.3.7) is

$$S(\Phi) = \int_\Sigma d\sigma^+ d\sigma^- d\theta^+ d\theta^- g_{ij} D_+\Phi^i D_-\Phi^j. \tag{2.3.9}$$

This action has by construction a manifest symmetry generated by Q;

$$\delta_\epsilon \Phi^i := \epsilon^\alpha Q_\alpha \Phi^i = \epsilon^+ Q_+ \Phi^i + \epsilon^- Q_- \Phi^i, \tag{2.3.10}$$

where ϵ is a constant spinor. We shall often abbreviate terms as $\epsilon^\alpha Q_\alpha$ by ϵQ. This symmetry is manifest, because the effect of this symmetry on the action leads to a total derivative. We can perform the odd integrations in (2.3.9) to rewrite everything in terms of the physical fields φ^i and ψ^i, where F is put on its equation of motion. To recall, this means that we determine via the action the equation of motion for F (obtained by varying the field F and looking for stationary points of S), which turns out to be exactly solvable in terms of the other fields, and then substitute the result back in the action. The end result is given by

$$S(\varphi,\psi) = \int_\Sigma d\tau d\sigma \left(g_{ij}\partial_+\varphi^i\partial_-\varphi^j + i g_{ij}\psi_+^i \nabla_-\psi_+^j + i g_{ij}\psi_-^i\nabla_+\psi_-^j + \frac{1}{2}R_{ijkl}\psi_+^i\psi_+^j\psi_-^k\psi_-^l \right), \tag{2.3.11}$$

with $\nabla_\pm \psi_\alpha^i = \partial_\pm\psi_\alpha^i + \Gamma^i_{jk}\partial_\pm\varphi^j\psi_\alpha^k$, Γ^i_{jk} the Christoffel symbols for the Levi-Cevita connection and $R^i{}_{jkl}$ its curvature. The supersymmetry in components is given by

$$\begin{aligned} \delta_\epsilon\varphi^i &= \epsilon\psi^i, \\ \delta_\epsilon\psi_+^i &= i\epsilon^+\partial_+\varphi^i - \epsilon^-\Gamma^i_{jk}\psi_-^j\psi_+^k, \\ \delta_\epsilon\psi_-^i &= i\epsilon^-\partial_-\varphi^i - \epsilon^+\Gamma^i_{jk}\psi_+^j\psi_-^k. \end{aligned} \tag{2.3.12}$$

From now on, such expressions will be abbreviated with $\pm$ symbols. For instance, the last two equations in (2.3.12) are combined into a single equation:

$$\delta_\epsilon\psi_\pm^i = i\epsilon^\pm\partial_\pm\varphi^i - \epsilon^\mp\Gamma^i_{jk}\psi_\mp^j\psi_\pm^k. \tag{2.3.13}$$

When reading this equation, the sign is always taken to be the top or the bottom one.

To understand the global form of (2.3.11), we first need to know what global object the $\psi_\pm^i$ represent. The combination $\theta^\pm\psi_\pm^i$ in the bosonic superfield (eqn. (2.3.8)) implies that these fields behave under Lorentz transformation as $\psi_\pm \mapsto e^{\mp\alpha/2}\psi_\pm$. By convention we allow spin indices to be raised and lowered using the rules $\theta^\alpha = C^{\alpha\beta}\theta_\beta$ and $\theta_\alpha = \theta^\beta C_{\beta\alpha}$, where

$$C_{\alpha\beta} = -C^{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

When calculating with these quantities, one should keep in mind that objects transform according to their index, with lower indices transforming oppositely to upper indices. As such, the $\psi^i_\pm$ form the components of a spinor ψ^i, and the $i, j, \ldots$ indices indicate that these spinors also form the components of an object with tangent indices to M. Putting this together, we deduce that the appropriate bundle for ψ to live in is

$$S_\Sigma \otimes \varphi^*(TM), \tag{2.3.14}$$

with S_Σ the trivial spin bundle of Σ with respect to the flat metric h. This spin bundle decomposes into $S_\Sigma = S^+ \oplus S^-$, so that in a local coordinate frame $\partial_i := \frac{\partial}{\partial x^i}$ on M we can write $\psi = \psi^i_+ \otimes \partial_i + \psi^i_- \otimes \partial_i$. As the components $\psi^i_\pm$ must be fermionic, i.e. anti–commuting with other fermionic objects, ψ is a section of the bundle $S_\Sigma \otimes \varphi^*(TM)$ with 'parity reversed fibers'. Parity in this context has nothing to do with the symmetry that goes by the name of parity, which is about reflections of the space coordinate on the world–sheet (cf. the text below Equation (2.4.3)). Instead, parity in this context refers to the fact that ψ is an anti–commuting object. Perhaps a good way to think about this, although mathematically not very rigorous, is as follows. Let $\{s_\alpha\}_{\alpha \in I}$ be a basis for the space of sections of $S_\Sigma \otimes \varphi^*(TM)$, I being some index set. An ordinary section can then be expressed as $s = \sum_{\alpha \in I} \lambda^\alpha s_\alpha$, where the sum is not really well defined in this setting. The λ^α are real numbers, and for a fermionic section we replace these by the odd elements from a Grassmann algebra

$$G := \mathbb{R}[\{\lambda_\alpha | \alpha \in I\}]/(\{\lambda_\alpha \lambda_\beta + \lambda_\beta \lambda_\alpha | \alpha, \beta \in I\}). \tag{2.3.15}$$

This way of looking at the fermionic fields will be convenient when discussing the path integral.

The covariant derivative ∇ appearing in (2.3.11) is the tensor product of the trivial connection on S_Σ (which for now is a trivial bundle) and the pullback of the Levi-Cevita connection on TM. The metric on TM provides a bilinear form on this bundle, which in coordinates looks like $g(\psi, \psi') = g_{ij}\psi^i \psi'^j$, which is in fact antisymmetric as ψ and ψ' anti–commute. Using this form we can write the terms with the covariant derivative in (2.3.11) as $ig(\psi_\pm, \nabla_\mp \psi_\pm)$. Similarly, the curvature tensor can be extended to the fermionic fields, enabling us to write the last term in (2.3.11) as $\frac{1}{2} g(R(\psi_+, \psi_+)\psi_-, \psi_-) = \frac{1}{2} g(R(\psi_-, \psi_-)\psi_+, \psi_+)$, where the last equality follows from the usual symmetry properties of the Riemann tensor. A coordinate free[7] expression of (2.3.11) is thus given by

$$S(\varphi, \psi) = \int_\Sigma d\sigma^\pm \bigg(g(d\varphi \hat{,} \star\, d\varphi) + ig(\psi_+, \nabla_- \psi_+) + ig(\psi_-, \nabla_+ \psi_-) + \frac{1}{2} g(R(\psi_+, \psi_+)\psi_-, \psi_-) \bigg). \tag{2.3.16}$$

We stress again that the metric on Σ is fixed to be the standard flat metric (2.3.6). The first term in (2.3.16) can be generalized to arbitrary surfaces Σ and metrics, in fact its form is already well defined for any of these choices. The other terms are not however, as for general metrics on surfaces with higher genus it can happen that the structure group of the tangent bundle does not reduce to $SO^+(1,1)$, which means that there is no globally defined notion of $+$ and $-$ directions (cf. Section 5.3 for more information about this).

[7] With respect to coordinates on the target.

To summarize; a representation of the $(1,1)$ extended Poincaré superalgebra on the sigma model of Section 2.1 requires fermionic fields $\psi \in S_\Sigma \otimes \varphi^*(TM)$, acting as the superpartners of $\varphi : \Sigma \to M$. This representation puts no constraints on M whatsoever. Furthermore, since the spin bundle decomposes into $S^\pm$, we can forget one of the $\pm$ components and obtain also a representation of the $(1,0)$ and $(0,1)$ algebras. From now on we will work with the $(1,1)$–supersymmetric sigma model.

2.4 Extended supersymmetry and complex geometry

In the previous section we saw that the sigma model, for any target space M, could be extended to a $(1,1)$ supersymmetric model. The next step is to enlarge these symmetries, e.g. define $(2,2)$–representations on it. However, the existence of more symmetries puts constraints on M, which turn out to be strongly related to complex geometry. The necessary computations to understand this relation are rather tedious, and we postpone them to Chapter 4. In this way, this section becomes easier to read and provides a better overview of the essential ingredients.

In [1], it was shown that the most general second symmetry commuting with the first one and preserving parity (cf. the text below Equation (2.4.3)) is of the form

$$\delta_\epsilon \Phi^i = I^i{}_j \epsilon D \Phi^j, \tag{2.4.1}$$

with I a complex structure. Note that it is a priori not clear whether a second supersymmetry could be expressed in superspace, but for these models this is possible. For the action to be invariant and the corresponding charges[8] $Q_\pm^2$ to satisfy the super-Poincaré algebra (M, g, I) must be a Kähler manifold. This means that I is orthogonal with respect to g:

$$g(Iv, Iw) = g(v, w) \qquad \forall v, w \in TM, \tag{2.4.2}$$

and that the *fundamental 2–form* $\omega := gI$ is closed. This last statement is equivalent to I being covariantly constant, in the sense that $\nabla I = 0$ where ∇ is the Levi-Cevita connection.

The precise calculation of the constraints on (M, g, I, ω) can be found in Chapter (4), but we can already try to argue why the symmetry must be of the form (2.4.1), by a 'what else can it be' method. As mentioned before, we will assume that the supersymmetry can be phrased in superspace language. In this case this assumption is correct[9], but there are also models in which this is not the case.

The only available tools in $(1,1)$–superspace are the coordinates and their derivatives, and of course the superfields Φ^i themselves. If we want to obey the algebra (2.2.4), the tools we pick must have spin $\pm 1/2$ and that restricts us to $Q_\pm$ and $D_\pm$. The number of these derivatives that can appear in

[8] Note that superscripts on the Q's always refer to the type of supercharge, and not to some actual power of the charge.

[9] In [1] the entire calculation was done in component fields, without using superspace tools. This thus proves that the most general form of symmetry respecting parity is indeed given by (2.4.1).

the symmetry is restricted for dimensional reasons: Every physical quantity has a *dimension*, which is a combination of the basic building blocks of nature, the relevant for us being mass, time and length. For instance, speed, which is the ratio of length and time, has dimension *length/time*. In order to express a dimension into an actual number one has to choose units, such as kilograms, meters and seconds. There is one convenient choice of units, in which the speed of light c and the Planck constant $\hbar$ are equal to 1. Since the dimension of c equals *length/time*, and that of $\hbar$ equals *mass* $\cdot$ *length*^{2}*/time*, with this choice we can express any dimension in terms of, say, the mass dimension, indicated by square brackets. Space and time coordinates have mass dimension $[\sigma^{\pm}] = -1$, because e.g. $[\textit{length}] = \hbar/(c \cdot \textit{mass}) = 1/\textit{mass}$. Hence space–time derivatives have dimension $[\partial_{\pm}] = +1$. As fermionic derivatives have opposite dimension to the fermionic coordinates, $D_{\pm}$ has a well–defined dimension only if $[\theta] = -[\frac{\partial}{\partial\theta}] = -\frac{1}{2}$, so $[D_{\pm}] = \frac{1}{2}$. Since the action appears in the exponential of the path–integral, it must be dimensionless (otherwise the exponential as a power series does not make sense), $[S] = 0$, and a quick perusal of (2.3.9) reveals that $[\Phi] = 0$. A variation of Φ^i must also be dimensionless, and since $[\epsilon] = -\frac{1}{2}$, we deduce that the second symmetry must be of the form

$$\delta_\epsilon \Phi^i = \epsilon^+ (I^i_{+j} D_+ \Phi^j + J^i_{+j} Q_+ \Phi^j) + \epsilon^- (I^i_{-j} D_+ \Phi^j + J^i_{-j} Q_- \Phi^j), \tag{2.4.3}$$

for certain tensors $I_{\pm}, J_{\pm}$. This form is further restricted by parity. In any field theory on space–time, a parity transformation denotes the reflection of the space like coordinates. In our case, parity acts as $(\tau, \sigma) \mapsto (\tau, -\sigma)$ and it effectively interchanges the $+$ and $-$ components of spinors. To understand this, observe that we can tell the two components apart by their behavior under a Lorentz boost:

$$\psi_{\pm} \mapsto e^{\mp \alpha/2} \psi_{\pm}. \tag{2.4.4}$$

However, if we interchange the space coordinate, we reflect the direction of the boost, and we change the $\mp$ sign in the exponent above. In this way, the $\pm$ components of ψ are interchanged. For our symmetry to preserve parity, everything must be symmetric in $\psi_{\pm}$, so in particular $I_+ = I_- =: I$ and $J_+ = J_- =: J$ in (2.4.3).

Finally, this symmetry must commute with the first one, which is given by $\delta_\epsilon \Phi^i = \epsilon Q \Phi^i$. In the commutator $[\delta^1, \delta^2]\Phi^i$ the following term occurs:

$$J^i_{\ j} \epsilon^\alpha_1 \epsilon^\beta_2 \{Q_\alpha, Q_\beta\} \Phi^j = -2i J^i_{\ j} \epsilon^\alpha_1 \epsilon^\beta_2 \delta_{\alpha\beta} \partial_\alpha \Phi^j, \tag{2.4.5}$$

where we used the $(2,2)$–algebra (Equation 2.2.4). Since this term cannot be canceled against any other terms in the commutator, the only way out is the constraint $J = 0$, and we are indeed left with a symmetry of the form (2.4.1).

Imposing $(2,2)$–supersymmetry is thus not as innocent as (1,1), and the underlying reason is the fact that $\{D_{\pm}, D_{\pm}\} = -P_{\pm}$, while $\{Q_{\pm}, Q_{\pm}\} = +P_{\pm}$. Therefore, if we had inserted a tensor in front of Q when defining the (1,1) symmetry, that tensor should square to the identity. There is then no harm to replace it by the identity operator, available on every manifold[10]. For the extra symmetries

[10]More generally, an endomorphism squaring to the identity is called a product structure. Given a product structure, the tangent bundle splits into ± 1 eigenbundles, and the algebra implies that the complex structures preserve those.

we have to use D, and the corresponding tensors have to square to minus the identity. This is the essential reason why complex geometry is needed for extended supersymmetries.

We can go even further, and try to define a representation of (p,p)–supersymmetry with $p \geq 3$. As for the $(2,2)$ case, these symmetries individually all have the form $\delta_\epsilon \Phi^i = (I^a)^i{}_j \epsilon D \Phi^j$ where $a \in \{1, \ldots, p-1\}$. The algebraic part of the superalgebra imposes the constraint

$$I^a I^b + I^b I^a = 0, \tag{2.4.6}$$

so the different I's have to anti–commute. In particular, once we have two extra symmetries given by I_1, I_2, we get a third one for free: $I_3 := I_1 I_2$. Now I_3 is covariantly constant since both I_1 and I_2 are, so in particular is integrable. These three complex structures form a quaterniotic algebra, and such a structure on M is called a *hyperkähler structure*. It turns out that one cannot go further than $(4,4)$, so that Table 2.1 gives a complete classification of parity preserving supersymmetries on the sigma model.

Table 2.1: Relation between supersymmetry and target space geometry.

(p,q) Susy	(1,1)	(2,2)	(4,4)
Geometry	Riemannian	Kähler	Hyperkähler

One might wonder why we did not use the higher (p,q)–superspaces to construct the associated higher representations. After all, this is the way we set up the general representation theory in Section 2.3. The problem is that it is not a priori clear what to write down for the action in higher superspace. Take for instance the $(2,2)$ case. On a general target space, it is not clear what to write down for the action in $(2,2)$–superspace. However, once we know that we are on a Kähler manifold, we can use the so–called Kähler potential in local coordinates, and then it turns out to be straightforward to write down an action [16]. The whole model can then be phrased in $(2,2)$–superspace, and the whole setup becomes manifest in the sense that almost no computations are needed to check all the algebra relations, as well as the invariance of the action. But to know that the Kähler condition is necessary, we first need to go through the tedious computations that we saved for Chapter 4.

Note that in these higher superspaces, there are many components of a superfield, and to reduce to the desired number of independent fields one usually imposes constraints on the fields, such as *chiral*, or *twisted chiral* constraints. For a definition of these and a more concise treatment on the matter we refer again to [16].

Chapter 3

Generalized geometry

Gates, Hull and Roček [16] realized that the presence of a B–field in the $(2,2)$–sigma model requires a different kind of geometry on the target space M than the usual Kähler geometry. These so–called *bi–hermitian* geometries generalize the concept of Kähler geometry, with the main difference that bi–hermitian structures consist of two complex structures instead of one, and that there are two connections with torsion instead of the usual, torsion-free, Levi-Cevita connection. Gualtieri showed in [18] that bi–hermitian structures are equivalent to so–called *generalized Kähler structures*. The latter is defined on the bundle $TM \oplus T^*M$ instead of TM, and it is this extra freedom that allows for a convenient description of the bi–hermitian picture. For instance, while the two complex structures do not necessarily commute, they can be described by two commuting *generalized complex structures* on $TM \oplus T^*M$.

The study of $T \oplus T^*$ is the main ingredient of *generalized complex geometry*, a field in differential geometry that unifies complex and symplectic structures. It was first introduced by Hitchin, and further developed by Gualtieri and Cavalcanti. In this chapter we give a self-contained introduction to this topic, emphasizing on the equivalence between generalized Kähler structures and bi–hermitian models. Most of the material is based on [18].

3.1 Linear algebra

Generalized geometry is the study of $TM \oplus T^*M$, where M is a smooth manifold of dimension n, and the geometric structures hereon. This bundle, which we abbreviate as $T \oplus T^*$, has some interesting structures on it, and we first focus on those that only involve the linear algebraic structure. Therefore, we forget for the moment about M and consider a real vector space V of dimension n, and study $V \oplus V^*$. Elements from V will be denoted by $X, Y, Z, \ldots$, elements from V^* will be denoted by $\xi, \eta, \zeta, \ldots$ and elements from $V \oplus V^*$ by $u, v, w, \ldots$.

The vector space $V \oplus V^*$ is equipped with a natural pairing, given by

$$\langle X + \xi, Y + \eta \rangle := \frac{1}{2}(\eta(X) + \xi(Y)). \tag{3.1.1}$$

It is natural in the sense that it does not use any choice of basis for V or V^*. This pairing has signature (n, n), for if e_i is a basis for V and e^i a dual basis for V^*, then $e_i \pm e^i$ are mutually orthogonal of length ± 1. The group of linear isomorphisms of $V \oplus V^*$ preserving this pairing is denoted by $SO(V \oplus V^*)$, with corresponding Lie algebra

$$\mathfrak{so}(V \oplus V^*) = \{T | \langle Tv, w\rangle + \langle v, Tw\rangle = 0 \; \forall x, y \in V \oplus V^*\}. \tag{3.1.2}$$

For an element

$$\begin{pmatrix} A & \beta \\ B & D \end{pmatrix}$$

to belong to this algebra, the anti–symmetry condition implies that B and β are skew, i.e. $B \in \wedge^2 V^*$ and $\beta \in \wedge^2 V$, while $D = -A^*$. The most important symmetry will be the B part, which, after exponentiating, acts as $\exp(B)(X + \xi) = X + \xi + \iota_X B$. We will call these transformations *B-transforms* following the usual nomenclature, but we stress that it is unrelated to the B–field that will be introduced in later sections.

The main objects of interest in generalized geometry are the maximal isotropic subbundles of $TM \oplus T^*M$, and so we first need a good understanding of the maximal isotropic subspaces of $V \oplus V^*$. A subspace $L < V \oplus V^*$ is called isotropic if the pairing is zero on L, i.e.

$$L < L^\perp := \{v \in V \oplus V^* | \langle v, w\rangle = 0 \; \forall w \in L\}. \tag{3.1.3}$$

If L is maximal with this property, it is called maximal isotropic, or a *linear Dirac structure*.

Given a subspace $E < V$ and an element $\epsilon \in \wedge^2 E^*$ we can define

$$L(E, \epsilon) := \{X + \xi \in E + V^* | \; \xi|_E = \iota_X \epsilon\}. \tag{3.1.4}$$

The maximal isotropics of $V \oplus V^*$ are then described by these spaces:

Theorem 3.1.1. *Every maximal isotropic is of the form $L(E, \epsilon)$.*

Proof. One quickly verifies that these spaces are maximal isotropic, so let L be any maximal isotropic and define $E := \pi_V(L)$. Since L is isotropic we have $L \cap V^* \subset Ann(E)$, where

$$Ann(E) := \{\xi \in V^* | \; \xi|_E = 0\}.$$

Clearly $L + Ann(E)$ is still isotropic, so we have $Ann(E) < L$ since L is maximal. Using the equality $E^* = V^*/Ann(E)$ we can define $\epsilon : E \to E^*$ as follows. For $X \in E$ there is by definition of E a $\xi \in V^*$ with $X + \xi \in L$, and we define $\epsilon(X) = [\xi] \in V^*/Ann(E)$. This is well defined, for if $\xi' \in V^*$ also satisfies $X + \xi' \in L$ then $\xi - \xi' \in L \cap V^* = Ann(E)$. Remains to verify that $L < L(E, \epsilon)$, the maximality of L then gives an equality. But by definition of ϵ any element in L can be written as $X + \iota_X \epsilon + \xi$ where $\xi \in Ann(E)$, and it is obvious that those elements lie in $L(E, \epsilon)$. $\square$

3.1.1 Spinors for $V \oplus V^*$

In view of Theorem 3.1.1 one would expect that a maximal subbundle of $T \oplus T^*$ is described by a subbundle of $E < T$ and a 2–form $\epsilon \in \Gamma(\wedge^2 E^*)$, but this is not always the case. In fact, if L is a maximal isotropic subbundle then the projection $\pi_T(L)$ is not necessarily of constant rank throughout M. Fortunately, there is an alternative description using spinors.

Let $CL(V \oplus V^*)$ denote the Clifford algebra associated to $V \oplus V^*$ with its natural pairing [1]. There is a natural choice of pinors for $V \oplus V^*$, presented by the space $\wedge^\bullet V^*$. The Clifford action is given by

$$(X + \xi) \cdot \varphi := \iota_X \varphi + \xi \wedge \varphi. \tag{3.1.5}$$

This is indeed well defined, since

$$(X + \xi)^2 \cdot \rho = (X + \xi) \cdot (\iota_X \rho + \xi \wedge \rho) = \iota_X(\xi \wedge \rho) + \xi \wedge \iota_X \rho = (\iota_X \xi)\rho = \langle X + \xi, X + \xi \rangle \rho.$$

One readily verifies that this representation is both faithful as irreducible, justifying the name 'pinors'. Following the nomenclature of the literature, we will abuse notation and call $\wedge^\bullet V^*$ the space of spinors, instead of pinors. These 'spinors' form one of the main tools of generalized geometry, as a lot of structures on $T \oplus T^*$ and properties thereof can be described purely on the level of differential forms.

In signature (n, n) the Clifford algebra has a volume element ω satisfying $\omega^2 = 1$. This gives a decomposition $S := \wedge V^* = S^+ \oplus S^-$ where $S^\pm$ is the $\pm$ eigenspace for ω, and can be identified with the actual space of spinors. Using an explicit basis $e_1, \ldots, e_n$ for V with dual basis $e^1, \ldots, e^n$ for V^*, the volume element is given by

$$\omega = (e_1 + e^1) \cdots (e_n + e^n)(e_1 - e^1) \cdots (e_n - e^n) \tag{3.1.6}$$

and indeed satisfies $\omega^2 = (-1)^{\frac{1}{2}(2n-1)(2n)}(-1)^n = (-1)^{2n^2} = 1$. Using this form for ω one quickly verifies that $S^{+\backslash -} = \wedge^{\text{even}\backslash\text{odd}} V^*$, where it depends on n whether the even or odd forms belong to the $+$ or $-$ eigenspace.

The double covering map $\rho : \text{Spin}(V \oplus V^*) \to SO(V \oplus V^*)$ given by $\rho(x)w = xwx^{-1}$ for $x \in Spin(V \oplus V^*)$ and $w \in V \oplus V^*$ induces an isomorphism on Lie algebras

$$d_e\rho : \mathfrak{spin}(V \oplus V^*) \to \mathfrak{so}(V), \tag{3.1.7}$$

given by $d_e\rho(X)(w) = [X, w]$. For later use we need to understand what the inverse image is of the element

$$\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \tag{3.1.8}$$

[1] We refer to appendix A.1 for definitions and properties of Clifford algebras, (s)pin-groups and (s)pinors.

under this isomorphism. Again let e_i be a basis for V and e^i a dual basis for V^*. Writing $B = \frac{1}{2}b_{ij}e^i \wedge e^j$, we claim that $B = d_e\rho(\frac{1}{2}b_{ij}e^je^i)$. Indeed,

$$d_e\rho\left(\frac{1}{2}B_{ij}e^je^i\right)(e_k) = \frac{1}{2}B_{ij}(e^je^ie_k - e_ke^je^i) = \frac{1}{2}B_{ij}(e^j(\delta^i_k - e_ke^i) - (\delta^j_k - e^je_k)e^i) = B_{kj}e^j = \iota_{e_k}B. \tag{3.1.9}$$

In particular, this inverse image acts on spinors via

$$(d_e\rho)^{-1}(B)\varphi = \frac{1}{2}B_{ij}e^j \wedge e^i \wedge \varphi = -B \wedge \varphi, \tag{3.1.10}$$

and taking exponentials we obtain $\exp(B)\varphi = e^{-B} \wedge \varphi$. One should keep in mind though that this equation is misleading, as strictly speaking it is the inverse image under ρ of $\exp(B)$ that is acting on spinors, and we have made a choice since there are two elements in this inverse image.

Every space of spinors comes equipped with a bilinear form which is invariant under the connected component of the identity of the corresponding spin group, see also [11]. For $\wedge^\bullet V^*$ we can give an explicit description. For forms $\alpha, \beta \in \wedge^\bullet V^*$, we define

$$(\alpha, \beta) := (\alpha^t \wedge \beta)_{\text{top}}, \tag{3.1.11}$$

where $^t : \wedge^\bullet V^* \to \wedge^\bullet V^*$ is given on decomposable forms by $(v^1 \wedge \cdots \wedge v^k)^t := v^k \wedge \cdots \wedge v^1$. The subscript *top* implies that we take the top degree part. This bilinear form is called the Chevalley pairing, or Mukai pairing. To verify that it is invariant under $Spin^0(V \oplus V^*)$, the connected component of the identity, we first write (3.1.11) in a way that reflects the Clifford action. Let f be a nonzero element in $\wedge^n V$, which acts on forms of top degree via inner contraction. More specifically, if we write f in terms of a basis, $f = e_1 \wedge \cdots \wedge e_n$, then $\iota_f(\varphi) = \iota_{e_n} \cdots \iota_{e_1}\varphi$. If we regard φ as an alternating multilinear map on V^n, then this latter expression equals $\varphi(e_1, \ldots, e_n)$. This defines a nondegenerate pairing between $\wedge^n V$ and $\wedge^n V^*$, and we have

$$(\iota_f(\alpha, \beta))f = (\iota_f(\alpha^t \wedge \beta))f = f^t \cdot \alpha^t \cdot \beta \cdot f = (\alpha \cdot f)^t \cdot \beta \cdot f. \tag{3.1.12}$$

Here $\cdot$ denotes Clifford multiplication, viewing $\wedge^\bullet V^*$ and $\wedge^\bullet V$ as subspaces of $CL(V \oplus V^*)$, which is allowed as both V and V^* are isotropic[2]. The second equality above is true because $\alpha(f) \cdot f = 0$, so that we can move $\alpha(f)$ along $\alpha(s) \cdot t$ at the cost of their graded commutator, which is exactly $\iota_f(\alpha(s) \wedge t)$, after which $\alpha(f)$ kills f. From (3.1.12) we see that for $v \in V \oplus V^*$ we have

$$\iota_f(v \cdot \alpha, v \cdot \beta)f = (v \cdot \alpha \cdot f)^t \cdot v \cdot \beta \cdot f = (\alpha \cdot f)^t \cdot v \cdot v \cdot \beta \cdot f = \langle v, v\rangle \iota_f(\alpha, \beta)f, \tag{3.1.13}$$

so that for a general element $x = v_1 \cdots v_{2r} \in Spin(V \oplus V^*)$ we have

$$(x \cdot \alpha, x \cdot \beta) = (v_1 \cdots v_{2r} \cdot \alpha, v_1 \cdots v_{2r} \cdot \beta) = \prod_{i=1}^{2r} \langle v_i, v_i\rangle(\alpha, \beta) = N(x)(\alpha, \beta). \tag{3.1.14}$$

[2]This condition is necessary as for a subspace $L < V \oplus V^*$, the map $v_1 \otimes \cdots \otimes v_k \mapsto v_1 \cdots v_k$ from $L^{\otimes k} \to CL(V \oplus V^*)$ factors through $\wedge^k L$ if and only if $v_i \cdot v_j = -v_j \cdot v_i$, which happens if and only if L is isotropic.

Here $N : Spin(V \oplus V^*) \to \{\pm 1\}$ is the norm function and takes the value 1 on $Spin^0(V \oplus V^*)$ (cf. Equation (A.0.2) in the appendix), so that the pairing (3.1.11) is indeed invariant.

To every nonzero spinor $\varphi \in \wedge^\bullet V^*$ we can associate its *null space*, which is defined by

$$L_\varphi := \{v \in V \oplus V^* |\, v \cdot \varphi = 0\}. \tag{3.1.15}$$

For $v, w \in L_\varphi$ we have $0 = (vw + wv)\varphi = 2\langle v, w\rangle\varphi$, and because $\varphi \neq 0$ we see that L_φ is isotropic. If L_φ is maximally isotropic, we call φ a *pure spinor*. We would like to know whether all maximal isotropic subspaces of $V \oplus V^*$ can be obtained as such a null space, and whether the associated pure spinor is unique. In order to answer these questions, we use the classification of maximal isotropics given in Theorem 3.1.1. First, note that if $v \in L_\varphi$ and $g \in Spin(V \oplus V^*)$, then $0 = v \cdot \phi = v \cdot g^{-1}g\varphi$ so that $gvg^{-1} \in L_{g\cdot\varphi}$. In other words, $L_{g\cdot\varphi} = \rho(g)L_\varphi$. Using this equivariance property, we can prove the following theorem, where we recall that for an m–dimensional vector space E, its determinant line is defined by $\wedge^m E$.

Theorem 3.1.2. *Let $L = L(E, \epsilon)$ be a maximal isotropic as in Theorem 3.1.1. There exists a pure spinor for which L is the associated null space, and any such spinor is given by a nonzero element in the line $U_L := \exp(B)\det(Ann(E))$. Here $B \in \wedge^2 V^*$ is any 2–form such that $i^*B = \epsilon$ with $i : E \to V$ the inclusion. $U_L < \wedge^\bullet V^*$ is called the canonical line bundle associated to L.*

Proof. First observe that $L(E, \epsilon) = \exp(B)L(E, 0)$. Indeed, an element in $\exp(B)L(E, 0)$ looks like $X + \xi + \iota_X B$ with $X \in E$ and $\xi \in Ann(E)$. Since $(\iota_X B + \xi)|_E = \iota_X \epsilon$ these elements lie in $L(E, \epsilon)$, and since both spaces have dimension n they are equal. Next we claim that $L(E, 0) = L_\varphi$ where φ is a nonzero element from $\det(Ann(E))$. Indeed, $X + \xi \in L(E, 0)$ if and only if $X \in E$ and $\xi|_E = 0$, and this happens if and only if $\iota_X\varphi = 0$ and $\xi \wedge \varphi = 0$. Conversely, if $L(E, 0) = L_\varphi$ for some φ, the same equations hold and one readily verifies that this implies $\varphi \in \det(Ann(E))\backslash\{0\}$. This proves the theorem for $\epsilon = 0$. The general case follows immediately from the equivariance property described above. More precisely, $L(E, \epsilon) = \exp(B)L(E, 0) = \exp(B)L_\varphi = L_{\exp(B)\cdot\varphi}$. Here $\exp(B) \cdot \varphi$ is the spinorial action of B on φ, which was calculated in (3.1.10) and is given by $e^{-B} \wedge \varphi$. □

Remark. From this theorem we also see that any pure spinor can be written as $\exp(B)\theta^1 \wedge \cdots \wedge \theta^k$ where B is a 2–form and θ^i are linearly independent 1–forms.

In the discussion above we only focused on vector spaces over $\mathbb{R}$, but in the context of generalized complex structures we will mostly be interested in maximal isotropic subspaces of $(V \oplus V^*) \otimes \mathbb{C}$. All the results above however continue to be true over the complex numbers, in fact over any field of characteristic zero.

We close the linear algebra part with a result which will be important when discussing complex structures later on.

Theorem 3.1.3. *Let L_φ and L'_φ be two maximal isotropics associated to pure spinors φ and φ'. Then $L_\varphi \cap L'_\varphi = \{0\}$ if and only if $(\varphi, \varphi') \neq 0$.*

Proof. According to Theorem I.4.1 in [11] the action of $SO(V \oplus V^*)$ on the set of maximal isotropics is transitive. Therefore, we can pick a $g \in Spin(V \oplus V^*)$ such that $\rho(g)V^* = L'_\varphi$, and by equivariance it follows that φ' is a multiple of $g \cdot e$, where e is any nonzero element from $\wedge^n V^*$. Since the pairing is $Spin(V \oplus V^*)$ invariant (up to a $\pm$ sign), we may as well assume that $L'_\varphi = V^* = L_e$. Writing $\varphi = e^B \wedge \theta_1 \wedge \cdots \wedge \theta_k$, we see that $(e, \varphi) = e^t \wedge \exp(B)\theta_1 \wedge \cdots \wedge \theta_k$. As e is already of maximal degree, this product is zero if $k > 0$, and equal to $\pm e \neq 0$ if $k = 0$. Recall that the θ_i span the space $Ann(\pi_V(L_\varphi))$, so that $(e, \varphi) \neq 0$ if and only if $Ann(\pi_V(L_\varphi)) = \{0\}$, but we saw in the proof of Theorem 3.1.1 that $L_\varphi \cap V^* = Ann(\pi_V(L_\varphi))$, which concludes the proof. □

3.2 The Courant bracket

Having discussed the relevant linear algebra, we will now turn to the differential geometric content of generalized geometry. Instead of giving straight away the definition of the Courant bracket, we derive it in a way that makes it clear where it comes from. Besides this philosophical reason, this alternative description is also convenient for actual computations.

3.2.1 Derived brackets

Let M be a smooth n–dimensional real manifold, and let $\mathcal{A}$ be the space of linear differential operators on $\Omega^\bullet(M)$, the space of sections of the bundle $\wedge^\bullet T^*M$. $\mathcal{A}$ is equipped with a $\mathbb{Z}_2$-grading, where an operator a is of degree $k \in \mathbb{Z}_2$ if $a(\Omega^l(M)) \subset \Omega^{l+k}(M)$, where the $\mathbb{Z}_2$ grading on forms is according to the parity of their degree. This grading induces a bracket on $\mathcal{A}$ given by

$$[a, b] := ab - (-1)^{|a||b|}ba, \tag{3.2.1}$$

where $|a|$ is the degree of a, and this bracket turns $\mathcal{A}$ into a Lie superalgebra. Suppose that D is an odd derivation, i.e. $D : \mathcal{A}^i \to \mathcal{A}^{i+1}$, $D([a, b]) = [Da, b] + (-1)^{|a|}[a, Db]$, and $D^2 = 0$. Then we can define another bracket on $\mathcal{A}$, given by

$$[a, b]_D := (-1)^{|a|+1}[Da, b]. \tag{3.2.2}$$

This bracket is called the *derived bracket* for D, and it is a degree 1 bracket, in the sense that $[\mathcal{A}^i, \mathcal{A}^j] \subset \mathcal{A}^{i+j+1}$. For a more concise treatment of the derived bracket we refer to [27]. A straightforward computation shows that $[,]_D$ satisfies the (graded) Jacobi identity:

$$[a, [b, c]_D]_D = [[a, b]_D, c]_D + (-1)^{(a+1)(b+1)}[b, [a, c]_D]_D. \tag{3.2.3}$$

However, in general this bracket is not skew–symmetric. A way to produce such a derivation D is by taking an element $d \in \mathcal{A}^{odd}$, such that $[d, d] = 0$, and then define D to be

$$Da := [d, a] = (-1)^{a+1}[a, d]. \tag{3.2.4}$$

It is a derivation because $[,]$ satisfies the Jacobi identity, and squares to zero because $[d, d] = 0$. The corresponding derived bracket will be denoted by $[,]_d$.

