The Gauss-Bonnet Theorem

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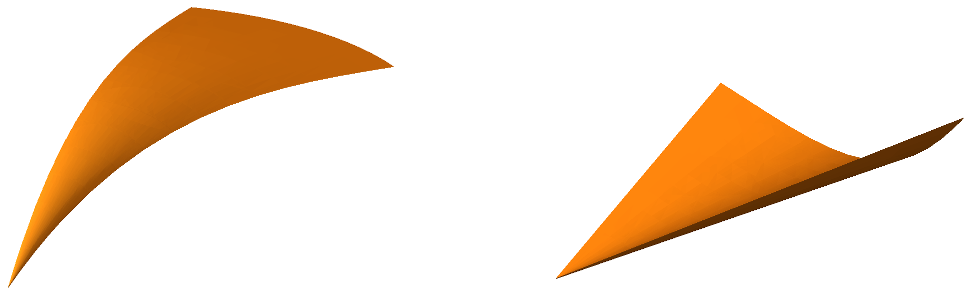
Introduction

Suppose you are hungry and you are not in the mood to spend hours in the kitchen. You also want to have the idea that you eat fresh vegetables. What would you do? You will buy a pizza.

But there is a problem. When you divided the pizza in slices and picked up a piece, all cheese and salami will fall to the ground due to gravity. This is shown in Figure 1a. It turns out that when you bend the back of the pizza slice, like in Figure 1b you prevent the point from tipping down [2].[1]



(a) *A slice of pizza where the topping will fall of.*

(b) *The correct way to hold a pizza slice. Bend the back a litte bit.*

Figure 1: *How to hold a pizza.*

To describe this problem mathematically, we see a pizza as a 2 dimensional surface. For surfaces we can define a notion of curvature that it preserved under isometric immersions. Therefore, both slices in Figure 1 have the same curvature. But when the point in Figure 1b bend down, the curvature – and thus the metric – must change. This only happens when the pizza is torn apart.

The main result in this thesis is the Gauss-Bonnet theorem. It states that the integral of the Gaussian curvature K over a surface Σ only depends on the topology of the surface. More precisely, it states that

$$\int_\Sigma K \mathrm{d}A = 2\pi\chi$$

[1]This will only work with Italian pizza, because these are well baked and don't have stretchy dough.

where χ is the Euler characteristic. This is interesting because the Gaussian curvature is a number that only depends on the metric and χ is a number that is purely topological in origin.

We will define another type of curvature, namely the geodesic curvature, k_g for the boundary of a surface. For surfaces with boundary the Gauss-Bonnet theorem states that

$$\int_{\Sigma} K \mathrm{d}A + \int_{\partial\Sigma} k_g \mathrm{d}s = 2\pi\chi.$$

In chapter 1 we will give the definition of a surface and recall some definitions from differential geometry. In chapter 2 we will define the notion of curvature. We will start with a geometric intuitive definition of curvature in $\mathbb{R}^3$ and we will extend to general smooth manifolds with Riemannian metric. We will also prove the 'Pizza theorem' stated above.

In chapter 3 we will investigate what happens to the curvature when we do a change of coordinates. It will turn out that it is related to another topological property called the index. In this chapter we will prove the Poincare-Hopf theorem which states that the sum of the indices of a vector field is equal to the Euler characteristic.

At that point we have all the ingredients to prove the Gauss-Bonnet theorem for manifolds *without* boundary. To prove the Gauss-Bonnet theorem for surfaces *with* boundary we will define the geodesic curvature k_g in chapter 4. In chapter 6 we will finally prove the Gauss-Bonnet theorem.

Contents

1 Preliminary definitions

The Gauss-Bonnet theorem relates the curvature of a surface to a topological property called the Euler characteristic. To properly state this theorem we have to first define what a surface is. In short, it is a 2-manifold (with or without boundary) which is equipped with a Riemannian metric. Readers familiar with Riemannian metrics and manifolds can skip this chapter.

1.1 An introduction to surfaces

In this section we will introduce the notion of surface. If we think about surfaces we think about something that is 2-dimensional, though doesn't necessary have to be flat. We can achieve this by requiring that a surface must be topological manifold of dimension 2.

Definition 1.1 (Lee [8], pg. 3)**.** *Suppose M is a topological space. We say that M is a* **topological n-manifold** *of topological manifold of dimension n if it has the following properties:*

1. *M is a Hausdorff space.*
2. *M is second countable.*
3. *M is locally Euclidean of dimension n; That is, each point $p \in M$ has a neighborhood which is homeomorphic to $\mathbb{R}^n$.*

Notice that when we have manifolds with boundary we loosen the third property such that points must have a neighborhood which is homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^{n+}$: $\{x \in \mathbb{R}^n \,|\, x_n \geq 0\}$

In this thesis we will do calculus on surfaces and so must be there some notion of differentiability. The usual way to define a differentiable structure is to check that compositions of local homeomorphisms are differentable. The following definitions state this more formally.

Definition 1.2 (Lee [8], pg. 4)**.** *Let M be a topological manifold of dimension n. A* **coordinate chart** *ϕ is a homeomorphism between the open set $U \subset M$ to a open subset in $\mathbb{R}^n$. a coordinate chart is centered at $p \in M$ when $\phi(p) = 0$.*

Definition 1.3 (Lee [8], pg. 12-13)**.** *A C^k-***structure** *or differential structure on a topological n-manifold is a collection of charts $\{(\phi_\alpha, U_\alpha) \,|\, \phi_\alpha\colon U_\alpha \to M, \alpha \in A\}$ such that*

1. $\bigcup_{\alpha\in\alpha} U_\alpha = M$
2. *For* $\alpha, \beta \in A$, $\phi_\alpha \circ \phi_\beta^{-1}$ *is* C^k *on the* $U_\alpha \cap U_\beta$.
3. *The collection of charts is maximal with relation to the second property. This means that if* ϕ *is a chart such that the composition between* ϕ *and all the charts in the collection is continuous differentiable, then* ϕ *is in the collection of charts.*

Definition 1.4 (Lee [8], pg. 13)**.** *A* C^k**-manifold** *of dimension* n *is a topological* n*-manifold endowed with a* C^k*-structure* F.

Recall that we can always extend a C^k-structure to a maximal structure [8, Prop. 1.17, pg. 13]. Therefore, the third property isn't an important one.

From the concept of differentiability, we get tangent vectors. Recall that the tangent space T_pM of M at p is the space of all paths through p with an equivalence relation such that paths which have the same speed, are in the same equivalence class [7, pg. 15]. Recall that tangent spaces are n-dimensional vector spaces.

On $\mathbb{R}^n$ we have the notion of inner products. There is a generalization on manifolds, namely the Riemannian metric:

Definition 1.5 (Lee [7], pg. 23)**.** *A Riemannian metric on a smooth manifold* M *is a 2-tensor field* $g\colon TM \times TM \to \mathbb{R}$ *that has the following two properties:*

1. *Symmetric:* $g(X, Y) = g(Y, X)$
2. *Positive definitive:* $g(X, X) > 0$ *if* $X \neq 0$

Remark: Usually we denote $g(X, Y)$ by $\langle X, Y\rangle$. This is because the default Riemannian metric of $\mathbb{R}^n$ is just the normal inner product, so we can interpret g as an inner product.

In this thesis a surface is defined as follows:

Definition 1.6. *A* **surface** *is a smooth 2-manifold (with or without boundary) equipped with a Riemannian metric* g.

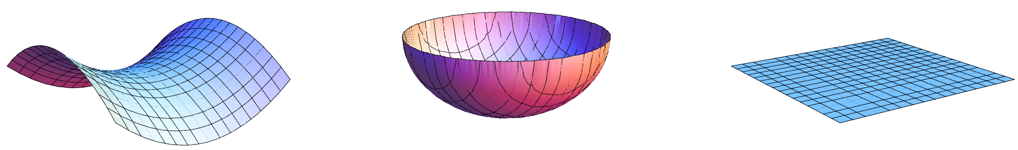
2 Gaussian curvature

In this chapter we will define the first type of curvature on a surface: the Gaussian curvature. We start by defining the curvature of an arc. Next we give a geometric definition of the Gaussian curvature. After that we reformulate the curvature as the determinant of a special matrix. When we arrive at this point, we can calculate the Gaussian curvature, but we are restricted to the case when our surface is immersed in $\mathbb{R}^3$.