The nice aspect about this construction is that some interesting subspaces of $\mathcal{A}$ turn out to be closed under the derived bracket, for certain choices of d. As a first example, we take d to be the ordinary exterior derivative on forms, which obviously satisfies $[d, d] = 2d^2 = 0$, and look at the subspace $TM < \mathcal{A}^{-1}$, where a vector field on M acts on $\Omega^\bullet(M)$ by inner contraction. Using the well–known identity $[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$, with $\mathcal{L}_X$ the Lie derivative in the direction of X, we obtain

$$[\iota_X, \iota_Y]_d = [[\iota_X, d], \iota_Y] = [\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}, \tag{3.2.5}$$

so that indeed the space of vector fields is closed under the derived bracket, and under the identification $X \leftrightarrow \iota_X$ the derived bracket is identified with the Lie bracket. This example can straightforwardly be generalized to the subspace of $\mathcal{A}$ given by all polyvector fields, i.e. sections of $\wedge^\bullet TM$. A similar computation can be done, and it turns out that the induced bracket on $\Gamma(\wedge^\bullet TM)$ is the Schouten-Nijenhuis bracket:

$$[X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_l] = \sum_{i,j} (-1)^{(i+j)} [X_i, Y_j] \wedge X_0 \wedge \cdots \hat{X}_i \cdots \wedge X_k \wedge Y_1 \wedge \cdots \hat{Y}_j \cdots \wedge Y_l. \tag{3.2.6}$$

As a second example, we consider the operator

$$d^H := d - H\wedge \in \mathcal{A}, \tag{3.2.7}$$

where H is a closed three–form on M. It is obvious that for the associated bracket the previous subspace is not closed, due to the presence of H, which turns a pair of vectors into a 1–form. Therefore, besides vector fields we allow also 1–forms, i.e. we look at the subspace $T \oplus T^* < \mathcal{A}$, where the inclusion is given by $X + \xi \mapsto \iota_X + \xi\wedge$. On this subspace the derived bracket takes the form

$$\begin{aligned} [\iota_X + \xi\wedge, \iota_Y + \eta\wedge]_{d^H} =& [[\iota_X + \xi\wedge, d^H], \iota_Y + \eta\wedge] \\ =& [\mathcal{L}_X - \iota_X H \wedge + d\xi\wedge, \iota_Y + \eta\wedge] \\ =& \iota_{[X,Y]} + \mathcal{L}_X\eta \wedge + \iota_Y\iota_X H \wedge - \iota_Y d\xi \wedge . \end{aligned} \tag{3.2.8}$$

In particular we see that $T \oplus T^* < \mathcal{A}$ is closed under the derived bracket, which induces on $T \oplus T^*$ the so–called Courant bracket

$$[\![X + \xi, Y + \eta]\!] := [X, Y] + L_X\eta - \iota_Y d\xi + \iota_Y\iota_X H. \tag{3.2.9}$$

We use explicit double brackets for the Courant bracket to distinguish it from the other brackets, although it should be clear from the context which bracket is meant. Also, the Courant bracket depends, besides the differential structure of M, also on the 3–form H, so in the literature one often finds an H-subscript to stress that fact. We shall not mention any dependence on H in the Courant bracket, as there will always be a flux present.

Note that a lot of authors take the anti–symmetrization of (3.2.9) as the definition of the Courant

bracket, which then fails to satisfy the Jacobi identity. In those texts, the bracket in (3.2.9) is called the Dorfman bracket. Instead, we will use the above definition which is more natural in the context of derived brackets, with the slight disadvantage that it lacks skew–symmetry. Indeed, the graded Jacobi identity for $[,]$ gives

$$\begin{aligned}[[\iota_X + \xi\wedge, d^H], \iota_Y + \eta\wedge] =&[\iota_X + \xi\wedge, [d^H, \iota_Y + \eta\wedge]] + [d^H, [\iota_X + \xi\wedge, \iota_Y + \eta\wedge]] \\ =&[\iota_X + \xi\wedge, [d^H, \iota_Y + \eta\wedge]] + d(\iota_X\eta + \iota_Y\xi),\end{aligned}$$

which implies for the Courant bracket

$$[\![X + \xi, Y + \eta]\!] + [\![Y + \eta, X + \xi]\!] = 2d\langle X + \xi, Y + \eta\rangle. \tag{3.2.10}$$

Here the bracket is the natural pairing as defined in (3.1.1). Besides this relation between the bracket and the pairing, we have the following identities.

Lemma 3.2.1. *For $u = X + \xi, v = Y + \eta, w = Z + \zeta \in \Gamma(T \oplus T^*)$ and $f \in C^\infty(M)$ we have*

i) $\pi(u)\langle v, w\rangle = \langle [\![u, v]\!], w\rangle + \langle v, [\![u, w]\!]\rangle$,

ii) $[\![u, fv]\!] = f[\![u, v]\!] + (\pi(u)f)v$,

where $\pi(u)\langle v, w\rangle = X(\langle v, w\rangle) = d\langle v, w\rangle(X)$.

Proof. For $i)$ we compute

$$\begin{aligned}\langle [\![u, v]\!], w\rangle + \langle v, [\![u, w]\!]\rangle =&\langle [X, Y] + \mathcal{L}_X\eta - \iota_Y d\xi + \iota_Y\iota_X H, Z + \zeta\rangle \\ &+ \langle Y + \eta, [X, Z] + \mathcal{L}_X\zeta - \iota_Z d\xi + \iota_Z\iota_X H\rangle \\ =&\frac{1}{2}\left(\iota_{[X,Y]}\zeta + \iota_Z(\mathcal{L}_X\eta - \iota_Y d\xi + \iota_Y\iota_X H)\right) \\ &+ \frac{1}{2}\left(\iota_{[X,Z]}\eta + \iota_Y(\mathcal{L}_X\zeta - \iota_Z d\xi + \iota_Z\iota_X H)\right) \\ =&\frac{1}{2}\left(\iota_Z\mathcal{L}_X\eta + \iota_{[X,Z]}\eta + \iota_Y\mathcal{L}_X\zeta + \iota_{[X,Y]}\zeta\right) \\ =&\frac{1}{2}(\mathcal{L}_X(\iota_Z\eta + \iota_Y\zeta)) \\ =&\pi(u)\langle v, w\rangle,\end{aligned}$$

where in the penultimate equality we used again $[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$. For $ii)$, we compute

$$\begin{aligned}[\![u, fv]\!] =&[X, fY] + \mathcal{L}_X(f\eta) - \iota_{fY}d\xi + \iota_{fY}\iota_X H \\ =&f[X, Y] + X(f)(Y + \eta) + f\mathcal{L}_X\eta - f\iota_Y d\xi + f\iota_Y\iota_X H \\ =&f[\![u, v]\!] + (\pi(u)f)v.\end{aligned}$$

□

Remark. The triple $(T \oplus T^*, \langle,\rangle, [\![,]\!])$ satisfies all the axioms of an *exact Courant algebroid*, for more information about these structures see [6].

Given the natural pairing and the bracket, we may wonder what the symmetries of these structures are, i.e. what bundle maps

$$\begin{array}{ccc} T \oplus T^* & \xrightarrow{F} & T \oplus T^* \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array} \tag{3.2.11}$$

preserve both the pairing as the bracket. To answer this question we first observe the following.

Lemma 3.2.2. *The only automorphisms of TM preserving the Lie bracket are maps of the form (f, f_*), where f_* is the tangent map of f.*

Proof. Given a general bundle automorphism (f, F), we can compose with $(f^{-1}, (f_*)^{-1})$, so that without loss of generality we may assume $f = id$. For $g \in C^\infty(M)$ and $X \in \Gamma(TM)$ we have

$$F[X, gX] = F(X(g)X) = X(g)F(X),$$

but on the other hand

$$F[X, gX] = [F(X), gF(X)] = (F(X)g)F(X).$$

As F is supposed to be an isomorphism, we see that X and $F(X)$ give the same derivations on functions, hence are equal, so that $F = Id$. □

For $T \oplus T^*$ we have more symmetries available, given by the B-transforms.

Theorem 3.2.3. *Let (f, F) be a bundle map as in (3.2.11), and suppose that F preserves both the pairing as the Courant bracket. Then there exists a 2–form B with $(f^*)^{-1}(H) = H + dB$, and F is the composition of $(f_*, (f^*)^{-1})$ and $\exp(B)$.*

Proof. Given (f, F), the map $\pi \circ F \circ i$ is a bundle map from T to itself, hence equals f_* by the previous lemma. Here $\pi : T \oplus T^* \to T$ is the projection and $i : T \to T \oplus T^*$ is the inclusion (which is not a map of Courant algebroids for nonzero H). Applying F to both sides of the equality (3.2.10) gives

$$F(d\langle v, w\rangle) = d\langle F(v), F(w)\rangle \qquad \forall v, w \in \Gamma(T \oplus T^*). \tag{3.2.12}$$

As F preserves the bracket, the right-hand side equals $d(f^{-1})^*\langle v, w\rangle = (f^{-1})^* d\langle v, w\rangle$, which implies that $F(\xi) = (f^{-1})^*(\xi)$ for $\xi \in \Gamma(T^*)$. This restricts F to be of the form

$$F = \begin{pmatrix} f_* & 0 \\ B \circ f_* & (f^*)^{-1} \end{pmatrix}. \tag{3.2.13}$$

However, such a transformation does not necessarily preserve the Courant bracket:

$$F[\![X + \xi, Y + \eta]\!] - [\![F(X + \xi), F(Y + \eta)]\!] = \iota_{f_*Y}\iota_{f_*X}\left((f^*)^{-1}(H) - dB - H\right), \tag{3.2.14}$$

which vanishes precisely under the condition of the theorem. □

3.3 Generalized complex structures

Consider again a finite dimensional vector space V over $\mathbb{R}$. We will first define the notion of a generalized complex structure on V, then extend it to $T \oplus T^*$.

Definition 3.3.1. A *generalized complex structure* on V is a linear map $\mathcal{J} : V \oplus V^* \to V \oplus V^*$ satisfying $\mathcal{J}^2 = -Id$ and $\langle \mathcal{J}v, \mathcal{J}w \rangle = \langle v, w \rangle$ for all $v, w \in V \oplus V^*$.

The first property mimics the definition of a complex structure, while the second condition, written as $\mathcal{J}^* = -\mathcal{J}$ (we identify $(V \oplus V^*)^*$ with $V \oplus V^*$ via the pairing), is the analogue of a symplectic structure. It is therefore natural to expect for those two structures to provide the basic examples, and indeed, if I is a complex structure and ω a symplectic structure on V, the following transformations are easily verified to define generalized complex structures:

$$\mathcal{J}_I := \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}, \qquad \mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{3.3.1}$$

If $\mathcal{J}$ is a generalized complex structure, its $+i$ eigenspace $L < (V \oplus V^*) \otimes \mathbb{C}$ is isotropic due to the orthogonality condition on $\mathcal{J}$. Since $\overline{L}$ is the $-i$ eigenspace of $\mathcal{J}$, we have $(V \oplus V^*) \otimes \mathbb{C} = L \oplus \overline{L}$, hence L and $\overline{L}$ are maximal isotropics. Conversely, given L maximal isotropic in $(V \oplus V^*) \otimes \mathbb{C}$ with $L \cap \overline{L} = \{0\}$, it defines a generalized complex structure on V by letting L and $\overline{L}$ be the $+i$ and $-i$ eigenspaces. The *type* of $\mathcal{J}$ is defined as the type of the corresponding maximal isotropic L, which was defined as the complex codimension of $\pi_{V \otimes \mathbb{C}}(L)$ in $V \otimes \mathbb{C}$.

In order to study generalized complex structures on V it thus suffices to look at maximal isotropics $L < (V \oplus V^*) \otimes \mathbb{C}$ satisfying $L \cap \overline{L} = \{0\}$. In terms of a representing pure spinor φ_L, this condition translates into $(\varphi_L, \overline{\varphi}_L) \neq 0$, as follows from Theorem 3.1.3. If we write $\varphi_L = e^{B+i\omega} \wedge \Omega$, we see that

$$(\varphi_L, \overline{\varphi}_L) = ((e^{B+i\omega} \wedge \Omega)^t \wedge e^{B-i\omega} \wedge \overline{\Omega})_{top} = ((\Omega)^t \wedge e^{-2i\omega} \wedge \overline{\Omega})_{top}. \tag{3.3.2}$$

In order for this to be nonzero, the top degree on M should be even, i.e. M itself should be even dimensional.

Remark. As it turns out, a generalized complex structure $\mathcal{J}$ can to some extent be regarded as the product of a complex and a symplectic structure. This fact remains true in the context of differential geometry (cf. definitions 3.3.4 and 3.3.5 below). There is a generalized Darboux Theorem (Theorem 4.35 in [18]), which states that in a regular neighborhood of a point, where the type k of $\mathcal{J}$ is constant, the whole generalized complex structure is isomorphic to a product of an open set in $\mathbb{C}^k$ and an open set in (R^{2n-2k}, ω_0) with ω_0 the standard symplectic form on R^{2n}. We will not go into this.

Since L and $\overline{L}$ are maximal isotropic and $L \cap \overline{L} = \{0\}$, $\overline{L}$ can be identified with L^* via $\iota_v w = 2\langle w, v \rangle$. The factor of 2 is conventional, so that we have the relation $\iota_v w = vw + wv$ as operators on spinors. This induces a decomposition of the space of differential forms. More precisely, define $U_n := U_L$ and $U_{n-k} := \wedge^k \overline{L} \cdot U_n$. Then we have

$$\wedge^\bullet T^* \otimes \mathbb{C} = U_{-n} \oplus U_{-n+1} \oplus \ldots \oplus U_n. \tag{3.3.3}$$

Indeed, if $v_1, \dots, v_{2n}$ is a (complex) basis for L and $v^1, \dots, v^{2n}$ a dual basis for $\overline{L}$, we have

$$v_{i_k} \cdots v_{i_1} \cdot v^{i_1} \cdots v^{i_k} \rho = \rho$$

for $\rho \in U_L$ and the indices i_j mutually distinct, as follows immediately from the defining anti–commutation relation $(v_i v^j + v^j v_i) = 2\langle v_i, v^j \rangle = \delta_i^j$. From this equality one straightforwardly verifies (3.3.3). In this sense elements from L ($\overline{L}$) act as raising (lowering) operators. Note that this grading is incompatible with the usual grading of forms by their degree, but there is still a $\mathbb{Z}_2$ grading, i.e. forms in U_n are even or odd depending on the type of $\mathcal{J}$.

The decomposition (3.3.3) behaves well with respect to the Mukai pairing in the following sense.

Lemma 3.3.2. *The Mukai pairing (3.1.11) is non–degenerate on $U_k \times U_{-k}$, while all other pairs are orthogonal to each other.*

Proof. The spaces U_k and U_l are spanned by elements of the form

$$\alpha = v_1 \cdots v_{n-k} \cdot \rho, \qquad \beta = w_1 \cdots w_{n+l} \cdot \overline{\rho},$$

where $\rho \in U_n \backslash \{0\}$, $v_i \in \overline{L}$ and $w_i \in L$. Using Equation (3.1.13) and the anti–commutation relation, we see that $(\alpha, \beta) = 0$ for $k \neq -l$, while for $k = -l$ we have

$$(\alpha, \beta) = 2^{n-k} \det(\langle v_i, w_j \rangle)(\rho, \overline{\rho}),$$

which is indeed non–degenerate because $L^* \cong \overline{L}$ and $(\rho, \overline{\rho}) \neq 0$, since $L \cap \overline{L} = \{0\}$ (see also Theorem 3.1.3). □

There is an alternative interpretation of the spaces U_k as eigenspaces for the Lie algebra action of $\mathcal{J}$. Note that $\mathcal{J}$ defines an element of $\mathfrak{so}(V \oplus V^*)$ (Equation (3.1.2)), and so it has an inverse under the map $d_e\rho : \mathfrak{spin}(V \oplus V^*) \to \mathfrak{so}(V \oplus V^*)$, which acts on $\wedge^\bullet V^*$, the space of spinors.

Lemma 3.3.3. *The space U_k is an eigenspace for the endomorphism $(d_e\rho)^{-1}(\mathcal{J})$ with eigenvalue ik.*

Proof. Denoting the inverse image of $\mathcal{J}$ by $\tilde{\mathcal{J}}$, let $\rho \in U_n$ and consider the element $\tilde{\mathcal{J}} \cdot \rho$. For $v \in L$ we have

$$v \cdot (\tilde{\mathcal{J}} \cdot \rho) = [v, \tilde{\mathcal{J}}] \cdot \rho = -iv \cdot \rho = 0,$$

so that $\tilde{\mathcal{J}} \cdot \rho \in U_n$, and since this is a line, it equals $\lambda\rho$ for some $\lambda \in \mathbb{C}$. Note that we used here the defining relation for $\tilde{\mathcal{J}}$, namely $[\tilde{\mathcal{J}}, v] = \mathcal{J}v$. For a general element $v_1 \cdots v_k \cdot \rho \in U_{n-k}$, where $v_i \in \overline{L}$, we have

$$\tilde{\mathcal{J}} v_1 \cdots v_k \cdot \rho = \sum_{i=1}^{k} v_1 \cdots v_{i-1} [\tilde{\mathcal{J}}, v_i] v_{i+1} \cdots v_k \cdot \rho + v_1 v_2 \cdots v_k \tilde{\mathcal{J}} \cdot \rho = (\lambda - ik)\rho.$$

Since $U_{n-k}^* = U_{k-n} = U_{n-(2n-k)}$ and $\tilde{\mathcal{J}}$ is real, it follows that $(\lambda - ik)^* = (\lambda - i(2n-k))$, which implies that $\lambda = \mu + ni$ where $\mu \in \mathbb{R}$, and it remains to show that $\mu = 0$. This follows from

Lemma 3.3.2 and the fact that $\tilde{\mathcal{J}}$ is anti–symmetric with respect to the Mukai pairing (this follows from differentiation of 3.1.14):

$$(\mu + ni - ik)(\alpha, \beta) = (\tilde{\mathcal{J}}\alpha, \beta) = -(\alpha, \tilde{\mathcal{J}}\beta) = -(\mu - ni + ki)(\alpha, \beta),$$

for $\alpha \in U_{n-k}$ and $\beta \in U_{-n+k}$. □

Having defined the linear algebraic content of a generalized complex structure, we can transport its definition to a manifold M, of even dimension $2n$. From the above discussion the proof of the next definition/proposition is straightforward.

Definition 3.3.4. A *generalized almost complex structure* on M is given by the following three equivalent data:

- An endomorphism $\mathcal{J}$ of $T \oplus T^*$ satisfying $\mathcal{J}^2 = -1$ and $\mathcal{J}^* = -\mathcal{J}$ with respect to the natural pairing.
- A maximal isotropic subspace $L < (T \oplus T^*) \otimes \mathbb{C}$ satisfying $L \cap \overline{L} = \{0\}$.
- A pure spinor line (complex rank 1 sub-bundle) $U < \wedge T^* \otimes \mathbb{C}$, satisfying $(\varphi, \overline{\varphi}) \neq 0$ at every point, for all $\varphi \in U$ nonzero.

Integrability of a generalized almost complex structure $\mathcal{J}$ is analogous to integrability in ordinary complex geometry, where we require that the $+i$ eigenbundle is involutive with respect to the Lie bracket.

Definition 3.3.5. A *generalized complex structure* on M is a generalized almost complex structure on M such that its $+i$ eigenbundle L is involutive with respect to the Courant bracket.

Remark. In case the 3–form H is not exact, the associated Courant bracket is often called *twisted*, with H defining the twist. Similarly, every structure where there is a notion of integrability with respect to the twisted Courant bracket is also called twisted. Note however that this name has nothing to do with the so–called topological twist which will be discussed in Section 5.3.

For the structures defined in (3.3.1), one readily verifies that $\mathcal{J}_I$ is integrable in the generalized setting if and only if H is of type $(2,1)+(1,2)$, and I is integrable in the classical setting. Similarly, $\mathcal{J}_\omega$ is integrable if and only if ω satisfies $id\omega = H$. If $H = 0$ this is nothing but the symplectic condition on ω.

Again it is natural to ask what condition integrability imposes on the canonical line bundle, and the answer is similar to the complex case. Recall that on a complex manifold we have a (p, q) decomposition of forms, and complex structures are integrable if and only if the equation $d = \partial + \overline{\partial}$ holds. We will now generalize this concept to the context of generalized complex structures. From (3.3.3) we know that the forms decompose as

$$\wedge^\bullet T^* \otimes \mathbb{C} = U_{-n} \oplus U_{-n+1} \oplus \ldots \oplus U_n. \tag{3.3.4}$$

In principle the operator d^H decomposes as

$$d^H = \sum_{k=-n}^{n} \pi_k \circ d^H,$$

with π_k the projection onto U_k, but we can say more than that. Define

$$\overline{\partial} := \pi_{k-1} \circ d^H : \Gamma(U_k) \to \Gamma(U_{k-1}) \qquad \partial := \pi_{k+1} \circ d^H : \Gamma(U_k) \to \Gamma(U_{k+1}), \tag{3.3.5}$$

and the Nijenhuis tensor

$$\begin{aligned} N :&\Gamma(\overline{L}) \times \Gamma(\overline{L}) \times \Gamma(\overline{L}) \to C^\infty(M) \\ &(u, v, w) \mapsto -2\langle [\![u, v]\!], w\rangle. \end{aligned} \tag{3.3.6}$$

Note that all spaces are over the complex numbers, e.g. $C^\infty(M)$ denotes all smooth complex valued functions. Using Lemma 3.2.1 we see that N defines an element of $\wedge^3 L$, again identifying L with $\overline{L}^*$. The general expression for d^H is given by the following result, which is Lemma 2.3 in [7].

Theorem 3.3.6. *Let $\mathcal{J}$ be a generalized almost complex structure, inducing the grading U_k of forms as in (3.3.4). The operator d^H satisfies $d^H = \partial + \overline{\partial} + N + \overline{N}$, where N and $\overline{N}$ act on spinors by the Clifford action.*

Corollary 3.3.7. *$\mathcal{J}$ is integrable if and only if $d^H = \partial + \overline{\partial}$.*

Note that $d^H = \partial + \overline{\partial}$ is equivalent to the inclusion $d^H(\Gamma(U_n)) \subset \Gamma(U_{n-1})$. This follows from Theorem 3.3.6, for the Clifford action of $\overline{N}$ on $\wedge^\bullet T^* \otimes \mathbb{C}$ is zero if and only if it is zero on U_n. In particular, a sufficient condition for integrability is the existence of a non–vanishing closed section of U_n, and these type of structures have their own name.

Definition 3.3.8. A generalized complex structure $\mathcal{J}$ on M is called *generalized Calabi–Yau* if there exists a global, nowhere vanishing section of U_n that is d^H closed.

3.4 Generalized Kähler manifolds

As already mentioned in Section 2.4, Kähler manifolds play an important role in extended supersymmetric sigma models, and in this section we generalize this concept to generalized complex manifolds. Recall that a Kähler structure on M consists of a complex structure I and a metric g, such that I is orthogonal with respect to g (cf. Equation (2.4.2)) and such that the fundamental 2–form $\omega = gI$ is closed. Now we want to extend this idea to $T \oplus T^*$. We already have the notion of a generalized complex structure, so what we need is a generalization of g.

Definition 3.4.1. A *generalized metric* on M is a map $\mathcal{G} : T \oplus T^* \to T \oplus T^*$ satisfying $\mathcal{G}^* = \mathcal{G}$ and $\mathcal{G}^2 = 1$. Furthermore, the bilinear form $\mathbb{G}(v, w) := \langle v, \mathcal{G}w\rangle$ should be positive definite on $T \oplus T^*$.

Remark. The idea behind this definition is that, as with all the structures in generalized geometry, the desired metric $\mathbb{G}(,)$ on $T \oplus T^*$ should be compatible with the natural pairing, which is expressed through the map $\mathcal{G}$. The property $\mathcal{G}^* = \mathcal{G}$ then ensures that the associated bilinear form $\mathbb{G}(,)$ is symmetric.

If $\mathcal{J}_1$ and $\mathcal{G}$ are a generalized complex structure and metric on M, the triple $(M, \mathcal{J}_1, \mathcal{G})$ is called generalized hermitian if $\mathcal{J}_1$ commutes with $\mathcal{G}$. This is equivalent to $\mathcal{J}_1$ being orthogonal with respect to the metric $\mathbb{G}(,)$. In this case, we can define a generalized almost complex structure by $\mathcal{J}_2 := \mathcal{G}\mathcal{J}_1$. For a generalized Kähler structure we require $\mathcal{J}_2$ to integrable, which is the analogue of the condition $d\omega = 0$ for ordinary Kähler structures.

Definition 3.4.2. A *generalized Kähler structure* on M consists of two commuting generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ such that $\mathcal{G} := -\mathcal{J}_1\mathcal{J}_2$ is a generalized metric on M.

Note that the only constraint on $\mathcal{G}$ as given in the above definition is the positive definite condition. To justify this language we first verify that ordinary Kähler satisfy this condition. If (M, I, g, ω) is a Kähler manifold, we obtain two generalized complex structures on $T \oplus T^*$ given by the formula (with $\mathcal{J}_I$ differing from the one in (3.3.1) by a minus sign):

$$\mathcal{J}_I := \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}, \qquad \mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{3.4.1}$$

A quick computation shows that

$$\mathcal{G} := -\mathcal{J}_I \cdot \mathcal{J}_\omega = -\mathcal{J}_\omega \cdot \mathcal{J}_I = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}, \tag{3.4.2}$$

defines a generalized metric on M.

If $\mathcal{G}$ is a generalized metric, it induces a decomposition $T \oplus T^* = C_+ \oplus C_-$, with $C_\pm$ the ± 1 eigenbundles of $\mathcal{G}$. By the positive definite condition on $\mathcal{G}$, it follows that C_+ is positive definite while C_- is negative definite for $\langle , \rangle$, and because the metric has signature (n, n) it follows that $C_\pm$ are both n–dimensional. Furthermore, since both T and T^* are isotropic while $C_\pm$ are definite, they intersect trivially and combining this fact with their dimensions we conclude that $C_\pm$ are both the graph of map $T \to T^*$. We can write this map for C_+ as the sum of a symmetric and antisymmetric[3] part, say $g + b$, and the positive definite condition for C_+ implies that g defines a metric on M. Since elements of the form $X + g(X) + b(X)$ are perpendicular to ones of the form $X - g(X) + b(X)$ we deduce that C_- is the graph of the map $-g + b$. These two maps give isomorphisms $T \to C_\pm$, with inverses the projections onto T, see also figure 3.1.

Now let $(M, \mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler manifold. Using the isomorphism $T \cong C_\pm$, we will translate the structure as it is defined on $T \oplus T^*$, to a set of geometrical data on T. Note that $\mathcal{G}$ is

[3]We use the letter b to denote the antisymmetric part of this map because this is standard notation, but note that it is not directly related to the B–field that we will define later. For instance, b in this context is globally defined, while the 2–forms in the definition of the B–field are not.

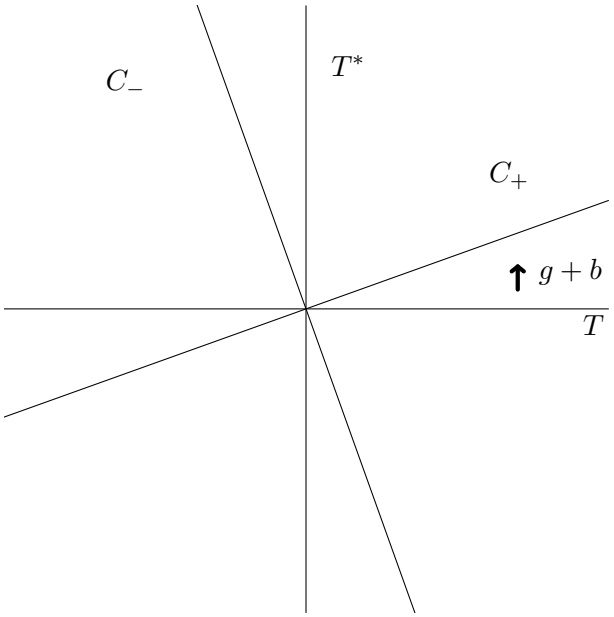

Figure 3.1: Decomposition $T \oplus T^* = C_+ \oplus C_-$.

completely determined by its eigenspaces $C_\pm$, which in turn are determined by g and b. Furthermore, since $\mathcal{J}_1$ and $\mathcal{J}_2$ commute, $(T \oplus T^*) \otimes \mathbb{C}$ decomposes into eigenbundles for both

$$(T \oplus T^*) \otimes \mathbb{C} = L_1 \oplus \overline{L_1} = L_1^+ \oplus L_1^- \oplus \overline{L_1^+} \oplus \overline{L_1^-}, \tag{3.4.3}$$

where L_1 is the $+i$ eigenbundle for $\mathcal{J}_1$ and $L_1 = L_1^+ \oplus L_1^-$ is the decomposition of L_1 into the $\pm i$ eigenbundles for $\mathcal{J}_2$. From the definition of $\mathcal{G}$ it is obvious that on $C_\pm$ we have $\mathcal{J}_1 = \pm\mathcal{J}_2$, hence $C_\pm \otimes \mathbb{C} = L_1^\pm \oplus \overline{L_1^\pm}$. Thus $\mathcal{J}_1$ together with $C_\pm$ determines $\mathcal{J}_2$. Let π_T denote the projection from $T \oplus T^*$ onto T, and define almost complex structures $I_\pm$ on T via the formula

$$I_\pm(\pi_T(v)) = \pi_T(\mathcal{J}_1(v)) \qquad \text{for } v \in C_\pm. \tag{3.4.4}$$

As $\mathcal{J}_1$ is orthogonal with respect to the natural pairing, it follows that $I_\pm$ are both orthogonal with respect to g, so $(g, b, I_\pm)$ defines what is called an almost bi–hermitian structure (almost referring to $I_\pm$ not necessarily being integrable). Conversely, $I_\pm$ determine $\mathcal{J}_1$ on $C_\pm$, hence on the whole of $T \oplus T^*$. So together, the almost bi–hermitian data $(g, b, I_\pm)$ contains all the algebraic content of the generalized Kähler structure, and we want to determine how the integrability conditions on $\mathcal{J}_1, \mathcal{J}_2$ translate into these data. The answer is given by the following Theorem, proved in [18] in Sections 6.3 and 6.4.

Theorem 3.4.3. *[18] The following four conditions are equivalent for an almost generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{G})$ and its associated data $(g, b, I_\pm)$:*

1. *$\mathcal{J}_1$ and $\mathcal{J}_2$ are integrable.*
2. *L_1^+ and L_1^- are involutive.*

3. *$I_\pm$ are integrable and $H + db = \mp d^c_\pm \omega_\pm$, where $d^c_\pm := i(\overline{\partial}_\pm - \partial_\pm)$ and $\partial_\pm$ is the usual ∂-operator for $I_\pm$.*

4. *$\nabla^\pm I_\pm = 0$, with $\nabla^\pm := \nabla \pm \frac{1}{2} g^{-1}(H + db)$ and $H + db$ is of type $(2,1) + (1,2)$ with respect to I_+ and I_-.*

To prove the equivalence of 3 and 4 we need a few equations that relate these different points of view, and we present them here in a lemma.

Lemma 3.4.4.

i) Let $N_\pm(X,Y) = [X,Y] - [I_\pm X, I_\pm Y] + I_\pm[I_\pm X, Y] + I_\pm[X, I_\pm Y]$ be the Nijenhuis tensor with respect to $I_\pm$. We have

$$\begin{aligned} N_\pm(X,Y) =& (\nabla^\pm_Y I_\pm) I_\pm X - (\nabla^\pm_X I_\pm) I_\pm Y + (\nabla^\pm_{I_\pm Y} I_\pm) X - (\nabla^\pm_{I_\pm X} I_\pm) Y \\ & \mp g^{-1}(\iota_Y \iota_X - \iota_{I_\pm Y} \iota_{I_\pm X})(H + db) \mp I_\pm g^{-1}(\iota_Y \iota_{I_\pm X} + \iota_{I_\pm Y} \iota_X)(H + db). \end{aligned} \tag{3.4.5}$$

ii) The forms $\omega_\pm = g I_\pm$ satisfy

$$d\omega_\pm(X,Y,Z) = g((\nabla^\pm_X I_\pm)Y, Z) \mp (H + db)(X, Y, I_\pm Z) + c.p., \tag{3.4.6}$$

where $c.p.$ stands for cyclic permutations in X, Y and Z.

Proof. $i)$: Using the fact that the Levi-Cevita connection is torsion–free, one readily verifies that

$$\nabla^\pm_X Y - \nabla^\pm_Y X - [X,Y] = \pm g^{-1}(H + db).$$

Using this we see that

$$\begin{aligned} N_\pm(X,Y) =& \left(\nabla^\pm_X Y - \nabla^\pm_Y X \mp g^{-1}\iota_Y \iota_X (H + db)\right) \\ & - \left(\nabla^\pm_{I_\pm X} I_\pm Y - \nabla^\pm_{I_\pm Y} I_\pm X \mp g^{-1}\iota_{I_\pm Y}\iota_{I_\pm X}(H + db)\right) \\ & + I_\pm\left(\nabla^\pm_{I_\pm X} Y - \nabla^\pm_Y I_\pm X \mp g^{-1}\iota_Y \iota_{I_\pm X}(H + db)\right) \\ & + I_\pm\left(\nabla^\pm_X I_\pm Y - \nabla^\pm_{I_\pm Y} X \mp g^{-1}\iota_{I_\pm Y}\iota_X (H + db)\right) \\ =& (\nabla^\pm_Y I_\pm) I_\pm X - (\nabla^\pm_X I_\pm) I_\pm Y + (\nabla^\pm_{I_\pm Y} I_\pm) X - (\nabla^\pm_{I_\pm X} I_\pm) Y \\ & \mp g^{-1}(\iota_Y \iota_X - \iota_{I_\pm Y} \iota_{I_\pm X})(H + db) \mp I_\pm g^{-1}(\iota_Y \iota_{I_\pm X} + \iota_{I_\pm Y} \iota_X)(H + db), \end{aligned}$$

where in the last equality we used $I_\pm \nabla^\pm I_\pm = -(\nabla^\pm I_\pm) I_\pm$.
$ii)$: We have

$$\begin{aligned} d\omega_\pm(X,Y,Z) =& X\omega_\pm(Y,Z) - \omega([X,Y],Z) + c.p. \\ =& X g(I_\pm Y, Z) - g(I_\pm[X,Y], Z) + c.p. \\ =& g(I_\pm \nabla^\pm_X Y, Z) + g((\nabla^\pm_X I_\pm)Y, Z) + g(I_\pm Y, \nabla^\pm_X Z) \\ & - g(I_\pm\left(\nabla^\pm_X Y - \nabla^\pm_Y X \mp g^{-1}\iota_Y \iota_X (H + db)\right), Z) + c.p. \\ =& g((\nabla^\pm_X I_\pm)Y, Z) \mp (H + db)(X, Y, I_\pm Z) + c.p. \end{aligned}$$

□

Proof. (of Theorem 3.4.3):
$1 \Rightarrow 2$: If both $\mathcal{J}_1$ and $\mathcal{J}_2$ are integrable, then $L_1 = L_1^+ \oplus L_1^-$ and $L_2 = L_1^+ \oplus \overline{L_1^-}$ are involutive. In particular

$$[\![\Gamma(L_1^+), \Gamma(L_1^+)]\!] \subset \Gamma(L_1^+ \oplus L_1^-) \cap \Gamma(L_1^+ \oplus \overline{L_1^-}) = \Gamma(L_1^+),$$

so L_1^+ is involutive. Applying the same argument with L_1 and $\overline{L}_2 = L_1^- \oplus \overline{L_1^+}$ yields the same result for L_1^-.

$2 \Rightarrow 1$: Suppose $L_1^\pm$ are involutive. To show that $L_1^+ \oplus L_1^-$ is involutive, we have to check

$$[\![u, v]\!] \in L_1$$

for $u \in \Gamma(L_1^+)$, $v \in \Gamma(L_1^-)$. Since L is maximal isotropic, it suffices to check that $[\![u, v]\!]$ is orthogonal to L_1. If $w \in \Gamma(L_1^+)$, we have

$$\langle [\![u, v]\!], w\rangle = -\langle [\![u, w]\!], v\rangle = 0$$

because $[\![u, w]\!] \in L_1^+$. In the final equation we used anti–symmetry of the tensor $\langle [\![,]\!], \rangle$.

$2 \Leftrightarrow 3$: By construction of $I_\pm$ we have

$$L_1^+ = \{X + (g + b)(X) | X \in T_+^{(1,0)}\} = \{X + (b - i\omega_+)X | X \in T_+^{(1,0)}\},$$

where we used $g(X) = -igI_+X = -i\omega_+(X)$. Likewise

$$L_1^- = \{X + (-g + b)(X) | X \in T_-^{(1,0)}\} = \{X + (b + i\omega_-)X | X \in T_-^{(1,0)}\}.$$

Observe that both bundles are of the form $F := \{X + c(X) | X \in \Gamma(E)\}$ for a complex 2–form c and some sub-bundle $E < T \otimes \mathbb{C}$. If $X + cX$,$Y + cY$ are two sections of this bundle we have

$$[\![X + cX, Y + cY]\!] = [X, Y] + \mathcal{L}_X \iota_Y c - \iota_Y d\iota_X c + \iota_Y \iota_X H = [X, Y] + \iota_{[X,Y]} c + \iota_Y \iota_X (dc + H),$$

where in the last equality we used $[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$. In particular, F is involutive if and only if E is involutive with respect to the Lie bracket and $\iota_Y \iota_X (dc + H) = 0$ for all $X, Y \in E$. So $L_1^\pm$ are involutive if and only if $T_\pm^{(1,0)}$ are involutive with respect to the Lie bracket (i.e. $I_\pm$ are integrable) and

$$\iota_Y \iota_X (db + H \mp id\omega_\pm) = 0 \quad \forall X, Y \in T_\pm^{(1,0)}. \tag{3.4.7}$$

Now recall that $\omega_\pm$ is of type $(1, 1)$ with respect to both I_+ and I_-, so that $d\omega_\pm = \partial_\pm \omega_\pm + \overline{\partial}_\pm \omega_\pm$ is of type $(2, 1) + (1, 2)$ if $I_\pm$ are integrable. The condition (3.4.7) is then equivalent to

$$\iota_Y \iota_X (db + H) = \pm i \iota_Y \iota_X \partial_\pm \omega_\pm \quad \forall X, Y \in T_\pm^{(1,0)},$$

and since $db + H$ is real[4] this latter is equivalent to $H + db = \mp d_\pm^c \omega_\pm$.

[4]Note that $d^c = i(\overline{\partial}_\pm - \partial_\pm)$ can be written as $d^c = -I^* dI^*$, hence is also a real operator.

$4 \Rightarrow 3$: If $\nabla^{\pm} I_{\pm} = 0$, Lemma 3.4.4 $i)$ implies

$$N_{\pm}(X,Y) = \mp g^{-1}[\iota_Y \iota_X - \iota_{I_{\pm}Y}\iota_{I_{\pm}X}](H+db) \mp I_{\pm} g^{-1}[\iota_Y \iota_{I_{\pm}X} + \iota_{I_{\pm}Y}\iota_X] g^{-1}(H+db). \tag{3.4.8}$$

The fact that $H + db$ is of type $(2,1)+(1,2)$ is equivalent to the equation

$$(H+db)(X,Y,Z) = (H+db)(I_{\pm}X, I_{\pm}Y, Z) + (H+db)(I_{\pm}X, Y, I_{\pm}Z) + (H+db)(X, I_{\pm}Y, I_{\pm}Z),$$

as one readily verifies. Using this together with the identity $I_{\pm} g^{-1} = -g^{-1} I_{\pm}^*$ we see that (3.4.8) vanishes, so $I_{\pm}$ are integrable. Furthermore, Lemma 3.4.4 $ii)$ implies

$$d\omega_{\pm}(X,Y,Z) = \mp\left((H+db)(I_{\pm}X,Y,Z) + (H+db)(X,I_{\pm}Y,Z) + (H+db)(X,Y,I_{\pm}Z)\right),$$

which is easily seen to be equivalent to $d^c\omega_{\pm} = \mp(H+db)$.

$3 \Rightarrow 4$: Clearly $H + db = \mp d^c_{\pm}\omega_{\pm}$ implies that $H + db$ is of type $(2,1)+(1,2)$ with respect to $I_{\pm}$. This fact, together with Lemma 3.4.4 $i)$ and $ii)$ imply

$$0 = (\nabla^{\pm}_Y I_{\pm})I_{\pm}X - (\nabla^{\pm}_X I_{\pm})I_{\pm}Y + (\nabla^{\pm}_{I_{\pm}Y} I_{\pm})X - (\nabla^{\pm}_{I_{\pm}X} I_{\pm})Y, \tag{3.4.9}$$

and

$$0 = g((\nabla^{\pm}_X I_{\pm})Y, Z) + g((\nabla^{\pm}_Y I_{\pm})Z, X) + g((\nabla^{\pm}_Z I_{\pm})X, Y), \tag{3.4.10}$$

see the proof of $4 \Rightarrow 3$ below for an explanation. We can use (3.4.9) to rewrite the term $g((\nabla^{\pm}_Y I_{\pm})Z, X)$ in (3.4.10), and using the fact that $\nabla^{\pm} I_{\pm}$ is skew–symmetric with respect to g we obtain

$$g((\nabla^{\pm}_X I_{\pm})Y, Z) - g((\nabla^{\pm}_{I_{\pm}Z} I_{\pm})I_{\pm}Y, Z) + g((\nabla^{\pm}_{I_{\pm}Y} I_{\pm})I_{\pm}Z, X) = 0.$$

Replacing Y and Z by $I_{\pm}Y$ and $I_{\pm}Z$, and using (3.4.10) again we obtain

$$g((\nabla^{\pm}_X I_{\pm})Y, Z) = 0,$$

so that indeed $\nabla^{\pm} I_{\pm} = 0$. □

3.4.1 Example of a generalized Kähler manifold

In this section we give an example of a compact six–dimensional (twisted) generalized Kähler manifold, which does not allow for any ordinary Kähler structure. This manifold will be the quotient of a Lie group G, so we first recall some basic Lie theory. The construction is based on [15], in communication with the authors.

A Lie group G is called nilpotent if its Lie algebra $\mathfrak{g}$ is nilpotent, which means that it is not abelian and that its lower central series

$$\mathfrak{g} \supset [\mathfrak{g},\mathfrak{g}] \supset [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] \supset \dots$$

becomes zero in a finite number of steps. Similarly, we have the notion of a solvable Lie group G, where we replace the lower central series by the derived series

$$\mathfrak{g} \supset [\mathfrak{g},\mathfrak{g}] \supset [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] \supset \dots$$

Given a nilpotent (solvable) group G, and a closed subgroup H, the quotient G/H is a manifold which is called a nilmanifold (solvmanifold). Note that some authors require H to be discrete and co-compact (i.e., G/H is compact) as part of the definition.

On such spaces we can try to construct invariant generalized complex structures, where invariant means invariant under the group translations. Such invariant structures are completely determined by their behavior on the Lie algebra $\mathfrak{g}$, which effectively reduces all computations to linear algebra on $\mathfrak{g}$. In [9] it was shown that all six–dimensional nilmanifolds admit generalized complex structures, but besides the torus, none of them admits an invariant generalized Kähler structure [5]. For solvmanifolds there are no such restrictions, as the example will show.

Every element $X \in \mathfrak{g}$ determines a left invariant vector field $\tilde{X}$ on G by left translation

$$\tilde{X}(g) := T_e l_g(X),$$

where l_g is the map $h \mapsto gh$, and clearly every left invariant vector field is obtained in this way. Similarly every element $\alpha \in T_e^*G$ determines and is determined by a left invariant form $\tilde{\alpha}$ on G, invariant in the sense that $l_g^*(\tilde{\alpha}) = \tilde{\alpha}$. In this way $\wedge^\bullet\mathfrak{g}^*$ becomes a subcomplex of $\wedge^\bullet T^*G$. If $\tilde{\alpha}$ is such an invariant 1–form, we can compute $d\tilde{\alpha}$, which we know must be invariant again, by evaluating it on invariant vector fields:

$$d\tilde{\alpha}(\tilde{X},\tilde{Y}) = \tilde{X}(\tilde{\alpha}(\tilde{Y})) - \tilde{Y}(\tilde{\alpha}(\tilde{X})) - \tilde{\alpha}([\tilde{X},\tilde{Y}]) = -\alpha([X,Y]), \tag{3.4.11}$$

since the functions $\tilde{\alpha}(\tilde{X})$ and $\tilde{\alpha}(\tilde{Y})$ are equal to $\alpha(X)$ and $\alpha(Y)$, so in particular are constant. Note the two different brackets in (3.4.11); the first is the Lie bracket of vector fields while the second is the bracket of the Lie algebra itself. These brackets are related, in the sense that the map $X \mapsto \tilde{X}$ is a homomorphism of Lie algebras, i.e. $[\tilde{X},\tilde{Y}] = \widetilde{[X,Y]}$. The differential d thus acts on $\wedge^\bullet\mathfrak{g}^*$ via

$$d\alpha(X,Y) = -\alpha([X,Y]), \tag{3.4.12}$$

and in particular contains all the information about the bracket on $\mathfrak{g}$. Indeed, if e_i is a basis for $\mathfrak{g}$, and e^i the dual basis in $\mathfrak{g}^*$, then $de^i = -\frac{1}{2} f_{jk}{}^i e^j \wedge e^k$, where $f_{jk}{}^i$ are the structure constants of the algebra ($[e_j, e_k] = f_{jk}{}^i e_i$). We will abbreviate $e^{ij} := e^i \wedge e^j$, $e^{ijk} := e^i \wedge e^j \wedge e^k$, etc.

The idea is now to construct a solvable Lie algebra, that admits a left–invariant generalized Kähler structure (note that this can be verified on the Lie algebra itself) and whose corresponding Lie group[5]

[5] By a classical theorem of Lie, every finite dimensional Lie algebra belongs to a Lie group, which can be chosen such that it is simply connected (in which case it is unique).

G has a discrete subgroup H that is co-compact, i.e. $M := G/H$ is compact. Note that this space has the structure of a manifold, and is in fact locally diffeomorphic to G. Then we can transfer the generalized Kähler structure from G to M, using the fact that the map $G \to M$ is a local diffeomorphism. It might sound rather difficult to find this example, but there is a more or less standard recipe to find it.

We first pick a matrix m in $SL(3,\mathbb{Z})$ that has two conjugate eigenvalues $\alpha, \overline{\alpha}$ which are not real, and a real eigenvalue $c > 1$, and let (c_1, c_2, c_3) and $(\alpha_1, \alpha_2, \alpha_3)$ be the eigenvectors corresponding to c and α respectively. Due to the condition $\det(m) = 1$, we have $|\alpha|^2 c = 1$. Note that a general element in $SL(3,\mathbb{Z})$ is of this form. With these constants we define a group structure on $G := \mathbb{R} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C}$ by

$$(t, u, z, w) \cdot (t', u', z', w') := (t + t', c^t u' + u, \alpha^t z' + z, e^{i\frac{\pi}{2}t} w' + w). \tag{3.4.13}$$

As a consequence of the equality $|\alpha|^2 c = 1$ this group is easily seen to be unimodular, which means that right multiplication preserves the left–invariant Haar measure (in this case just the ordinary Lebesgue measure) of the group. This property is necessary for a group to admit a cocompact discrete group, see Section 9.1 in [13] for more details. This explains the choice of the matrix m in $SL(3,\mathbb{Z})$. To construct this subgroup, first observe that the three vectors (c_i, α_i) in $\mathbb{R} \times \mathbb{C}$ are linearly independent over $\mathbb{R}$. For if $\sum_i \lambda^i (c_i, \alpha_i) = 0$, we would also have $\sum_i \lambda^i (c_i, \alpha_i, \overline{\alpha}_i) = 0$ because the λ^i are real. This would imply that the matrix

$$\begin{pmatrix} c_1 & \alpha_1 & \overline{\alpha}_1 \\ c_2 & \alpha_2 & \overline{\alpha}_2 \\ c_3 & \alpha_3 & \overline{\alpha}_3 \end{pmatrix} \tag{3.4.14}$$

is singular, which contradicts the fact that its columns are eigenvectors with distinct eigenvalues. So the (c_i, α_i) span $\mathbb{R} \times \mathbb{C}$, and the lattice[6] generated by the elements

$$(1,0,0,0), \quad (0, c_i, \alpha_i, 0) \;\; i = 1,2,3, \quad (0,0,0,1) \quad \text{and} \quad (0,0,0,i), \tag{3.4.15}$$

is a subgroup of G, which we denote by H. Indeed, looking at the group structure, we only have to check that the element $(1,0,0,0)$ preserves this lattice under the group multiplication, and this follows directly from

$$(1,0,0,0) \cdot (0, c_i, \alpha_i, 0) = (1, cc_i, \alpha\alpha_i, 0) = \sum_j m_{ij} (0, c_j, \alpha_j, 0) + (1,0,0,0), \tag{3.4.16}$$

and

$$(1,0,0,0) \cdot (0,0,0,1) = (1,0,0,i), \qquad (1,0,0,0) \cdot (0,0,0,i) = (1,0,0,-1). \tag{3.4.17}$$

H is discrete, hence closed, so that the quotient will be a manifold, and for this we use the fact that $c > 1$ and that$|\alpha| < 1$. The quotient $M = G/H$ is indeed compact, and to see this, let us perform the

[6] Here we really mean lattice, in the sense that we take the $\mathbb{Z}$-linear span of them in $\mathbb{R}^2 \oplus \mathbb{C}^2$

quotient in two steps: First, we quotient out the lattice Γ_1, which equals the above lattice without the $(1,0,0,0)$ direction. The group structure on Γ_1 is just the addition in $\mathbb{R}\oplus\mathbb{C}^2$, as follows directly from (3.4.13). So if $\pi_1 : G \to \mathbb{R}$ is the projection map onto the first factor, we see that $\pi_1^{-1}(t)/\Gamma_1$ equals $(\{t\}\oplus\mathbb{R}\oplus\mathbb{C}^2)/\Gamma_1$, and this is just the usual quotient of $\mathbb{R}^5$ by a lattice, so G/Γ_1 is identified with a five–dimensional torus, fibered over $\mathbb{R}$. In performing the last quotient in the $(1,0,0,0)$ direction, the line $\mathbb{R}$ becomes a circle, and in the identification $x \sim x+1$ the two fibers above x (isomorphic to a 5-torus) are identified with itself via

$$u \sim cu, \qquad z \sim \alpha z, \qquad w \sim iw, \tag{3.4.18}$$

so in particular M is compact.

To construct the associated Lie algebra $\mathfrak{g}$ we use the relation between the Lie bracket and the exterior derivative as in (3.4.11), so that it suffices to find 6 linearly independent left–invariant 1–forms and compute their exterior derivatives. The following forms are left–invariant:

$$dt, \quad c^{-t}du, \quad \alpha^{-t}dz, \quad e^{-i\frac{\pi}{2}t}dw. \tag{3.4.19}$$

For instance, if $g := (t', u', z', w')$ then

$$l_g^*(c^{-t}du) = (l_g^*(c^{-t}))d(l_g^*u) = c^{-t-t'}d(c^{t'}u + u') = c^{-t}du.$$

Now we split the complex forms up into their real and imaginary parts. To this end, write $\alpha = c^{-1/2}e^{i\theta}$, $z = x + iy$ and $w = a + ib$. Then (3.4.19) gives the following real forms on G that are left–invariant:

$$\begin{aligned}
e^1 &:= dt & e^4 &:= c^{t/2}(\cos(t\theta)dy - \sin(t\theta)dx)\\
e^2 &:= c^{-t}du & e^5 &:= \cos(\frac{\pi t}{2})dx + \sin(\frac{\pi t}{2})dy\\
e^3 &:= c^{t/2}(\cos(t\theta)dx + \sin(t\theta)dy) & e^6 &:= \cos(\frac{\pi t}{2})dy - \sin(\frac{\pi t}{2})dx
\end{aligned} \tag{3.4.20}$$

They satisfy the following algebra

$$\begin{aligned}
de^1 &= 0 & de^4 &= \frac{1}{2}\ln(c)e^{14} - \theta e^{13}\\
de^2 &= -\ln(c)e^{12} & de^5 &= \frac{\pi}{2}e^{16}\\
de^3 &= \frac{1}{2}\ln(c)e^{13} + \theta e^{14} & de^6 &= -\frac{\pi}{2}e^{15}
\end{aligned} \tag{3.4.21}$$

From these equations it follows that $[e_i, e_j] = 0$ if both i and j are not equal to 1, and that $e_1 \notin [\mathfrak{g},\mathfrak{g}]$, so that $[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] = 0$ and $\mathfrak{g}$ is indeed solvable.

We define two left–invariant almost complex structures $I_\pm$ on G by defining their $(1,0)$–forms to be

$$f_\pm^1 := e^1 \pm ie^2, \quad f_\pm^2 := e^3 + ie^4, \quad f_\pm^3 := e^5 + ie^6. \tag{3.4.22}$$

To show that both $I_\pm$ are integrable, we have to show that $d = \partial_\pm + \overline{\partial}_\pm$, which amounts to showing that the forms $df^i_\pm$ are of type $(2,0)+(1,1)$ with respect to $I_\pm$. Using (3.4.21) we compute

$$\begin{aligned} df^1_\pm &= \frac{1}{2}\ln(c) f^1_\pm \wedge \overline{f}^1_\pm \\ df^2_\pm &= \frac{1}{2}(\frac{1}{2}\ln(c) - i\theta)(f^1_\pm \wedge f^2_\pm + \overline{f}^1_\pm \wedge f^2_\pm) \\ df^3_\pm &= -\frac{i\pi}{4}(f^1_\pm \wedge f^3_\pm + \overline{f}^1_\pm \wedge f^3_\pm), \end{aligned} \tag{3.4.23}$$

so that indeed both $I_\pm$ are integrable. Let g be the standard Euclidean metric on G, i.e. $g = \sum_i e^i \otimes e^i$, for which $I_\pm$ are obviously orthogonal. The 2–forms $\omega_\pm = gI_\pm$ are given by

$$\omega_\pm = \pm e^{12} + e^{34} + e^{56}, \tag{3.4.24}$$

and satisfy

$$\mp d^c_\pm \omega_\pm = \mp (I^*_\pm)^{-1} d I^*_\pm \omega_\pm = \pm I^*_\pm d(\pm e^{12} + e^{34} + e^{56}) = \pm I(\ln(c) e^{134}) = -\ln(c) e^{234} =: H. \tag{3.4.25}$$

Note that $dH = -\ln(c)(-\ln(c)e^{1234} + \frac{1}{2}\ln(c)e^{1234} + \frac{1}{2}\ln(c)e^{1234}) = 0$, so H is a closed 3–form, hence the data $(g, I_\pm, H)$ defines a generalized Kähler structure by condition 3 of Theorem 3.4.3, with b equal to 0. We will see later that H is not exact, and that the dimension of the first cohomology group of M is 1, so that in particular M does not admit any Kähler structure[7]. Let us determine the canonical line bundles associated to the generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$. From the discussion in the previous section we know that

$$L^\pm_1 = \{X \pm g(X) |\ X \in T^{(1,0)}_\pm\}, \tag{3.4.26}$$

and a global frame for $T^{(1,0)}_\pm M$ is given by the left–invariant vector fields associated to the Lie algebra elements

$$e_1 \mp i e_2, \qquad e_3 - i e_4, \qquad e_5 - i e_6. \tag{3.4.27}$$

Using (3.4.26), (3.4.27) and the equalities $L_1 = L^+_1 \oplus L^-_1$, $L_2 = L^+_1 \oplus \overline{L^-_1}$ one readily verifies that the canonical line bundles for $\mathcal{J}_1$ and $\mathcal{J}_2$ are generated by the (left–invariant) forms

$$\begin{aligned} \rho_1 &:= (1 + ie^{12})(e^3 - ie^4)(e^5 - ie^6), \\ \rho_2 &:= (e^1 - ie^2)(1 + ie^{34})(1 + ie^{56}). \end{aligned} \tag{3.4.28}$$

Let us check that $\mathcal{J}_1$ and $\mathcal{J}_2$ are integrable using corollary 3.3.7. For this it suffices to show that $d^H \rho_i = v \cdot \rho_i$ for some $v \in (T \oplus T^*) \otimes \mathbb{C}$, because of the remark made below corollary 3.3.7.
A straightforward calculation gives

$$\begin{aligned} d^H \rho_1 &= (\frac{1}{2}\ln(c) + i\theta - i\frac{\pi}{2}) e^1 \cdot \rho_1, \\ d^H \rho_2 &= -\ln(c) e^1 \cdot \rho_2, \end{aligned} \tag{3.4.29}$$

so indeed $\mathcal{J}_1$ and $\mathcal{J}_2$ are integrable, as they should be. We also see that both $\mathcal{J}_1$ and $\mathcal{J}_2$ are not generalized Calabi–Yau (Definition 3.3.8).

[7] Due to Hodge-decomposition, the odd dimensional cohomology groups on a Kähler manifold are even dimensional.

3.5 Lie algebroids

The $+i$ eigen-bundle L of a generalized complex structure is isotropic and as a consequence the Courant bracket restricted to it is skew–symmetric (cf. (3.2.10)), hence defines a Lie bracket on $\Gamma(L)$. Furthermore, L comes with a projection map $\pi : L \to T$, which besides being a homomorphism of Lie algebras, satisfies $[\![u, fv]\!] = f[\![u, v]\!] + \pi(u)(f)v$ for $u, v \in \Gamma(L)$, $f \in C^\infty(M)$. These properties are important enough to deserve a special name.

Definition 3.5.1. A vector bundle L over M is called a *Lie algebroid* if there is a Lie bracket on $\Gamma(L)$ (i.e., a skew symmetric bracket $[\![,]\!]$ satisfying the Jacobi identity) and there is a map $\pi : L \to TM$ that induces a homomorphism of Lie algebras $\Gamma(L) \to \Gamma(TM)$, and satisfies

$$[\![u, fv]\!] = f[\![u, v]\!] + \pi(u)(f)v \qquad \forall u, v \in \Gamma(L),\ f \in C^\infty(M). \tag{3.5.1}$$

Similarly, we have the notion of a complex Lie algebroid, where we replace T by $T \otimes \mathbb{C}$.

The bracket on a Lie algebroid L endows the exterior algebra of its dual L^* with a differential $d_L : \Gamma(\wedge^k L^*) \to \Gamma(\wedge^{k+1} L^*)$, defined by

$$\begin{aligned} d_L(\alpha)(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} \pi(X_i)\alpha(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}) \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \alpha([\![X_i, X_j]\!], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}). \end{aligned} \tag{3.5.2}$$

The fact that $d_L^2 = 0$ is readily verified, which is actually the same computation as for the ordinary exterior derivative. The cohomology of d_L is called the *Lie algebroid cohomology*, and it will be important later on when discussing the cohomology of physical observables.

Let us turn our attention again to generalized complex structures, so let L be the $+i$ eigen–bundle for a generalized complex structure $\mathcal{J}$. The Courant bracket endows $\Gamma(L)$ with a Lie bracket, since the bracket is skew–symmetric on isotropic bundles. Together with the projection onto $T \otimes \mathbb{C}$, L has the structure of a complex Lie algebroid. We want to relate the Lie algebroid cohomology of L, which at the moment seems rather abstract, to some kind of cohomology on the space of differential forms on M. Recall that L induces the decomposition

$$\wedge^\bullet T^* \otimes \mathbb{C} = U_{-n} \oplus \ldots \oplus U_n,$$

where U_n is the pure spinor line of L and $U_{n-k} = \wedge^k \overline{L} \cdot U_n$. On the space of forms we have the operator $d^H = d - H\wedge$, and integrability of $\mathcal{J}$ is by corollary 3.3.7 equivalent to the decomposition $d^H = \partial + \overline{\partial}$. The relation between d_L and d_H is expressed through the following lemma.

Lemma 3.5.2. *For $\alpha \in \Gamma(\wedge^\bullet L^*)$ and $\rho \in \wedge^\bullet T^*$ we have*

$$[\overline{\partial}, \alpha] \cdot \rho = (d_L \alpha) \cdot \rho. \tag{3.5.3}$$

Here $[,]$ stands for the graded commutator for operators acting on spinors.

Proof. First recall that we identify $\overline{L}$ with L^* via the natural pairing: $\beta(v) = 2\langle \beta, v\rangle$ for $\beta \in \overline{L}$ and $v \in L$. It suffices to prove (3.5.3) for $\alpha \in \overline{L}$, since both sides satisfy the same sort of Leibniz rule for k–forms:

$$[\overline{\partial}, \alpha_1 \wedge \cdots \wedge \alpha_k] = \sum_{i=1}^{k} (-1)^i \alpha_1 \wedge \cdots \wedge [\overline{\partial}, \alpha_i] \wedge \cdots \wedge \alpha_k,$$

and

$$d_L(\alpha_1 \wedge \cdots \wedge \alpha_k) = \sum_{i=1}^{k} (-1)^i \alpha_1 \wedge \cdots \wedge d_L(\alpha_i) \wedge \cdots \wedge \alpha_k.$$

So let $\alpha \in \overline{L}$ and $\rho \in U_k$, and pick any $v \in L$. We have

$$[v, [\overline{\partial}, \alpha]] \cdot \rho = \pi_{k-1}([v, [d_H, \alpha]] \cdot \rho) = \pi_{k-1}(-[\![\alpha, v]\!] \cdot \rho) = (\pi_{\overline{L}}(-[\![\alpha, v]\!])) \cdot \rho,$$

and

$$[v, d_L\alpha] \cdot \rho = (\iota_v d_L \alpha) \cdot \rho,$$

where we used again the rule $(v\beta + \beta v) \cdot \rho = (\iota_v \beta) \cdot \rho$ for an arbitrary form $\beta \in \wedge^\bullet \overline{L}$, see the proof of Theorem 3.3.6 for an explanation. We claim that $\pi_{\overline{L}}(-[\![\alpha, v]\!]) = \iota_v d_L \alpha$. Indeed, for $w \in L$ we have

$$\begin{aligned} 2\langle w, -[\![\alpha, v]\!]\rangle &= 2\langle w, [\![v, \alpha]\!] - 2d\langle \alpha, v\rangle\rangle \\ &= 2\pi_T(v)(\langle w, \alpha\rangle) - 2\langle [\![v, w]\!], \alpha\rangle - 2\pi_T(w)(\langle \alpha, v\rangle) \\ &= d_L\alpha(v, w). \end{aligned}$$

Again, note the various factors of 2 appearing everywhere. We see that $[v, [\overline{\partial}, \alpha] - d_L\alpha] = 0$ as operators on spinors, for all $v \in L$. Looking at the decomposition of $\wedge T^* \otimes \mathbb{C} = U_{-n} \oplus \ldots \oplus U_n$, v acts as a degree 1 operator, while both $[\overline{\partial}, \alpha]$ and $d_L\alpha$ are degree -2 operators. It follows that for $\rho \in U_n$,

$$0 = [v, [\overline{\partial}, \alpha] - d_L\alpha] \cdot \rho = v \cdot ([\overline{\partial}, \alpha] - d_L\alpha) \cdot \rho.$$

Since v is arbitrary, we conclude that

$$[\overline{\partial}, \alpha] \cdot \rho = d_L\alpha \cdot \rho \qquad \forall \rho \in U_n.$$

For $\rho \in U_{n-1}$ we can proceed with the same trick, since we now know that on U_n the two operators are equal. Continuing with this procedure, we see that $[\overline{\partial}, \alpha] = d_L\alpha$ as operators on U_k for all k. □

If $\mathcal{J}$ is generalized Calabi–Yau and $\rho \in \Gamma(U_n)$ is the corresponding d^H–closed form, consider the following map:

$$\begin{aligned} \wedge^\bullet \overline{L} &\longrightarrow \wedge^\bullet T^* \otimes \mathbb{C} \\ \alpha &\mapsto \alpha \cdot \rho. \end{aligned} \tag{3.5.4}$$

This map is an isomorphism because ρ is nowhere vanishing, and because $d^H\rho = \overline{\partial}\rho = 0$, Lemma 3.5.2 implies that $\overline{\partial}(\alpha \cdot \rho) = (d_L\alpha) \cdot \rho$. In other words, the map (3.5.4) intertwines the operators d_L and $\overline{\partial}$, so that it induces an isomorphism on cohomology of the two operators. So for generalized Calabi–Yau spaces, Lie algebroid cohomology can be computed in terms of differential forms with the help of the $\overline{\partial}$ operator.

Example 3.5.3. Suppose $\mathcal{J}$ is associated to an ordinary complex structure, with $H = 0$:

$$\mathcal{J}_I := \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}. \tag{3.5.5}$$

We have $L = T^{(0,1)} \oplus (T^*)^{(1,0)}$, so that $U_n = \wedge^{(n,0)} T^*$. It follows that

$$\wedge^k \overline{L} = \bigoplus_{p=0}^{k} \wedge^p T^{(1,0)} \otimes \wedge^{k-p} (T^*)^{(0,1)}, \tag{3.5.6}$$

hence

$$U_{n-k} = \wedge^k \overline{L} \cdot U_n = \bigoplus_{p=0}^{k} \wedge^{(n-p,k-p)} T^*. \tag{3.5.7}$$

The exterior derivative can be written as $d = \partial_I + \overline{\partial}_I$ where ∂_I is the ordinary operator from complex geometry, hence it maps $\wedge^{(n-p,k-p)} T^*$ into $\wedge^{(n-p+1,k-p)} T^* \oplus \wedge^{(n-p,k-p+1)} T^*$. The first summand lies in $U_{n-(k-1)} = U_{n-k+1}$, while the second lies in $U_{n-(k+1)} = U_{n-k-1}$, so that the operators ∂ and $\overline{\partial}$ indeed agree with the 'generalized' ∂ and $\overline{\partial}$. For $\mathcal{J}$ to be generalized Calabi–Yau, there must exist a nowhere-vanishing section of $U_n = \wedge^{(n,0)} T^*$, that is d–closed. This is precisely the usual definition for I to be Calabi–Yau, and the isomorphism in (3.5.4) is well–known, and usually denoted by $\alpha \mapsto \alpha \vdash \Omega$, with Ω the Calabi–Yau $(n,0)$–form. So, Lie algebroid cohomology of Calabi–Yau manifolds is given by the ordinary Dolbeault cohomology.

Example 3.5.4. Now let $\mathcal{J}$ be associated to a symplectic structure (again, $H = 0$):

$$\mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{3.5.8}$$

The associated eigenbundle is given by $L = \{X - i\iota_X \omega |\ X \in T\}$, but the U_k decomposition is a bit more complicated.

Lemma 3.5.5. *Let $\Lambda := -\omega^{-1}$ be the Poisson bi-vector corresponding to ω, and let $d^{\mathcal{J}} := [d, \tilde{\mathcal{J}}]$, where $\tilde{\mathcal{J}}$ is the lift of $\mathcal{J}$ to the Lie algebra of the spin group, see also Lemma 3.3.3. Then*

i) $d^{\mathcal{J}}$ lowers the degree of a form by 1,

ii) $U_k = \{e^{i\omega} e^{\Lambda/2i} \alpha |\ \alpha \in \wedge^{n-k} T^* \otimes \mathbb{C}\}$,

iii) $d(e^{i\omega} e^{\Lambda/2i} \alpha) = e^{i\omega} e^{\Lambda/2i} (d\alpha - \frac{1}{2i} d^{\mathcal{J}} \alpha)$,

where Λ acts on forms by inner contraction.

Proof. See Lemma 2.2 and Theorem 2.3 in [8]. □

In particular, we see that U_n has a global non–vanishing section given by $e^{i\omega}e^{\Lambda/2i}\cdot 1 = e^{i\omega}$, which is obviously closed. The generalized Calabi–Yau condition thus imposes no extra conditions on symplectic manifolds. Note that $iii)$ implies that under the identification $\wedge^k T^* \otimes \mathbb{C} \leftrightarrow U_{n-k}$ induced by the operator $e^{i\omega}e^{\Lambda/2i}$, the $\overline{\partial}$ operator is identified with the usual exterior derivative, so that the Lie algebroid cohomology of symplectic manifolds equals the de Rham cohomology of the underlying manifold.

Remark. In the context of twisting (Chapter 5), the above two examples are often referred to as the B and A model respectively.

Example 3.5.6. In the example from Section 3.4.1, both $\mathcal{J}_1$ and $\mathcal{J}_2$ are not generalized Calabi–Yau, as follows from (3.4.29). Nevertheless, let us compute the d^H–cohomology of M. We will first compute it on the Lie algebra, and then show that it equals the actual d^H–cohomology. Consider first the d–cohomology of $\mathfrak{g}$.

Degree 1: We know that e^1 is closed, while for the others only a combination of e^3 and e^4 has a chance of being closed. We have

$$d(ae^3 + be^4) = (\frac{a}{2}\ln(c) - b\theta)e^{13} + (a\theta + \frac{b}{2}\ln(c))e^{14},$$

which is zero if and only if

$$a = \frac{2b\theta}{\ln(c)} \quad \text{and} \quad b(2\theta^2 + \frac{1}{2}\ln^2(c)) = 0.$$

The second equation implies $b = 0$, and the first then implies $a = 0$. So there is no closed 1–form besides e^1.

Degree 2: The exact 2–forms are spanned by:

$$\{e^{12}, e^{13}, e^{14}, e^{15}, e^{16}\}. \tag{3.5.9}$$

The remaining closed forms are of the form

$$\beta = a_1e^{23} + a_2e^{24} + a_3e^{25} + a_4e^{26} + a_5e^{34} + a_6e^{35} + a_7e^{36} + a_8e^{45} + a_9e^{46} + a_{10}e^{56}. \tag{3.5.10}$$

Using (3.4.21) we can compute $d\beta$, which vanishes when $a_i = 0$ for $i = 1, 2, \ldots, 9$, while a_{10} can be arbitrary. Hence the closed 2–forms are spanned by

$$\{e^{12}, e^{13}, e^{14}, e^{15}, e^{16}, e^{56}\}. \tag{3.5.11}$$

Degree 3: The space of exact 3–forms is spanned by

$$\{e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{145}, e^{146}, e^{135}, e^{136}\}. \tag{3.5.12}$$

The remaining closed forms must be of the form

$$\begin{aligned}\beta =& a_1 e^{156} + a_2 e^{234} + a_3 e^{235} + a_4 e^{236} + a_5 e^{245} + a_6 e^{246} + a_7 e^{256} + a_8 e^{345} \\ &+ a_9 e^{346} + a_{10} e^{356} + a_{11} e^{456}.\end{aligned} \tag{3.5.13}$$

$d\beta = 0$ when $a_i = 0$ for $i = 3, 4, \ldots, 11$, while a_1, a_2 can be arbitrary. So the closed 3–forms are spanned by

$$\{e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{145}, e^{146}, e^{135}, e^{136}, e^{156}, e^{234}\}. \tag{3.5.14}$$

Degree 4: The space of exact 4–forms is spanned by

$$\{e^{1235}, e^{1236}, e^{1245}, e^{1246}, e^{1256}, e^{1345}, e^{1346}, e^{1356}, e^{1456}\}. \tag{3.5.15}$$

Remaining closed forms are of the form

$$\beta = a_1 e^{1234} + a_2 e^{2345} + a_3 e^{2346} + a_4 e^{2356} + a_5 e^{2456} + a_6 e^{3456}, \tag{3.5.16}$$

which is closed when $a_i = 0$ for $i = 2, 3, 4, 5, 6$, with no restriction on a_1. Hence the closed 4–forms are spanned by

$$\{e^{1235}, e^{1236}, e^{1245}, e^{1246}, e^{1256}, e^{1345}, e^{1346}, e^{1356}, e^{1456}, e^{1234}\}. \tag{3.5.17}$$

Degree 5: The space of exact 5–forms is spanned by

$$\{e^{12345}, e^{12346}, e^{12356}, e^{12456}, e^{13456}\}. \tag{3.5.18}$$

There is one 5–form left, e^{23456}, which is closed. So all 5–forms are closed.

Degree 6: There are no exact forms, and there is only one 6–form, given by e^{123456} which is of course closed.

We want to relate this to the de Rham cohomology of M, for which we need the following result

Theorem 3.5.7. *[30] For every solvmanifold $M = G/H$, the inclusion $\wedge^\bullet(\mathfrak{g}^*) \subset \wedge^\bullet T^*(G/H)$ induces an injective map on cohomology.*

Therefore, to show that the above cohomology equals the de Rham cohomology of M we only need to know the Betti-numbers of M (the dimensions of the cohomology groups). To compute these, we use the observation made above Equation (3.4.18), which states that M is a torus fibration over a circle. We will make use of the *Mayer-Vietoris sequence* in cohomology:

Theorem 3.5.8. *[21] Let X be a topological space and A, B be two subspaces whose interiors together cover X. Then we have the following long exact sequence in cohomology:*

$$\ldots \to H^i(X, \mathbb{R}) \xrightarrow{(i_A^*, i_B^*)} H^i(A, \mathbb{R}) \oplus H^i(B, \mathbb{R}) \xrightarrow{j_A^* - j_B^*} H^i(A \cap B, \mathbb{R}) \xrightarrow{\delta} H^{i+1}(X, \mathbb{R}) \to \ldots \tag{3.5.19}$$

where $i_{A/B}$ is the inclusion of A/B into X, $j_{A/B}$ is the inclusion of $A \cap B$ into A/B, and δ is a co-boundary map (its precise definition will not be important to us).

We apply this to M with $A := \pi_1^{-1}([-\frac{1}{8}, \frac{5}{8}])$ and $B := \pi_1^{-1}([\frac{3}{8}, \frac{9}{8}])$, where $\pi_1 : M \to \mathbb{R}/\mathbb{Z}$ is the projection onto the first factor. Both A and B are then homotopic to the 5-torus, whose cohomology in degree k has dimension $\binom{5}{k}$. An easy way to see this is to view an n-torus as the n-fold product $(S^1)^{\times n}$ and use the Künneth formula

$$H^k(X \times Y, \mathbb{R}) \cong \bigoplus_{i+j=k} (H^i(X, \mathbb{R}) \otimes H^i(Y, \mathbb{R})). \tag{3.5.20}$$

Note that $A \cap B$ consists of two connected components, both homotopic to the 5-torus. For the map $j_A^* - j_B^*$, we identify the fiber of A with that of B using the identity map at one of the connected components of $A \cap B$, and by the map (3.4.18) on the other. Now we have to fill in the long exact sequence, which starts with the terms

$$\begin{aligned} 0 \to H^0(M) = \mathbb{R} \to H^0(A) \oplus H^0(B) = \mathbb{R} \oplus \mathbb{R} \xrightarrow{f} H^0(A \cap B) = \mathbb{R} \oplus \mathbb{R} \to \\ \to H^1(M) \to H^1(A) \oplus H^1(B) = \mathbb{R}^5 \oplus \mathbb{R}^5 \xrightarrow{g} H^1(A \cap B) = \mathbb{R}^5 \oplus \mathbb{R}^5 \to \dots \end{aligned} \tag{3.5.21}$$

The map f is given by $(a, b) \mapsto (a - b, a - b)$ and has one–dimensional co–kernel. The map g is given by (cf. (3.4.18))

$$\begin{aligned} &(du, 0) \mapsto (du, du) && (0, du) \mapsto (-du, -cdu) \\ &(dz, 0) \mapsto (dz, dz) && (0, du) \mapsto (-dz, -\alpha dz) \\ &(d\overline{z}, 0) \mapsto (d\overline{z}, d\overline{z}) && (0, d\overline{z}) \mapsto (-d\overline{z}, -\overline{\alpha} d\overline{z}) \\ &(dw, 0) \mapsto (dw, dw) && (0, dw) \mapsto (-dw, -idw) \\ &(d\overline{w}, 0) \mapsto (d\overline{w}, d\overline{w}) && (0, d\overline{w}) \mapsto (-d\overline{w}, id\overline{w}) \end{aligned} \tag{3.5.22}$$

and so has, because $c \neq 1$, zero kernel. It follows that $b^1(M) := \dim(H^1(M)) = 1$. The computation of the other dimensions is similar, and we only need to know the kernels of the maps $H^i(A) \oplus H^i(B) \to H^i(A \cap B)$. For $i = 2$ we have $H^i(A) = \mathbb{R}^{10}$, spanned by $du \wedge dz, du \wedge d\overline{z}, \dots$ and it is easily verified that the kernel is spanned by $(dw \wedge d\overline{w}, dw \wedge d\overline{w})$. For $i = 3$ the kernel is also one–dimensional, with basis $(du \wedge dz \wedge d\overline{z}, du \wedge dz \wedge d\overline{z})$, as a consequence of the identity $c|\alpha|^2 = 1$. Using the long exact sequence and Poincaré duality[8] we obtain all the Betti-numbers of M:

$$b^0 = 1, \qquad b^1 = 1, \qquad b^2 = 1, \qquad b^3 = 2, \qquad b^4 = 1, \qquad b^5 = 1, \qquad b^6 = 1. \tag{3.5.23}$$

These dimensions are the same as we found for the Lie algebra cohomology, hence both cohomologies are equal by Theorem 3.5.7. Now finally we are ready to compute the d^H–cohomology. For this we consider d^H as a map from even/odd forms to odd/even forms, and we start with the even ones. Recall that $H = -\ln(c)e^{234}$.