To solve this problem we stop and introduce the concept of frames. We will then find relations called the structure relations. From this we can describe the Gaussian curvature using the notion of frames. We then also find that the curvature doesn't depend on how a surface is embedded, but it only depends on the metric of the tangent space. At last we can define the Gaussian curvature without using any notion of ambient space.

2.1 The geometric definition of curvature

The next topic we will cover is the notion of curvature. Intuitively curvature measures how much a surface moves away from the tangent plane. You could define the curvature by patching the holes and test if it is usable as coffee-cup. For example Figure 2a would make a lot of mess and Figure 2c would make drinking really challenging, but the half-sphere in Figure 2b won't spill a single droplet. To give a rigorous mathematical definition of curvature. First we will define the curvature of a path. Next we will define curvature for surfaces embedded in $\mathbb{R}^3$. Our goal is to generalize it for Riemannian manifolds.



(a) *A saddle parametrized by* $z = y^2 - x^2$.

(b) *A Cup parametrized by* $z = -\sqrt{1 - y^2 - x^2}$.

(c) *A Plane parametrized by* $z = 0$.

Figure 2: *Some samples of curved surfaces.*

2.1.1 Surfaces and paths in $\mathbb{R}^3$

To define curvature on surfaces we first need to define the curvature of a path $C \subset \mathbb{R}^3$.

Definition 2.1. *A* **regular path** *on a surface Σ is an immersion of the open interval in Σ, $\gamma : I \to \Sigma$.*

Let $\gamma : I \to C$ be a regular path in $\mathbb{R}^3$. We want to measure how fast the path moves away from the tangent plane at some $t_0 \in I$. Note that $\gamma'(t_0)$ will be in the tangent plane, so this will not measure the curvature. The length of the second derivative will be a good candidate. But by the chain rule, the second derivative is dependent on the choice of parameterization. Luckily, there is a natural way to parameterize a path such that we can get a well defined notion of curvature.

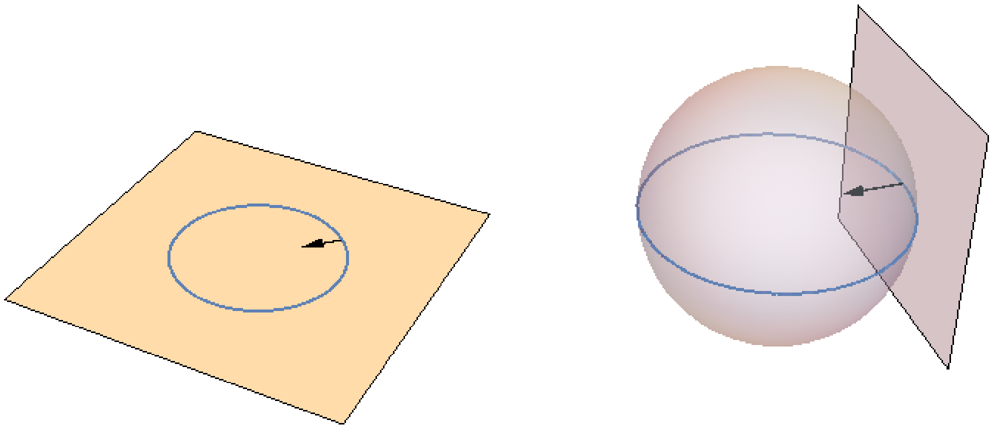
Lemma 2.2. *Every parametrization $\gamma\colon I \to \mathbb{R}$ can be changed to a* **parametrization by arc length** *$\tilde{\gamma}\colon J \to \Sigma$. That is, when $\tilde{s}(t)$ denotes the arc length of the path between $\tilde{\gamma}(t)$ and a fixed point $\tilde{\gamma}(t_0)$, then $\tilde{s}(t) = t - t_0$.*

Proof: Given $t \in I$ we define the arc length of a path γ from the point t_0 by $s(t) = \int_{t_0}^{t} |\gamma'(\tau)| \mathrm{d}\tau$. For a continuous path the integral is well defined. Note that $s(t)$ is a strict increasing function. So $s(t)$ has an inverse. Let $\tilde{\gamma} = \gamma \circ s^{-1}$. This has the same image as γ, because s is bijective. Also, note that $|\tilde{\gamma}'(x)| = \frac{|\gamma'(s^{-1}(x))|}{|\gamma'(s^{-1}(x))|} = 1$. Therefore, it follows that $\tilde{s}(t) = \int_{t_0}^{t} |\tilde{\gamma}(\tau)| \mathrm{d}\tau = t - t_0$. $\square$

From the fact that $|\tilde{\gamma}'(x)| = 1$ follows that $\frac{\partial}{\partial x}\langle \tilde{\gamma}'(x), \tilde{\gamma}'(x)\rangle = 0$. This is equal to $2\langle \tilde{\gamma}''(x), \tilde{\gamma}'(x)\rangle = 0$. Therefore, we can always change a regular path such that the second derivative is orthogonal to the tangent plane without changing the image. Now we have everything to define the curvature of a regular path.

Definition 2.3. *Let Σ be a surface embedded in $\mathbb{R}^3$. Let $\gamma\colon I \to \Sigma$ be a regular path parametrized by arc length. We call the number $k(s) = |\gamma''(s)|$ the* **curvature** *of γ at s.*

The curvature of a path is not enough to measure the rate a surface moves away from the tangent plane. Look at Figure 3. In Figure 3a the second derivative is in the tangent plane. It this is useless for the definition of curvature. In Figure 3b it is orthogonal to the tangent plane. We would like that all second derivatives are of this form. To solve this we need to use the fact that we are embedding in $\mathbb{R}^3$, because now we can define the normal vector.



(a) *The curve of a circle in a plane. The second derivative lies inside the tangent plane.*

(b) *The curve of a circle in a sphere. The second derivative lies orthogonal to the tangent plane.*

Figure 3: *Some samples of curved surfaces.*

We define the normal N of the tangent plane as follows: given a locally orientable surface Σ. Given a basis $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ of the tangent space at a point $p \in \Sigma$, we can compute the vector product $\frac{\partial}{\partial x} \times \frac{\partial}{\partial y}$. This is normal to the tangent space and depends smooth on the coordinates. Rescale this vector so that the length is equal to 1. Notice that the orientation is induced from the orientation of the chart. When the orientation induced by the chart is different than the orientation of the surface, we multiply by -1. This map $N\colon \Sigma \to S^2 \subset \mathbb{R}^3$ gives the normal at each point of the surface and we call it the **Gauss map**.

When we project the second derivative to the normal space, we can measure how fast a path on a surface moves away from the tangent plane.

Lemma 2.4. *Let $\gamma\colon I \to \Sigma$be a regular path parameterized by arc length passing through $\gamma(t_0) = p$ and let N be the Gauss map. We define the* **normal curvature** *k_n of the path γ at p as*

$$k_n = \langle N, \gamma''(t_0)\rangle.$$

In particular if two paths through p have the same speed, then they have the same normal curvature at p.

Proof: Suppose that there are 2 regular paths parameterized by arc length. $\gamma\colon I \to \Sigma, \tilde{\gamma}\colon J \to \Sigma$ such that for some point $p \in \Sigma$, $\gamma(t_0) = \tilde{\gamma}(\tilde{t_0}) = p$ and $\gamma'(t_0) = \tilde{\gamma}'(\tilde{t_0})$. Because the normal is orthogonal to the tangent space the following are equal to zero:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t_0}\langle N(\gamma(s)), \gamma'(s)\rangle =& \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t_0}\langle N(\tilde{\gamma}(s)), \tilde{\gamma}'(s)\rangle \\
\langle \mathrm{d}N\gamma'(t_0), \gamma'(t_0)\rangle + \langle N(\gamma(t_0)), \gamma''(t_0)\rangle =& \langle \mathrm{d}N\tilde{\gamma}'(t_0), \tilde{\gamma}'(t_0)\rangle + \langle N(\tilde{\gamma}(t_0)), \tilde{\gamma}''(t_0)\rangle \\
\langle N(\gamma(t_0)), \gamma''(t_0)\rangle =& \langle N(\tilde{\gamma}(t_0)), \tilde{\gamma}''(t_0)\rangle \\
k_n(\gamma(t_0)) =& k_n(\tilde{\gamma}(t_0))
\end{aligned}
$$

□

This result was first proven by Jean Baptiste Meusnier in 1785 and is called the Meusnier Theorem [10]. Also, from this lemma follows that k_n can be written as $\langle \mathrm{d}N\gamma'(t_0), \gamma'(t_0)\rangle$.