Even forms: For an even form $\phi^0 + \phi^2 + \phi^4 + \phi^6$ to be d^H–closed we must have

$$d\phi^0 = 0, \qquad H \wedge \phi^0 = d\phi^2, \qquad H \wedge \phi^2 = d\phi^4. \tag{3.5.24}$$

[8]Poincaré duality on an oriented compact m–dimensional manifold M states that there is a non–degenerate pairing between $H^i(M)$ and $H^{m-i}(M)$, implying that both have the same dimension.

Since H is not exact, the second equation implies $\phi^0 = 0$ so that ϕ^2 is closed. Then ϕ^2 is a linear combination of the forms in (3.5.11), but observe that the forms e^{12}, e^{13} and e^{14} are already d^H exact, because $de^i = d^H e^i$ for $i = 2, 3, 4$. Since we are dividing out those classes anyway, we assume that $\phi^2 = ae^{15} + be^{16} + ce^{56}$. The third equation above then implies

$$\ln(c)(ae^{12345} + be^{12346} - ce^{23456}) = d\phi^4.$$

In particular the left-hand side is exact, and looking at (3.5.18) we see that $c = 0$, so that ϕ^4 is of the form $\phi^4 = -\frac{2a\ln(c)}{\pi}e^{2346} + \frac{2b\ln(c)}{\pi}e^{2345} + \beta$ where β is a closed 4–form. Including ϕ^6, which is just a multiple of e^{123456}, the even d^H–closed forms are spanned by (modulo some forms that were already seen to be d^H–exact)

$$\{e^{15} - \frac{2\ln(c)}{\pi}e^{2346}, e^{16} + \frac{2\ln(c)}{\pi}e^{2345}, e^{123456}\} \cup \{\text{closed 4–forms}\}. \tag{3.5.25}$$

The forms in the first set are all d^H–exact:

$$\begin{aligned} e^{15} - \frac{2\ln(c)}{\pi}e^{2346} &= d^H(-\frac{2}{\pi}e^6) \\ e^{16} + \frac{2\ln(c)}{\pi}e^{2345} &= d^H(\frac{2)}{\pi}e^5) \\ e^{123456} &= d^H(-\frac{1}{\ln(c)}e^{156}). \end{aligned} \tag{3.5.26}$$

We claim that the closed 4–forms are also d^H–closed. For the exact ones, this is easy to see, because the forms in (3.5.15) are the d of an element of the form (3.5.13) with $a_1 = a_2 = 0$. All these forms however carry at least one of the indices 2, 3 or 4, so give zero when wedged with H, and so d^H applied to such a form gives the same as d applied to it. The only remaining closed 4–form is e^{1234}, but we have $d^H(-\frac{1}{\ln(c)}e^1) = e^{1234}$. We conclude that there is no d^H–cohomology in even degree.

Odd forms: For an odd form $\phi^1 + \phi^3 + \phi^5$ to be d^H–closed we must have

$$d\phi^1 = 0, \qquad H \wedge \phi^1 = d\phi^3, \qquad H \wedge \phi^3 = d\phi^5. \tag{3.5.27}$$

The first equation implies that ϕ^1 is proportional to e^1, but since e^{1234} is not exact, the second equation implies $\phi^1 = 0$. Then ϕ^3 must be closed, so belongs to the span of (3.5.14). From these, the exact ones are also d^H–exact, precisely for the same reason as for the exact 4–forms. We may therefore take $\phi^3 = ae^{156} + be^{234}$, so that the third equation gives $d\phi^5 = a\ln(c)e^{123456}$, which implies $a = 0$ as the right-hand side is not exact. Therefore, again modulo some d^H–exact ones, the d^H–closed odd forms are given by

$$\{e^{234}\} \cup \{\text{closed 5–forms}\}. \tag{3.5.28}$$

Again, the exact 5–forms are also d^H–exact, and the one that is not exact, e^{23456}, is in fact d^H–exact:

$$d^H(\frac{1}{\ln(c)}e^{56}) = e^{23456}.$$

Finally, we have the obvious identity $d^H(\frac{1}{\ln(c)}) = e^{234}$, so we conclude that there is also no d^H–cohomology in odd degree.

3.6 Gerbes

The essential new feature of the bi–hermitian model is the so–called "B–field", and the mathematical theory behind it is that of gerbes. We will give a short introduction to the concept of gerbes, and explain what role it plays in generalized geometry and string theory. Most of the material is based on [10] and [28].

3.6.1 Complex line bundles

Let M be a smooth manifold. Before we introduce the structure of a gerbe on M, we first recall some facts about complex line bundles (complex vector bundles of rank 1). These will be denoted by L, not to be confused with any Lie algebroid on M. If L is a line bundle over M, a local non–vanishing section of L, also called a *local frame*, defines a local trivialization. Choosing a hermitian metric and restricting to orthonormal frames, the structure group of L can always be restricted to $U(1)$, which is to say that transition maps between different trivializations take values in $U(1)$, and in this sense the theory of complex line bundles is equivalent to the theory of principal $U(1)$-bundles. In what follows we shall assume that an explicit choice of hermitian metric has been made. We will denote such a metric by $(.,.)$, L_x will denote the fiber of L over $x \in M$ and $L|_U$ the restriction of L over a set $U \subset M$.

Given a line bundle L, we can always choose an open cover $\{U_\alpha\}_{\alpha\in I}$ of M such that L over each U_α is trivial. If e_α are the frames defining these trivializations, i.e. $e_\alpha : U_\alpha \to L$ are sections with $(e_\alpha, e_\alpha) = 1$ at every point of U_α, on an overlap $U_{\alpha\beta} := U_\alpha \cap U_\beta$ there must be a function $g_{\alpha\beta} : U_{\alpha\beta} \to U(1)$ such that $e_\beta = g_{\alpha\beta} e_\alpha$ on $U_{\alpha\beta}$. Obviously these functions satisfy

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1} \quad \text{and} \quad g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1. \tag{3.6.1}$$

Conversely, L is completely determined by the data $\{U_\alpha, g_{\alpha\beta}\}$ with $g_{\alpha\beta}$ satisfying the above two conditions, as we can obtain L by gluing the sets $U_\alpha \times \mathbb{C}$ with the 'gluing data' $g_{\alpha\beta}$. If $\{U_\alpha, g'_{\alpha\beta}\}$ is another set of transition functions, one readily verifies that it defines a bundle that is isomorphic to L if and only if there exist $h_\alpha : U_\alpha \to U(1)$, satisfying $g_{\alpha\beta}(g'_{\alpha\beta})^{-1} = h_\beta / h_\alpha$. Isomorphism classes of line bundles are thus classified by elements in the first Čech cohomology $\check{H}^1(M, U(1))$.

A unitary connection ∇ on L is a connection in the usual sense, such that in addition

$$X(v_1, v_2) = (\nabla_X v_1, v_2) + (v_1, \nabla_X v_2) \qquad \text{for } X \in TM,\ v_1, v_2 \in \Gamma(L). \tag{3.6.2}$$

In terms of a local frame e_α that satisfies $(e_\alpha, e_\alpha) = 1$, we can define a 1–form A_α such that $\nabla_X e_\alpha = -iA_\alpha(X) e_\alpha$. The above condition becomes

$$(-iA_\alpha(X) e_\alpha, e_\alpha) + (e_\alpha, -iA_\alpha(X) e_\alpha) = 0, \tag{3.6.3}$$

and we see that the 1–form iA_α takes values in the Lie algebra $\mathfrak{u}(1) = i\mathbb{R}$, so in particular A_α is real. In an overlap $U_{\alpha\beta}$ we see that $\nabla e_\beta = -iA_\beta e_\beta = -iA_\beta g_{\alpha\beta} e_\alpha$, but on the other hand $\nabla e_\beta =$

Local data	Relations	Gauge transformations
$g_{\alpha\beta}$	$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$	$g_{\alpha\beta} \mapsto h_\alpha h_\beta^{-1} g_{\alpha\beta}$
A_α	$d\log(g_{\alpha\beta}) = i(A_\alpha - A_\beta)$	$A_\alpha \mapsto A_\alpha - id\log(h_\alpha)$
F	$F\|_{U_\alpha} = dA_\alpha$	$F \mapsto F$

Table 3.1: Data specifying a line bundle with connection. The first two columns describe the line bundle in terms of local data, and the third column expresses the effect of a gauge transformation associated with (h_α).

$\nabla(g_{\alpha\beta}e_\alpha) = dg_{\alpha\beta}e_\alpha - g_{\alpha\beta}iA_\alpha e_\alpha$ from which we deduce that

$$i(A_\alpha - A_\beta) = g_{\alpha\beta}^{-1} dg_{\alpha\beta}. \tag{3.6.4}$$

In particular there is a globally defined 2–form F which on U_α is given by dA_α, and one can show that the cohomology class of $F/2\pi$ is integral, i.e. the integral of $F/2\pi$ over any closed surface gives an integer[9].

As is the case for any connection, there is the usual notion of parallel transport, which assigns to each path $\gamma : [0,1] \to M$ a linear map $L_{\gamma(0)} \to L_{\gamma(1)}$. Since the connection is metric this parallel transport preserves the inner product, so in particular we obtain an element in $U(1)$ for each loop in M, that describes the parallel transport around it. This number is called the *holonomy* of the loop, denoted by $\mathrm{Hol}(\gamma)$.

If γ is a loop, we can subdivide it into a finite number of segments γ_α where γ_α is contained in a local trivialization U_α, and we denote the point where two segments γ_α and γ_β meet by $\gamma_{\alpha\beta}$. One can show that

$$\mathrm{Hol}(\gamma) = \exp\left(i\sum_{\gamma_\alpha} \int_{\gamma_\alpha} A_\alpha - \sum_{\gamma_{\alpha\beta}} \log g_{\alpha\beta}(\gamma_{\alpha\beta}) \right). \tag{3.6.5}$$

It is not obvious that this is indeed the holonomy along γ, in fact it is not even clear why (3.6.5) is well–defined (i.e., independent of all choices involved). The interpretation of (3.6.5) is however relatively easy to understand. Consider parallel transport in one local trivialization (U_α, e_α). A path in $L|_{U_\alpha}$ covering the segment γ_α in U_α looks like $f_\alpha(t)e_\alpha(\gamma_\alpha(t))$ for some function $f_\alpha : I_\alpha \to \mathbb{C}$ where $I_\alpha \subset [0,1]$ is the interval corresponding to γ_α, and this path is horizontal if and only if $\frac{df_\alpha}{dt} - iA_\alpha(\gamma_\alpha(t)) = 0$. Clearly, the solution to this equation takes the form

$$f(t)e_\alpha = f_\alpha(t_0)\exp(i\int_{t_0}^{t} A_\alpha(\gamma_\alpha(t))dt)e_\alpha,$$

where t_0 is the starting point of γ_α. At the endpoint $\gamma_{\alpha\beta}$ we have to switch to the frame e_β, producing the factor $g_{\alpha\beta}^{-1}(\gamma_{\alpha\beta})$, and continue the transport along γ_β. This explains Equation (3.6.5), and

[9]More precisely, the class $F/2\pi$ can be identified with the first Chern class of the line bundle, which is an integral cohomology class. See [35] Chapter III for more details.

in particular shows that it is well defined and independent of all choices, because holonomy itself is something intrinsic to (L, ∇).

Remark. In practice, one can always 'gauge-away' the terms involving the co–cycle g in (3.6.5). More precisely, a complex line bundle over a circle is always trivial, because its Chern class is a degree 2 cohomology class and thus vanishes on S^1. Note that this heavily relies on the fact that we are working over the complex numbers, as over the real numbers we have the Möbius band as counterexample. Pictorially, the difference between complex and real numbers is that $\mathbb{C}^*$ is connected, while $\mathbb{R}^*$ is not, so that we can 'undo' any twisting of the line along a circle over the complex numbers, but not over the reals. The expression in (3.6.5) is among physicists known as a *Wilson loop*.

3.6.2 $U(1)$-Gerbes

Now we come to the definition of a gerbe, or more precisely a $U(1)$-gerbe. To give a motivation for the defining axioms for a gerbe, think of a line bundle L not as some total space over the manifold, but rather as an assignment that assigns to each member of a particular open cover $\{U_\alpha\}_{\alpha\in I}$ of M a line bundle L_α, and assigns to each intersection $U_{\alpha\beta}$ an isomorphism between L_α and L_β. These should be compatible on triple intersections, and if the U_α are such that L_α are trivial, we recover the transition functions $g_{\alpha\beta}$ of the previous section. Of course this description looks rather silly, but it helps to understand the following definition.

Definition 3.6.1. Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of M. A $U(1)$-*gerbe* G over M with respect to this open cover is a rule that assigns to each double intersection $U_{\alpha\beta}$ a line bundle $L_{\alpha\beta}$, and to each triple intersection $U_{\alpha\beta\gamma}$ a trivialization (local frame) $\sigma_{\alpha\beta\gamma}$ of the bundle $L_{\alpha\beta\gamma} := L_{\alpha\beta}L_{\beta\gamma}L_{\gamma\alpha}$. These data are subject to the following axioms:

1 $L_{\alpha\beta} = L_{\beta\alpha}^{-1}$.

2 $\sigma_{s(\alpha)s(\beta)s(\gamma)} = \sigma_{\alpha\beta\gamma}^{\mathrm{sign}(s)}$ for all permutations $s \in S_3$.

3 $\delta\sigma_{\alpha\beta\gamma\delta} = \sigma_{\beta\gamma\delta}\sigma_{\alpha\gamma\delta}^{-1}\sigma_{\alpha\beta\delta}\sigma_{\alpha\beta\gamma}^{-1} = 1$.

To understand this last condition observe that the bundle $L_{\beta\gamma\delta}L_{\alpha\gamma\delta}^{-1}L_{\alpha\beta\delta}L_{\alpha\beta\gamma}^{-1}$ is already trivial due to condition 1, and condition 3 merely requires that $\delta\sigma_{\alpha\beta\gamma\delta}$ coincides with the standard trivialization 1.

Remark. There is also the more general notion of a p-gerbe with values in $U(1)$, that assigns to each $(p+2)$-tuple $U_{\alpha_1}, \ldots, U_{\alpha_{p+2}}$ a $p-1$ gerbe over the intersection $U_{\alpha_1} \cap \ldots \cap U_{\alpha_{p+2}}$, for which line bundles ($p = 0$) and our definition of a gerbe ($p = 1$) are special cases. See [10] for more details.

The definition above is not very useful to work with in practice, but we can describe a gerbe in terms of some other local data, similarly as a line bundle can be described by local transition functions. To this end, we refine our cover[10] such that the line bundles $L_{\alpha\beta}$ are all trivial, and denote by $\sigma_{\alpha\beta}$ a frame

[10]There is an obvious notion of refinement for a gerbe, by refining its defining open cover and restricting all structures to the smaller open sets.

for $L_{\alpha\beta}$ with $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}^{-1}$. With this choice there exist maps $g_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to U(1)$ with the property that

$$\sigma_{\alpha\beta\gamma} = g_{\alpha\beta\gamma}\sigma_{\alpha\beta}\sigma_{\beta\gamma}\sigma_{\gamma\alpha}. \tag{3.6.6}$$

The conditions 1 and 2 above for σ translate into

$$g_{s(\alpha)s(\beta)s(\gamma)} = g_{\alpha\beta\gamma}^{\text{sign}(s)} \quad \forall s \in S_3, \qquad \delta g_{\alpha\beta\gamma\delta} = g_{\beta\gamma\delta}g_{\alpha\gamma\delta}^{-1}g_{\alpha\beta\delta}g_{\alpha\beta\gamma}^{-1} = 1. \tag{3.6.7}$$

As such, g defines an element of $\check{H}^2(M, U(1))$, and we want to know whether different cohomology classes define different gerbes. However, we have not yet defined what it means for two gerbes to be isomorphic.

Definition 3.6.2. Two gerbes G and G' are isomorphic if there is a common refinement $\{U_\alpha\}$ for both the open cover of G and of G', on which there are isomorphisms $L_{\alpha\beta} \to L'_{\alpha\beta}$ over $U_{\alpha\beta}$, such that the induced maps $L_{\alpha\beta\gamma} \to L'_{\alpha\beta\gamma}$ take $\sigma_{\alpha\beta\gamma}$ to $\sigma'_{\alpha\beta\gamma}$.

Now if G and G' are isomorphic, after refining the open cover $\{U_\alpha\}$ if necessary, we may choose frames $\sigma_{\alpha\beta}, \sigma'_{\alpha\beta}$ for $L_{\alpha\beta}$ and $L'_{\alpha\beta}$, so that the isomorphisms $L_{\alpha\beta} \to L'_{\alpha\beta}$ are given by $\sigma_{\alpha\beta} \mapsto h_{\alpha\beta}\sigma'_{\alpha\beta}$ for certain functions $h_{\alpha\beta} : U_{\alpha\beta} \to U(1)$. The last condition of the definition then implies

$$g'_{\alpha\beta\gamma}g_{\alpha\beta\gamma}^{-1} = h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}, \tag{3.6.8}$$

meaning that g and g' define the same Čech cohomology class. The converse is also true, i.e. two cohomologous elements define the same gerbe, and the proof is straightforward. Hence, isomorphism classes of gerbes are classified by $\check{H}^2(M, U(1))$. The cohomology class of $g_{\alpha\beta\gamma}$ is denoted by $c(G)$, and is called the *Dixier-Douady class* of the gerbe. We also have the notion of the trivial gerbe, assigning the trivial bundle to each $U_{\alpha\beta}$ and $\sigma_{\alpha\beta\gamma} = 1$.

There is an alternative description for two gerbes to be isomorphic, in terms of what [10] calls an 'object'. Although this looks somewhat more involved than the definition above, it is more convenient in the context of connections.

Lemma 3.6.3. *Two gerbes G and G' are isomorphic if and only if there exists line bundles K_α on U_α (refining the U_α if necessary) and isomorphisms $m_{\alpha\beta} : K_\beta \to L'_{\alpha\beta}L_{\alpha\beta}^{-1} \otimes K_\alpha$ over $U_{\alpha\beta}$ satisfying*

$$m_{\alpha\beta} \circ m_{\beta\gamma} \circ m_{\gamma\alpha} = \sigma'_{\alpha\beta\gamma}\sigma_{\alpha\beta\gamma}^{-1} \otimes 1. \tag{3.6.9}$$

This last map is defined by $(\sigma'_{\alpha\beta\gamma}\sigma_{\alpha\beta\gamma}^{-1} \otimes 1)(v) = (\sigma'_{\alpha\beta\gamma}\sigma_{\alpha\beta\gamma}^{-1})(\pi_\alpha(v)) \otimes v$ for $v \in K_\alpha$ and π_α the line bundle projection $K_\alpha \to U_\alpha$.

Proof. Assuming the cover is such that all bundles involved are trivial, pick frames k_α for K_α and $\sigma_{\alpha\beta}$ for $L_{\alpha\beta}$. The relation between the $m_{\alpha\beta}$ of this lemma and the $h_{\alpha\beta}$ of Equation (3.6.8) is given by $m_{\alpha\beta} : k_\beta \mapsto h_{\alpha\beta}\sigma'_{\alpha\beta}\sigma_{\alpha\beta}^{-1} \otimes k_\alpha$. It is readily verified that the $m_{\alpha\beta}$ satisfy the conditions of the lemma if and only if $h_{\alpha\beta}$ satisfies $g'_{\alpha\beta\gamma}g_{\alpha\beta\gamma}^{-1} = h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}$. □

Next we come to the definition of connections.

Definition 3.6.4. Let G be a gerbe as in Definition 3.6.1. A connection ∇ on G is given by connections $\nabla_{\alpha\beta}$ on each $L_{\alpha\beta}$ subject to the conditions:

1 $\nabla_{\alpha\beta} + \nabla_{\beta\alpha}$ is the trivial connection on the trivial bundle $L_{\alpha\beta} \otimes L_{\beta\alpha}$.

2 $\nabla_{\alpha\beta\gamma}\theta_{\alpha\beta\gamma} = 0$ where $\nabla_{\alpha\beta\gamma} = \nabla_{\alpha\beta} + \nabla_{\beta\gamma} + \nabla_{\gamma\alpha}$.

This can be represented in terms of local data just as for line bundles. Assume that the open cover for the gerbe is refined such that each $L_{\alpha\beta}$ is trivial with trivializing frame $\sigma_{\alpha\beta}$, and let $g_{\alpha\beta\gamma}$ be the corresponding co–cycle. The connections $\nabla_{\alpha\beta}$ on $L_{\alpha\beta}$ are represented by real 1–forms $A_{\alpha\beta}$ via $\nabla_{\alpha\beta}\sigma_{\alpha\beta} = -iA_{\alpha\beta}\sigma_{\alpha\beta}$. The two conditions on $\nabla_{\alpha\beta}$ imply the constraints

$$A_{\alpha\beta} = -A_{\beta\alpha} \quad \text{and} \quad d\log(g_{\alpha\beta\gamma}) = i(A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha}). \tag{3.6.10}$$

Here $d\log(g) = dg/g$ is well defined, even though $\log(g)$ perhaps is not. In particular it follows that $dA_{\alpha\beta}$ defines a co–cycle, i.e. an element in $\check{H}^1(M, \Omega^2(M))$. However, $\Omega^2(M)$ is a fine sheaf, i.e. it allows for partitions of unity, and so has no cohomology in degrees greater than 0. Therefore, $dA_{\alpha\beta}$ is a co-boundary, meaning that (after possibly again a refinement of the open cover) we can find 2–forms B_α on U_α with

$$dA_{\alpha\beta} = (\delta B)_{\alpha\beta} = B_\alpha - B_\beta. \tag{3.6.11}$$

This gives a globally defined 3–form H, which on U_α is given by dB_α, called the *curvature* of the connection.

For line bundles we noted that the curvature 2–form defines an integral cohomology class, and one may wonder what kind of closed 3–forms can arise as the curvatures of a gerbe. The answer is similar as that for line bundles. Firstly, it was mentioned above that isomorphism classes of gerbes are in bijective correspondence with $H^2(M, U(1))$, which is isomorphic to $H^3(M, \mathbb{Z})$. To see this, consider the exponential sequence of sheaves

$$0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{e^{2\pi i \cdot}} U(1) \to 0, \tag{3.6.12}$$

where $\mathbb{Z}$, $\mathbb{R}$, and $U(1)$ stand for the sheaves of smooth functions with values in these groups, all of them equipped with their standard smooth structure. The long exact sequence in cohomology then gives an isomorphism $H^2(M, U(1)) \cong H^3(M, \mathbb{Z})$, since the sheaf $\mathbb{R}$ is fine and has no cohomology in positive degrees. Furthermore, one can show that $H/2\pi$ defines the same cohomology class as the image of $c(G)$ under this isomorphism, so that the condition on H to be the curvature for a gerbe is that $H/2\pi$ is integral.

Given the additional structure of connection, we have to extend the notion of isomorphism to gerbes with connection.

Definition 3.6.5. Two gerbes with connections (G, ∇) and (G', ∇') are isomorphic if they are isomorphic as gerbes, and if $(U_\alpha, K_\alpha, m_{\alpha\beta})$ is the data giving the isomorphism as in Lemma 3.6.3, there should exist connections ∇_α on K_α such that:

1 The maps $m_{\alpha\beta}$ are connection preserving: $\nabla_\beta \mapsto \nabla'_{\alpha\beta} + \nabla^*_{\alpha\beta} + \nabla_\alpha$, where $\nabla^*_{\alpha\beta}$ denotes the dual connection on $L^{-1}_{\alpha\beta}$.

2 The corresponding 2–forms B'_α and B_α satisfy $B_\alpha = B'_\alpha + K(\nabla_\alpha)$ with $K(\nabla_\alpha)$ the curvature 2–form of ∇_α.

If the K_α are trivial with frames k_α, the ∇_α are given by 1–forms A_α, and the maps $m_{\alpha\beta}$ are represented by $h_{\alpha\beta}$. The two conditions above then translate into

1 $i(A'_{\alpha\beta} - A_{\alpha\beta} + A_\alpha - A_\beta) = d\log(h_{\alpha\beta})$.

2 $B_\alpha - B'_\alpha = dA_\alpha$.

There is a lot of notation involved in all these definitions, but most of them are for theoretical purposes. For computational use, we summarize the most relevant information in Table 3.2.

Local data	Relations	Gauge transformations
$g_{\alpha\beta\gamma}$	$g_{\beta\gamma\delta}g^{-1}_{\alpha\gamma\delta}g_{\alpha\beta\delta}g^{-1}_{\alpha\beta\gamma} = 1$	$g_{\alpha\beta\gamma} \mapsto h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}g_{\alpha\beta\gamma}$
$A_{\alpha\beta}$	$i(A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha}) = d\log(g_{\alpha\beta\gamma})$	$A_{\alpha\beta} \mapsto A_{\alpha\beta} + A_\beta - A_\alpha - id\log(h_{\alpha\beta})$
B_α	$B_\alpha - B_\beta = dA_{\alpha\beta}$	$B_\alpha \mapsto B_\alpha - dA_\alpha$
H	$H\|_{U_\alpha} = dB_\alpha$	$H \mapsto H$

Table 3.2: Data specifying a Gerbe with connection. The first two columns describe the gerbe in terms of local data, and the third column expresses the effect of a gauge transformation associated with $(h_{\alpha\beta}, A_\alpha)$.

The following result will prove useful to relate gerbes to line bundles on the loop space of M later on.

Lemma 3.6.6. *Let G be a trivial gerbe, i.e. $c(G) = 0$, and suppose that $(U_\alpha, K_\alpha, m_{\alpha\beta})$ defines a trivialization as in Lemma 3.6.3. If ∇ is a connection on G, there are connections ∇_α on K_α such that the $m_{\alpha\beta}$ are connection preserving in the above sense. If there is another such choice $(U_\alpha, K'_\alpha, m'_{\alpha\beta}, \nabla'_\alpha)$, the difference between the two trivializations defines a line bundle on M with connection.*

Proof. Refining the cover, the maps $m_{\alpha\beta}$ are of the form $k_\beta \mapsto h_{\alpha\beta}\sigma_{\alpha\beta} \otimes k_\alpha$, with $(\delta h)_{\alpha\beta\gamma} = g_{\alpha\beta\gamma}$. We have to find 1–forms A_α, such that $-iA_\beta h_{\alpha\beta} = dh_{\alpha\beta} - ih_{\alpha\beta}(A_{\alpha\beta} + A_\alpha)$ or

$$A_\alpha - A_\beta = -A_{\alpha\beta} - id\log h_{\alpha\beta}.$$

The right hand side is a co–cycle, i.e. an element of $\check{H}^1(M, \Omega^1(M))$, which is zero because $\Omega^1(M)$ is a fine sheaf, hence we can always find a solution to this equation. If $(U_\alpha, K'_\alpha, m'_{\alpha\beta}, \nabla'_\alpha)$ is a second

trivialization, consider the line bundle $K_\alpha^{-1} \otimes K'_\alpha$ over the open set U_α. On an overlap $U_{\alpha\beta}$ we have transition maps

$$m_{\alpha\beta}^{-1} \otimes m'_{\alpha\beta} : K_\beta^{-1} \otimes K'_\beta \longrightarrow K_\alpha^{-1} \otimes L_{\alpha\beta}^{-1} \otimes L_{\alpha\beta} \otimes K'_\alpha = K_\alpha^{-1} \otimes K'_\alpha,$$
$$k_\beta^{-1} \otimes k'_\beta \mapsto h_{\alpha\beta}^{-1} h'_{\alpha\beta} k_\alpha^{-1} k'_\beta.$$

Since $(\delta h)_{\alpha\beta\gamma} = (\delta h')_{\alpha\beta\gamma} = g_{\alpha\beta\gamma}$, these transition maps satisfy the co–cycle condition, hence define a line bundle on M. Furthermore, each $K_\alpha^{-1} \otimes K'_\alpha$ is equipped with the connection $\nabla_\alpha^* + \nabla'_\alpha$, and the transition functions $m_{\alpha\beta}^{-1} \otimes m'_{\alpha\beta}$ are connection preserving because of condition 1 in Definition 3.6.5, and because $\nabla_{\alpha\beta} + \nabla_{\alpha\beta}^*$ is the trivial connection on $L_{\alpha\beta} \otimes L_{\alpha\beta}^{-1}$ (condition 1 of Definition 3.6.4). The line bundle constructed thus has a globally defined connection, and the connection 1–forms are given by $A'_\alpha - A_\alpha$ on U_α. □

Finally, we come to what for us will be the most important notion for a gerbe with connection, namely its holonomy. The holonomy for a line bundle with connection has a nice geometrical interpretation in terms of parallel transport, but unfortunately for gerbes the notion of holonomy is less intuitive.
Let (G, ∇) be a gerbe with connection, with corresponding local data $(U_\alpha, g_{\alpha\beta\gamma}, A_{\alpha\beta}, B_\alpha)$. If Σ is a closed oriented surface[11] in M, we can restrict G to Σ, which by dimensional reasons must be trivial. Using Lemma 3.6.6 we can find data $(h_{\alpha\beta}, A_\alpha)$ such that on Σ we have

$$\begin{aligned} (\delta h)_{\alpha\beta\gamma} &= g_{\alpha\beta\gamma} \\ A_\alpha - A_\beta &= -A_{\alpha\beta} - id \log h_{\alpha\beta}. \end{aligned} \tag{3.6.13}$$

We can then define a 2–form ϵ on Σ by $\epsilon|_{U_\alpha} = B_\alpha + dA_\alpha$, which is globally defined because of the above relations and the defining relation $B_\alpha - B_\beta = dA_{\alpha\beta}$. The holonomy of the gerbe along Σ is then defined by

$$\mathrm{Hol}(\Sigma) = \exp(i \int_\Sigma \epsilon). \tag{3.6.14}$$

One can show that (3.6.14) is independent of the chosen local data, see [10] or [28] for more details. To gain more insight in this rather abstract quantity we give a more local expression. Suppose we have a triangulation of Σ such that each face lies in some U_α, and such that each vertex meets only three triangles. We denote these faces by Σ_α, and we denote by $\Sigma_{\alpha\beta}$ the common edge of the faces Σ_α and Σ_β. The orientation of $\Sigma_{\alpha\beta}$ is such that it is negative with respect to Σ_α and positive with respect to Σ_β, when considering $\Sigma_{\alpha\beta}$ as the boundary of Σ_α and Σ_β. Finally, we denote by $\Sigma_{\alpha\beta\gamma}$ the common vertex of the faces Σ_α, Σ_β and Σ_γ, which are aligned along the orientation of Σ. In other words, when tracing a loop around $\Sigma_{\alpha\beta\gamma}$ along the orientation, we pass through the associated faces in that particular order. Part of this triangulation is shown in figure 3.2, and computing the integral of

[11] Closed in this context means compact without boundary.

ϵ on this piece gives

$$\begin{aligned}\int \epsilon &= \int_{\Sigma_1}(B_1+dA_1)+\int_{\Sigma_2}(B_2+dA_2)+\int_{\Sigma_3}(B_3+dA_3)\\ &= \int_{\Sigma_1}B_1+\int_{\Sigma_2}B_2+\int_{\Sigma_3}B_3+\int_{ac}(A_2-A_1)+\int_{ab}(A_1-A_3)+\int_{ad}(A_3-A_2)\\ &= \int_{\Sigma_1}B_1+\int_{\Sigma_2}B_2+\int_{\Sigma_3}B_3+\int_{ac}A_{12}+\int_{ab}A_{31}+\int_{ad}A_{23}-i\log(g_{123}(a)),\end{aligned} \tag{3.6.15}$$

where we ignored contributions coming from the boundary, and used Stokes' Theorem. We thus see that the holonomy in terms of this data is given by

$$\mathrm{Hol}(\Sigma) := \exp\left(i\sum_{\Sigma_\alpha}\int_{\Sigma_\alpha}B_\alpha + i\sum_{\Sigma_{\alpha\beta}}\int_{\Sigma_{\alpha\beta}}A_{\alpha\beta}+\sum_{\Sigma_{\alpha\beta\gamma}}\log(g_{\alpha\beta\gamma}(\Sigma_{\alpha\beta\gamma}))\right). \tag{3.6.16}$$

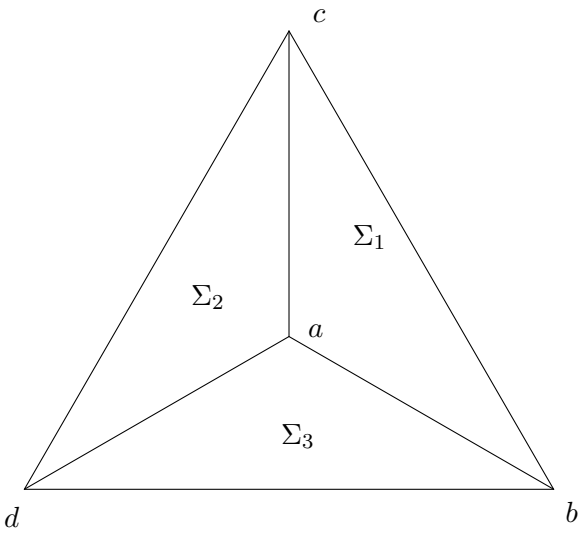

Figure 3.2: Part of the triangulation Σ_α. The orientation is counterclockwise.

Remark. Just as for line bundles, in practice one does not work with the formula (3.6.16). Instead, one uses a gauge choice for which the g's and A's drop out of the formula. Indeed, as we already mentioned above, the gerbe restricted to the surface Σ is trivial so that we can gauge away the $g's$. Next, we want to gauge away the $A_{\alpha\beta}$'s, and to do this we follow the same steps as in the proof of Lemma 3.6.6.

3.6.3 Relation with generalized geometry

When we introduced the Courant bracket we mentioned that when H is non–zero, the bracket is called twisted. In this section we will give a precise geometric interpretation of this 'twist', via the theory of

gerbes. In this section only, $[\![,]\!]$ denotes the Courant bracket with zero H.

Let $(U_\alpha, g_{\alpha\beta\gamma}, A_{\alpha\beta}, B_\alpha)$ be the local data of a gerbe G. Let E be the vector bundle over M which on U_α is given by $(T \oplus T^*)|_{U_\alpha}$, and has transition maps

$$\begin{pmatrix} 1 & 0 \\ dA_{\alpha\beta} & 1 \end{pmatrix}. \tag{3.6.17}$$

on $U_{\alpha\beta}$. Since $dA_{\alpha\beta}$ is a co–cycle (see Equation (3.6.10)), these maps satisfy the co–cycle condition and so indeed form transition functions. Moreover, since $dA_{\alpha\beta}$ is closed, Theorem 3.2.3 implies that these transition maps are automorphisms of the Courant bracket (with zero H) on $(T \oplus T^*)|_{U_\alpha}$. In particular E itself is a Courant algebroid, i.e. there is a globally defined bracket on E that satisfies similar properties as the Courant bracket on $T \oplus T^*$. Over U_α, E is just given by $T \oplus T^*$ with the untwisted bracket, but globally the bracket is twisted.
To explicitly determine this bracket, observe that we have an exact sequence

$$0 \to T^* \xrightarrow{i} E \xrightarrow{p} T \to 0, \tag{3.6.18}$$

where i is the inclusion, and p is the projection, defined in the following way. Over U_α we have the canonical inclusion i and projection p

$$T^*|_{U_\alpha} \xrightarrow{i} T \oplus T^*|_{U_\alpha} \xrightarrow{p} T|_{U_\alpha}, \tag{3.6.19}$$

and we have to verify that they extend to the whole of E. For this observe that for $\xi \in T^*$,

$$\begin{pmatrix} 1 & 0 \\ dA_{\alpha\beta} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ \xi \end{pmatrix}, \tag{3.6.20}$$

so that the inclusion indeed extends globally. Likewise, as the transitions do not affect the 'T part', the projection is also well defined.

The existence of the 2–forms B_α has the nice feature that this sequence splits. Again, we first define this split locally, and then extend it globally. On U_α we define it by

$$\begin{aligned} T|_{U_\alpha} &\to E_{U_\alpha} = (T \oplus T^*)_{U_\alpha}, \\ X &\mapsto X + \iota_X B_\alpha. \end{aligned} \tag{3.6.21}$$

On overlaps $U_{\alpha\beta}$ we have

$$\begin{pmatrix} X \\ \iota_X B_\alpha \end{pmatrix} = \begin{pmatrix} X \\ \iota_X B_\beta + (\iota_X B_\alpha - \iota_X B_\beta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ dA_{\alpha\beta} & 1 \end{pmatrix} \begin{pmatrix} X \\ \iota_X B_\beta \end{pmatrix}, \tag{3.6.22}$$

so indeed our local expression agrees on overlaps. Now that we know that $E \cong T \oplus T^*$, we can compute what the bracket of E becomes on $T \oplus T^*$ under this isomorphism. Denoting the splitting

$T^* \to E$ by s, we have $s + i : T \oplus T^* \xrightarrow{\sim} E$ and we have

$$\begin{aligned}
[\![X+\xi, Y+\eta]\!]_{\text{"Twisted"}} =& (s+i)^{-1}\left([\![(s+i)(X+\xi),(s+i)(Y+\eta)]\!]\right) \\
=& e^{-B_\alpha}[\![e^{B_\alpha}(X+\xi), e^{B_\alpha}(Y+\eta)]\!] \\
=& [\![X+\xi, Y+\eta]\!] + \iota_Y\iota_X H \\
=& [\![X+\xi, Y+\eta]\!]_H,
\end{aligned} \tag{3.6.23}$$

Thus a Gerbe with connection induces a deformation from the Courant bracket with zero 3–form to the Courant bracket with 3–form H. However, one should keep in mind that the gerbe is not completely determined by its curvature, so that not all the structure of the gerbe is contained in the Courant bracket.

3.6.4 Relation with string theory: Charged strings

Now that we have a basic understanding of gerbes, we turn our attention to the role they play in physics. Let us quickly recall the theory of electromagnetism for point–particles, which for simplicity we take in Minkowski space–time $M = \mathbb{R}^{(1,3)}$. To define the electromagnetic field, one introduces a line bundle L with connection ∇ on the configuration space of the particle, which equals M. As M is contractible, we may assume L to be trivial so that the connection ∇ is described by a global 1–form, usually denoted A, with associated curvature 2–form F called the 'field strength'. In four dimensions, F has six independent components which are identified with the electric field E and the magnetic field B. To couple A with the particle, one introduces in the action the term

$$\int_{x([0,1])} A = \int_{[0,1]} \dot{x}^\mu A_\mu, \tag{3.6.24}$$

where $x : [0,1] \to M$ is the trajectory of the particle. The equations of motion for the particle turn out to depend only on the field strength F, which is mathematically clear because for a small deformation $x'(t)$ of the path $x(t)$ we have

$$\delta S = -e\int_{x([0,1])} A - (-e\int_{x'([0,1])} A) = -e\int_X dA = \int_X F, \tag{3.6.25}$$

where X is a surface with $\partial X = x([0,1]) - x'([0,1])$. To determine the equations of motion it suffices to consider infinitesimal variations of the path, so classically only the form of F matters. However, in the Bohm-Aharonov experiment it was shown that charged particles can be affected by an electromagnetic field, even if it is identically zero in the neighborhood of the particle. What happens is that the field A can be non–zero because of the presence of a electromagnetic field elsewhere, and cannot be gauged to zero. Classically this makes no difference, but quantum-mechanically the entire shape of the action is important and not just its stationary points, and so the presence of A becomes visible through the holonomy of the line bundle. This holonomy can be non–trivial, for instance if we form a loop around the region with the electromagnetic field. It is a fact that a line bundle with connection is completely determined by its holonomy, in fact there are explicit formulas to recover the transition functions and connection forms out of the holonomy.

Now we want to define a similar electromagnetic theory for closed strings, and the most natural way to generalize the concept described above is by introducing a line bundle on the configuration space of the string, which is given by the loop space $LM := \{\gamma : S^1 \to M\}$. The interaction of the string with this electromagnetic field takes place through a collection of locally defined 2–forms B_α, in direct analogy with the interaction for point–particles and electromagnetism, which takes place through the locally defined 1–forms A_α. To distinguish between point–particles and strings, the electromagnetic field for strings is called the B–field.

To construct this line bundle on LM we first have to discuss very briefly the topology of LM. One natural choice of topology on such a space of maps is the *compact open topology*, which has as a basis for its topology sets of the form

$$L(K,U) := \{\gamma \in LM | \gamma(K) \subset U\} \tag{3.6.26}$$

where K runs over the compact sets of S^1, and U over the open sets of M. Since S^1 is compact, it suffices to take K closed. An open set U in M thus produces an open set in LM, consisting of all the loops contained in U, and we will denote this set by $\hat{U}$.

Given a gerbe G with connection ∇, we can pick an open cover U_i such that $G|_{U_i}$ is trivial, and such that the $\hat{U}_i$ cover LM. This is indeed possible, for if γ is a loop in M, we can thicken it to a solid open torus in M. This set is homotopic to γ itself, hence has zero third cohomology groups. As γ runs over all possible loops, these open sets cover M and the corresponding open sets in LM cover LM. We labeled the cover with Latin indices to stress that it has nothing to do with a cover for G, i.e. there is not necessarily a line bundle defined over the intersection U_{ij}.

Now we construct the line bundle on LM. On each $\hat{U}_i$ consider the trivial bundle $\hat{U}_i \times \mathbb{C}$. We have to specify gluing data, i.e. we have to define functions $\hat{g}_{ij} : \hat{U}_i \cap \hat{U}_j \to \mathbb{C}^*$ satisfying the co–cycle condition. Since we have trivializations of (G, ∇) (in the sense of Lemma 3.6.6) both on U_i and on U_j, Lemma 3.6.6 gives us a line bundle L_{ij} with connection on U_{ij} (again, this is not the line bundle that the gerbe itself associates to U_{ij}, for this last set may not even be in the open cover of the gerbe). This provides an obvious candidate for $\hat{g}$, namely

$$\hat{g}_{ij}(\gamma) = \mathrm{Hol}_{ij}(\gamma) \qquad \text{for } \gamma \in \hat{U}_i \cap \hat{U}_j, \tag{3.6.27}$$

where Hol_{ij} is the holonomy of the connection on L_{ij}.

Theorem 3.6.7. *The transition maps $\hat{g}_{ij}$ satisfy the co–cycle condition, hence define a line bundle on LM.*

Proof. We have to verify the constraints

$$\hat{g}_{ij} = \hat{g}_{ji}^{-1} \qquad \text{and} \hat{g}_{ij}\hat{g}_{jk}\hat{g}_{ki} = 1.$$

Let $(V_\alpha, K_\alpha, \nabla_\alpha, m_{\alpha\beta})$ and $(V_\alpha, K'_\alpha, \nabla'_\alpha, m'_{\alpha\beta})$ be the two trivializations of (G, ∇) on U_{ij} coming from U_i and U_j respectively. The corresponding line bundle on U_{ij} is locally on V_α given by $K_\alpha^{-1} \otimes K'_\alpha$, has transition functions $m_{\alpha\beta}^{-1} \otimes m'_{\alpha\beta}$ and the 1–forms defining the connection are given by $A'_\alpha - A_\alpha$. If we do the same construction for U_{ji} we get the dual bundle with the dual connection, and so it follows from Equation (3.6.5) that $\hat{g}_{ij} = \hat{g}_{ji}^{-1}$. Now suppose $(V_\alpha, K''_\alpha, \nabla''_\alpha, m''_{\alpha\beta})$ is another trivialization of (G, ∇) on U_{ijk}, coming from U_k. Again by inspecting (3.6.5) we deduce that $\hat{g}_{ij}\hat{g}_{jk}\hat{g}_{ki} = 1$, because

$$(A'_\alpha - A_\alpha) + (A''_\alpha - A'_\alpha) + (A_\alpha - A''_\alpha) = 0 \qquad \text{and} \qquad (h_{\alpha\beta}^{-1}h'_{\alpha\beta})(h_{\alpha\beta}^{\prime -1}h''_{\alpha\beta})(h_{\alpha\beta}^{\prime\prime -1}h_{\alpha\beta}) = 1.$$

Here the $h_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$ correspond to $m_{\alpha\beta}$ as in the proof of Lemma 3.6.6. □

One could expect the holonomy of the gerbe to be related to some connection on this line bundle, but there are some subtleties. First, we have not discussed any smooth structure on LM, so we cannot use the familiar differential geometric concepts. Second, to make this correspondence we have to view a closed surface as a path in LM. It is clear that every path of loops defines a closed surface in M, but the converse is not obvious, as a surface can have holes for instance. Intuitively, such surfaces describe loops that somewhere split into multiple loops, and these do play an important role in string theory, but as such it is difficult to relate gerbe holonomy with the above constructed line bundle. We will not go further into this.

Chapter 4

The general supersymmetric sigma model

In Chapter 2 we saw the construction of the $(2,2)$–supersymmetric sigma model, that was invariant under parity. One can generalize the model by ignoring this symmetry, and this was first investigated by Gates, Hull and Roček in [16]. These new models involve a B–field, and are only defined on manifolds admitting a bi–hermitian structure, which we know from Chapter 3 to be equivalent to a generalized Kähler structure. In this chapter we reproduce the results from [16], and give the relation between the B–field appearing in the action, and gerbes.

4.1 $(2,2)$–sigma models with B–field

Once parity can be broken, there are two things that allow for a modification. The first one is the appearance of an antisymmetric tensor in the action (2.3.9):

$$S(\Phi) = \int d^2\sigma d\theta^+ d\theta^- (g_{ij} + b_{ij}) D_+\Phi^i D_-\Phi^j, \tag{4.1.1}$$

where the bosonic part of the integral is taken over the world–sheet Σ, which equals a cylinder. The term involving b breaks parity, because when we integrate out the θ's there is a term $b_{ij}\partial_+\varphi^i\partial_-\varphi^j$. In terms of σ and τ, this term is proportional to $b_{ij}\partial_\tau\varphi^i\partial_\sigma\varphi^j$ by anti–symmetry of b. Clearly this term is not invariant under the parity transformation $(\tau,\sigma)\mapsto(\tau,-\sigma)$.

To define (4.1.1) globally, it is not necessary for the 2–form b to be globally defined on M, and we will see later that this term represents a gerbe with connection, for which b is a local potential in the sense of (3.6.11). The only restriction on b is then that the 3–form $H = db$ *is* globally defined, and that $H/2\pi$ is integral (cf. the discussion below (3.6.11)).

The next change is the form of the second supersymmetry. Recall from Section 2.4 that the second supersymmetry is given by

$$\delta_\epsilon\Phi^i = I^i{}_j\epsilon^+ D_+\Phi^j + I^i{}_j\epsilon^- D_-\Phi^j,$$

where I is a complex structure. Parity, which on spinors interchanges the $\pm$ components (recall our discussion about parity from Section 2.4), forces the two I's above to be equal. Now this can be

generalized to

$$\delta_\epsilon \Phi^i = I^i_{+j}\epsilon^+ D_+\Phi^j + I^i_{-j}\epsilon^- D_-\Phi^j, \tag{4.1.2}$$

with $I_\pm$ not necessarily equal. Before we investigate whether (4.1.2) satisfies the $(2,2)$–algebra (eqn. 2.2.4) and leaves the action in (4.1.1) invariant, we look at the equations of motion implied by (4.1.1). After all, we are looking for on–shell supersymmetry so it is important to know which conditions are vacuous for on–shell fields[1].

The equations of motion can be computed using the stationary point principle as we discussed in Chapter 2. Under a small deformation $\delta\Phi^i$ of the superfield the action changes as:

$$\delta S = \int \Big((g_{ij,k} + b_{ij,k}) D_+\Phi^i D_-\Phi^j \delta\Phi^k + (g_{ij} + b_{ij}) \left(D_+\delta\Phi^i D_-\Phi^j + D_+\Phi^i D_-\delta\Phi^j \right) \Big), \tag{4.1.3}$$

where we suppressed the integration measure $d^2\sigma d\theta^+ d\theta^-$. We can perform a partial integration using the identity $\int D_\pm(\ldots) = 0$, which is a consequence of

$$\int d\theta \frac{d}{d\theta} = 0 \tag{4.1.4}$$

and the fact that there are no boundary terms in the ordinary, bosonic integral, if we assume that the fields vanish at infinity. Keeping in mind that $D_\pm$ anti–commutes with other fermionic objects, we have

$$\begin{aligned}
\delta S = \int \Big(& (g_{ij,k} + b_{ij,k}) D_+\Phi^i D_-\Phi^j \delta\Phi^k - (g_{ij,k} + b_{ij,k}) D_+\Phi^k \delta\Phi^i D_-\Phi^j \\
& - (g_{ij} + b_{ij}) \delta\Phi^i D_+ D_-\Phi^j + (g_{ij,k} + b_{ij,k}) D_-\Phi^k D_+\Phi^i \delta\Phi^j + (g_{ij} + b_{ij}) D_- D_+\Phi^i \delta\Phi^j \Big) \\
= \int & \left((-2\Gamma_{ij,k} + H_{ijk}) D_+\Phi^i D_-\Phi^j + 2 g_{jk} D_- D_+\Phi^j \right) \delta\Phi^k.
\end{aligned} \tag{4.1.5}$$

For this to vanish for all infinitesimal deformations $\delta\Phi^i$, Φ must satisfy the following equations of motion

$$D_- D_+\Phi^k - \Gamma^{-k}_{ij} D_+\Phi^i D_-\Phi^j = 0. \tag{4.1.6}$$

Here $\Gamma^{\pm i}_{jk} = \Gamma^i_{jk} \pm \frac{1}{2} g^{ri} H_{rjk}$ are the Christoffel symbols for the connections $\nabla^\pm = \nabla \pm \frac{1}{2} g^{-1} H$, and $H = db$. We take the convention that n–forms are written as $\alpha = \frac{1}{n!}\alpha_{i_1 \ldots i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n}$, so that $H_{ijk} = b_{ij,k} + b_{ki,j} + b_{jk,i}$.

Now we are ready to look at the constraints on $I_\pm$. First of all, observe that $I^i_{\pm j}$ form the components of a tensor, i.e. $I_\pm$ are sections of $TM \otimes T^*M$. This follows from (4.1.2), since both $\delta_\epsilon\Phi^i$ and

[1] Strictly speaking, we have to determine the equations of motion but also determine whether all our symmetries that we define respect these equations. Mathematically, we need to verify whether our symmetries induce well–defined maps $X/I \to X/I$, where X is the space of fields and I is the ideal generated by the equations of motion. An example of such an explicit calculation can be found in Appendix B in [37].

$D_\pm\Phi^i$ have the same behavior under a change of coordinates. Now let us see whether (4.1.2) satisfies the $(2,2)$–algebra. Recall that the supersymmetry charges $Q^1_\pm$ and $Q^2_\pm$ are defined via

$$\delta^a_\epsilon = \epsilon Q^a, \;\; a = 1, 2. \tag{4.1.7}$$

We explicitly label the symmetries δ^1 and δ^2 to distinguish them, this should not be confused with applying the symmetry δ once or twice. It is important to realize (4.1.7) in actual computations, as the Q^a are only implicitly defined on the Φ^i. It implies for instance that Q^a acts via the chain rule on any function of the Φ^i:

$$Q^a(f(\Phi)) = \frac{\partial f}{\partial \Phi^i} Q^a \Phi^i. \tag{4.1.8}$$

Furthermore, we have the following identity:

$$\begin{aligned}
[\delta^a_{\epsilon_1}, \delta^b_{\epsilon_2}]\Phi^i =& \delta^a_{\epsilon_1}(\epsilon_2 Q^b\Phi) - \delta^b_{\epsilon_2}(\epsilon_1 Q^a\Phi) \\
=& \epsilon_2 Q^b(\delta^a_{\epsilon_1}\Phi) - \epsilon_1 Q^a(\delta^b_{\epsilon_2}\Phi) \\
=& \epsilon_2 Q^b(\epsilon_1 Q^a\Phi) - \epsilon_1 Q^a(\epsilon_2 Q^b\Phi) \\
=& \epsilon_1^\alpha \epsilon_2^\beta \{Q^a_\alpha, Q^b_\beta\}\Phi,
\end{aligned} \tag{4.1.9}$$

where once again we stress that ϵ and Q are fermionic objects, hence anti–commute. According to (2.2.4) and (2.3.1) we can write (4.1.9) as

$$[\delta^a_{\epsilon_1}, \delta^b_{\epsilon_2}]\Phi^i = -2i\delta^{ab}\left(\epsilon_1^+\epsilon_2^+\partial_+ + \epsilon_1^-\epsilon_2^-\partial_-\right)\Phi^i, \tag{4.1.10}$$

and this imposes a constraint on the left hand side. We know this to be true for $a = b = 1$ as it follows straightforwardly from the definition of Q^1 in (2.3.4). For $a = b = 2$ we have

$$\begin{aligned}
[\delta^2_{\epsilon_1}, \delta^2_{\epsilon_2}]\Phi^i =& \delta^2_{\epsilon_1}\left(\epsilon_2^+ I^i_{+j} D_+\Phi^j + \epsilon_2^- I^i_{-j} D_-\Phi^j\right) - (1 \leftrightarrow 2) \\
=& \epsilon_2^+ I^i_{+j,k}\left(\epsilon_1^+ I^k_{+l} D_+\Phi^l + \epsilon_1^- I^k_{-l} D_-\Phi^l\right) D_+\Phi^j + \epsilon_2^+ I^i_{+j} D_+\left(\epsilon_1^+ I^j_{+l} D_+\Phi^l + \epsilon_1^- I^j_{-l} D_-\Phi^l\right) \\
&+ \epsilon_2^- I^i_{-j,k}\left(\epsilon_1^+ I^k_{+l} D_+\Phi^l + \epsilon_1^- I^k_{-l} D_-\Phi^l\right) D_-\Phi^j + \epsilon_2^- I^i_{-j} D_-\left(\epsilon_1^+ I^j_{+l} D_+\Phi^l + \epsilon_1^- I^j_{-l} D_-\Phi^l\right) \\
&- (1 \leftrightarrow 2) \\
=& 2\epsilon_1^+\epsilon_2^+\left((I^i_{+j,k} I^k_{+l} + I^i_{+k} I^k_{+l,j}) D_+\Phi^j D_+\Phi^l + I^i_{+j} I^j_{+l} D^2_+\Phi^l\right) \\
&+ \left(\epsilon_1^+\epsilon_2^- + \epsilon_1^-\epsilon_2^+\right)\Bigg((I^i_{+j} I^j_{-l} - I^i_{-j} I^j_{+l}) D_+ D_-\Phi^l + \\
&\qquad (I^i_{-j,k} I^k_{+l} + I^i_{-k} I^k_{+l,j} - I^i_{+l,k} I^k_{-j} - I^i_{+k} I^k_{-j,l}) D_-\Phi^j D_+\Phi^l\Bigg) \\
&+ 2\epsilon_1^-\epsilon_2^-\left((I^i_{-j,k} I^k_{-l} + I^i_{-k} I^k_{-l,j}) D_-\Phi^j D_-\Phi^l + I^i_{-j} I^j_{-l} D^2_-\Phi^l\right).
\end{aligned} \tag{4.1.11}$$

Using $D^2_+ = i\partial_+$, and the definition of the Nijenhuis tensor

$$N^\pm(X,Y) := N_{I_\pm}(X,Y) = [X,Y] - [I_\pm X, I_\pm Y] + I_\pm[I_\pm X, Y] + I_\pm[X, I_\pm Y], \tag{4.1.12}$$

which in components looks like

$$N^{\pm k}_{ij} = I^l_{\pm i} I^k_{\pm j,l} + I^k_{\pm l} I^l_{\pm i,j} - (i \leftrightarrow j), \tag{4.1.13}$$

we can rewrite (4.1.11) as

$$\begin{aligned}[\delta^2_{\epsilon_1}, \delta^2_{\epsilon_2}]\Phi^i =&\epsilon_1^+\epsilon_2^+\left(2iI^i_{+j}I^j_{+l}\partial_+\Phi^l - N^{+i}_{jl}D_+\Phi^j D_+\Phi^l\right)\\ &+\epsilon_1^-\epsilon_2^-\left(2iI^i_{-j}I^j_{-l}\partial_-\Phi^l - N^{-i}_{jl}D_-\Phi^j D_-\Phi^l\right)\\ &+(\epsilon_1^+\epsilon_2^- + \epsilon_1^-\epsilon_2^+)\Big((I^i_{+j}I^j_{-l} - I^i_{-j}I^j_{+l})D_+D_-\Phi^l +\\ &\qquad (I^i_{-j,k}I^k_{+l} + I^i_{-k}I^k_{+l,j} - I^i_{+l,k}I^k_{-j} - I^i_{+k}I^k_{-j,l})D_-\Phi^j D_+\Phi^l\Big).\end{aligned} \tag{4.1.14}$$

This last expression is not very promising, but with the help of (4.1.6) we can rewrite the term $D_+D_-\Phi^l$, and we can replace derivatives by covariant derivatives using

$$I^i_{\pm j,k} = \nabla^\pm_k I^i_{\pm j} - \Gamma^{\pm i}_{kr}I^r_{\pm j} + \Gamma^{\pm r}_{kj}I^i_{\pm r}. \tag{4.1.15}$$

Why we use ∇^+ for I_+ and ∇^- for I_- will become clear later, but for now we observe that it transforms (4.1.14) into the more readable equation

$$\begin{aligned}[\delta^2_{\epsilon_1}, \delta^2_{\epsilon_2}]\Phi^i =&\epsilon_1^+\epsilon_2^+\left(2iI^i_{+j}I^j_{+l}\partial_+\Phi^l - N^{+i}_{jl}D_+\Phi^j D_+\Phi^l\right)\\ &+\epsilon_1^-\epsilon_2^-\left(2iI^i_{-j}I^j_{-l}\partial_-\Phi^l - N^{-i}_{jl}D_-\Phi^j D_-\Phi^l\right)\\ &+(\epsilon_1^+\epsilon_2^- + \epsilon_1^-\epsilon_2^+)\\ &\cdot\left[(\nabla^-_k I^i_{-j})I^k_{+l} - (\nabla^+_k I^i_{+l})I^k_{-j} + I^i_{-k}(\nabla^+_j I^k_{+l}) - I^i_{+k}(\nabla^-_l I^k_{-j})\right]\cdot D_-\Phi^j D_+\Phi^l.\end{aligned} \tag{4.1.16}$$

As this must equal (4.1.10) for all choices of ϵ_1, ϵ_2, we obtain three independent equations:

1. $I^2_\pm = -1$, so $I_\pm$ must be almost complex structures on M. This puts already topological restrictions on M, e.g. it must be even dimensional and orientable.

2. $N^\pm = 0$, implying that $I_\pm$ are both integrable complex structures.

3. $(\nabla^-_k I^i_{-j})I^k_{+l} - (\nabla^+_k I^i_{+l})I^k_{-j} + I^i_{-k}(\nabla^+_j I^k_{+l}) - I^i_{+k}(\nabla^-_l I^k_{-j}) = 0$. At this stage it is not clear what this equation tells us, but we will see in a moment what to do with it (cf. Equation (4.1.24)).

Remark. The second condition about integrability is necessary to close the $(2,2)$–algebra on–shell, but one could look at more general models where $N^\pm$ is not necessarily zero. If the complex structures satisfy $g_{ij}I^i_{\pm k}I^j_{\pm l} = g_{kl}$ and $\nabla^\pm I_\pm = 0$ (cf. Equation (4.1.19) and (4.1.24)), it can be shown that $\nabla^\pm N^\pm = 0$, which implies that the terms $N^{\pm i}_{jl}D_\pm\Phi^j D_\pm\Phi^l$ define a symmetry of the action. So although the algebra does not close, the commutator of two symmetries is again a (new) symmetry. In [29] these models are investigated in full generality, and in [14] they are worked out in superspace together with a nontrivial example, in which it turns out that the obtained symmetry algebra is infinite dimensional. We shall not investigate these kind of models, but always assume the second condition to be fulfilled.

We have not computed the full algebra yet, as we still have to verify the constraint $[\delta^1_{\epsilon_1}, \delta^2_{\epsilon_2}]\Phi^i = 0$. To keep the computation short we take $\epsilon^-_1 = \epsilon^-_2 = 0$, the general case is entirely similar.

$$\begin{aligned}
[\delta^1_{\epsilon_1}, \delta^2_{\epsilon_2}]\Phi^i =& \delta^1_{\epsilon_1}\left(\epsilon^+_2 I^i_{+j} D_+\Phi^j\right) - \delta^2_{\epsilon_2}\left(\epsilon^+_1 Q_+\Phi^i\right) \\
=& \epsilon^+_2 I^i_{+j,k}\left(\epsilon^+_1 Q_+\Phi^i\right) D_+\Phi^j + \epsilon^+_2 I^i_{+j} D_+\left(\epsilon^+_1 Q_+\Phi^i\right) - \epsilon^+_1 Q_+(\epsilon^+_2 I^i_{+j} D_+\Phi^j) \\
=& \epsilon^+_1\epsilon^+_2\left(-I^i_{+j,k} Q_+\Phi^k D_+\Phi^j + I^i_{+j} D_+ Q_+\Phi^j + I^i_{+j,k} Q_+\Phi^k D_+\Phi^j + I^i_{+j} Q_+ D_+\Phi^j\right) \\
=& 0, \qquad (4.1.17)
\end{aligned}$$

since $\{Q_+, D_+\} = 0$. Thus the two supersymmetries already commute, without any assumptions on the target space.

Beside the algebraic relations, the action must also be invariant under the symmetry δ^2_ϵ, where again we take $\epsilon^- = 0$ for aesthetic reasons. In dealing with the commutators we worked with fields on shell, but here we do not. Indeed, fields on shell are by definition those for which the action is invariant under *any* variation, so in particular a supersymmetry variation. We have

$$\begin{aligned}
\delta^2_\epsilon S =& \int \Big((g_{ij,k} + b_{ij,k})\epsilon^+ I^k_{+l} D_+\Phi^l D_+\Phi^i D_-\Phi^j + (g_{ij} + b_{ij}) D_+(\epsilon^+ I^i_{+k} D_+\Phi^k) D_-\Phi^j \\
&\quad + (g_{ij} + b_{ij}) D_+\Phi^i D_-(\epsilon^+ I^j_{+k} D_+\Phi^k)\Big) \\
=& \int \epsilon^+\Big(-\Big((g_{ij} + b_{ij}) I^i_{+k} + (g_{ki} + b_{ki}) I^i_{+j}\Big) D^2_+\Phi^k D_-\Phi^j \\
&\quad + \Big((g_{ij,k} + b_{ij,k}) I^k_{+l} - (g_{kj} + b_{kj}) I^k_{+i,l} + (g_{ik} + b_{ik}) I^k_{+l,j} - (g_{ik,l} + b_{ik,l}) I^k_{+j} \\
&\qquad - (g_{ik} + b_{ik}) I^k_{+j,l}\Big) D_+\Phi^l D_+\Phi^i D_-\Phi^j\Big), \qquad (4.1.18)
\end{aligned}$$

which must vanish for all ϵ^+ and all superfields Φ^i. In the second equality we used again a partial integration. Observe that the term $D^2_+\Phi^k D_-\Phi^j = i\partial_+\Phi^k D_-\Phi^j$ is independent of the term $D_+\Phi^l D_+\Phi^i D_-\Phi^j$, i.e. we can find superfields for which the first expression is zero while the second is not, and vice versa. Therefore the two expressions in (4.1.18) must separately vanish. Vanishing of the first term gives anti–symmetry:

$$(g + b)(I_+ v, w) + (g + b)(v, I_+ w) = 0, \qquad (4.1.19)$$

and since g is symmetric and b is anti–symmetric this holds also for g and b separately[2].

As for the second term in (4.1.18), rewriting derivatives of g in terms of the Christoffel symbols and derivatives of I_+ as covariant derivatives (with respect to the torsion-free Levi-Cevita connection) most terms cancel against each other or vanish, because they are contracted with the term

[2]This is a standard polarization argument: for $v = w$, the equation gives $g(I_+ v, v) = 0$ and using bi-linearity of g this implies $g(I_+ v, w) + g(v, I_+ w) = 0$. Together with (4.1.19) this must then also be true for b.

$D_+\Phi^l D_+\Phi^i D_-\Phi^j$ which is antisymmetric in l and i. After that, the constraint on the second term can be written as

$$\begin{aligned}
0 &= \left(g_{ik}\nabla_j I^k_{+l} + b_{ij,k}I^k_{+l} - \partial_l(b_{ik}I^k_{+j}) + \partial_j(b_{ik}I^k_{+l}) - b_{ik,j}I^k_{+l} - b_{kj}I^k_{+i,l}\right) D_+\Phi^l D_+\Phi^i D_-\Phi^j \\
&= \left(g_{ik}\nabla_j I^k_{+l} + (b_{ij,k} + b_{jk,i} + b_{ki,j})I^k_{+l}\right) D_+\Phi^l D_+\Phi^i D_-\Phi^j \\
&= \left(g_{ik}\nabla_j I^k_{+l} + \frac{1}{2}H_{ijk}I^k_{+l} - \frac{1}{2}H_{ljk}I^k_{+i}\right) D_+\Phi^l D_+\Phi^i D_-\Phi^j \\
&= g_{ik}\left(\nabla_j I^k_{+l} + \frac{1}{2}H^k{}_{jr}I^r_{+l} - \frac{1}{2}H^r{}_{jl}I^k_{+r}\right) D_+\Phi^l D_+\Phi^i D_-\Phi^j \\
&= g_{ik}\nabla^+_j I^k_{+l} D_+\Phi^l D_+\Phi^i D_-\Phi^j. \qquad (4.1.20)
\end{aligned}$$

In the second line we used the identity $b_{ik}I^k_{+j} = b_{jk}I^k_{+i}$, and in the third line we used the covariant derivative of I_+

$$\nabla^+_j I^k_{+l} = I^k_{+l,j} + \Gamma^{+k}_{jr}I^r_{+l} - \Gamma^{+r}_{jl}I^k_{+r} = \nabla_j I^k_{+l} + \frac{1}{2}H^k{}_{jr}I^r_{+l} - \frac{1}{2}H^r{}_{jl}I^k_{+r}, \qquad (4.1.21)$$

together with the identity

$$H_{ljk}I^k_{+i} = g_{sk}H_{lj}{}^s I^k_{+i} = -g_{ik}H_{lj}{}^s I^k_{+s} = g_{ik}H^s{}_{jl}I^k_{+s}, \qquad (4.1.22)$$

using anti–symmetry of I_+ with respect to g. Therefore, for the action to be invariant we must have

$$\nabla^+ I_+ = 0. \qquad (4.1.23)$$

A similar calculation can be done for I_-, but we can read off the result straight away from (4.1.1). Commuting $D_+\Phi^i$ with $D_-\Phi^j$ in the action and interchanging the indices i and j we see that the action gets an overall minus sign and b changes into $-b$. Then $\delta^2_\epsilon S$ for $\epsilon^+ = 0$ is given by (4.1.18) with $-b$ instead of b and $+$ interchanged with $-$. Therefore, every constraint on I_+ can be translated to one on I_- by replacing b by its negative, so the total constraints are

$$\nabla^\pm I_\pm = 0. \qquad (4.1.24)$$

Observe that (4.1.24) implies that the third condition coming from the $(2,2)$–algebra is automatically fulfilled, and so gives no extra constraint. We summarize the above discussion in the following theorem.

Theorem 4.1.1. *([16]) Let (M, g) be a Riemannian manifold equipped with an $U(1)$-gerbe, with curvature 3–form H. The $(1,1)$–supersymmetric sigma model on a flat world–sheet Σ (a cylinder) with target space M, admits an extension to $(2,2)$ on–shell supersymmetry if and only if M admits two integrable complex structures $I_\pm$ that satisfy the constraints*

1. *$g(I_\pm v, w) = -g(v, I_\pm w)$, and if b is a local potential for the gerbe then $b(I_\pm v, w) = -b(v, I_\pm w)$,*
2. *$\nabla^\pm I_\pm = 0$.*

Equivalently, from Theorem (3.4.3), M must be twisted generalized Kähler with H the form defining the twist of the Courant bracket.

Remark. Note that the condition $b(I_\pm v, w) = -b(v, I_\pm w)$ is not gauge–independent, i.e. for different choices of b, in which the holonomy could be more complicated (involving g's and A's) this condition could be different. What is gauge–independent is the fact that H is of type $(2,1)+(1,2)$ with respect to both $I_\pm$, which is a direct consequence of the integrability of $I_\pm$ and the constraint $b(I_\pm v, w) = -b(v, I_\pm w)$.

To end this section we discuss briefly some other types of representations one can consider besides $(2,2)$. First of all, since we do not require any relation between the $+$ and $-$ sectors, we can drop one of the complex structures and consider $(2,1)$ or $(1,2)$ representations. In effect, half of the data and the constraints above drops out, and what is left is called a SKT structure(Kähler with strong torsion). For more information about those we refer to [7].

If we look for $(3,3)$ representations, we obtain a second pair of complex structures $J_\pm$ satisfying the same constraints as above, together with the constraint that $J_\pm$ must anti–commute with $I_\pm$. We can then form a third pair $K_\pm := I_\pm J_\pm$, which satisfies $\nabla^H K_\pm = 0$ by the Leibniz rule. If H would be $(2,1)+(1,2)$ for $K_\pm$, we would conclude that $K_\pm$ are both integrable, but at present the author does not know whether this is always the case. Certainly this seems very likely, but in any case we know that $(4,4)$ supersymmetry corresponds to a generalized Hyperkähler structure: three generalized complex structures that mutually anti–commute, and commute with one fixed generalize metric. This discussion is summarized in Table 4.1, which is the extension of Table 2.1.

Table 4.1: Relation supersymmetry and target space geometry.