Example: Let us look back at Figure 3. In Figure 3a the second derivative lies inside the tangent plane and is perpendicular to the normal. In this situation k_n is 0. In Figure 3b the second derivative is parallel to the normal. Now $k_n > 0$.

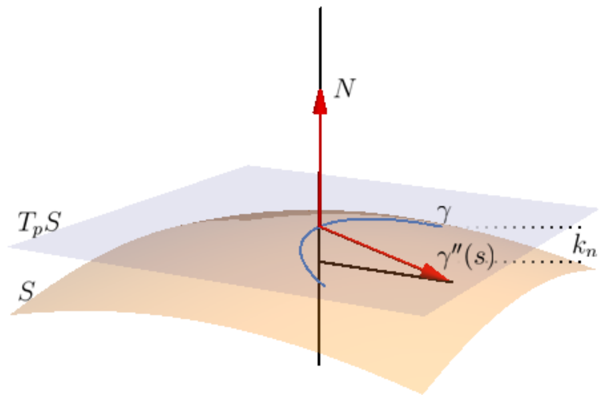


Figure 4: *A graphical illustration of the construction of the normal curvature k_n*

2.1.2 The curvature at a point on a surface

In this section we will define the curvature at a point on a surface. Recall that the normal curvature in the direction $v \in T_p\Sigma$ is defined as $k_n = \langle \mathrm{d}Nv, v\rangle$. We will examine the properties of the map $\mathrm{d}N$. From this we will develop the tools required to define the Gaussian curvature.

Lemma 2.5. *For surfaces embedded in $\mathbb{R}^3$, the map $\mathrm{d}N$ at $p \in \Sigma$ is a map whose domain and codomain is $T_p\Sigma$*

Proof: By definition the domain of N at p is the tangent space at p. Notice that in $\mathbb{R}^3$ the tangent space is the vector space which is normal to the Gauss map. In other words:

$$T_p\Sigma = \{v \in \mathbb{R}^3 \colon \langle N(p), v\rangle = 0\}.$$

By definition of the Gauss map, the codomain of N is the sphere $S^2 \subset \mathbb{R}^3$. Thus, the codomain of $\mathrm{d}N$ is $T_{N(p)}S^2$. When we use the fact that the tangent plane of the sphere is orthogonal to the line through the origin, we get that

$$\mathrm{Im}(\mathrm{d}N_p) \subseteq T_{N(p)}S^2 = \{v \in \mathbb{R}^3 \colon \langle N(p), v\rangle = 0\}.$$

Because both spaces are the same we get that $\mathrm{d}N_p \colon T_p\Sigma \to T_p\Sigma$. □

Definition 2.6 (do Carmo [5], pg. 82). *Suppose that $\iota \colon \Sigma \to \mathbb{R}^n$ is an immersion. We say that $\langle \cdot, \cdot \rangle_p$ is a* **metric induced by the immersion** *ι when it is equal to the pullback of the metric in $\mathbb{R}^n$.*

Lemma 2.7. *The map $\mathrm{d}N$ is a symmetric linear map.*

Proof: Pick a parametrization of our surface $\phi \colon U \subset \mathbb{R}^2 \to \Sigma \subset \mathbb{R}^3$ with $\phi(0,0) = p$ for some point $p \in \Sigma$. Recall that $\frac{\partial\phi}{\partial x}\big|_{(0,0)}$ and $\frac{\partial\phi}{\partial y}\big|_{(0,0)}$ is form basis of $T_p\Sigma$. Because $N(p)$ is normal to the tangent space the following holds:

$$\langle N(\phi(x,y)), \frac{\partial\phi}{\partial x}\rangle = \langle N(\phi(x,y)), \frac{\partial\phi}{\partial y}\rangle = 0$$

Differentiating the first part with respect to x and the second part with respect to y at $(0,0)$ we get the following result:

$$\begin{aligned} \frac{\partial}{\partial y}\Big|_{(0,0)}\langle N(\phi(x,y)), \frac{\partial\phi}{\partial x}\rangle =& \frac{\partial}{\partial x}\Big|_{(0,0)}\langle N(\phi(x,y)), \frac{\partial\phi}{\partial y}\rangle \\ \langle \mathrm{d}N_p\frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial x}\rangle + \langle N(p), \frac{\partial^2\phi}{\partial x\partial y}\rangle =& \langle \mathrm{d}N_p\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\rangle + \langle N(p), \frac{\partial^2\phi}{\partial x\partial y}\rangle \end{aligned}$$

From this we can conclude that $\langle \mathrm{d}N_p\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\rangle = \langle \mathrm{d}N_p\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle$. For arbitrary vectors in $v, w \in T_p\Sigma$ where $v = v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}$ and $w = w_1\frac{\partial}{\partial x} + w_2\frac{\partial}{\partial y}$, we can now check if $\mathrm{d}N$ is symmetric:

$$\begin{aligned}
\langle \mathrm{d}Nv, w\rangle =&\langle \mathrm{d}N(v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}), (w_1\frac{\partial}{\partial x} + w_2\frac{\partial}{\partial y})\rangle \\
=&v_1w_1\langle \mathrm{d}N\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\rangle + v_1w_2\langle \mathrm{d}N\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle + v_2w_1\langle \mathrm{d}N\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\rangle + v_2w_2\langle \mathrm{d}N\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\rangle \\
=&v_1w_1\langle \mathrm{d}N\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\rangle + v_1w_2\langle \mathrm{d}N\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\rangle + v_2w_1\langle \mathrm{d}N\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle + v_2w_2\langle \mathrm{d}N\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\rangle \\
=&\langle \mathrm{d}N(w_1\frac{\partial}{\partial x} + w_2\frac{\partial}{\partial y}), (v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y})\rangle \\
=&\langle \mathrm{d}Nw, v\rangle
\end{aligned}$$

From this we can conclude the lemma. □

Because the map $\mathrm{d}N$ is symmetric, $\mathrm{d}N$ has eigenvalues. Next we will show that there is a correspondence between the extreme values of k_n and the eigenvalues of $\mathrm{d}N$:

Lemma 2.8. *The map $k_n(v)$ is a symmetric quadratic map on the circle which has either four critical points: 2 identical maxima and 2 minima or is constant.*

Proof: Let $p_1 \in \mathbb{R}$ be the maximum of k_n at a given point. Let v_1 be a unit vector in $T_p\Sigma$ such that $k_n(v_1) = p_1$. Let v_2 be any unit vector normal to v_1. Define $\lambda_2 = k_n(v_2)$. Because $\{v_1, v_2\}$ is an orthonormal basis, we can describe any vector on the circle using its angle:

$$u(\vartheta) = \cos\vartheta v_1 + \sin\vartheta v_2$$

Notice that $u(0) = v_1$ is a global maximum of k_n. Taking the derivative with respect to ϑ at $\vartheta = 0$ must yield zero. Differentiation k_n gives:

$$\begin{aligned}
0 =&\frac{\partial}{\partial\vartheta}\Big|_0 k_n(u(\vartheta)) \\
=&-\frac{\partial}{\partial\vartheta}\Big|_0 \langle \mathrm{dN}u(\vartheta), u(\vartheta)\rangle \\
=&-\frac{\partial}{\partial\vartheta}\Big|_0 \langle \mathrm{dN}\cos\vartheta v_1 + \sin\vartheta v_2, \cos\vartheta v_1 + \sin\vartheta v_2\rangle \\
=&-\frac{\partial}{\partial\vartheta}\Big|_0 \left(p_1\cos^2\vartheta + 2\cos\vartheta\sin\vartheta\langle \mathrm{d}Nv_1, v_2\rangle + \lambda_2\sin^2\vartheta\right) \\
=&-2\langle \mathrm{d}Nv_1, v_2\rangle
\end{aligned}$$

We can conclude from this that $\langle \mathrm{d}Nv_1, v_2\rangle = 0$. Straight forward calculation of k_n yields:

$$\begin{aligned}
k_n(u(\vartheta)) =& \langle \mathrm{d}Nu(\vartheta), u(\vartheta)\rangle \\
=& \langle \mathrm{d}N(\cos\vartheta v_1 + \sin\vartheta v_2), \cos\vartheta v_1 + \sin\vartheta v_2)\rangle \\
=& \cos^2\vartheta\langle \mathrm{d}Nv_1, v_1\rangle + \sin^2\vartheta\langle \mathrm{d}Nv_2, v_2\rangle \\
=& p_1\cos^2\vartheta + \lambda_2\sin^2\vartheta
\end{aligned}$$

Therefore, k_n is a quadratic map. To find the extreme values of k_n we will differentiate $k_n(u(\vartheta))$ with respect to ϑ.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\vartheta}k_n(u(\vartheta)) =& -2p_1\cos\vartheta\sin\vartheta + \lambda_2\cos\vartheta\sin\vartheta \\
=& (\lambda_2 - p_1)\sin 2\vartheta
\end{aligned}$$

This is equal to zero when $\lambda_2 = p_1$ or when ϑ is a multiple of $\frac{\pi}{2}$. When $p_1 = \lambda_2$, the map is constant. When $p_1 \neq \lambda_2$, extreme values are twice p_1, or twice λ_2. Because we assumed that p_1 is the global maximum and they are not equal, λ_2 must be a global minimum. □

Corollary 2.9. *The critical points of the map k_n on the circle are eigenvectors of $\mathrm{d}N$ and the corresponding eigenvalues are the maxima and minima of k_n.*

Proof: We use the variables of the previous lemma. The critical points of k_n are

$$\begin{aligned}
k_n(\pm v_1) =& p_1 \\
k_n(\pm v_2) =& \lambda_2.
\end{aligned}$$

Because $\{v_1, v_2\}$ is an orthonormal basis, we can write $\mathrm{d}N$ as

$$\mathrm{d}Nw = \langle \mathrm{d}Nw, v_1\rangle v_1 + \langle \mathrm{d}Nw, v_2\rangle v_2.$$

In the cases $w = v_1$ and $w = v_2$, we have

$$\begin{aligned}
\mathrm{d}Nv_1 &= p_1 v_1 + \langle \mathrm{d}Nv_1, v_2\rangle v_2 \\
\mathrm{d}Nv_2 &= \langle \mathrm{d}Nv_1, v_2\rangle v_1 + \lambda_2 v_2.
\end{aligned}$$

From $\langle \mathrm{d}Nv_1, v_2\rangle = 0$ it follows that $\mathrm{d}Nv_1 = p_1 v_1$ and $\mathrm{d}Nv_2 = \lambda_2 v_2$. □

Definition 2.10. *The eigenvalues of $\mathrm{d}N$ are called the* **principal curvatures**.

We have shown that p_1 and λ_2 are eigenvalues of $\mathrm{d}N$. Therefore, $\mathrm{d}N$ in the eigenbasis is equal to a diagonal matrix with principal curvatures on the diagonal. The determinant of $\mathrm{d}N$ gives thus information about the curvature at a point $p \in \Sigma$. Therefore, we define the curvature at a point as follows:

Definition 2.11. *Let Σ be a surface immersed in $\mathbb{R}^3$ and let $p \in \Sigma$. Let N be the Gauss map. Then the* **Gaussian curvature** *K is defined as*

$$K = \det \mathrm{d}N$$

which is independent of basis.

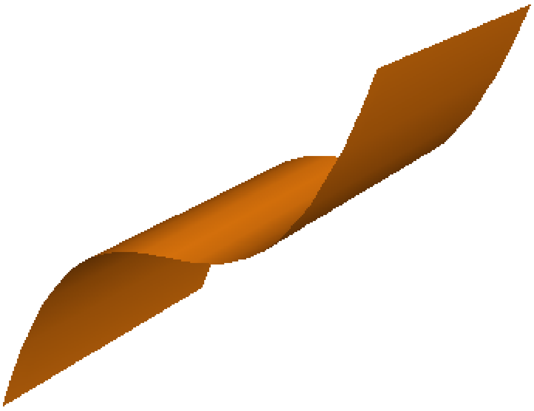
Note that this definition does not depend on the orientation of Σ, because when the orientation change, both principal curvatures change sign and so the minus signs cancel each other.

Definition 2.12. *A point in an embedded surface is called* **elliptic**, **parabolic** *or* **hyperbolic** *if the Gaussian curvature is respectively positive, negative or zero. When the normal curvatures are zero in each direction, we call such point* **planar***.*

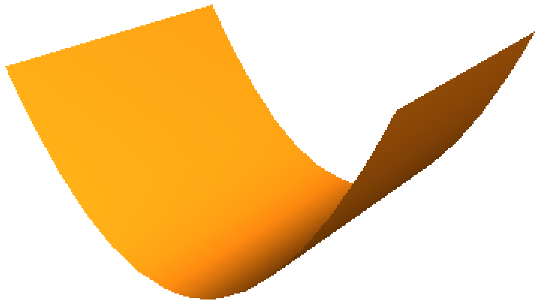
Example: In Figure 2a the Gaussian curvature at every point is negative: if we take a path along the x-axis, we get a downward pointing path. Thus, the second derivative is downward pointing. In the y-direction, we only get upward pointing paths. Therefore, the principal curvatures have opposite sign and so is this surface hyperbolic at each point.

Example: In Figure 2b all paths lay at the same side of the tangent plane. Therefore, all normal curves has the same sign and so is the Gaussian curvature positive. Thus this surface is elliptic at each point. Notice that this example is a subset of S^2. From straightforward calculation can be shown that the Gaussian curvature for the sphere is equal to 1.

Example: In Figure 2c all the second derivatives lie inside the plane. In this case the normal curvatures is zero in all directions and so is the plane 'planar'. Another example of a planar point is the center of Figure 5a. Note that in Figure 5b the Gaussian curvature is zero, but k_n is non-zero along the x-axis. This point is thus a parabolic, but not a planar point.



(a) *The parametrization of $z = x^3$. Note that the normal curvature at the centre is zero in all directions.*



(b) *The parametrization of $z = x^2$. The Gaussian curvature at the centre is zero, but the normal curvature of the path along the x-axis is non-zero.*

Figure 5: *Some examples of surfaces where $K = 0$ at the centre. In the first figure the normal curvature at $(0, 0, 0)$ is zero in each direction, but in the second figure the normal curvature in the x-direction is non-zero.*

2.2 Intermezzo: orthonormal frames

Before we continue, we will introduce the notion of frames. The proofs and definitions of this chapter comes from Lee in [8] and can be skipped if the reader is already familiar with this topic. Next section we will find relations from which the Gaussian curvature can be recovered.

Definition 2.13 (Lee [8], pg. 174)**.** *A* **vector field** *is a smooth map $X: M \to TM$ such that $\pi \circ X$ is the identity function.*

Just as in linear algebra we can define a basis on the tangent bundle. Let $U \subseteq M$ be open and let $(X_1, \ldots, X_k)$ be an ordered set of vector fields on U. If $(X_1, \ldots, X_k)$ forms a basis for T_pM for all $p \in U$, then we say that $(X_1, \ldots, X_k)$ is a local frame of M. If $U = M$, we speak of a global frame.