(p,q) Susy	(1,1)	(2,2)	(2,1)	(4,4)
Geometry	Riemannian	Gen.Kähler	SKT	Gen.Hyperkähler

4.2 The action in terms of the physical fields

To investigate the structure of the generalized sigma model we only worked in superspace, which is convenient for computations but has the drawback that the physical content of the theory is hidden in the components of the superfields. Also, it is not immediately obvious what the coordinate free[3] expression for (4.1.1) is without looking at these components, so we first perform the integration over

[3]With respect to coordinates on M.

the Grassmann variables $\theta^{\pm}$. Using the identities

$$\begin{aligned} D_+\Phi^i &= \psi_+^i - \theta^- F^i + i\theta^+\partial_+\varphi^i + i\theta^+\theta^-\partial_+\psi_-^i, \\ D_-\Phi^j &= \psi_-^j + \theta^+ F^j + i\theta^-\partial_-\varphi^j + i\theta^-\theta^+\partial_-\psi_+^j, \\ g_{ij}(\Phi) + b_{ij}(\Phi) &= (g_{ij}(\varphi) + b_{ij}(\varphi)) + (g_{ij,k}(\varphi) + b_{ij,k}(\varphi))(\theta^+\psi_+^k + \theta^-\psi_-^k + \theta^-\theta^+ F^k) \\ &\quad + \frac{1}{2}(g_{ij,kl}(\varphi) + b_{ij,kl}(\varphi))(\theta^+\psi_+^k + \theta^-\psi_-^k)(\theta^+\psi_+^l + \theta^-\psi_-^l), \end{aligned} \tag{4.2.1}$$

we compute

$$\begin{aligned} S = \int d^2\sigma \Big(& (g_{ij} + b_{ij})(i\psi_+^i\partial_-\psi_+^j - F^i F^j + \partial_+\varphi^i\partial_-\varphi^j + i\psi_-^j\partial_+\psi_-^i) \\ & + (g_{ij,k} + b_{ij,k})(-i\psi_+^k\psi_+^i\partial_-\varphi^j - \psi_+^k\psi_-^j F^i + \psi_-^k\psi_+^i F^j - i\psi_-^k\psi_-^j\partial_+\varphi^i + F^k\psi_+^i\psi_-^j) \\ & + \frac{1}{2}(g_{ij,kl} + b_{ij,kl})(\psi_+^k\psi_-^l\psi_+^i\psi_-^j - \psi_-^k\psi_+^l\psi_+^i\psi_-^j)\Big). \end{aligned} \tag{4.2.2}$$

We can partially integrate some of the terms above, e.g.

$$\int d^2\sigma b_{ij}\psi_+^i\partial_-\psi_+^j = -\int d^2\sigma\big(b_{ij,k}\partial_-\varphi^k\psi_+^i\psi_+^j + b_{ij}\partial_-\psi_+^i\psi_+^j\big),$$

allowing us to write $b_{ij}\psi_+^i\partial_-\psi_+^j = -\frac{1}{2}b_{ij,k}\partial_-\varphi^k\psi_+^i\psi_+^j$ in the integral. Together with the identity

$$g_{ij,k} = \Gamma_{ik,j} + \Gamma_{jk,i},$$

the action simplifies to

$$\begin{aligned} S = \int d^2\sigma \Big(& (g_{ij} + b_{ij})\partial_+\varphi^i\partial_-\varphi^j + ig_{ij}\psi_+^i\nabla_-^+\psi_+^j + ig_{ij}\psi_-^i\nabla_+^-\psi_-^j \\ & + (g_{ij,kl} + b_{ij,kl})(\psi_+^k\psi_-^l\psi_+^i\psi_-^j) + (-g_{ij}F^iF^j + 2\Gamma_{ik,j}^-\psi_-^k\psi_+^i F^j)\Big). \end{aligned} \tag{4.2.3}$$

Here ∇_-^+ is an abbreviation for $\nabla^+_{\frac{\partial}{\partial\sigma^-}}$, so that

$$\nabla_-^+\psi_+^j = \partial_-\psi_+^j + \Gamma_{kl}^{+j}\partial_-\varphi^k\psi_+^l, \tag{4.2.4}$$

and similarly for ∇_+^-. Equation (4.2.3) immediately yields the equation of motion for F:

$$F^i = \Gamma_{jk}^{-i}\psi_-^k\psi_+^j. \tag{4.2.5}$$

This equation for F allows one in principle to eliminate F from the classical theory, but is this also allowed in the quantum theory? The answer is yes, because of the following. The F dependent term in the path integral[4] schematically looks like

$$\int DF\exp(-aF^2 + bF),$$

[4]See Chapter 5 for the definition of the path integral.

with a and b functions of the other fields. This is a gaussian integral whose value we know exactly to be

$$\int DF \exp(-aF^2 + bF) = \int DF \exp(-a(F - \frac{b}{2a})^2 + \frac{b^2}{4a}) \sim \exp(\frac{b^2}{4a}),$$

where we dropped an irrelevant constant from the integration. However, the equation of motion for F is $F = \frac{b}{2a}$ and if we substitute that for F in the action we get the term

$$-a(\frac{b}{2a})^2 + b(\frac{b}{2a}) = \frac{b^2}{4a},$$

so indeed, evaluating the path integral for F simply amounts to put it on shell. Since the symmetry algebra closes only on shell, there is no harm to integrate out F. After that, the action is given by

$$\begin{aligned} S = \int d^2\sigma \Big(& (g_{ij} + b_{ij})\partial_+\varphi^i\partial_-\varphi^j + ig_{ij}\psi_+^i\nabla_-^+\psi_+^j + ig_{ij}\psi_-^i\nabla_+^-\psi_-^j \\ & - (g_{ik,jl} + b_{ik,jl} + \Gamma^-_{ik,r}\Gamma^{-r}_{jl})(\psi_+^i\psi_+^j\psi_-^k\psi_-^l)\Big). \end{aligned} \tag{4.2.6}$$

The last term looks like a curvature term so let us compute

$$\begin{aligned} R^{\pm r}_{jkl} =& \partial_k\Gamma^{\pm r}_{lj} + \Gamma^{\pm s}_{lj}\Gamma^{\pm r}_{ks} - (k \leftrightarrow l) \\ =& \partial_k\left(\frac{1}{2}g^{rs}(g_{ls,j} + g_{js,l} - g_{jl,s} \pm H_{slj})\right) + \Gamma^{\pm s}_{lj}\Gamma^{\pm r}_{ks} - (k \leftrightarrow l) \\ =& -\frac{1}{2}g^{rm}g^{sn}g_{mn,k}\,(g_{ls,j} + g_{js,l} - g_{jl,s} \pm H_{slj}) + \frac{1}{2}g^{rs}\,(g_{ls,jk} + g_{js,lk} - g_{jl,sk} \pm H_{slj,k}) \\ & + \Gamma^{\pm s}_{lj}\Gamma^{\pm r}_{ks} - (k \leftrightarrow l). \end{aligned} \tag{4.2.7}$$

Hence,

$$\begin{aligned} R^{\pm}_{ijkl}\psi_+^i\psi_+^j\psi_-^k\psi_-^l =& \Big(- (\Gamma_{ik,n} + \Gamma_{nk,i})(2\Gamma^n_{lj} \pm H^n_{lj}) + (g_{li,jk} - g_{jl,ik} \pm H_{ilj,k}) \\ & + 2\Gamma^{\pm s}_{lj}\Gamma^{\pm}_{ks,i}\Big)\psi_+^i\psi_+^j\psi_-^k\psi_-^l \\ =& -2(g_{lj,ik} \pm b_{jl,ik} + \Gamma^{\pm}_{ki,r}\Gamma^{\pm\ r}_{lj})\psi_+^i\psi_+^j\psi_-^k\psi_-^l. \end{aligned} \tag{4.2.8}$$

So finally, we conclude that the action is given by

$$S = \int d^2\sigma\Big((g_{ij} + b_{ij})\partial_+\varphi^i\partial_-\varphi^j + ig_{ij}\psi_+^i\nabla_-^+\psi_+^j + ig_{ij}\psi_-^i\nabla_+^-\psi_-^j + \frac{1}{2}R^+_{ijkl}\psi_+^i\psi_+^j\psi_-^k\psi_-^l\Big). \tag{4.2.9}$$

By construction this action is invariant under the global symmetry $\delta\Phi^i = \epsilon Q^1\Phi^i + \tilde{\epsilon}Q^2\Phi^i$ that induces a symmetry on the components via

$$\delta\Phi^i = \delta\varphi^i + \theta^+\delta\psi_+^i + \theta^-\delta\psi_-^i + \theta^-\theta^+\delta F^i. \tag{4.2.10}$$

In order to determine these variations, we have to expand $\delta\Phi^i$ in the Grassmann variables, keeping in mind that $I_\pm$ also depends on these:

$$I^i_{\pm j}(\Phi) = I^i_{\pm j}(\varphi) + \theta^+ I^i_{\pm j,k}\psi_+^k + \theta^- I^i_{\pm j,k}\psi_-^k + \theta^-\theta^+(I^i_{\pm j,kl}\psi_+^k\psi_-^l + I^i_{\pm j,k}F^k). \tag{4.2.11}$$

Using this we compute

$$\begin{aligned}
\delta\Phi^i =&\epsilon^+ Q_+\Phi^i + \epsilon^- Q_-\Phi^i + \tilde\epsilon^+ I^i_{+j} D_+\Phi^j + \tilde\epsilon^- I^i_{-j} D_-\Phi^j \\
=&\left(\epsilon^+\psi^i_+ + \epsilon^-\psi^i_- + \tilde\epsilon^+ I^i_{+j}\psi^j_+ + \tilde\epsilon^- I^i_{-j}\psi^j_-\right) \\
&+\theta^+\left(i\epsilon^+\partial_+\varphi^i - \epsilon^- F^i + \tilde\epsilon^+(-iI^i_{+j}\partial_+\varphi^j - I^i_{+j,k}\psi^k_+\psi^j_+) + \tilde\epsilon^-(-I^i_{-j}F^j - I^i_{-j,k}\psi^k_+\psi^j_-)\right) \\
&+\theta^-\left(\epsilon^+ F^i + i\epsilon^-\partial_-\varphi^i + \tilde\epsilon^+(I^i_{+j}F^j - I^i_{+j,k}\psi^k_-\psi^j_+) + \tilde\epsilon^-(-iI^i_{-j}\partial_-\varphi^j - I^i_{-j,k}\psi^k_-\psi^j_-)\right) \\
&+\theta^-\theta^+\Big(i\epsilon^+\partial_+\psi^i_- - i\epsilon^-\partial_-\psi^i_+ \\
&\qquad+\tilde\epsilon^+\left(-iI^i_{+j}\partial_+\psi^j_- - I^i_{+j,k}\psi^k_+F^j - iI^i_{+j,k}\psi^k_-\partial_+\varphi^j + I^i_{+j,kl}\psi^k_+\psi^l_-\psi^j_+ + I^i_{+j,k}F^k\psi^j_+\right) \\
&\qquad+\tilde\epsilon^-\left(iI^i_{-j}\partial_-\psi^j_+ + iI^i_{-j,k}\psi^k_+\partial_-\varphi^j - I^i_{-j,k}\psi^k_-F^j + I^i_{-j,kl}\psi^k_+\psi^l_-\psi^j_- + I^i_{-j,k}F^k\psi^j_-\right)\Big).
\end{aligned} \tag{4.2.12}$$

The variation for F is not very elegant, but fortunately we are only interested in φ^i and $\psi^i_\pm$, whose variations with F on shell (Equation (4.2.5)) are given by:

$$\begin{aligned}
\delta\varphi^i =&\epsilon^+\psi^i_+ + \epsilon^-\psi^i_- + \tilde\epsilon^+ I^i_{+j}\psi^j_+ + \tilde\epsilon^- I^i_{-j}\psi^j_-, \\
\delta\psi^i_+ =&i\epsilon^+\partial_+\varphi^i - \epsilon^-\Gamma^{+i}_{jk}\psi^j_-\psi^k_+ + \tilde\epsilon^+\left(-iI^i_{+j}\partial_+\varphi^j - I^i_{+j,k}\psi^k_+\psi^j_+\right) \\
&+\tilde\epsilon^-\left(-I^i_{-j}\Gamma^{+j}_{kl}\psi^k_-\psi^l_+ - I^i_{-j,k}\psi^k_+\psi^j_-\right), \\
\delta\psi^i_- =&\epsilon^+\Gamma^{+i}_{jk}\psi^j_-\psi^k_+ + i\epsilon^-\partial_-\varphi^i + \tilde\epsilon^+\left(I^i_{+j}\Gamma^{+j}_{kl}\psi^k_-\psi^l_+ - I^i_{+j,k}\psi^k_-\psi^j_+\right) \\
&+\tilde\epsilon^-\left(-iI^i_{-j}\partial_-\varphi^j - I^i_{-j,k}\psi^k_-\psi^j_-\right),
\end{aligned} \tag{4.2.13}$$

where we have rewritten derivatives on $I_\pm$ using (4.1.24).

The action and the symmetries as they are given in (4.2.9) and (4.2.13) depend on the coordinates on M so we still have to understand their global, intrinsic definitions. In the action the only term that needs explanation is the kinetic term $b_{ij}\partial_+\varphi^i\partial_-\varphi^j$, as the other ones were already discussed in Chapter 2, see also Equation (2.3.16). Interpreting b_{ij} as a local potential of a gerbe, the term $b_{ij}\partial_+\varphi^i\partial_-\varphi^j$ in the action is not well defined, but once we look at the term e^{iS}, it will correspond exactly to the gerbe holonomy as given in Equation (3.6.16), in a gauge where the co–cycle $g_{\alpha\beta\gamma} = 1$ and $A_{\alpha\beta} = 0$, see also the remark at the end of Section 3.6.2.

To verify that the symmetry, as given in (4.2.13), extends globally, we have to explicitly perform a coordinate transformation $\varphi \mapsto \varphi'(\varphi)$ and determine whether both sides transform in the same way.

The transformation rules of the relevant quantities are given by

$$\begin{aligned}
\psi'^{i}_{\pm} &= \frac{\partial \varphi'^{i}}{\partial \varphi^{j}} \psi^{j}_{\pm}, \\
I'^{i}_{\pm j} &= \frac{\partial \varphi'^{i}}{\partial \varphi^{k}} \frac{\partial \varphi^{l}}{\partial \varphi'^{j}} I^{k}_{\pm l}, \\
\Gamma'^{\pm i}_{jk} &= \frac{\partial \varphi'^{i}}{\partial \varphi^{l}} \frac{\partial^2 \varphi^{l}}{\partial \varphi'^{m} \partial \varphi'^{n}} + \frac{\partial \varphi^{m}}{\partial \varphi'^{j}} \frac{\partial \varphi^{n}}{\partial \varphi'^{k}} \frac{\partial \varphi' i}{\partial \varphi^{r}} \Gamma^{\pm r}_{mn}.
\end{aligned} \tag{4.2.14}$$

With these equations one can compare the transformation behavior of both sides of (4.2.13), which we will not explicitly do here.

Chapter 5

The topological model

So far we have been studying the classical theory of the sigma model, mostly focussing under which conditions it allows for $(2,2)$–supersymmetries. These models were only defined on a cylinder, and in this section we will use a method called *the topological twist* to extend everything to arbitrary worldsheets. Under certain conditions, this procedure transforms the sigma model into a *topological field theory*. Before we come to the twist we briefly discuss the concept of quantization, in order to explain the definition of the *path–integral* which is important to understand topological field theories from the physical point of view.

5.1 Quantization

Quantization is the process of transforming a classical theory into a quantum theory. Instead of diving into the general theory, which would be outside the scope of this thesis, we start with a classical relativistic point–particle moving in space–time, which we take to be a manifold M, and then generalize to strings.

The time evolution of the physical system is described by a Lagrangian, which in this setting is a function on TM (it depends both on the position and the velocity of the particle). Note that time in this context refers to a parametrization of the path the particle follows in space–time. As explained in Chapter 2, the action S becomes a functional on the space of all paths in M defined by $S(\gamma) = \int_{[0,1]} L(\gamma(\tau), \dot{\gamma}(\tau))d\tau$, and it determines the evolution of the system completely by picking out the path for which it has an extremal value. It is often convenient to pass from the Lagrangian formalism to the Hamiltonian formalism, which is to say that we replace TM by T^*M. This is done by means of the *momentum* map $p : TM \to T^*M$, given in local coordinates on TM and T^*M by

$$p_i(x,y) = \frac{\partial L}{\partial y^i}, \tag{5.1.1}$$

where x^i are local coordinates on M, y^i the coordinates on the fibers of TM (i.e., a point in TM is described by (x^i, y^i)) and p_i are coordinates on the fibers of T^*M. Due to the transformation rule of the y^i and the chain rule, one can verify that p_i transform in the right way (oppositely to the y^i) so that

this map is globally well defined. Note that this map is not necessarily an isomorphism, as the matrix $\frac{\partial p_i}{\partial y^j}$ can be singular.

The Hamiltonian is defined locally by

$$H(x,p) := p_i y^i - L(x,y), \tag{5.1.2}$$

assuming that we can express this in terms of the momenta p_i (which is the case if, for instance, (5.1.1) is an isomorphism, but this is not a necessary condition). Passing to the Hamiltonian formalism has the advantage that T^*M is a symplectic manifold, whose symplectic form in local coordinates (x,p) is given by $\sum_i dp_i \wedge dx^i$. Note that we always have, in any physical system, the relation $\{x^i, p_j\} = \delta^i_j$, but the form of p depends on the Lagrangian and so the Poisson bracket in this sense is different for different physical systems.

The classical deterministic point of view is changed drastically in quantum mechanics. Instead of having a well defined location in M, the state of the particle is described by a *wave–function*, a square integrable $\mathbb{C}-$valued function $\Psi \in L^2(M)$, that is normalized in the sense that $\int_M |\Psi|^2 = 1$. We can no longer speak about a particle's position, but only about the probability of finding it in a region $U \subset M$, which is given by $\int_U |\Psi|^2$. There are basically two different approaches for a mathematical setup of quantum mechanics. One of them is called the *canonical formalism*, the other is called the *path–integral* formalism. Continuing with our example, in the canonical formalism, quantization replaces the classical configuration space M by the Hilbert space $\mathcal{H} := L^2(M)$, and the functions on T^*M (called observables in the classical theory) by operators on $\mathcal{H}$. These operators can be unbounded, i.e. they are not necessarily bounded nor defined on the whole Hilbert space, and we will denote the set of unbounded operators by $O(\mathcal{H})$. This map $C^\infty(T^*M) \to O(\mathcal{H})$ is usually denoted by $f \mapsto \hat{f}$, and is required to satisfy

$$\widehat{\{f,g\}} = -i[\hat{f},\hat{g}], \tag{5.1.3}$$

the bracket being defined on the common domain of $\hat{f}$ and $\hat{g}$. The reason for the factor i is that physical operators should be hermitian[1], $\hat{f}^\dagger = \hat{f}$, so that the commutator must satisfy $[\hat{f},\hat{g}]^\dagger = -[\hat{f},\hat{g}]$. If there were no i in (5.1.3), we would have $[\hat{f},\hat{g}]^\dagger = (\widehat{\{f,g\}})^\dagger = \widehat{\{f,g\}} = [\hat{f},\hat{g}]$, which would be inconsistent. In the canonical formalism itself there are different points of view, the most important two are called the Schrödinger picture and Heisenberg picture. In the Schrödinger picture the operators are fixed and the states (the wave functions) are time evolving, and their evolution is governed by the quantized Hamiltonian $\hat{H}$, by means of

$$|\Psi(\tau_f)\rangle = e^{-i\hat{H}(\tau_f - \tau_i)}|\Psi(\tau_i)\rangle, \tag{5.1.4}$$

so that wave functions satisfy the familiar Schrödinger Equation:

$$i\frac{d}{d\tau}|\Psi(\tau)\rangle = \hat{H}|\Psi(\tau)\rangle. \tag{5.1.5}$$

[1]The eigenvalues of these operators correspond to the physical measurable quantities, which must be real.

In the Heisenberg picture the states are time independent, while the operators are time dependent and evolve according to

$$\frac{d}{d\tau}O = \frac{\partial}{\partial\tau}O + i[\hat{H}, O]. \tag{5.1.6}$$

The Heisenberg picture resembles more the classical point of view, as classically the states are the points on M (clearly time independent) and the time evolution of the observables is governed by the Hamiltonian and the Poisson bracket via

$$\frac{df}{d\tau} = \frac{\partial f}{\partial\tau} - \{H, f\}. \tag{5.1.7}$$

Despite the differences, the Heisenberg and Schrödinger picture give the same physical information, which is all contained in the *expectation values*. The expectation value of an operator O on a system in a state Ψ is defined by

$$\langle O\rangle_\Psi := \langle\Psi|O|\Psi\rangle, \tag{5.1.8}$$

which in our example is just $\int_M dx\Psi^*(x)(O\Psi)(x)$ (dx being the density used to define the space $L^2(M)$), and agrees with the probabilistic interpretation of expectation value.

The other approach to quantum mechanics is via the path–integral, and to introduce this technique let us define, still in the context of the classical relativistic particle, for each point $x \in M$ a special state $|x\rangle$ corresponding to the delta function based at x. This is of course not an element in $\mathcal{H} = L^2(M)$, but rather a linear functional on $\mathcal{H}$ given by evaluation of functions in the point x. Note that this functional is not continuous, hence can not be represented by an element from $\mathcal{H}$ via the inner product[2]. For a state $|\Psi\rangle$ representing the function $\Psi \in L^2(M)$ we define $\langle x|\Psi\rangle := \Psi(x)$, and we formally write

$$|\Psi\rangle = \int_M dx\langle x|\Psi\rangle|x\rangle = \int_M dx\Psi(x)|x\rangle, \tag{5.1.9}$$

which is often written as $\int_M dx|x\rangle\langle x| = 1$. The state $|x\rangle$ can be thought of as representing the classical state x, with the quantum state being smeared out over the various classical states. Note that $|x\rangle$ is an eigenstate of the position operator acting on functionals by transposition.

The fundamental identity in the path–integral approach is the following:

$$\langle y|e^{-i\hat{H}\tau_f}\hat{O}_n(\tau_n)\cdots\hat{O}_1(\tau_1)e^{i\hat{H}\tau_i}|x\rangle = \int_{\varphi(\tau_i)=x}^{\varphi(\tau_f)=y} D[\varphi]e^{iS(\varphi)}O_n(\tau_n)\cdots O_1(\tau_1), \tag{5.1.10}$$

the integral being taken over all possible paths in M with starting point x and ending point y, and $\tau_n \geq \ldots \geq \tau_1$. We end our discussion about the point–particle with the observation that Equations

[2]More rigorously, the correct setup here is that of a 'rigged Hilbert space'. This consists of a so–called 'nuclear' subspace $S \subset \mathcal{H}$ (the term nuclear referring to some technical conditions which we will not discuss), that is in the common domain of the operators of interest and represents the set of physical states. The space S^* (the space of linear functions on S), which itself is not contained in $\mathcal{H}$, contains the eigenstates of the physical operators, like the $|x\rangle$ discussed above.

(5.1.4), (5.1.9) and (5.1.10) imply that the evolution of wave functions is given by

$$\Psi(y,\tau_f) = \int_M dx \int_{\varphi(\tau_i)=x}^{\varphi(\tau_f)=y} D[\varphi] e^{iS(\varphi)} \Psi(x,\tau_i), \tag{5.1.11}$$

and in this sense the path–integral acts as an integral operator describing the evolutions of wave functions.

5.1.1 Strings and the operator state correspondence

Now consider a closed string moving in space–time, again denoted by M. Its configuration space is not M, as it was for the point–particle, but equals $LM := \{\gamma : S^1 \to M\}$, which is the space of all possible string shapes in M. The phase space can also be defined, namely T^*LM which has as fiber over a loop γ the space $T^*_\gamma LM := \Gamma(\gamma^* T^* M)$. As before, canonical quantization requires us to replace this configuration space by the Hilbert space of square integrable wave functions on it, i.e. $\mathcal{H} := L^2(LM)$. However, to the author's knowledge there is no natural choice of measure on LM for general M, but we will assume that this is not a serious problem and proceed as for the point particle. A state Ψ is thus a functional, assigning a complex number to each possible loop, giving a probability for the string to be in a certain shape.

Analogously to the state $|x\rangle$ we have for each loop γ a 'state' $|\gamma\rangle$, and the analogue of (5.1.9) is

$$|\Psi\rangle = \int_{LM} d\gamma \; \Psi(\gamma)|\gamma\rangle. \tag{5.1.12}$$

The time evolution in terms of these states is again described by

$$\langle \gamma_f | e^{-iH(\tau_f - \tau_i)} | \gamma_i \rangle = \int D[\varphi] e^{iS}, \tag{5.1.13}$$

the integral taking place over all paths in LM with start and ending points γ_i and γ_f. Such a path can be described on M as a particular type of cobordism between the image of γ_i and γ_f in M. With particular type we mean that such a cobordism can not have a hole in it, as this is not interpreted as a path in LM. Here we see already a big difference with point–particle physics, for if we consider all possible cobordisms between γ_i and γ_f, we already see string interactions taking place. In contrast, for point particles one has to add appropriate interaction terms to the lagrangian to create these interactions. This is a fortunate coincidence, as adding such interaction terms for strings would endanger the conformal symmetry, and if present, also the supersymmetry.

If we allow for string interactions to take place, the relevant quantities to compute are

$$\langle \gamma_f^1, \ldots, \gamma_f^n | e^{-iH(\tau_1 - \tau_0)} | \gamma_i^1, \ldots, \gamma_i^m \rangle, \tag{5.1.14}$$

for which the path–integral formalism tells us to sum over all possible surfaces Σ with boundary $\partial\Sigma = \Sigma_i \cup \overline{\Sigma_f}$, where the bar stands for reversed orientation and Σ_i, Σ_f are the unions of the γ_i and γ_f respectively. The precise meaning of the measure $D[\varphi]$ depends on the particular type of theory

one looks at. For the closed superstring, it involves also an integral over the metric h on the world–sheet, as well as an integral over the fields $\psi_\pm$. Intuitively it is clear that the space of cobordisms falls into different connected components, corresponding to the number of holes in the surface establishing the cobordism, and so the final amplitude becomes

$$\langle \gamma_f^1, \dots, \gamma_f^n | e^{-i\hat{H}(\tau_f - \tau_i)} | \gamma_i^1, \dots, \gamma_i^m \rangle = \sum_{g \geq 0} \int D[h_g] D[\varphi] D[\psi_\pm] e^{iS(\varphi, h_g, \psi_\pm)}, \tag{5.1.15}$$

where the sum is over all genera, and each path integral is over all cobordisms with g holes in it. There are two subtleties in this equation. First, the term $\tau_f - \tau_i$ in this equation is no longer well–defined, as there is no longer a global τ coordinate on the worldsheet. We will therefore write $\langle \gamma_f^1, \dots, \gamma_f^n | \gamma_i^1, \dots, \gamma_i^m \rangle$ for the amplitude instead of the formula above, and not mention the evolution operator that is associated to the cobordism. Secondly, for reasons that will be explained in Section 5.3, in order to properly define the action over a general cobordism we first have to Wick rotate $\tau \mapsto t = i\tau$, so that the factor iS in the path–integral becomes $-S$.

Now in an actual process the states are given by wave functionals and not by the states $|\gamma\rangle$, so what we really have to calculate are amplitudes of the form

$$\begin{aligned} \langle \Psi_f^1, \dots, \Psi_f^n | \Psi_i^1, \dots, \Psi_i^m \rangle = \int_{LM} d\gamma_f^1 \cdots d\gamma_f^n d\gamma_i^1 \cdots d\gamma_i^m (\Psi_f^1(\gamma_f^1))^* \cdots (\Psi_f^n(\gamma_f^n))^* \Psi_i^1(\gamma_i^1) \cdots \Psi_i^m(\gamma_i^m) \cdot \\ \cdot \langle \gamma_f^1, \dots, \gamma_f^n | \gamma_i^1, \dots, \gamma_i^m \rangle. \end{aligned} \tag{5.1.16}$$

Fortunately, conformal symmetry of the theory allows for a simplification of this formula. The analogue of (5.1.11) for a noninteracting string is

$$\Psi(\tau_f, \gamma) = \int_{LM} d\gamma' \int_{\varphi(\tau_i) = \gamma'}^{\varphi(\tau_f) = \gamma} D\varphi e^{-S(\varphi)} \Psi(\tau_i, \gamma'), \tag{5.1.17}$$

where the integral over φ is taken over all $\varphi \in \mathrm{Map}(S^1 \times [\tau_i, \tau_f], M)$ restricting to γ and γ' on the boundaries. For notational convenience we suppress the additional integrals over the metric h and the fields $\psi_\pm$. Such a cylinder is conformally equivalent to an annulus in the complex plane, by mapping (τ, σ) to $e^{\tau + i\sigma}$. In the plane the time direction is replaced by the radial direction, so the limit $\tau_i \to -\infty$ corresponds to shrinking the inner circle of the annulus to a point. In taking this limit, (5.1.17) reduces to

$$\Psi(\tau_f, \gamma) = \int^{\varphi(\tau_f) = \gamma} D\varphi e^{-S(\varphi)} O_\Psi(0), \tag{5.1.18}$$

where O_Ψ is a local operator whose value at the point 0 in the plane replaces the factor $\Psi(\tau_i, \gamma')$ in (5.1.17), as we take the limit $\tau_i \to -\infty$. Note that this operator, besides being a functional of the field φ, can also depend on the fields $\psi_\pm$, whose contribution we suppressed in (5.1.17). Hence asymptotic states, i.e. states whose corresponding boundary is stretched out to an infinitely long cylinder, are represented by a path integral over a hemisphere with a local operator put at the top, where the hemisphere is put in the place of the infinite cylinder. This identification is called the

'operator–state correspondence'. In particular we can take as local operator just the identity, and we call the corresponding state the vacuum $|0\rangle$. For a general (local) operator O the corresponding state is denoted by $|O\rangle = O|0\rangle$. We will only look at these asymptotic states, because in physical scattering processes these are the only states that can be detected. The string interactions, which take place through all these cobordisms, are not directly detectable. So the basic observables that one computes in string theory are the so–called *n–point correlation functions*:

$$\langle O_1 \cdots O_n \rangle := \sum_{g \geq 0} \int_{\Sigma_g} dx_1 \cdots dx_n \langle O_1(x_1) \cdots O_n(x_n) \rangle_g, \tag{5.1.19}$$

with Σ_g a compact connected oriented surface[3] of genus g, and the last term is defined as

$$\langle O_1(x_1) \cdots O_n(x_n) \rangle_g := \int D[h]D[\varphi]D[\psi_\pm] e^{-S(\varphi,h,\psi_\pm)} O_1(x_1) \cdots O_n(x_n), \tag{5.1.20}$$

where the integral over h is over all possible metrics on Σ_g (Riemannian metrics as we performed a Wick rotation), the integral over φ is over all possible maps $\varphi : \Sigma_g \to M$, and the fermionic integral over $\psi_\pm$ is over all possible fermionic sections of $S_{\Sigma_g} \otimes \varphi^* TM$ with S_{Σ_g} the spinor bundle of Σ_g. There is an integration over the insertion points in (5.1.19) because the strings can hit the surface at all possible points. An analogous treatment can be done for open strings, albeit that the operator–state correspondence is a bit more subtle, as instead of closed surfaces one obtains surfaces with boundary.

Due to conformal invariance, a large part of the integration over h (namely those in the direction of conformally equivalent metrics) is trivial, and after a procedure called 'Fadeev–Popov gauge–fixing', the integral reduces to an integral over all conformally inequivalent metrics, which is precisely the space of complex structures on Σ_g. This goes at the cost of a Jacobian due to the change of variables in the integral, which gives rise to some extra terms, called ghosts, in the action.

Remark. To set up a perturbation theory for the term in (5.1.19), one usually adds an additional kinetic term for the metric h to the action, given by the Einstein–Hilbert term $\lambda \int_\Sigma R_h$ with R_h the Ricci scalar and λ a coupling constant. In four dimensions such a term is non–trivial and leads to the Einstein Equations for gravity, but in two dimensions this term is by Gauss' Theorem proportional to the Euler characteristic of Σ, given by $\chi(\Sigma) = 2 - 2g$. Therefore, the contribution of a genus g surface to the path integral is weighted by the factor $e^{-4\pi\lambda(2-2g)}$, so for e^λ small (5.1.19) is indeed a perturbative expansion in the genus.

5.2 Topological field theories

The difficulty of string theory lies in its integral over all worldsheet metrics. One way to handle this problem is by considering actions whose path–integral in (5.1.20), before doing the integral over all metrics, is actually independent of the metric. Quantum field theories whose physical outcome only depends on the global properties of the space on which they are defined, are called *topological field*

[3] Recall that we assumed our world–sheets to be oriented.

theories. This name can be very misleading, as usually it means that the theory is independent of the metric but can still depend on other quantities, like for instance the differentiable structure on the space. For a concise treatment about topological field theories we refer to [3].

Topological theories are classified into two different classes, namely those of *Schwarz type* and those of *Witten type*. Roughly speaking, in a theory of Schwarz type (also called quantum–type) the theory is metric independent because the metric does not appear in the action at all. This sounds rather trivial, but it can lead to interesting (and certainly non–trivial) theories, one of the most familiar one being Chern–Simons theory.

For the supersymmetric sigma model however, the relevant type is that of Witten. In these theories, which are also referred to as 'cohomological field theories', the topological invariance is achieved with the aid of a nilpotent fermionic symmetry, and the physical observables are those invariant under this symmetry. One of the first models of this type were Witten's topological strings of type A and B ([36]), which we will discuss later as special cases of our sigma model with H–flux.

5.2.1 Cohomological field theories

From Noethers Theorem we know that a symmetry of a quantum system leads to a conserved charge Q, that generates the symmetry on an operator O via

$$\delta_\epsilon O = \epsilon[Q, O],$$

where the bracket is graded, as operators can be either fermionic or bosonic, and the parameter ϵ is either a complex number or a grassmann number depending on whether the symmetry is bosonic or fermionic. A *cohomological field theory* is a field theory with a fermionic symmetry whose corresponding charge Q satisfies the following axioms.

1 Q is nilpotent: $Q^2 = 0$.

2 The vacuum is invariant: $Q|0\rangle = 0$.

3 The energy momentum tensor is Q–exact: $T_{\alpha\beta} := \frac{\delta S}{\delta h^{\alpha\beta}} = [Q, G_{\alpha\beta}]$ for some (necessarily fermionic) operator $G_{\alpha\beta}$.

The second property can be understood by writing the symmetry generated by Q as $\exp(\epsilon Q)$, with ϵ the Grassmann symmetry variable. The symmetry is then invariant, $\exp(\epsilon Q)|0\rangle = |0\rangle$, if and only if $Q|0\rangle = 0$.

The third property is usually ensured by an action that is Q–exact modulo metric independent terms, i.e.

$$S = \{Q, V\} + S_{top} \tag{5.2.1}$$

for some functional V, and S_{top} denotes a topological term, i.e. independent of the worldsheet metric. One readily verifies the Q–exactness of the energy momentum tensor in this case. The operator Q is

often called the BRST operator, as such an operator usually arises in the BRST procedure in gauge theories.

The physical observables of the theory are defined to be the h–independent operators that are symmetry invariant, i.e. satisfy $[Q, O] = 0$. If an operator can be written as $[Q, O]$, all expectation values involving this operator will be zero, as follows from

$$\langle 0|O_1 \cdots O_r([Q, O])O_{r+1} \cdots O_n|0\rangle = 0,$$

since we can (anti–) commute the Q past all the O_i, eventually annihilating the vacuum. Note that we suppress the dependence on the insertion points in this correlator, for in a topological theory the correlators are independent of those. Under a variation of the metric $\delta h^{\alpha\beta}$, we have

$$\frac{\delta}{\delta h^{\alpha\beta}}\langle O_1 \cdots O_n\rangle = \int D[\varphi]O_1 \cdots O_n(\frac{\delta}{\delta h^{\alpha\beta}}(-S))e^{-S(\varphi)} = -\langle O_1 \cdots O_n[Q, G_{\alpha\beta}]\rangle = 0.$$

So indeed, a cohomological theory is topological.

5.3 Twisting the $(2, 2)$–model

In Chapter 4 we constructed a model with $(2, 2)$–supersymmetry in the case of a flat worldsheet, i.e. a cylinder for the closed string. We would like to extend it to arbitrary surfaces, for example to describe interacting strings. After performing a Wick rotation, which we will define in a moment, the action can straightforwardly be extended to arbitrary surfaces, but for the symmetries it is not that simple. In all computations we used Grassmannian variables, ϵ and $\tilde{\epsilon}$, which we assumed to be constant. However, on a general surface these parameters are not just functions, instead they are sections of some vector bundle. To demand that they are covariantly constant with respect to some connection can be too restrictive. However to obtain a topological field theory the nilpotent symmetry is crucial, so fortunately there exists a trick called 'topological twisting', which was first introduced by Witten in [36], that allows us to avoid this problem. To explain this trick, and make the preceding discussion more precise let us look at the action of the generalized $(2, 2)$–sigma model, which we recall for the reader's convenience:

$$S = \int d^2\sigma \left\{ (g_{ij} + b_{ij})\partial_+\varphi^i\partial_-\varphi^j + ig_{ij}\psi_+^i\nabla_-^+\psi_+^j + ig_{ij}\psi_-^i\nabla_+^-\psi_-^j + \frac{1}{2}R_{ijkl}^+\psi_+^i\psi_+^j\psi_-^k\psi_-^l \right\}. \tag{5.3.1}$$

The presence of the metric h on Σ is somewhat hidden in this equation, but that is due to the fact that until now we defined it to be flat. The generalization to arbitrary metrics is not straightforward, as the spinor bundles depend on the metric, so that the kinetic terms of the fermions obtain nontrivial corrections (due to spin–connections) when passing to a general spinor bundle. Furthermore, in all previous discussions the Lorentz transformations on Σ belonged to $SO^+(1, 1)$, the connected component of $SO(1, 1)$, which is to say that they are of the form $\sigma^\pm \mapsto e^{\pm\alpha}\sigma^\pm$. Under these transformations the fermionic fields transform as $\psi_\pm \mapsto e^{\mp\alpha/2}\psi_\pm$ (see appendix A) and for this reason a quantity such as

$\psi_+\nabla_-^+\psi_+$ is Lorentz invariant. Under the other type of Lorentz transformations, $\sigma^\pm \mapsto -e^{\pm\alpha/2}\sigma^\pm$ so that the kinetic terms for the fermions are not globally defined. Now on an arbitrary surface with a metric of signature $(1,1)$, it is not possible to restrict the structure group of the tangent bundle to $SO^+(1,1)$. For if it were possible, we could choose local orthonormal frames (U_i, e_i^1, e_i^2) with U_i an open cover of Σ and e_i^1, e_i^2 a frame on U_i in which $h = diag(-1,1)$. Suppose that the transition functions of the tangent bundle belong to $SO^+(1,1)$, i.e. the frames e_i are related to each other in overlaps by Lorentz transformations of the form

$$\begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}. \tag{5.3.2}$$

If we define $e_i^\pm := e_i^1 \pm e_i^2$, then on $U_i \cap U_j$ we have $e_i^\pm = e^{\pm\alpha} e_j^\pm$ for some α. We can then define a real line bundle on Σ, which on U_i is given by the line spanned by e_i^+. This is indeed globally defined, by the relation between the different e_i^α on overlaps. In particular this line bundle is oriented, as the transition functions are positive ($e^\alpha > 0$). An oriented line bundle is trivial, hence admits a global nowhere vanishing section, so there should be a nowhere vanishing vector field on Σ. This gives restrictions on the genus g, as the theorem of Poincaré–Hopf tells us that

$$\sum_{i\in I} \text{index}_{x_i}(v) = \chi(\Sigma) = 2 - 2g, \tag{5.3.3}$$

where v is an arbitrary vector field on Σ with an isolated set of zeroes $\{x_i\}_{i\in I}$, and the index at such a zero is defined as follows. Embed Σ in some Euclidean space, and pick a small circle around the zero such that v is nonzero on that circle. Then $\frac{v}{\|v\|}$ is a map $S^1 \to S^1$, and the index is defined as the winding number[4] of that map. In particular, we see that a restriction of the structure group of the tangent bundle to $SO(1,1)^+$ restricts the genus to 1.

To avoid this difficulty we perform a Wick rotation $\tau \mapsto t = i\tau$. On a cylinder, where we have global coordinates, we know how to make sense of this, but on an arbitrary surface there is to the author's knowledge no obvious generalization of this concept. Therefore, we perform a Wick rotation for the action as given in (5.3.1), and try to extend the result to an arbitrary Riemann surface. In this step we gain a positive definite metric, which does not present the problems discussed above, but has the drawback that we lose the distinction between the time and space like directions. Recall that the time direction on the surface gives the direction in which strings are propagating while the space direction gives coordinates on the strings 'frozen in time'. Such a time direction on a curved surface will be impossible to define at some points, which is exactly what the Poincaré–Hopf Theorem tells us.