Some examples of (local) frames are:

1. The standard basis of $\mathbb{R}^n$.

2. The induced basis $\{\frac{\partial}{\partial x_i}\}$ of T_pM. This frame is called the **coordinate frame**.

In the case we have a Riemannian metric on our manifold we can also define orthonormal frames:

Definition 2.14 (Lee [8], pg. 178). *We say that a local frame $(e_1, \dots e_n)$ for M on an open subset $U \subseteq M$ is an* **orthonormal frame** *if it is an orthonormal basis at each point $p \in U$.*

Lemma 2.15 ([8] Lemma 8.13, page 179). *For every smooth manifold there always exists a local orthonormal frame in a neighborhood of $p \in M$.*

Proof: We will use the Gramm-Schmidt procedure: Choose an arbitrary a local frame $(X_1, \dots, X_n)$ around $p \in M$ defined on the open $U \subseteq M$. Take for example the coordinate frame. We will use induction. Suppose our frame consists only of one vector field X_1. Then define the smooth vector field

$$e_1 = \frac{X_1}{\|X_1\|_p}$$

Notice that this is an orthonormal frame on an 1-manifold. Now suppose we have a local frame $(X_1, \cdots X_{k+1})$ and suppose that we found $(e_1, \dots, e_k)$ independent vector fields such that $\mathrm{span}(X_1\big|_p, \dots X_k\big|_p) = \mathrm{span}(e_1\big|_p, \dots, e_k\big|_p)$ for all $p \in U$ Define the vector field e_{k+1} as follows:

$$e_{k+1}\big|_p = \frac{X_{k+1}\big|_p - \sum_{i=1}^k \alpha_i(p) e_i\big|_p}{\|X_{k+1}\big|_p - \sum_{i=1}^k \alpha_i(p) e_i\big|_p\|_p}$$

Here α_i is a real valued differentiable function defined on the manifold, which is to be defined. Because X_{k+1} is independent of $\mathrm{span}(X_1\big|_p, \dots X_k\big|_p)$, so is e_{k+1} to $\mathrm{span}(e_1\big|_p, \dots e_k\big|_p)$. From this it follows that the denominator is nowhere zero. Thus e_{k+1} is well defined. We also know that $\langle e_{k+1}, e_{k+1}\rangle = 1$. We only have to pick α_i such that e_{k+1} is tangent to all other e_i. Using the given Riemannian metric we can calculate that for $l \leq k$

$$\langle e_{k+1}|_p, e_l|_p\rangle_p = \frac{1}{\|\dots\|_p}\left(\langle X_{k+1}|_p, e_l|_p\rangle_p - \sum_{i=0}^k \alpha_i \langle e_i|_p, e_l|_p\rangle_p\right) = 0.$$

Notice that $\langle e_i|_p, e_l|_p\rangle_p = \delta_{i,l}$. This simplifies to

$$\langle X_{k+1}|_p, e_l|_p\rangle_p - \alpha_l = 0$$

When we pick $\alpha_i = \langle X_{k+1}, e_i\rangle$, we have found an local orthonormal frame for our manifold. ☐

2.3 The Gaussian curvature is independent of the normal vector

In this paragraph we will find an intrinsic expression of K, i.e. an expression which only depend on the induced Riemannian metric on the tangent plane. We will use a technique which is called the method of moving frames [13]. We will follow do Carmo [5, chapter 5] and use Lee in [7] for the notion of connections.

In calculus, we had a notion of directional derivative. Recall that the directional derivative in $\mathbb{R}^n$ is a bilinear function which has 2 arguments: a direction and a multilinear map to differentiate. Notice that $D\colon \Gamma(\mathbb{R}^n) \times \Gamma(\mathbb{R}^n) \to \Gamma(\mathbb{R}^n)$ can be rewritten as $D\colon \Gamma(\mathbb{R}^n) \to (\Gamma(\mathbb{R}^n) \to \Gamma(\mathbb{R}^n))$ or $D\colon \Gamma(\mathbb{R}^n) \to \Gamma(\mathbb{R}^n)^* \otimes \Gamma(\mathbb{R}^n)$. It is possible to extend this definition to vector bundles. Given a vector bundle $\pi\colon E \to M$ over a smooth n-manifold, we want to make in analogy to the directional derivative, the following linear map

$$\nabla\colon \Gamma(E) \to \Gamma(E \otimes T^*M).$$

Recall that the directional derivative has also a Leibniz rule. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function and $X, Y\colon \mathbb{R}^n \to \mathbb{R}^n$ be vector fields. Then $D_X(fY) = \mathrm{d}f(X) \cdot Y_p + f(p)D_XY$, We require that ∇ has the same property. Then

$$\nabla(fX) = \mathrm{d}f \otimes X + f\nabla X$$

where X is a section of E and f is a smooth function on M. When ∇ satisfies these requirements, we call ∇ a **connection**. When $E = TM$ we call ∇ an affine or **linear connection**. Before we continue we define some terminology that are related to connections:

Definition 2.16. *Given a connection ∇, $X \in \Gamma(TM)(M)$ and $s \in \Gamma(E)$. Then ∇s returns a linear function $\Gamma(TM) \to \Gamma(E)$. Define the* **covariant derivative** *$\nabla_X s$ of s in the direction of X to be*

$$\nabla_X s = (\nabla s)(X).$$

Notice that for any smooth function f on M we get that $\nabla_{fX} s = f\nabla_X s$ and $\nabla_X(fs) = \mathrm{d}f(X) \otimes s + f\nabla_X s$. Some writers like Lee define connections using these properties [7, pg. 49-50]. Both definitions are equivalent.

Theorem 2.17 (Lee [7], Th. 5.4, pg. 68)**.** *If (M, g) is a Riemannian manifold, then there exists a unique linear connection such that the following two properties holds:*

1. *Symmetric: The* **torsion tensor** *$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ is equal to zero.*

2. Compatible with g: For all vector fields X, Y, Z the following relation holds:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

We call this unique connection the Riemannian connection or the **Levi-Civita connection**

Proof: Let X, Y, Z be vector fields on M. Because ∇ is compatible with g, we get the following relations:

$$\begin{aligned} X\langle Y, Z\rangle =& \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ Y\langle Z, X\rangle =& \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \\ Z\langle X, Y\rangle =& \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \end{aligned}$$

Next we introduce the formula $\tau(X, Y) = 0$:

$$\begin{aligned} X\langle Y, Z\rangle =& \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle + \langle Y, [X, Z]\rangle \\ Y\langle Z, X\rangle =& \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle + \langle Z, [Y, X]\rangle \\ Z\langle X, Y\rangle =& \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Y Z\rangle + \langle X, [Z, Y]\rangle \end{aligned}$$

We can solve this for $\langle \nabla_X Y, Z\rangle$. As a result we get:

$$\begin{aligned} \langle \nabla_X Y, Z\rangle = & \tfrac{1}{2}\left(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \right. \\ - & \left. \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle\right) \end{aligned} \tag{1}$$

This uniquely determines ∇. Also, this connection exists. Since, for every coordinate chart U_α we can pick a local frame. Using this frame we can define ∇ on each U_α using Equation 1. Also, Formula 1 ensures us that ∇ is well defined on overlaps. Hence, ∇ exists.

We only have to prove that ∇ is a connection. All terms on the right are $\mathbb{R}$-linear over X and Y. Thus,

$$\begin{aligned} \nabla_X(aY_1 + bY_2) =& a\nabla_X Y_1 + b\nabla_X Y_2 \qquad \text{for } a, b \in \mathbb{R} \text{ and } Y_1, Y_2 \in \Gamma(TM) \\ \nabla_{aX_1 + bX_2}(Y) =& a\nabla_{X_1} Y + b\nabla_{X_2} Y \qquad \text{for } a, b \in \mathbb{R} \text{ and } X_1, X_2 \in \Gamma(TM). \end{aligned}$$

Next we check that $\nabla_X Y$ is C^∞-linear over X: Let $f \in C^\infty(M)$, then:

$$\begin{aligned}
2\langle \nabla_{fX} Y, Z\rangle =& fX\langle Y, Z\rangle + Y\langle Z, fX\rangle - Z\langle fX, Y\rangle - \\
&-\langle Y, [fX, Z]\rangle - \langle Z, [Y, fX]\rangle + \langle fX, [Z, Y]\rangle \\
=& fX\langle Y, Z\rangle + Y(f)\langle Z, X\rangle + fY\langle Z, X\rangle - Z(f)\langle X, Y\rangle - fZ\langle X, Y\rangle - \\
&-Z(f)\langle Y, X\rangle + f\langle Y, [Z, X]\rangle - Y(f)\langle Z, X\rangle - f\langle Z, f[Y, X]\rangle + \langle fX, [Z, Y]\rangle \\
=& fX\langle Y, Z\rangle + fY\langle Z, X\rangle - fZ\langle X, Y\rangle - \\
&-f\langle Y, [X, Z]\rangle - f\langle Z, [Y, X]\rangle + f\langle X, [Z, Y]\rangle \\
=& 2\langle f\nabla_X Y, Z\rangle
\end{aligned}$$

Because Z can be chosen arbitrary, we get that $\nabla_{fX} Y = f\nabla_X Y$.