After this Wick rotation, the action becomes

$$S = \int_\Sigma dz d\bar{z} \left\{ (g_{ij} + b_{ij})\partial_z\varphi^i \partial_{\bar{z}}\varphi^j + i g_{ij}\psi_+^i \nabla_{\bar{z}}^+ \psi_+^j + i g_{ij}\psi_-^i \nabla_z^- \psi_-^j + \frac{1}{2} R^+_{ijkl}\psi_+^i \psi_+^j \psi_-^k \psi_-^l \right\}, \tag{5.3.4}$$

[4] More precisely, the Brouwer degree.

where $z := \sigma - it = \sigma + \tau = \sigma^+$, $\nabla^+_{\overline{z}}$ is an abbreviation of $\nabla^+_{\frac{\partial}{\partial \overline{z}}}$ and similarly for ∇^-_z. So for instance, $\nabla^+_{\overline{z}}\psi^j_+ = \partial_{\overline{z}}\psi^j_+ + \Gamma^{+j}_{kl}\partial_{\overline{z}}\varphi^k\psi^l_+$.

Now that we have somewhat artificially changed the signature we can extend (5.3.4) to arbitrary surfaces. First, note that a Riemannian metric h, or rather its conformal class, together with an orientation determines a complex structure I_Σ. In a local positively oriented orthonormal frame e_1, e_2, it is given by $I_\Sigma(e_1) = e_2$ and $I_\Sigma(e_2) = -e_1$. Since the orientation of Σ is defined locally by t, σ, this implies that the complex coordinates z and $\overline{z}$ are indeed the complex coordinates associated to I_Σ. Since the metric is positive definite, Lorentz transformations are now rotations, i.e. $z \mapsto e^{i\alpha}z$ for some $\alpha \in [0, 2\pi)$. This suggests that $\psi_\pm$ now transform as $e^{\mp\frac{i\alpha}{2}}\psi_\pm$. Indeed, while in signature $(1,1)$ spinors are real, in signature $(2,0)$ they are complex and transform in the way just described. Again, see appendix A for the details.

These transformations for the spinors remind us to the way differential forms transform. On a Riemann surface we have the notion of a canonical and anti–canonical line bundle K and $\overline{K}$, which are the bundles of 1–forms of type $(1,0)$ and $(0,1)$ respectively. Note that their definitions are metric dependent; different conformal classes give rise to different complex structures, hence different K's. Under a rotation with angle α, a $(1,0)$–form dz transforms as $dz \mapsto e^{i\alpha}dz$ and a $(0,1)$–form $d\overline{z}$ transforms as $d\overline{z} \mapsto e^{-i\alpha}d\overline{z}$, so that the component of a 1–form in these local coordinates transforms in the opposite way. Combining these two facts, we see that the spinor bundle on a Riemann surface is given by $K^{1/2} \oplus \overline{K}^{1/2}$, where for a line bundle L we denote by $L^{1/2}$ a line bundle[5] that satisfies $L^{1/2} \otimes L^{1/2} = L$. On a Riemann surface both $K^{1/2}$ and $\overline{K}^{1/2}$ exist. The easiest way to see this is to consider their Chern classes $c_1(K), c_1(\overline{K}) \in H^2(\Sigma) \cong \mathbb{Z}$, which are even. Since line bundles are classified by their Chern class, and products of line bundles corresponds to addition of Chern classes we can take any bundle whose Chern class is half of that of K and $\overline{K}$.
The fermionic fields are then sections of these bundles, and if σ and $\overline{\sigma}$ are local non–vanishing sections of $K^{1/2}$ and $\overline{K}^{1/2}$ respectively, these sections are given by $\psi_+\sigma$ and $\psi_-\overline{\sigma}$.

As the bundle $K^{1/2}$ is holomorphic and $\overline{K}^{1/2}$ is anti–holomorphic, there are canonical (anti–)holomorphic connections defined on them, given in local trivializations σ and $\overline{\sigma}$ as above, by $\overline{\partial}(\psi_+\sigma) = (\overline{\partial}\psi_+)\sigma$ and $\partial(\psi_+\overline{\sigma}) = (\partial\psi_+)\overline{\sigma}$ respectively. One easily verifies that this gives globally defined operators

$$\begin{aligned} \overline{\partial} &: \Gamma(K^{1/2}) \to \Omega^{(0,1)}(K^{1/2}), \\ \partial &: \Gamma(\overline{K}^{1/2}) \to \Omega^{(1,0)}(\overline{K}^{1/2}), \end{aligned} \tag{5.3.5}$$

because the transition maps are (anti–) holomorphic. If E is any vector bundle on Σ, then $\Omega^{(p,q)}(E) = \Gamma(\wedge^{(p,q)}T^*\Sigma \otimes E)$ denotes the (p,q)–forms with values in the vector bundle E. The operators ∇^+

[5]Note that choosing a square root, just as for ordinary numbers, involves a choice. After twisting however, any explicit dependence on the choice disappears.

and ∇^- on the bundles $K^{1/2}\otimes\varphi^*(TM)$ and $\overline{K}^{1/2}\otimes\varphi^*(TM)$ are then defined via

$$\nabla^+ := \overline{\partial}\otimes 1 + 1\otimes\pi^{(0,1)}\circ\varphi^*(\nabla^+) : \Gamma(K^{1/2}\otimes\varphi^*(TM)) \to \Omega^{(0,1)}(K^{1/2}\otimes\varphi^*(TM)),$$
$$\nabla^- := \partial\otimes 1 + 1\otimes\pi^{(1,0)}\circ\varphi^*(\nabla^-) : \Gamma(\overline{K}^{1/2}\otimes\varphi^*(TM)) \to \Omega^{(1,0)}(\overline{K}^{1/2}\otimes\varphi^*(TM)). \quad (5.3.6)$$

Here $\pi^{(p,q)}$ denotes the projection of forms onto their (p,q)–part, and $\varphi^*(\nabla^\pm)$ denotes as usual the pull–back of the connections $\nabla^\pm$ on TM. If s and t are sections of $K^{1/2}\otimes\varphi^*(TM)$ and if $\partial_1,\ldots,\partial_n$ is a frame for TM, then s and t can be written as $s = s^i\partial_i$ and $t = t^i\partial_i$, with s^i and t^i sections of $K^{1/2}$. $\nabla^+ t^j$ is then a section of $\wedge^{(0,1)}T^*\Sigma\otimes K^{1/2}$, so that $g_{ij}s^i\nabla^+t^j$ is a section of $\wedge^{(0,1)}T^*\Sigma\otimes K = \wedge^{(1,1)}T^*\Sigma$. In other words, $g_{ij}s^i\nabla^+t^j$ is a $(1,1)$–form on Σ and can be integrated, and its local expression is precisely the kinetic term for the ψ_+ fermion in (5.3.4). A similar conclusion holds for the kinetic term of ψ_-. Thus indeed, after performing the Wick rotation the action becomes globally defined for any worldsheet Σ.

Now that the action has been taken care of, the issue of the symmetry remains. From the above discussion it follows that the spinorial parameters ϵ^+ and $\tilde{\epsilon}^+$ are sections of $K^{-1/2}$, while ϵ^- and $\tilde{\epsilon}^-$ are sections of $\overline{K}^{-1/2}$. For the action to remain invariant under the symmetries, we need $\overline{\partial}\epsilon^+ = \overline{\partial}\tilde{\epsilon}^+ = 0$ and $\partial\epsilon^- = \partial\tilde{\epsilon}^- = 0$, in order for the calculations in Chapter 4 to remain correct. So what we need for the symmetry to survive are holomorphic sections of $K^{-1/2}$ and anti–holomorphic sections of $\overline{K}^{-1/2}$. However, for a vector bundle E on a Riemann surface Σ, the number of zeroes minus the number of poles (counted with multiplicities) of any non–zero meromorphic section s is given by an invariant of E, called its *degree*. In particular, if the degree is negative, there are no holomorphic sections. For the particular case of the holomorphic tangent bundle $T\Sigma$, the degree equals the Euler degree which equals $2-2g$, where g is the genus of Σ. Since K is the dual of $T\Sigma$, $K^{-1/2} = T^{1/2}$, hence its degree equals $1-g$. In particular for $g>1$, there are no holomorphic sections. Similarly, for $g>1$, there are no anti–holomorphic sections of $\overline{K}^{-1/2}$. One may wonder what kind of model can arise in genus 0 and 1, since there non–trivial sections do exist. Besides the fact that these theories would probably be more difficult, physically it seems unnatural to define a theory only for surfaces of genus 0 and 1.

Here is where the topological twist comes in. Instead of working with the square roots of K and $\overline{K}$, let us redefine the fields such that half of them become sections of K and $\overline{K}$, while the other half become sections of the trivial bundle, i.e. functions on Σ. If we do this in the right way we can define half of the symmetry parameters to be zero, and the other half to be constant, as we will see shortly. The difficulty lies in finding the right field redefinition, but the answer is presented by the bi–hermitian structure on M. The two complex structures $I_\pm$ define a splitting of $TM\otimes\mathbb{C}$ into $\pm i$ eigenspaces:

$$TM\otimes\mathbb{C} = T^{(1,0)}_\pm M \oplus T^{(0,1)}_\pm M, \quad (5.3.7)$$

where the subscript reminds us with respect to which complex structure we are decomposing TM. We can thus choose two decompositions for $\psi_\pm$, giving 4 choices in total. However, in the action $\psi_\pm$ is accompanied with the covariant derivative $\nabla^\pm$, and we know that $\nabla^\pm I_\pm = 0$. Since it is very convenient to be able to commute the eigenspace projections with the covariant derivatives, the

natural choice is to decompose ψ_+ with respect to I_+ and ψ_- with respect to I_-. Using the projection operators $P_\pm := \frac{1}{2}(1 + iI_\pm)$ and $\overline{P}_\pm := \frac{1}{2}(1 - iI_\pm)$, we define

$$\begin{aligned} \chi &:= P_+\psi_+ \in \Gamma\left(K^{1/2} \otimes T_+^{(0,1)}M\right) \qquad \overline{\chi} := \overline{P}_+\psi_+ \quad \in \Gamma\left(K^{1/2} \otimes T_+^{(1,0)}M\right) \\ \lambda &:= iP_-\psi_- \in \Gamma\left(\overline{K}^{1/2} \otimes T_-^{(0,1)}M\right) \qquad \overline{\lambda} := -i\overline{P}_-\psi_- \in \Gamma\left(\overline{K}^{1/2} \otimes T_-^{(1,0)}M\right). \end{aligned} \tag{5.3.8}$$

The factor i in the definition of λ is for later convenience. Using that $\nabla^\pm$ are metric ($\nabla^\pm g = 0$) and commute with $I_\pm$, plus the fact that $T^{(1,0)}M$ and $T^{(0,1)}M$ are both isotropic with respect to g, we have

$$g(\psi_+, \nabla^+\psi_+) = 2g(\overline{\chi}, \nabla^+\chi) = 2g(\chi, \nabla^+\overline{\chi}), \tag{5.3.9}$$

and similarly for ψ_-. In other words, we can choose which of the fields in the decomposition are 'acted upon by ∇'. The next step is then to pick one of $\chi, \overline{\chi}$ and one of $\lambda, \overline{\lambda}$ to become scalars on Σ, which in the action we choose to be the fields on which the covariant derivatives acts. The role of the other fields is then completely determined, as the products $\chi\overline{\chi}$ and $\lambda\overline{\lambda}$ must still lie in K and $\overline{K}$ respectively, for the action to remain well–defined. These choices give 4 different twists, which is related to the natural $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry on the generalized Kähler structure. Let us for definiteness make the following choice:

$$\begin{aligned} \chi &\in \Gamma(T_+^{(0,1)}M), \qquad \overline{\chi} \in \Gamma(K \otimes T_+^{(1,0)}M), \\ \lambda &\in \Gamma(T_-^{(0,1)}M), \qquad \overline{\lambda} \in \Gamma(\overline{K} \otimes T_-^{(1,0)}M). \end{aligned} \tag{5.3.10}$$

The model this choice leads to is often referred to as the *B–model*, giving yet another meaning to the letter B. The other three choices for the scalar fields are[6]

- $\chi, \overline{\lambda}$, corresponding to $I_\pm \mapsto \pm I_\pm$, i.e.: $\mathcal{J}_1 \leftrightarrow \mathcal{J}_2$,
- $\overline{\chi}, \lambda$, corresponding to $I_\pm \mapsto \mp I_\pm$, i.e.: $\mathcal{J}_1 \mapsto -\mathcal{J}_2$, $\mathcal{J}_2 \mapsto -\mathcal{J}_1$,
- $\overline{\chi}, \overline{\lambda}$, corresponding to $I_\pm \mapsto -I_\pm$, i.e.: $\mathcal{J}_1 \mapsto -\mathcal{J}_1$, $\mathcal{J}_2 \mapsto -\mathcal{J}_2$.

Note that $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ is invariant under these transformations of $\mathcal{J}_1$ and $\mathcal{J}_2$. For our choice the action (5.3.4) becomes

$$S = 2\int dzd\overline{z}\left\{\frac{1}{2}(g_{ij} + b_{ij})\partial_z\varphi^i\partial_{\overline{z}}\varphi^j + ig_{ij}\overline{\chi}^i\nabla_{\overline{z}}^+\chi^j + ig_{ij}\overline{\lambda}^i\nabla_z^-\lambda^j + R^+_{ijkl}\chi^i\overline{\chi}^j\lambda^k\overline{\lambda}^l\right\}, \tag{5.3.11}$$

where we used (5.3.9) and the two identities

$$g(R^\pm(X,Y)V,W) = -g(R^\pm(X,Y)W,V), \qquad g(R^\pm(X,Y)V,W) = g(R^\mp(V,W)X,Y), \tag{5.3.12}$$

which follow immediately from the facts that $\nabla^\pm$ are metric and have torsion $\pm g^{-1}H$.

[6] Recall from Section 3.4 the relation between $\mathcal{J}_1$, $\mathcal{J}_2$ and $I_\pm$.

Next we have to rewrite the symmetries in terms of the new fields, and we first work this out before performing the twist. Define the following complex combinations

$$\begin{aligned}\alpha^+ &:= \epsilon^+ - i\tilde{\epsilon}^+ \in \Gamma(K^{-1/2}), & \overline{\alpha}^+ &:= \epsilon^+ + i\tilde{\epsilon}^+ \in \Gamma(K^{-1/2}),\\ \alpha^- &:= -i\epsilon^- - \tilde{\epsilon}^- \in \Gamma(\overline{K}^{-1/2}), & \overline{\alpha}^- &:= i\epsilon^- - \tilde{\epsilon}^- \in \Gamma(\overline{K}^{-1/2}).\end{aligned} \tag{5.3.13}$$

In terms of these parameters, the symmetry (4.2.13) becomes

$$\begin{aligned}\delta\varphi^i =&\alpha^+(P_+\psi_+)^i + \overline{\alpha}^+(\overline{P}_+\psi_+)^i + \alpha^-(iP_-\psi_-)^i + \overline{\alpha}^-(-i\overline{P}_-\psi_-)^i,\\ \delta\psi_+^i =&i\alpha^+\left((\overline{P}_+\partial_z\varphi)^i - \frac{1}{2}I^i_{+j,k}\psi_+^k\psi_+^j\right) + i\overline{\alpha}^+\left((P_+\partial_z\varphi)^i + \frac{1}{2}I^i_{+j,k}\psi_+^k\psi_+^j\right)\\ &+ i\alpha^-\left(-P^i_{-j}\Gamma^{+j}_{kl}\psi_-^k\psi_+^l - \frac{i}{2}I^i_{-j,k}\psi_+^k\psi_-^j\right) + i\overline{\alpha}^-\left(\overline{P}^i_{-j}\Gamma^{+j}_{kl}\psi_-^k\psi_+^l - \frac{i}{2}I^i_{-j,k}\psi_+^k\psi_-^j\right),\\ \delta\psi_-^i =&\alpha^+\left(P^i_{+j}\Gamma^{+j}_{kl}\psi_-^k\psi_+^l - \frac{i}{2}I^i_{+j,k}\psi_-^k\psi_+^j\right) + \overline{\alpha}^+\left(\overline{P}^i_{+j}\Gamma^{+j}_{kl}\psi_-^k\psi_+^l + \frac{i}{2}I^i_{+j,k}\psi_-^k\psi_+^j\right)\\ &+ \alpha^-\left(-(\overline{P}_-\partial_{\overline{z}}\varphi)^i + \frac{1}{2}I^i_{-j,k}\psi_-^k\psi_-^j\right) + \overline{\alpha}^-\left((P_-\partial_{\overline{z}}\varphi)^i + \frac{1}{2}I^i_{-j,k}\psi_-^k\psi_-^j\right).\end{aligned} \tag{5.3.14}$$

The variations of χ, $\overline{\chi}$, λ and $\overline{\lambda}$ will involve derivatives of the projection operators, but before going into that we first observe that

$$\delta\varphi = \alpha^+\chi^i + \overline{\alpha}^+\overline{\chi}^i + \alpha^-\lambda^i + \overline{\alpha}^-\overline{\lambda}^i, \tag{5.3.15}$$

which is only well defined after the twist if α^+ and α^- are functions, while $\overline{\alpha}^+ \in \Gamma(K^{-1})$ and $\overline{\alpha}^- \in \Gamma(\overline{K}^{-1})$. Requiring them to be (anti–)holomorphic implies for the first two that they are constant, while the last two must be zero for genus bigger than 1. We can thus put the first two equal to a (not necessarily equal) constant, and the last two equal to zero. This solves the issue of the covariant constant parameters, and simplifies computations considerably. The symmetries of the other fields can now straightforwardly be computed.

$$\begin{aligned}\delta\chi^i =&P^i_{+j}\delta\psi_+^j + P^i_{+j,k}\delta\varphi^k\psi_+^j\\ =&\alpha^+\left(\frac{i}{2}I^i_{+j,k}\chi^k\psi_+^j - \frac{i}{2}P^i_{+j}I^j_{+k,l}\psi_+^l\psi_+^k\right)\\ &+\alpha^-\left(\frac{i}{2}I^i_{+j,k}\lambda^k\psi_+^j - iP^i_{+j}P^j_{-k}\Gamma^{+k}_{lm}\psi_-^l\psi_+^m + \frac{1}{2}P^i_{+j}I^j_{-k,l}\psi_+^l\psi_-^k\right).\end{aligned} \tag{5.3.16}$$

Using the fact that the Nijenhuis tensor of I_+ is zero (Equation (4.1.13)), we can rewrite

$$I^i_{+j}I^j_{+k,l}\psi_+^l\psi_+^k = \frac{1}{2}(I^i_{+j}I^j_{+k,l} - I^i_{+j}I^j_{+l,k})\psi_+^l\psi_+^k = \frac{1}{2}(I^j_{+l}I^i_{+k,j} - I^j_{+k}I^i_{+l,j})\psi_+^l\psi_+^k = I^i_{+k,j}I^j_{+l}\psi_+^l\psi_+^k,$$

hence

$$P^i_{+j}I^j_{+k,l}\psi_+^l\psi_+^k = I^i_{+k,j}P^j_{+l}\psi_+^l\psi_+^k = I^i_{+k,j}\chi^j\psi_+^k,$$

which implies that the term proportional to α^+ in (5.3.16) vanishes. Next, we can rewrite derivatives on $I_\pm$ using (4.1.24) and substitute $I_\pm = -i(P_\pm - \overline{P}_\pm)$, so that (5.3.16) becomes

$$\begin{aligned}\delta\chi^i =&\alpha^-\left(\frac{1}{2}\left(\Gamma^{+r}_{kj}(P_+ - \overline{P}_+)^i_{+r} - \Gamma^{+i}_{kr}(P_+ - \overline{P}_+)^r{}_j\right)\lambda^k\psi^j_+ - iP^i_{+j}P^j_{-k}\Gamma^{+k}_{lm}\psi^l_-\psi^m_+\right.\\ &\left.-\frac{i}{2}P^i_{+j}\left(\Gamma^{-r}_{lk}(P_- - \overline{P}_-)^j_{-r} - \Gamma^{-j}_{lr}(P_- - \overline{P}_-)^r_{-k}\right)\psi^l_+\psi^k_-\right)\\ =&-\alpha^-\Gamma^{+i}_{jk}\lambda^j\chi^k.\end{aligned} \tag{5.3.17}$$

This last step can readily be verified by the reader, using only the definition of $P_\pm$ and the decompositions $\psi_+ = \chi + \overline{\chi}$ and $\psi_- = -i\lambda + i\overline{\lambda}$. Exactly the same computation can be done for $\delta\lambda$, and the result is (by symmetry we actually know this without doing any computation)

$$\delta\lambda^i = -\alpha^+\Gamma^{-i}_{jk}\chi^j\lambda^k. \tag{5.3.18}$$

The variations of $\overline{\chi}$ and $\overline{\lambda}$ are a bit more involved:

$$\begin{aligned}\delta\overline{\chi}^i =&\overline{P}^i_{+j}\delta\psi^j_+ + \overline{P}^i_{+j,k}\delta\varphi^k\psi^j_+\\ =&\alpha^+\left(i\overline{P}^i_{+j}\partial_z\varphi^j - \frac{i}{2}I^i_{+k,l}\psi^l_+\psi^k_+\right)\\ &+\alpha^-\left(-i\overline{P}^i_{+j}P^j_{-k}\Gamma^{+k}_{lm}\psi^l_-\psi^m_+ + \frac{1}{2}\overline{P}^i_{+j}I^j_{-k,l}\psi^l_+\psi^k_- - \frac{i}{2}I^i_{+j,k}\lambda^k\psi^j_+\right).\end{aligned} \tag{5.3.19}$$

Again, we can remove the derivatives on $I_\pm$ using (4.1.24), and a few simple algebraic manipulations yield

$$\delta\overline{\chi}^i =\alpha^+\left(i\overline{P}^i_{+j}\partial_z\varphi^j - \frac{1}{4}(P_+ - \overline{P}_+)^i{}_jH^j_{lk}\psi^l_+\psi^k_+ + \frac{1}{2}\Gamma^{+i}_{jk}\psi^j_+(\chi^k - \overline{\chi}^k)\right) - \alpha^-\left(\Gamma^{+i}_{jk}\lambda^j\overline{\chi}^k\right). \tag{5.3.20}$$

Since H is of type $(2,1)+(1,2)$ with respect to both complex structures, we have

$$H_{ijk}P^i_{\pm a}P^j_{\pm b}P^k_{\pm c} = 0 \quad \text{and complex conjugate.}$$

With the aid of $g_{ij}I^i_{\pm k} = -g_{ik}I^i_{\pm j}$, this translates into

$$\overline{P}^i_{\pm a}H^a_{jk}P^j_{\pm b}P^k_{\pm c} = 0 \quad \text{and complex conjugate.}$$

Using these equalities we conclude that

$$\delta\overline{\chi}^i =\alpha^+\left(i\overline{P}^i_{+j}\partial_z\varphi^j + \Gamma^{+i}_{jk}\overline{\chi}^j\chi^k - P^i_{+j}H^j_{kl}\chi^k\overline{\chi}^l\right) - \alpha^-\left(\Gamma^{+i}_{jk}\lambda^j\overline{\chi}^k\right). \tag{5.3.21}$$

By symmetry arguments the variation of $\overline{\lambda}$ can be deduced from (5.3.21), keeping in mind that ∂_z needs to be replaced by $\partial_{\bar{z}}$ and H by $-H$:

$$\delta\overline{\lambda}^i =\alpha^-\left(i\overline{P}^i_{-j}\partial_{\bar{z}}\varphi^j + \Gamma^{-i}_{jk}\overline{\lambda}^j\lambda^k + P^i_{-j}H^j_{kl}\lambda^k\overline{\lambda}^l\right) - \alpha^+\left(\Gamma^{-i}_{jk}\chi^j\overline{\lambda}^k\right). \tag{5.3.22}$$

We can identify two global symmetry charges Q_L and Q_R, defined by $\delta = \alpha^+ Q_L + \alpha^- Q_R$. In terms of the old charges they read $Q_L = \frac{1}{2}(Q^1_+ + iQ^2_+)$ and $Q_R = \frac{i}{2}(Q^1_- + iQ^2_-)$, and it follows from the $(2,2)$–algebra that $Q_L^2 = Q_R^2 = \{Q_L, Q_R\} = 0$. Their action on the fields is summarized below:

$$\begin{aligned}
&\{Q_L, \varphi^i\} = \chi^i && \{Q_R, \varphi^i\} = \lambda^i \\
&\{Q_L, \chi^i\} = 0 && \{Q_R, \chi^i\} = -\Gamma^{+i}_{jk}\lambda^j\chi^k \\
&\{Q_L, \lambda^i\} = -\Gamma^{-i}_{jk}\chi^j\lambda^k && \{Q_R, \lambda^i\} = 0 \\
&\{Q_L, \overline{\chi}^i\} = i\overline{P}^i_{+j}\partial_z\varphi^j + \Gamma^{+i}_{jk}\overline{\chi}^j\chi^k - P^i_{+j}H^j_{kl}\chi^k\overline{\chi}^l && \{Q_R, \overline{\chi}^i\} = -\Gamma^{+i}_{jk}\lambda^j\overline{\chi}^k \\
&\{Q_L, \overline{\lambda}^i\} = -\Gamma^{-i}_{jk}\chi^j\overline{\lambda}^k && \{Q_R, \overline{\lambda}^i\} = i\overline{P}^i_{-j}\partial_{\overline{z}}\varphi^j + \Gamma^{-i}_{jk}\overline{\lambda}^j\lambda^k + P^i_{-j}H^j_{kl}\lambda^k\overline{\lambda}^l
\end{aligned} \tag{5.3.23}$$

We define

$$Q := Q_L + Q_R, \tag{5.3.24}$$

and this will be the BRST operator for the model. To show that the theory is topological, i.e. that Q satisfies the axioms for a cohomological field theory, we need to find a Q–invariant vacuum and an operator V, such that

$$S = \{Q, V\},$$

modulo some topological terms. A choice for V that works in the case when there is no B–field (see [36]) is given by

$$V := -i\int_\Sigma d^2z g_{ij}\left(\overline{\chi}^i\partial_{\overline{z}}\varphi^j + \overline{\lambda}^i\partial_z\varphi^j\right). \tag{5.3.25}$$

However, in the more general models with B–field this expression does not work, and the solution to this problem is not yet solved. There are two special cases in which the answer is known:

- The first generalized complex structure $\mathcal{J}_1$ is of symplectic type, i.e. its pure spinor is of the form $\rho = e^\omega$ for ω a complex 2–form. This case is worked out in [37].
- I_+ and I_- commute. These include for example the classical A and B models, in which $I_+ = I_-$ and $H = 0$. This case is worked out in [25].

We will say more about the Q–invariant vacuum in Section 5.5.

5.4 Anomalies

To obtain a $(2,2)$–supersymmetric sigma model on an arbitrary Riemann surface we needed to twist the original model. Mathematically, all this amounts to is to change some of the input of the theory, i.e. change some of the vector bundles that are involved. Physically, the situation is more subtle, as the procedure of twisting is not always well defined at the quantum level. This has to do with a specific type of symmetry that is used to perform the twist, called R–symmetry, and there are some constraints on the model in order for R–symmetry to survive the quantization process.

If a classical action is invariant under some symmetry, it is not guaranteed that after the quantization the symmetry is still present. Whenever this happens we say that the symmetry has an *anomaly*. Usually the way this happens is that although the action itself is invariant, the measure used to define the path integral is not invariant. If a symmetry is anomalous, there are essentially two ways to proceed. Either the symmetry is not important and could be ignored from the beginning, or there are some constraints that one imposes on the model, in order for the symmetry to remain present. A well–known example happens in string theory, where the conformal symmetry has an anomaly that constrains the dimension of the target space to be 26–dimensional for the bosonic string, and 10–dimensional for the superstring. There are many more examples of anomalies in physics, and for a nice treatment of the subject we refer to [2].

Back to the sigma model. Besides Lorentz symmetry, which on the fermionic fields is given by

$$L(\alpha): \ \chi^i \mapsto e^{-i\alpha/2}\chi^i, \ \overline{\chi}^i \mapsto e^{-i\alpha/2}\overline{\chi}^i, \ \lambda^i \mapsto e^{+i\alpha/2}\lambda^i, \ \overline{\lambda}^i \mapsto e^{+i\alpha/2}\overline{\lambda}^i, \tag{5.4.1}$$

we have two more independent $U(1)$–symmetries, commonly known as R–symmetries:

$$\begin{aligned} R_V(\alpha): \ & \chi^i \mapsto e^{+i\alpha/2}\chi^i, \ \overline{\chi}^i \mapsto e^{-i\alpha/2}\overline{\chi}^i, \ \lambda^i \mapsto e^{+i\alpha/2}\lambda^i, \ \overline{\lambda}^i \mapsto e^{-i\alpha/2}\overline{\lambda}^i, \\ R_A(\alpha): \ & \chi^i \mapsto e^{+i\alpha/2}\chi^i, \ \overline{\chi}^i \mapsto e^{-i\alpha/2}\overline{\chi}^i, \ \lambda^i \mapsto e^{-i\alpha/2}\lambda^i, \ \overline{\lambda}^i \mapsto e^{+i\alpha/2}\overline{\lambda}^i, \end{aligned} \tag{5.4.2}$$

where the subscripts stand for *Vector* and *Axial*. Note that this symmetry does nothing with the bosonic fields, and on superspace they correspond to rotations in the odd coordinates, leaving the even coordinates fixed. Comparing (5.3.10) with (5.4.1), it follows that in the twisted model the Lorentz symmetry is given by

$$L^{tw}(\alpha) = L(\alpha) \circ R_A(\alpha) = R_A(\alpha) \circ L(\alpha). \tag{5.4.3}$$

In other words, to obtain the twisted model we have to use the axial R–symmetry, so to check whether our models have quantum anomalies, we have to inspect if (5.4.1) and (5.4.3) are also symmetries of the path–integral measure.

As the bosonic fields φ^i are Lorentz scalars both in the twisted as untwisted theory, we will only focus on the fermionic fields. Recalling our discussion in Section 2.3 about fermionic sections of vector bundles, we write

$$\begin{aligned} \chi &= \sum_{a\in I} \chi^a s_a, & \overline{\chi} &= \sum_{a\in I} \overline{\chi}^a \overline{s}_a, \\ \lambda &= \sum_{a\in I} \lambda^a t_a, & \overline{\lambda} &= \sum_{a\in I} \overline{\lambda}^a \overline{t}_a, \end{aligned} \tag{5.4.4}$$

where $\{s_a\}_{a\in I}$ and $\{t_a\}_{a\in I}$ form a basis for the space of sections of $\overline{K}^{\frac{1}{2}} \otimes \varphi^*(T_+^{(0,1)}M)$ and $K^{\frac{1}{2}} \otimes \varphi^*(T_-^{(0,1)}M)$ respectively, and similarly for the complex conjugates. Note that χ^a, $\overline{\chi}^a$, λ^a and $\overline{\lambda}^a$ are Grassmann numbers (cf. (2.3.15)). The measure $D\chi D\overline{\chi} D\lambda D\overline{\lambda}$ in the path–integral can be written as

$$D\chi D\overline{\chi} D\lambda D\overline{\lambda} = \prod_{a\in I} d\chi^a d\overline{\chi}^a d\lambda^a d\overline{\lambda}^a, \tag{5.4.5}$$

which is interpreted as an infinite limit. The measure (5.4.5) itself is perfectly invariant under the transformations (5.4.1) and (5.4.2), but this is not the end of the story. One of the peculiar things about Grassmann integration is the identity

$$\int d\theta(\text{"independent of } \theta\text{"}) = 0.$$

It implies for instance, that if the integrand in the path integral misses some χ^a, λ^a, $\overline{\chi}^a$ or $\overline{\lambda}^a$, then the whole integral vanishes. If we insert (5.4.4) in the fermionic part of the action (Equation (5.3.11)), and ignore the curvature term[7] we get

$$\sum_{a,a' \in I} \left(\overline{\chi}^{a'} \chi^a g(\overline{s}_{a'}, \nabla^+_{\overline{z}} s_a) + \overline{\lambda}^{a'} \lambda^a g(\overline{t}_{a'}, \nabla^-_z t_a) \right). \tag{5.4.6}$$

Considering the discussion above, we see that if $\nabla^{\pm}$ have a nonzero kernel, the path integral without any operator insertions vanishes. Elements in the kernel of $\nabla^{\pm}$ are called *zero modes*, and if we define

$$\begin{aligned}
l_+ :=& \dim \left(\text{Ker}[\nabla^+_{\overline{z}} : \Gamma(K^{1/2} \otimes T^{(0,1)}_+ M) \to \Gamma(K^{1/2} \otimes T^{(0,1)}_+ M)] \right), \\
\overline{l}_+ :=& \dim \left(\text{Ker}[\nabla^+_{\overline{z}} : \Gamma(K^{1/2} \otimes T^{(1,0)}_+ M) \to \Gamma(K^{1/2} \otimes T^{(0,1)}_+ M)] \right), \\
l_- :=& \dim \left(\text{Ker}[\nabla^-_{z} : \Gamma(\overline{K}^{1/2} \otimes T^{(0,1)}_- M) \to \Gamma(\overline{K}^{1/2} \otimes T^{(0,1)}_- M)] \right), \\
\overline{l}_- :=& \dim \left(\text{Ker}[\nabla^-_{z} : \Gamma(\overline{K}^{1/2} \otimes T^{(1,0)}_- M) \to \Gamma(\overline{K}^{1/2} \otimes T^{(0,1)}_- M)] \right),
\end{aligned} \tag{5.4.7}$$

then the only operators with possibly nonzero correlators are the ones whose fermionic part is of the form

$$\langle \chi^{l_+} \overline{\chi}^{\overline{l}_+} \lambda^{l_-} \overline{\lambda}^{\overline{l}_-} (\ldots) \rangle. \tag{5.4.8}$$

Here the part between the brackets can be anything, but the important part is the specific combination of fields in front of the brackets. Writing each field in the decomposition (5.4.4), this factor produces precisely the correct amount of zero modes, in order for the correlator to have a chance of being nonzero. Note that the superscripts refer to actual powers, and not to space–time indices on M.

Now, looking at (5.4.2), these operators are invariant under the axial R–symmetry only if

$$l_+ - \overline{l}_+ - l_- + \overline{l}_- = 0. \tag{5.4.9}$$

So in order for the quantum anomaly to vanish, we have to impose this constraint on the sigma model. At first sight these numbers seem impossible to calculate in practice, but fortunately the combinations $l_+ - \overline{l}_+$ and $l_- - \overline{l}_-$ turn out to depend only on some topological data on M. In order to explain this, we need a result known as the *Hirzebruch–Riemann–Roch Theorem*.

[7] The limit in which this is allowed, is when the string size is very small compared to the size of the target space, in the sense that the curvature term acts as a very small perturbation. However, the author does admit that mathematically this statement is not very convincing, and a more rigorous argumentation is needed.

Theorem 5.4.1. *[22] Let E be a holomorphic vector bundle on a compact complex manifold X of complex dimension n. Then we have the following equality:*

$$\sum_{i=0}^{2n}(-1)^i dim\ H^i(X,E) = \int_X ch(E)Td(X), \tag{5.4.10}$$

where the integral is extended to arbitrary forms by defining it to be zero on forms not of highest degree. Here $H^i(X,E)$ denote the cohomology groups associated to the sheaf of holomorphic sections of E, $ch(E)$ denotes the Chern character of E and $Td(X)$ denotes the Todd polynomial in the Chern classes of the holomorphic tangent bundle of X.

We will not give the explicit definitions of the Chern character and the Todd polynomial, as for us it will suffice to know that they are of the form

$$\begin{aligned} ch(E) =& \text{rank}(E) + c_1(E), \\ Td(X) =& 1 + \frac{c_1(TX)}{2}, \end{aligned} \tag{5.4.11}$$

where we ignored all cohomology classes of degree greater than 2, because in our case X will be two–dimensional.