At last we have to check the Leibniz rule: Let $f \in C^\infty(M)$. Then:

$$\begin{aligned}
2\langle \nabla_X fY, Z\rangle =& X\langle fY, Z\rangle + fY\langle Z, X\rangle - Z\langle X, fY\rangle - \\
&-\langle fY, [X, Z]\rangle - \langle Z, [fY, X]\rangle + \langle X, [Z, fY]\rangle \\
=& X(f)\langle Y, Z\rangle + fX\langle Y, Z\rangle + fY\langle Z, X\rangle - Z(f)\langle X, Y\rangle - fZ\langle X, Y\rangle - \\
&-f\langle Y, [X, Z]\rangle + X(f)\langle Z, Y\rangle + f\langle Z, [X, Y]\rangle + Z(f)\langle X, Y\rangle + f\langle X, [Z, Y]\rangle \\
=& 2\langle f\nabla_X Y, Z\rangle + 2X(f)\langle Y, Z\rangle
\end{aligned}$$

Again, Z is arbitrary. Thus, $\nabla_X(fY) = f\nabla_X Y + X(f)Y$ and we conclude that ∇ is a connection. □

We work in $\mathbb{R}^n$. On an open set $U \subseteq \mathbb{R}^n$ let $\{e_1, \ldots, e_n\}$ be an orthonormal frame. We can define a set of 1-forms $\{w_1, \ldots, w_n\}$ such that $w_i(e_j) = \delta_{ij}$. The idea is that we take the cobasis of our frame. We call $\{w_1, \ldots, w_n\}$ the coframe associated to $\{e_i\}$. We want to investigate how $\{w_i\}$ behaves when we smoothly change frames. Let $\widetilde{\nabla}$ be the Levi-Civita connection for $\mathbb{R}^n$. Define the following 1-form w_{ij} as follows:

$$w_{ij}(X) = \langle \widetilde{\nabla}_X e_i, e_j\rangle$$

Lemma 2.18. *The 1-form w_{ij} is anti-symmetric, i.e. $w_{ij} = -w_{ji}$*

Proof: Let X be a vector field. Because $\widetilde{\nabla}$ is compatible with the metric and $\langle e_i, e_j\rangle$ is constant, we get that

$$0 = X\langle e_i, e_j\rangle = \langle \widetilde{\nabla}_X e_i, e_j\rangle + \langle e_i, \widetilde{\nabla}_X e_j\rangle = w_{ij}(X) + w_{ji}(X).$$

So we can conclude that $w_{ij} = -w_{ji}$ □

Theorem 2.19 (The structure relations of Cartan)**.** *Let $U \subseteq \mathbb{R}^n, \{e_i\}$, $\{w_i\}$ and w_{ij} be as above. Then the following relations hold for all $1 \leq i, j \leq n$:*

$$\mathrm{d}w_i = \sum_{k=1}^{n} w_k \wedge w_{ki} \tag{2}$$

$$\mathrm{d}w_{ij} = \sum_{k=1}^{n} w_{ik} \wedge w_{kj} \tag{3}$$

Proof: Recall that the exterior derivative of an 1-form w is given by

$$\mathrm{d}w(X,Y) = Xw(Y) - Yw(X) - w([X,Y])$$

When we apply this formula on w_i for the generators (e_a, e_b) we get:

$$\begin{aligned} \mathrm{d}w_i(e_a, e_b) =& e_a w_i(e_b) - e_b w_i(e_a) - w_i([e_a, e_b]) \\ =& e_a \delta_{ib} - e_b \delta_{ia} - w_i([e_a, e_b]) \\ =& - w_i([e_a, e_b]) \end{aligned}$$

In the last step we used that the derivative of δ is zero. Using the fact that $\widetilde{\nabla}$ is torsion free, we get

$$\begin{aligned} \mathrm{d}w_i(e_a, e_b) =& - w_i(\widetilde{\nabla}_{e_a} e_b - \widetilde{\nabla}_{e_b} e_a) \\ =& - w_i(\widetilde{\nabla}_{e_a} e_b) + w_i(\widetilde{\nabla}_{e_b} e_a) \\ =& - w_{bi}(e_a) + w_{ai}(e_b). \end{aligned}$$

On the other hand we have

$$\begin{aligned} \sum_{k=1}^{n} w_k \wedge w_{ki}(e_a, e_b) =& \sum_{k=1}^{n} (w_k(e_a) w_{ki}(e_b) - w_k(e_b) w_{ki}(e_a)) \\ =& w_{ai}(e_b) - w_{bi}(e_a) \\ =& \mathrm{d}w_i(e_a, e_b). \end{aligned}$$

Using the fact that $d^2 = 0$ and using Relation (2) we get that:

$$
\begin{aligned}
0 =& \mathrm{d}^2 w_i \\
=& \sum_{j=1}^{n} \mathrm{d}(w_{ij} \wedge w_j) && \text{Using 2 and anti-symmetry} \\
=& \sum_{j=1}^{n} (\mathrm{d}w_{ij} \wedge w_j - w_{ij} \wedge \mathrm{d}w_j) && \text{Leibniz rule} \\
=& \sum_{j=1}^{n} (\mathrm{d}w_{ij} \wedge w_j - \sum_{k=1}^{n} w_{ij} \wedge w_{jk} \wedge w_k) && \text{Using 2 and anti-symmetry} \\
=& \sum_{j=1}^{n} (\mathrm{d}w_{ij} - \sum_{k=1}^{n} w_{ik} \wedge w_{kj}) \wedge w_j && \text{Reordering}
\end{aligned}
$$

Because all w_j are independent, $\mathrm{d}w_{ij} = \sum_{k=1}^{n} w_{ik} \wedge w_{kj}$. □

We have found a set of relations that describes the structure on $U \subseteq \mathbb{R}^n$. If we have a surface Σ immersed in $\mathbb{R}^n$, we might wonder how the structure relations look like on our manifold. Suppose Σ is a 2-manifold and $\iota\colon \Sigma \to \mathbb{R}^3$ is an immersion. To find the structure relations for Σ we take $p \in \Sigma$ and let $U \subseteq \Sigma$ be a neighborhood of $\iota(p)$, such that the restriction of U the immersion is an embedding. Let $V \subseteq \mathbb{R}^3$, such that V intersected with $\iota(\Sigma)$ is the image $\iota(U)$. For V we van pick an orthonormal frame $\{e_i\}$ and calculate the dual frame $\{w_i\}$. Choose the frame such that e_1 and e_2 are tangent to Σ. The structure relations for V are:

$$
\begin{aligned}
\mathrm{d}w_1 =& w_2 \wedge w_{21} + w_3 \wedge w_{31} \\
\mathrm{d}w_2 =& w_1 \wedge w_{12} + w_3 \wedge w_{32} \\
\mathrm{d}w_3 =& w_1 \wedge w_{13} + w_2 \wedge w_{23} \\
\mathrm{d}w_{12} =& w_{13} \wedge w_{32} \\
\mathrm{d}w_{13} =& w_{12} \wedge w_{23} \\
\mathrm{d}w_{23} =& w_{21} \wedge w_{13}
\end{aligned}
$$

We can use the pullback to find a frame on Σ. Recall that a pullback x^* is defined on an n-form w at p as

$$
x^*(w)_p(v_1, \ldots, v_n) = w_{x(p)}(\mathrm{d}x(v_1), \ldots, \mathrm{d}x(v_n))
$$

and recall that the pullback commutes with d and distributes with $\wedge$. So for all $i, j \in \{1,2,3\}$, the pullback $\iota^*(w_i)$ and $\iota^*(w_{ij})$ defines the structure relations on Σ.