We would like to apply this theorem to $X = \Sigma$, and E being one of the bundles belonging to the fermionic fields. There is one problem however; since the map φ can be arbitrary, it does not need to be holomorphic, so the bundles $\varphi^*(T_\pm M)$ are generically not holomorphic over Σ, preventing us to apply the theorem. To avoid this problem, we will simply give the bundles a holomorphic structure such that their degree zero cohomology groups are precisely the kernels we are interested in. To be more precise, let E be a complex vector bundle over Σ, not necessarily holomorphic, and suppose we have a connection $\nabla : \Gamma(E) \to \Omega^1(E)$. By projecting onto the $(0,1)$–forms, it gives a map

$$\nabla^{(0,1)} : \Gamma(E) \to \Omega^{(0,1)}(E). \tag{5.4.12}$$

Suppose that we can find, in a neighborhood around every point, a frame $s_1, \ldots, s_k$ where k is the complex rank of E, with the property that

$$\nabla^{(0,1)} s_i = 0. \tag{5.4.13}$$

In local coordinates $(z, \overline{z})$ on Σ, this means $\nabla_{\overline{z}} s_i = 0$. If s_i' is another such frame in that same neighborhood, it is related to s_i by means of $s_i' = g_{ij}s_j$ for some transition function g. Applying $\nabla^{(0,1)}$ to both sides we get

$$0 = \nabla^{(0,1)} s_i' = \nabla^{(0,1)}(g_{ij}s_j) = (\overline{\partial} g_{ij})s_j + g_{ij}\nabla^{(0,1)} s_j = (\overline{\partial} g_{ij})s_j, \tag{5.4.14}$$

which implies that g_{ij} is holomorphic. We can thus cover Σ with frames whose transition functions are holomorphic, hence E is holomorphic. A section of E can locally be written as $f_i s_i$, and is holomorphic precisely when $\overline{\partial} f_i = 0$, which can be written as $\nabla^{(0,1)}(f_i s_i) = 0$. So $H^0(\Sigma, E)$, which

can be identified with the space of holomorphic sections of E, is nothing else as the kernel of $\nabla^{(0,1)}$. The question thus remains, can we find local frames satisfying (5.4.13)? This is a local question, so we choose coordinates $(z, \overline{z})$ on some open set, and take an arbitrary local frame e_i. Let A_{ij} be the matrix of 1–forms on Σ such that $\nabla e_i = A_{ij} e_j$, and let $A^{(0,1)}$ be its $(0,1)$–part. Writing our desired frame $s_i = g_{ij} e_j$, the condition $\nabla^{(0,1)} s_i = 0$ translates into

$$(\frac{\partial g_{ij}}{\partial \overline{z}} + g_{ik} A^{\overline{z}}_{kj}) = 0, \tag{5.4.15}$$

where we write $A^{(0,1)}_{kj} = A^{\overline{z}}_{kj} d\overline{z}$. Since we can always find local solutions to (5.4.15), we can indeed find the s_i we need.

Applying this procedure to our bundles of interest, we obtain[8]

$$\begin{aligned} l_+ =& \mathrm{dim} H^0(\Sigma, K^{1/2} \otimes T^{(0,1)}_+ M), \qquad \bar{l}_+ = \mathrm{dim} H^0(\Sigma, K^{1/2} \otimes T^{(1,0)}_+ M), \\ l_- =& \mathrm{dim} H^0(\Sigma, \overline{K}^{1/2} \otimes T^{(0,1)}_- M), \qquad \bar{l}_- = \mathrm{dim} H^0(\Sigma, \overline{K}^{1/2} \otimes T^{(1,0)}_- M). \end{aligned} \tag{5.4.16}$$

There is one more result that we need to conclude our computation, which is the concept of *Serre duality*.

Theorem 5.4.2. *[35] Let E be a holomorphic vector bundle over a compact manifold X of complex dimension n. Then there is an isomorphism*

$$H^p(X, \wedge^{(q,0)} T^* \otimes E) \cong (H^{n-p}(X, \wedge^{(n-q,0)} T^* \otimes E^*))^*. \tag{5.4.17}$$

For $X = \Sigma$, which has complex dimension 1, this implies that $\mathrm{dim} H^0(\Sigma, E) = \mathrm{dim} H^1(\Sigma, K \otimes E^*)$ (recall that by definition $K = \wedge^{(1,0)} T^* \Sigma$). In particular, we obtain

$$\begin{aligned} \mathrm{dim} H^0(\Sigma, K^{1/2} \otimes T^{(1,0)}_+ M) =& \mathrm{dim} H^1(\Sigma, K \otimes (K^{1/2})^* \otimes (T^{(1,0)}_+ M)^*) \\ =& \mathrm{dim} H^1(\Sigma, K^{1/2} \otimes T^{(0,1)}_+ M), \end{aligned} \tag{5.4.18}$$

where in the last equality we identified $(T^{(1,0)}_+ M)^*$ with $T^{(0,1)}_+ M$ using the metric g on M. So finally, we come to the conclusion that

$$\begin{aligned} l_+ - \bar{l}_+ =& \mathrm{dim} H^0(\Sigma, K^{1/2} \otimes T^{(0,1)}_+ M) - \mathrm{dim} H^1(\Sigma, K^{1/2} \otimes T^{(0,1)}_+ M) \\ =& \int_\Sigma ch(K^{1/2} \otimes T^{(0,1)}_+ M) Td(\Sigma), \end{aligned} \tag{5.4.19}$$

where in the last equality we used Theorem 5.4.15. Using (5.4.11) we get

$$\begin{aligned} ch(K^{1/2}) &= 1 + c_1(K^{1/2}) = 1 + \frac{1}{2} c_1(K), \\ ch(\varphi^*(T^{(0,1)}_+ M)) &= n + \varphi^* c_1(T^{(0,1)}_+ M), \\ Td(\Sigma) &= 1 + \frac{1}{2} c_1(T\Sigma), \end{aligned} \tag{5.4.20}$$

[8] Note that the above considerations can equally well be applied for ∂ instead of $\overline{\partial}$, by symmetry between z and $\overline{z}$.

so we obtain

$$\begin{aligned} l_+ - \bar{l}_+ &= \int_\Sigma \left((1 + \frac{1}{2}c_1(K))(n + \varphi^* c_1(T_+^{(0,1)}M))(1 + \frac{1}{2}c_1(T\Sigma)) \right) \\ &= \int_{\varphi(\Sigma)} c_1(T_+^{(0,1)}M) + n(1-g) + n(g-1) \\ &= \int_{\varphi(\Sigma)} c_1(T_+^{(0,1)}M). \end{aligned} \tag{5.4.21}$$

A similar computation can be done for l_-, except that we have to interchange the roles of z and $\overline{z}$. The computations are then exactly the same, but in the end result the integral over Σ gets an extra minus sign because the orientation has been reversed. Therefore, the result is

$$l_- - \bar{l}_- = -\int_{\phi(\Sigma)} c_1(T_+^{(0,1)}M). \tag{5.4.22}$$

Coming back to our anomaly condition (5.4.9), the constraint takes the form

$$0 = \int_{\varphi(\Sigma)} (c_1(T_+^{(0,1)}M) + c_1(T_-^{(0,1)}M)) = \int_{\varphi(\Sigma)} c_1(L_1), \tag{5.4.23}$$

where L_1 is the $+i$ eigenbundle for $\mathcal{J}_1$. In this last equality we used the isomorphisms $L_1^+ \cong T_+^{(0,1)}M$ and $L_1^- \cong T_-^{(0,1)}M$, which follows from the correspondence between bi–hermitian structures and generalized Kähler structures discussed in Section 3.4. Since (5.4.23) must hold for all maps $\varphi : \Sigma \to M$, we deduce that the constraint on $(M, \mathcal{J}_1, \mathcal{J}_2)$ in order for the twist to be well–defined at the quantum level is given by

$$c_1(L_1) = 0. \tag{5.4.24}$$

5.5 Local observables and Q–cohomology

The local operators, also called the local observables of the theory, are constructed out of the fields φ^i, χ^i and λ^i, since $\overline{\chi}^i$ and $\overline{\lambda}^i$ are sections of K and $\overline{K}$ which depend on the metric. We assume these observables to be analytic in the fermionic fields, which implies that such an operator is given by

$$f = f_{a_1 \dots a_p b_1 \dots b_q}(\varphi) \chi^{a_1} \cdots \chi^{a_p} \lambda^{b_1} \cdots \lambda^{b_q}, \tag{5.5.1}$$

where $f_{a_1 \dots a_p b_1 \dots b_q}$ is completely anti–symmetric in the indices a_i and b_i. The action of Q_L and Q_R on f is given by

$$\begin{aligned} Q_L(f) = \big(& f_{a_1 \dots a_p b_1 \dots b_q, c} - \Gamma^{-d}_{cb_1} f_{a_1 \dots a_p d b_2 \dots b_q} - \Gamma^{-d}_{cb_2} f_{a_1 \dots a_p b_1 d b_3 \dots b_q} - \cdots \\ & \cdots - \Gamma^{-d}_{cb_q} f_{a_1 \dots a_p b_1 \dots b_{q-1} d} \big) \chi^c \chi^{a_1} \cdots \chi^{a_p} \lambda^{b_1} \cdots \lambda^{b_q}, \end{aligned} \tag{5.5.2}$$

$$\begin{aligned} Q_R(f) = \big(& f_{a_1 \dots a_p b_1 \dots b_q, c} - \Gamma^{+d}_{ca_1} f_{d a_2 \dots a_p b_1 \dots b_q} - \Gamma^{+d}_{ca_2} f_{a_1 d a_3 \dots a_p b_1 \dots b_q} - \cdots \\ & \cdots - \Gamma^{+d}_{ca_p} f_{a_1 \dots a_{p-1} d b_1 \dots b_q} \big) \lambda^c \chi^{a_1} \cdots \chi^{a_p} \lambda^{b_1} \cdots \lambda^{b_q}. \end{aligned} \tag{5.5.3}$$

These expressions very much resemble a covariant derivative acting on some kind of tensors on M. Indeed, we can make an identification

$$\text{Local operators} \longleftrightarrow \Omega_+^{(0,p)}(M) \otimes \Omega_-^{(0,q)}(M), \tag{5.5.4}$$

by mapping χ^i to the 1–form dual to $\frac{1}{2}(1 + iI_+)^i{}_j\partial_j$, and λ^i to the 1–form dual to $\frac{1}{2}(1 + iI_-)^i{}_j\partial_j$. One should view the χ^i as the 1–forms $d\overline{z}^i$ in complex coordinates for I_+, and similarly for λ^i. To see what the forms of the operators Q_L and Q_R are under this isomorphism we have to recall some basic facts about covariant derivatives.

If two bundles E and E' are equipped with connections ∇ and ∇', the tensor product $E \otimes E'$ induces a product connection $\nabla \otimes 1 + 1 \otimes \nabla'$. Also, if E^* is the dual of E, it carries a connection ∇^* defined by the formula

$$X(\alpha(v)) = \nabla^*_X\alpha(v) + \alpha(\nabla_X(v)) \qquad \text{for } X \in TM,\ v \in \Gamma(E),\ \alpha \in \Gamma(E^*). \tag{5.5.5}$$

These two constructions applied to the connections $\nabla^\pm$ on TM yield two connections on $\bigotimes_{i=1}^n T^*M$, and after taking the usual quotient, they induce connections on $\wedge^n T^*M$. Now this last space has a (p, q) decomposition with respect to both $I_\pm$, which we claim is preserved under $\nabla^\pm$, i.e. $\nabla^\pm_Y$ preserves the space $\Omega_\pm^{(p,q)}$ for all vectors $Y \in TM$. To see this, let $\alpha \in \Omega_\pm^{(p,q)}$ and pick $Y, X_1, \ldots, X_n \in TM$, $n = p + q$, where the X_i are either holomorphic or anti–holomorphic. We compute

$$\nabla^\pm_Y\alpha(X_1, \ldots, X_n) = Y(\alpha(X_1, \ldots, X_n)) - \alpha(\nabla^\pm_Y X_1, X_2, \ldots, X_n) - \ldots - \alpha(X_1, \ldots, \nabla^\pm_Y X_n). \tag{5.5.6}$$

Since $\nabla^\pm$ commutes with $I_\pm$, it preserves the $\pm i$ eigenspaces. Therefore, the right hand side is possibly nonzero only if p of the X_i are holomorphic, and q of them are anti–holomorphic.

Now recall that if ∇ is a connection on a vector bundle E, it has a natural extension to the space of p–forms with values in E, i.e. the space $\Omega^p(E) = \Gamma(\wedge^p T^*M \otimes E)$, by imposing the Leibniz rule

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^{\deg(\alpha)}\alpha \wedge \nabla s. \tag{5.5.7}$$

More explicitly, the map $\nabla : \Omega^p(E) \to \Omega^{p+1}(E)$ is given by the formula

$$\begin{aligned}\nabla\alpha(X_1, \ldots, X_{p+1}) = &\sum_{i=1}^{p+1}(-1)^{i+1}\nabla_{X_i}\alpha(X_1, \ldots, \widehat{X_i}, \ldots, X_{p+1}) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{p+1}).\end{aligned} \tag{5.5.8}$$

Applying this to the bundle $\wedge^{(0,q)}T^*_-M$, which by the arguments preceding Equation (5.5.6), has the connection ∇^- on it, we obtain the map

$$\nabla^- : \Omega^p \otimes \Omega_-^{(0,q)} \to \Omega^{p+1} \otimes \Omega_-^{(0,q)}. \tag{5.5.9}$$

Using (5.5.8) and involutivity of $T_+^{(0,1)}M$ it is readily verified that this map satisfies

$$\nabla^- : \Omega_+^{(0,p)} \otimes \Omega_-^{(0,q)} \longrightarrow \left(\Omega_+^{(1,p)} \otimes \Omega_-^{(0,q)}\right) \oplus \left(\Omega_+^{(0,p+1)} \otimes \Omega_-^{(0,q)}\right). \tag{5.5.10}$$

We define $\overline{D}_+$ by the composition of ∇^- and the projection onto $\Omega_+^{(0,p+1)} \otimes \Omega_-^{(0,q)}$. Similarly, we obtain a map

$$\overline{D}_- : \Omega_+^{(0,p)} \otimes \Omega_-^{(0,q)} \to \Omega_+^{(0,p)} \otimes \Omega_-^{(0,q+1)}. \tag{5.5.11}$$

Under the identification (5.5.4) we claim that Q_L and Q_R are identified with $\overline{D}_+$ and $\overline{D}_-$ respectively. Let us check this for Q_L. Pick $\alpha \in \Omega_+^{(0,p)} \otimes \Omega_-^{(0,q)}$, let $X_1, \ldots, X_{p+1}$ be anti–holomorphic vector fields for I_+ and let $Y_1, \ldots, Y_q$ be anti–holomorphic vector fields for I_-. By definition, we have

$$\begin{aligned}
\overline{D}_+\alpha(X_1, \ldots, X_{p+1}, Y_1, \ldots, Y_q) = &\sum_{i=1}^{p+1}(-1)^{i+1}\nabla^-_{X_i}\alpha(X_1, \ldots, \widehat{X_i}, \ldots, X_{p+1})(Y_1, \ldots, Y_q) \\
&+ \sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{p+1})(Y_1, \ldots, Y_q) \\
= &\sum_{i=1}^{p+1}(-1)^{i+1}\Big\{X_i\left(\alpha(X_1, \ldots, \widehat{X_i}, \ldots, X_{p+1})(Y_1, \ldots, Y_q)\right) \\
&\qquad - \sum_{j=1}^{q}\alpha(X_1, \ldots, \widehat{X_i}, \ldots, X_{p+1})(Y_1, \ldots, \nabla^-_{X_i}Y_j, \ldots, Y_q)\Big\} \\
&+ \sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{p+1})(Y_1, \ldots, Y_q).
\end{aligned} \tag{5.5.12}$$

In local coordinates, the terms involving the Lie bracket drop out, and using anti–symmetry of α (5.5.12) reduces precisely to (5.5.2).

Having determined the cohomology of the operators Q_L and Q_R in terms of intrinsic, coordinate free objects, we now relate it to the Lie algebroid cohomology[9] of $\mathcal{J}_1$. Recall that the relation between $\mathcal{J}_1$ and $I_\pm$ is given by the isomorphism $L_1^\pm \cong T_\pm^{(1,0)}$, where $L_1 = L_1^+ \oplus L_1^-$. As such, we have

$$\wedge^k\overline{L} = \sum_{i=0}^{k} \wedge^i\overline{L_1^+} \otimes \wedge^{k-i}\overline{L_1^-}, \tag{5.5.13}$$

and we can identify $\wedge^p\overline{L_1^+} \otimes \wedge^q\overline{L_1^-}$ with $\Omega_+^{(0,p)} \otimes \Omega_-^{(0,q)}$, and we will denote this space by $\wedge^{(p,q)}\overline{L_1}$. Under this identification we claim that the differential d_L on the exterior algebra $\wedge^\bullet\overline{L}$ corresponds to $Q = Q_L + Q_R$. First, using Equation (3.5.2) and the involutivity of L_1^+ and L_1^-, we observe that d_L satisfies

$$d_L|_{\wedge^{(p,q)}\overline{L_1}} = \pi^{(p+1,q)} \circ d_L + \pi^{(p,q+1)} \circ d_L. \tag{5.5.14}$$

[9] See Section 3.5 for the definition.

We will verify that $\pi^{(p+1,q)} \circ d_L$ corresponds with $\overline{D}_+$, which in turn is related to Q_L, and that $\pi^{(p,q+1)} \circ d_L$ corresponds to $\overline{D}_-$, hence to Q_R. Observe that all the operators involved satisfy the same kind of Leibniz rule, so to relate them it suffices to look at 1–forms. First, for $X \in T_+^{(1,0)}$ and $\alpha \in \Omega_+^{(0,1)} = (T_+^{(1,0)})^*$ we denote by X_+ and α_+ their images in L_1^+ and $\overline{L_1^+}$ respectively, under the isomorphism discussed above. Similar notation will be used for the $-$ components. For $\alpha \in \Omega_+^{(0,1)}$ and $X_1, X_2 \in T_+^{(0,1)}M$, using (5.5.12) we get

$$\overline{D}_+\alpha(X_1, X_2) = X_1(\alpha(X_2)) - X_2(\alpha(X_1)) - \alpha([X_1, X_2]). \tag{5.5.15}$$

On the other hand, we have

$$\begin{aligned} d_L\alpha_+(X_{1+}, X_{2+}) =& \pi(X_{1+})\alpha_+(X_{2+}) - \pi(X_{2+})\alpha_+(X_{1+}) - \alpha_+([\![X_{1+}, X_{2+}]\!]) \\ =& X_1(\alpha(X_2)) - X_2(\alpha(X_1)) - \alpha([X_1, X_2]), \end{aligned} \tag{5.5.16}$$

where in the last equality we used the equality $[\![X_{1+}, X_{2+}]\!] = [\![X_1, X_2]\!]_+$. For $\alpha \in \Omega_-^{(0,1)}$, $X \in T_+^{(0,1)}M$ and $Y \in T_-^{(0,1)}M$, we have

$$\overline{D}_+\alpha(X, Y) = X(\alpha(Y)) - \alpha(\nabla_X^- Y), \tag{5.5.17}$$

as follows from (5.5.12). At the Lie algebroid level we have

$$d_L\alpha_-(X_+, Y_-) = \pi(X_+)(\alpha_-(Y_-)) - \alpha_-([\![X_+, Y_-]\!]) = X(\alpha(Y)) - \alpha(\nabla_X^- Y), \tag{5.5.18}$$

where in the last equality we used the following equality to relate the covariant derivative with the Courant bracket:

$$\nabla_X^{\pm}(Y) = \pi_T\left(([\![X_{\mp}, Y_{\pm}]\!])_{\pm}\right), \tag{5.5.19}$$

where for $X \in T$, we denote by $X_\pm$ its image in $C_\pm$ under the map $\pm g + b$, and for $v \in T \oplus T^*$, we denote by $v_\pm$ its decomposition into $C_\pm$ components. To prove this, it suffices to check that the righthand side defines a connection, which is metric and has $\pm g^{-1}H$ as torsion. These facts follow directly from the properties of the Courant bracket, and for more specific details we refer to [24]. Similar equalities hold for $\overline{D}_-$, so indeed the operator $Q = Q_L + Q_R$ is identified with d_L, hence both have the same cohomology.

Given the Q–cohomology of local operators, one may wonder how it relates to the cohomology of states. In [26] this has been worked out, and it turns out that the Hilbert space of states is isomorphic to the space of differential forms, and the action of the local operators on them is given by the Clifford action (3.1.5). The quantization of Q is computed, and turns out to agree with $\overline{\partial}$. Recall that d^H can be written as $d^H = \partial + \overline{\partial}$, with respect to the decomposition

$$\wedge^\bullet T^* \otimes \mathbb{C} = U_{-n} \oplus U_{-n+1} \oplus \ldots \oplus U_n. \tag{5.5.20}$$

The local operators, being elements of $\wedge^\bullet \overline{L}$, act as lowering operators on these states. In particular, a vacuum vector for the theory is a nowhere-vanishing section of U_n. For the vacuum to be Q–invariant (axiom 2 of a cohomological field theory), it must be $\overline{\partial}$ closed, so that the topological twist only makes sense on generalized Calabi–Yau spaces! For these, there is an isomorphism between the cohomology of operators and the cohomology of states, as follows directly from our discussion in Section 3.5.

Chapter 6

Outlook

As is probably the case in any thesis, due to the limited amount of time there are a lot of questions that remain unanswered. The most important ones are summarized below.

- The main open problem in the twisted models on generalized Calabi–Yau spaces is the one addressed in Section 5.3, and amounts to solving the equation $S = \{Q, V\} + S_{top}$. For general spaces this remains unsolved.

- A lot of the computations, especially those in Chapter 4, seem to underly a more intrinsic geometric computation. Obviously, it is a 'remarkable' coincidence that commutator brackets of the symmetries lead to expressions involving only the Nijenhuis tensors and covariant derivatives with closed skew torsion. Since this is precisely the data that is encoded in the Courant bracket, we expect that supersymmetry can be formulated in pure geometric terms on $T \oplus T^*$. However, most likely this description will use the concept of supermanifolds, a concept we have carefully avoided in this thesis to reduce the amount of technicalities.

- Perhaps the most interesting feature of topological field theories is that the correlation functions of the physical quantum model provide differential geometric invariants of topological spaces[1]. This completely reverses the roles of physics and mathematics, in the sense that the physical model is now used to study geometric properties, instead of the other way around. The $(2, 2)$–supersymmetric sigma model with flux could thus serve as a tool to study e.g. generalized Kähler geometry, as a generalization of the classical Gromov–Witten invariants in symplectic geometry. Although the previous problems could possibly have been answered, this problem is most likely beyond the level of this thesis. Nevertheless, for the particular type of models that have been proved to be topological, there is nothing that obstructs us to find these invariants, and we hope to come back to this aspect in the near future.

[1] Numbers that depend only on the smooth structure of the space.

Appendix A

Spin groups and spinors

In this section we give a brief overview of the theory of spin and state some results that are used throughout the text. We will first recall the basics of Clifford algebras, through which we then construct the spin groups and the corresponding spinors.

Let V be a finite dimensional vector space over $\mathbb{R}$ and $\langle , \rangle$ a symmetric bilinear form on V, not necessarily non–degenerate or positive definite. We can form the tensor algebra $TV := \bigoplus_{i=0}^{\infty} V^{\otimes i}$, and look at the quotient by the ideal generated by elements of the form

$$v \otimes v - \langle v, v \rangle, \quad v \in V. \tag{A.0.1}$$

This quotient is called the Clifford algebra of the pair $(V, \langle , \rangle)$ and is denoted by $CL(V)$ (we suppress the dependence on the bilinear form, as it is always clear from the context which one is used). In terms of a basis $e_1, \ldots, e_n$ of V this is nothing else as the algebra freely generated by 1 and the e_i's, subject to the anti–commutation relations $e_i e_j + e_j e_i = 2\langle e_i, e_j \rangle$. One readily verifies that $\dim_{\mathbb{R}} CL(V) = 2^n$, basically by the same argument that is used to prove $\dim \wedge^{\bullet} V = 2^n$. Since we quotient out a non–homogeneous ideal, $CL(V)$ does not inherit the $\mathbb{Z}-$grading from TV, but it does have a $\mathbb{Z}_2$ grading. Indeed, the relation $v \otimes v = \langle v, v \rangle$ identifies elements of different degrees but does not change the parity (the degree mod 2) of an element. Hence we can write $CL(V) = CL_0(V) \oplus CL_1(V)$, $CL_0(V)$ denoting the even part and $CL_1(V)$ the odd part, and algebras with this property are called *superalgebras*.

For later use we define a couple of natural operations available on $CL(V)$:

$$\begin{aligned} &\text{Transposition :} && (v_1 \cdots v_r)^t := v_r \cdots v_1, \\ &\text{Involution :} && \alpha(\phi_0 + \phi_1) := \phi_0 - \phi_1, \text{ where } \phi_i \in CL(V)_i, \\ &\text{Norm :} && N(\phi) = \phi \cdot \alpha(\phi^t). \end{aligned} \tag{A.0.2}$$

Now assume that the bilinear form on V is non–degenerate. As is well known, any such form over the reals is classified by its signature, which is a pair of integers (r, s) with $r + s = d = dim(V)$, such that there are subspaces V^+ and V^- of dimensions r and s respectively, where $\langle , \rangle$ is positive definite

on V^+ and negative definite on V^-. Clearly any two forms of the same signature are equivalent, and we will denote the Clifford algebra associated to a form of signature (r, s) by $CL(r, s)$. The structure of $CL(r, s)$ and its even subalgebra $CL_0(r, s)$ can be classified completely, and without proof we state the result in Table A.1 (for a proof we refer to [4], Section I.4).

$r - s$ mod 8	$CL(V)$	$CL_0(V)$
0	$\mathbb{R}(2^{d/2})$	$\mathbb{R}(2^{(d-2)/2}) \oplus \mathbb{R}(2^{(d-2)/2})$
1	$\mathbb{R}(2^{(d-1)/2}) \oplus \mathbb{R}(2^{(d-1)/2})$	$\mathbb{R}(2^{(d-1)/2})$
2	$\mathbb{R}(2^{d/2})$	$\mathbb{C}(2^{(d-2)/2})$
3	$\mathbb{C}(2^{(d-1)/2})$	$\mathbb{H}(2^{(d-3)/2})$
4	$\mathbb{H}(2^{(d-2)/2})$	$\mathbb{H}(2^{(d-4)/2}) \oplus \mathbb{H}(2^{(d-4)/2})$
5	$\mathbb{H}(2^{(d-3)/2}) \oplus \mathbb{H}(2^{(d-3)/2})$	$\mathbb{H}(2^{(d-3)/2})$
6	$\mathbb{H}(2^{(d-2)/2})$	$\mathbb{C}(2^{(d-2)/2})$
7	$\mathbb{C}(2^{(d-1)/2})$	$\mathbb{R}(2^{(d-1)/2})$

Table A.1: Classification table for $CL(r, s)$, where $d = r + s$. We use the notation $R(m)$ to denote the ring of $m \times m$ matrices with entries in the ring R.

A consequence of this classification is that the irreducible representations [1] for $CL(r, s)$ are also completely classified. Indeed, a basic result in linear algebra tells us that the only irreducible representation (up to isomorphism) of matrix ring $K(m)$, with K a field, is given by K^m with the natural action. Furthermore, one can easily show that for two rings R_1 and R_2, the irreducible representations for $R_1 \times R_2$ are just the irreducible representations for one of the two factors with the other factor acting trivially. Combining these two facts, we see that $CL(r, s)$ has one or two irreducible representations, depending on whether it is of the form $K(m)$ or $K(m) \oplus K(m)$.

Now that we understand the basics of Clifford algebras, we can introduce the concepts of *Pin* and *Spin*. Consider the following two subsets of $CL(V)$:

$$\begin{aligned} Pin(V) &:= \{v_1 \cdots v_r |\ r \in \mathbb{Z}_{\geq 0},\ v_i \in V,\ \langle v_i, v_i \rangle = \pm 1\}, \\ Spin(V) &:= \{v_1 \cdots v_{2r} |\ r \in \mathbb{Z}_{\geq 0},\ v_i \in V,\ \langle v_i, v_i \rangle = \pm 1\}. \end{aligned} \tag{A.0.3}$$

Both of them are in fact groups, because for $v \in V$ of norm ± 1 we have $v \cdot \frac{v}{\langle v,v \rangle} = 1$. These groups are important, because there exists a map

$$\begin{aligned} \rho :& Pin(V) \to O(V), \\ \rho :& Spin(V) \to SO(V), \end{aligned} \tag{A.0.4}$$

where $O(V)$ denotes the group of transformations of V that are orthogonal with respect to $\langle , \rangle$, and $SO(V)$ its subgroup consisting of maps with unit determinant. For $v \in V$ with $\langle v, v \rangle = \pm 1$, we

[1] Since $CL(r, s)$ is not a group but a ring, the correct terminology would be 'simple module' instead of 'irreducible representation', but this is just a matter of language. Henceforth we will call a module simply a representation.

define the value of ρ on v via

$$\rho(v)w := -vwv^{-1} = -vw\frac{v}{\langle v,v\rangle} = w - \frac{\langle w,v\rangle}{\langle v,v\rangle}v \quad \text{for } w \in V, \tag{A.0.5}$$

where in the last equality we used the Clifford relation, and then extend ρ to $Pin(V)$ and $Spin(V)$ in the obvious way. From the definition above it is clear that ρ maps $Pin(V)$ into $O(V)$, as the map $\rho(v)$ given in (A.0.5) is simply the orthogonal reflection in the plane orthogonal to $v \in V$. By a theorem of Cartan–Dieudonné, every orthogonal map can be written as the decomposition of at most $\dim(V)$ orthogonal reflections, and so the map ρ is in fact surjective. Moreover, the restriction of ρ to $Spin(V)$ indeed maps into $SO(V)$, and this is also a surjective map. It is not difficult to compute the kernel of ρ, and it turns out to consist only of the elements ± 1 (note that -1 is indeed an element of $Spin(V)$). Therefore, ρ gives double–coverings of $O(V)$ and $SO(V)$, which are nontrivial (i.e., not of the form $O(V) \coprod O(V)$ or $SO(V) \coprod SO(V)$) for $\dim(V) \geq 2$ and $(r,s) \neq (1,1)$.

There are a lot of irreducible representations of $Pin(r,s)$ and $Spin(r,s)$ in general, but there is one type that deserves a special name.

Definition A.0.1. A representation $\pi : Pin(r,s) \to GL(W)$ for some vector space W is called a *pinor representation* if it is the restriction of an irreducible representation of $CL(r,s)$ on W. In this case W is called a space of pinors. Similarly, a representation $\pi : Spin(r,s) \to GL(W)$ is called a *spinor representation* if it is the restriction of an irreducible representation for $CL_0(r,s)$ on W, and W is called a space of spinors.

One readily verfies that both pinor and spinor representations are irreducible, for $CL(r,s)$ is generated as a vector space by $Pin(r,s)$ as well as $CL(r,s)_0$ is generated by $Spin(r,s)$. One should think about pinors and spinors as being those irreducible representations of $Pin(r,s)$ and $Spin(r,s)$, that also have a well defined action of the whole $CL(r,s)$ and $CL_0(r,s)$ respectively on them.

Due to the classification given in Table A.1 we can easily compute the spaces of pinors and spinors for $CL(r,s)$, with the use of our remarks above about the representation theory of $CL(r,s)$. The result is given in Table A.2 below.
The spaces P and S are closely related to each other by means of the volume element of the Clifford algebra. This volume element, denoted by ω, is defined in terms of an orthogonal basis $e_1, \ldots, e_d$ (orthogonal in the sense that $\langle e_i, e_j\rangle = \pm\delta_{ij}$) by

$$\omega := e_1 \cdots e_d,$$

and satisfies $\omega^2 = (-1)^{s+d(d-1)/2}$. If we choose another such basis, it must be related to the old one by means of an orthogonal transformation, so ω is determined up to sign. If we also fix an orientation, ω is completely fixed, and one can then use the volume element to relate P and S. For details see [4], Section I.5.

Next we illustrate the general theory by some examples, which are the cases we need for the text.

$r-s$ mod 8	Pinors	Spinors
0	$P \cong \mathbb{R}^{2^{d/2}}$	$S_\pm \cong \mathbb{R}^{2^{(d-2)/2}}$
1	$P_\pm \cong \mathbb{R}^{2^{(d-1)/2}}$	$S \cong \mathbb{R}^{2^{(d-1)/2}}$
2	$P \cong \mathbb{R}^{2^{d/2}}$	$S \cong \mathbb{C}^{2^{(d-2)/2}}$
3	$P \cong \mathbb{C}^{2^{(d-1)/2}}$	$S \cong \mathbb{H}^{2^{(d-3)/2}}$
4	$P \cong \mathbb{H}^{2^{(d-2)/2}}$	$S_\pm \cong \mathbb{H}^{2^{(d-4)/2}}$
5	$P_\pm \cong \mathbb{H}^{2^{(d-3)/2}}$	$S \cong \mathbb{H}^{2^{(d-3)/2}}$
6	$P \cong \mathbb{H}^{2^{(d-2)/2}}$	$S \cong \mathbb{C}^{2^{(d-2)/2}}$
7	$P \cong \mathbb{C}^{2^{(d-1)/2}}$	$S \cong \mathbb{R}^{2^{(d-1)/2}}$

Table A.2: The space of pinors and spinors for $CL(r,s)$, where again $d = r+s$. If there is a $\pm$ subscript on P or S, then this space has two representations on it, as the corresponding (even) Clifford algebra has the form $K(m) \oplus K(m)$.

Example A.0.2. Consider the case $(r,s) = (1,1)$. Let us first work out an explicit isomorphism $CL(1,1) \cong \mathbb{R}(2)$ and $CL(1,1)_0 \cong \mathbb{R}(1) \oplus \mathbb{R}(1)$, which will be needed for an explicit form of the Lorentz transformation of spinors on a world–sheet of signature $(1,1)$. We start with a two–dimensional vector space spanned by e_1 and e_2, with $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = 1$. Then $CL(1,1) = \mathbb{R} \oplus \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_1e_2$, and one readily verifies that the algebra map $CL(1,1) \to \mathbb{R}(2)$, given on the generators by

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.0.6}$$

is an isomorphism. Now $CL(1,1)_0$ is generated by 1 and elements of the form

$$(ae_1 + be_2)\cdot(ce_1 + de_2) = (ac - bd) + (ad - bc)e_1e_2,$$

which under the isomorphism above is mapped to the matrix

$$\begin{pmatrix} (ac-bd) - (ad-bc) & 0 \\ 0 & (ac-bd) + (ad-bc) \end{pmatrix}. \tag{A.0.7}$$

So we see that under the identification $CL(1,1) \cong \mathbb{R}(2)$, $CL(1,1)_0$ is identified with the diagonal matrices, and indeed $CL(1,1)_0 \cong \mathbb{R}(1) \oplus \mathbb{R}(1)$.

Obviously P, the space of pinors, in this signature is equal to $\mathbb{R}^2$, and splits into ± 1 eigenspaces of the volume element

$$\omega = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and these eigenspaces are preserved by $CL(1,1)_0$. Moreover, these eigenspaces correspond exactly to the two inequivalent spinor representations! This holds more generally for signature (r,r), as the volume element squares to 1 and commutes with $CL(r,r)_0$.

Now we work out the double cover $Spin(1,1) \to SO(1,1)$. First, observe that $SO(1,1)$ is given by matrices

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \text{ with } a^2 - b^2 = 1. \tag{A.0.8}$$

Obviously this has two connected components, corresponding to $a = \pm\sqrt{b^2+1}$. Observe that $Spin(1,1)$ is generated by elements of the form (A.0.7), with $a^2 - b^2 = \pm 1 = c^2 - d^2$. One easily checks that in that case (A.0.7) is of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \pm\lambda^{-1} \end{pmatrix} \text{ for } \lambda \in \mathbb{R}\backslash\{0\}, \tag{A.0.9}$$

and that every matrix of that form is an element in $Spin(1,1)$, which therefore has four connected components and therefore is a trivial double cover of $SO(1,1)$. To actually compute this map $Spin(1,1) \to SO(1,1)$, we have to compute the matrix associated with the conjugation of the matrices e_1 and e_2 by (A.0.9), which is just a straightforward calculation and it turns out that the map ρ in (A.0.4) is given by

$$\begin{pmatrix} \lambda & 0 \\ 0 & \pm\lambda^{-1} \end{pmatrix} \mapsto \begin{pmatrix} \pm\left(\frac{\lambda^2+\lambda^{-2}}{2}\right) & \pm\left(\frac{\lambda^2-\lambda^{-2}}{2}\right) \\ \pm\left(\frac{\lambda^2-\lambda^{-2}}{2}\right) & \pm\left(\frac{\lambda^2+\lambda^{-2}}{2}\right) \end{pmatrix}. \tag{A.0.10}$$

If we restrict to the connected component of the identity in $Spin(1,1)$, and write $\lambda = e^{\alpha/2}$ for $\alpha \in \mathbb{R}$, the map above simplifies and takes the form

$$\begin{pmatrix} e^{\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{pmatrix} \mapsto \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}. \tag{A.0.11}$$

Example A.0.3. After Wick–rotating the world–sheet, the signature changes from $(1,1)$ into $(2,0)$, so we will also take a closer at this second case now. It is a little different than the previous one, as we can already see from the table. Indeed, the volume element now squares to -1, so that the space of pinors P, which is real, splits into $\pm i$ eigenspaces of ω only after complexification. This can be made explicit as follows. Again we have a two–dimensional vector space spanned by e_1 and e_2, but this time with $e_1^2 = e_2^2 = 1$. There is an isomorphism $CL(2,0) \otimes \mathbb{C} \to \mathbb{C}(2)$ given by

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

extended complex linearly to the whole of $CL(2,0) \otimes \mathbb{C}$. Now $CL(2,0)_0$ is generated the elements

$$\begin{pmatrix} (ac+bd) + i(ad-bc) & 0 \\ 0 & (ac+bd) - i(ad-bc) \end{pmatrix}, \tag{A.0.12}$$

which shows that indeed $CL(2,0)_0 \cong \mathbb{C}$. Then $Spin(2,0)$ is described by these elements but with $a^2 + b^2 = c^2 + d^2 = 1$, so that we can write them as

$$\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$$

for some $\alpha \in [0, 4\pi)$. The double cover $Spin(2,0) \to SO(2,0)$ is then given by

$$\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \mapsto \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}. \tag{A.0.13}$$

Note that now both groups are connected and the double cover is just the map of the circle to itself with winding number 2.

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