Let's calculate the pullback of w_3. We will show that $\iota^*(w_3) = 0$. Indeed, let $p \in \Sigma$ and $v \in T_p\Sigma$. Then $\mathrm{d}\iota(v) = a_1e_1 + a_2e_2$ for some $a_1, a_2 \in \mathbb{R}$. Following the definition of the pullback we get

$$\iota^*(w_3)_p(v) = (w_3)_{\iota(p)}(\mathrm{d}\iota(v)) = (w_3)_{\iota(p)}(a_1e_1 + a_2e_2) = 0.$$

When we restrict ι to U we get that the pullback is a composition of injective functions. Therefore, on $\iota(U)$, the 1-form w_3 is zero. Using this and the structure relations, we get on $\iota(U)$

$$\mathrm{d}w_3 = w_1 \wedge w_{13} + w_2 \wedge w_{23} = 0.$$

Now we need the lemma of Cartan:

Lemma 2.20 (do Carmo [5], Lemma 1, pg. 80)**.** *Let V^n be an n-dimensional vector space and let $w_1, \ldots w_r : V^n \to \mathbb{R}$, $r \leq n$ be 1-forms such that all w_i are independent. Assume that there exists 1-forms $\theta_1, \ldots \theta_r$ such that $\sum_{i=0}^r w_i \wedge \theta_i = 0$. Then*

$$\theta_i = \sum_{j=1}^r a_{ij} w_j \quad \textit{with} \quad a_{ij} = a_{ji}.$$

Proof: Extend $w_1, \ldots w_r$ to a basis $\{w_1, \ldots, w_n\}$ of V^*. Then for some a_{ij} and b_{ij}, θ_i van be written as

$$\theta_i = \sum_{j=1}^r a_{ij} w_j + \sum_{l=r+1}^n b_{il} w_l.$$

Using $\sum_{i=0}^r w_i \wedge \theta_i = 0$, we get that

$$\begin{aligned}\sum_{i=1}^r w_i \wedge \theta_i =& \sum_{i=1}^r \sum_{j=1}^r a_{ij} w_i \wedge w_j + \sum_{i=1}^r \sum_{l=r+1}^n b_{il} w_i \wedge w_l \\ =& \sum_{i=1}^r \sum_{j=i+1}^r (a_{ij} - a_{ji}) w_i \wedge w_j + \sum_{i=1}^r \sum_{l=r+1}^n b_{il} w_i \wedge w_l \\ =& 0.\end{aligned}$$

Because w_i are independent, $a_{ij} = a_{ji}$ and $b_{il} = 0$. □

Applying Cartan's lemma to $\mathrm{d}w_3$ we get that

$$\begin{aligned} w_{13} =& h_{11}w_1 + h_{12}w_2 \\ w_{23} =& h_{21}w_1 + h_{22}w_2 \end{aligned}$$

for some $h_{ij} \in \mathbb{R}$.

We will relate the values h_{ij} to the Gaussian curvature. Recall that the Levi-Civita connection is unique. In $\mathbb{R}^3$ the differential $\mathrm{d}N(X)$ is equal to $\widetilde{\nabla}_X e_3$. Denote $\mathrm{d}N$ as a matrix in the frame $\{e_1, e_2\}$:

$$\begin{aligned} \mathrm{d}N &= \begin{pmatrix} \langle \mathrm{d}Ne_1, e_1\rangle & \langle \mathrm{d}Ne_2, e_1\rangle \\ \langle \mathrm{d}Ne_1, e_2\rangle & \langle \mathrm{d}Ne_2, e_2\rangle \end{pmatrix} \\ &= \begin{pmatrix} \langle \widetilde{\nabla}_{e_1} e_3, e_1\rangle & \langle \widetilde{\nabla}_{e_2} e_3, e_1\rangle \\ \langle \widetilde{\nabla}_{e_1} e_3, e_2\rangle & \langle \widetilde{\nabla}_{e_2} e_3, e_2\rangle \end{pmatrix} \\ &= \begin{pmatrix} w_{31}(e_1) & w_{32}(e_1) \\ w_{31}(e_2) & w_{32}(e_2) \end{pmatrix} \\ &= - \begin{pmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{pmatrix} \end{aligned}$$

Recall that the Gaussian curvature is defined as the determinant of $\mathrm{d}N$. We get that $K = h_{11}h_{22} - h_{12}^2$. To get a more convenient formulation we can calculate the following:

$$\mathrm{d}w_{12} = w_{13} \wedge w_{32} = -(h_{11}h_{22} - h_{12}^2)w_1 \wedge w_2 = -Kw_1 \wedge w_2$$

At the first glance it looks like that the Gaussian curvature does not depend how it is immersed in $\mathbb{R}^3$, but only depends on the induced metric on $T_p\Sigma$. This is indeed true and this result is called the Egregium theorem of Gauss[2]. We will prove this formally:

Theorem 2.21 (do Carmo [5], Proposition 2, pg. 92)**.** *If $\iota, \tilde{\iota}\colon \Sigma \to \mathbb{R}^3$ are two immersions with the same induced metric, then at each point $p \in \Sigma$ the Gaussian curvature are the same for both surfaces.*

Proof: Let $U \subseteq \Sigma$ be a neighborhood of p and pick an orthonormal frame $\{e_1, e_2\}$ in U. The frame $\{\iota^*(e_1), \iota^*(e_2)\}$ can be extended to an orthonormal frame in a subset of $\mathbb{R}^3$. Similarly the frame $\{\tilde{\iota}^*(e_1), \tilde{\iota}^*(e_2)\}$ can also be extended to a local orthonormal frame in $\mathbb{R}^3$. Let $w_i, \tilde{w}_i$ be the corresponding coframes and let $w_{ij} = \langle \nabla e_i, e_j\rangle$,

[2]Egregium is an adjective conjugation of Egregius, which means Extraordinary [9]

$\tilde{w}_{ij} = \langle \widetilde{\nabla} \tilde{e}_i, \tilde{e}_j \rangle$. Because the metric is the same for both immersions, we have that $w_1 = \tilde{w}_1$ and $w_2 = \tilde{w}_2$. Because the Levi-Civita connection is unique, we get $w_{12} = \tilde{w}_{12}$. Now the following holds:

$$-K w_1 \wedge w_2 = \mathrm{d}w_{12} = \mathrm{d}\tilde{w}_{12} = -\tilde{K} w_1 \wedge w_2$$

We can conclude that $K = \tilde{K}$ $\square$.

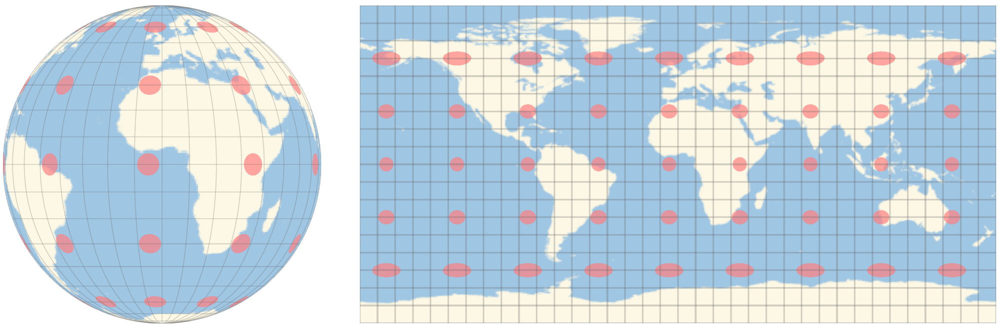


Figure 6: *According to Gauss 'remarkable' theorem, it is impossible to make global map which preserves length and area. The pink disks are in both images the same. As you can see, this global map is distorted.*

Example: As described by an article in Wired [2] we can use this theorem in everyday life. Take for example the art of drawing world maps. Look at Figure 6. To get a useful map, it must give an accurate representation of length and area. In other words. we would like that the global map is an isometric immersion. But if we could define such map, we would by Theorem 2.21 get that planes and spheres have the same Gaussian curvature. We know from earlier example, that this isn't true. Therefore, global maps will always have artifacts.

Example: Recall the pizza example from the introduction. This is also described in Wired [2]. Pizza slices are originally flat surfaces. Thus, the Gaussian curvature is zero. According to Theorem 2.21, K is conserved under isometric immersions. In Figure 1 both pizza slices are isometrically immersed. But if in Figure 1b the point will tip down, the curvature will become negative and so this immersion will change into a non-isometric immersion. In everyday life this will mean that the pizza-dough will stretch or that the pizza slice will be torn apart. For well baked pizza's this wont happen, so by bending you prevent the point from tipping down.

2.4 The Gaussian curvature for non-embedded 2-manifolds

As we have seen in last section, K is independent of isometric immersions. This raises the question: Is it possible to extend the definition of K to surfaces that can't be embedded in $\mathbb{R}^3$? Take for example the Klein bottle. It is a 2-manifold, but we don't have a curvature defined for it.

Look back at the structure relations. For $U \in \Sigma$ the were:

$$\begin{aligned} \mathrm{d}w_1 =& w_{12} \wedge w_2 \\ \mathrm{d}w_2 =& w_{21} \wedge w_1 \\ \mathrm{d}w_{12} =& w_{13} \wedge w_{32} \\ \mathrm{d}w_{13} =& w_{12} \wedge w_{23} \\ \mathrm{d}w_{23} =& w_{21} \wedge w_{13} \end{aligned}$$

Notice that $\mathrm{d}w_1$ and $\mathrm{d}w_2$ does not depend on the space the surface is embedded in. We might wonder if it is possible to define w_{12} for non-embedded space, such that the first two relations above holds. We will use the next lemma:

Lemma 2.22 (The Gauss formula, Lee [7], Th. 8.2, pg. 135)**.** *Let (M, g) be a Riemannian manifold immersed in $(\tilde{M}, \tilde{g})$. Assume that M is immersed isometrically into $\tilde{M}$. Let ∇ and $\widetilde{\nabla}$ be the corresponding Levi-Civita connections and let X and Y be vector fields. Then the tangential projection of $\widetilde{\nabla}_X Y$ to TM is equal to $\nabla_X Y$*

Proof: Denote the projection of $\widetilde{\nabla}_X Y$ to TM by $^\top$. Notice that this is a connection itself. If we show that it is symmetric and compatible with g, we can use the uniqueness of the Riemannian connection to prove the equality.

First we prove symmetry: because X, Y are vector fields on M, it follows that $[X, Y]$ is also a vector field on M, so $[X, Y]^\top = [X, Y]$. Using the symmetry of $\widetilde{\nabla}$, we get that

$$\begin{aligned} [X, Y] =& [X, Y]^\top \\ =& (\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X)^\top \\ =& (\widetilde{\nabla}_X Y)^\top - (\widetilde{\nabla}_Y X)^\top \end{aligned}$$

Thus, $(\widetilde{\nabla}_X Y)^\top$ is symmetric. Next we will show compatibility with g: Using the compatibility of $\tilde{g}$ we get:

$$X\tilde{g}(Y, Z) = \tilde{g}(\widetilde{\nabla}_X Y, Z) + \tilde{g}(Y, \widetilde{\nabla}_X Z)$$

Notice that the normal component will cancel, because Y and Z are in TM. Therefore,

$$X\tilde{g}(Y,Z) = \tilde{g}((\widetilde{\nabla}_X Y)^\top, Z) + \tilde{g}(Y, (\widetilde{\nabla}_X Z)^\top)$$

Using the fact that M is isometrically immersed into $\tilde{M}$, we can conclude that $(\widetilde{\nabla}_X Y)^\top$ is compatible with g. Because the Riemannian connection is unique, we get that $(\widetilde{\nabla}_X Y)^\top = \nabla_X Y$ □

Using this theorem we can relate w_{12} to the immersed connection. Let X a vector field on Σ. Recall that $w_{12}(X) = \langle \widetilde{\nabla}_X e_1, e_2 \rangle$. When we decompose $\widetilde{\nabla}$ into a tangent part and a normal part, we get that $w_{12}(X) = \langle \nabla_X e_1, e_2 \rangle$.

We can use this to define w_{12} for non-embedded surfaces. Let Σ be a Riemannian 2-manifold. Then Σ has a unique Levi-Civita connection ∇. Let $\{w_1, w_2\}$ be a coframe on $U \subseteq \Sigma$. Define w_{ij} as $\langle \nabla_X e_i, e_j \rangle$ for $i, j \in \{1, 2\}$. Notice that we didn't use anything about $\mathbb{R}^n$ when we defined the structure relations. The structure relations of Σ are:

$$\begin{aligned} \mathrm{d}w_1 &= w_{12} \wedge w_2 \\ \mathrm{d}w_2 &= w_{21} \wedge w_1 \end{aligned}$$

We can also define the Gaussian curvature on U:

Definition 2.23. *Using the definitions above. The* **Gaussian curvature** K *is defined by the relation*

$$\mathrm{d}w_{12} = -K w_1 \wedge w_2.$$

We need to check if this definition is independent of the choice of frame. The rest of this section will be dedicated to this.

Suppose $\{e_1, e_2\}$ and $\{\tilde{e}_1, \tilde{e}_2\}$ are two orthonormal frames. Let $\{w_1, w_2\}$ and $\{\tilde{w}_1, \tilde{w}_2\}$ be the corresponding dual frames and let w_{12} and $\tilde{w}_{12}$ be the corresponding connection forms. Let $f, g \colon \Sigma \to \mathbb{R}$ be functions such that

$$\tilde{e}_1 = f e_1 + g e_2.$$

From orthonormality and straightforward calculation follows that

$$\begin{aligned} \tilde{e}_2 &= \pm(-g e_1 + f e_2) \\ 1 &= f^2 + g^2 \end{aligned}$$

The plus-minus sign comes from the fact that both frames may have different orientation: Use the plus sign in the case they have the same orientation. Using this we can calculate the wedge product of $\tilde{w}_1$ and $\tilde{w}_2$:

$$\begin{aligned}\tilde{w}_1 \wedge \tilde{w}_2 &= \pm\,(fw_1 + gw_2) \wedge (-gw_1 + fw_2)\\ &= \pm\,(f^2+g^2) w_1 \wedge w_2\\ &= \pm\, w_1 \wedge w_2\end{aligned}$$

The wedge product is thus the same up to orientation. Now we will check if w_{12} change under orientation. We will do this by differentiating w_1 and w_2 and using the structure relations. Notice that

$$\begin{aligned}w_1 =& f\tilde{w}_1 \mp g\tilde{w}_2\\ w_2 =& g\tilde{w}_1 \pm f\tilde{w}_2.\end{aligned}$$

Exterior derivation of w_1 and w_2 yields

$$\begin{aligned}\mathrm{d}w_1 =& \mathrm{d}f \wedge \tilde{w}_1 + f\mathrm{d}\tilde{w}_1 \mp \mathrm{d}g \wedge \tilde{w}_2 \mp g\mathrm{d}\tilde{w}_2\\ \mathrm{d}w_2 =& \mathrm{d}g \wedge \tilde{w}_1 + g\mathrm{d}\tilde{w}_1 \pm \mathrm{d}f \wedge \tilde{w}_2 \pm f\mathrm{d}\tilde{w}_2.\end{aligned}$$

Next we fill in the structure relations and the anti-symmetry of $\tilde{w}_{12}$:

$$\begin{aligned}\mathrm{d}w_1 =& \mathrm{d}f \wedge \tilde{w}_1 + f\tilde{w}_{12} \wedge \tilde{w}_2 \mp \mathrm{d}g \wedge \tilde{w}_2 \pm g\tilde{w}_{12} \wedge \tilde{w}_1\\ \mathrm{d}w_2 =& \mathrm{d}g \wedge \tilde{w}_1 + g\tilde{w}_{12} \wedge \tilde{w}_2 \pm \mathrm{d}f \wedge \tilde{w}_2 \mp f\tilde{w}_{12} \wedge \tilde{w}_1\end{aligned}$$

Next we use the change of coordinates formulas for $\tilde{w}_1$ and $\tilde{w}_2$:

$$\begin{aligned}\mathrm{d}w_1 =& (f\mathrm{d}f + g\mathrm{d}g) \wedge \tilde{w}_1 - (f\mathrm{d}g - g\mathrm{d}f) \wedge \tilde{w}_2 \pm (f^2+g^2)\tilde{w}_{12} \wedge w_2\\ \mathrm{d}w_2 =& (f\mathrm{d}f + g\mathrm{d}g) \wedge \tilde{w}_2 + (f\mathrm{d}g - g\mathrm{d}f) \wedge \tilde{w}_1 \mp (f^2+g^2)\tilde{w}_{12} \wedge w_1\end{aligned}$$

Recall that $f^2 + g^2 = 1$, so $f\mathrm{d}f + g\mathrm{d}g = 0$. Define $\tau = f\mathrm{d}g - g\mathrm{d}f$. Filling this in we get

$$\begin{aligned}\mathrm{d}w_1 =& (\pm\tilde{w}_{12} - \tau) \wedge w_2\\ \mathrm{d}w_2 =& -(\pm\tilde{w}_{12} - \tau) \wedge w_1.\end{aligned}$$

From the uniqueness of ∇ follows that

$$w_{12} = \pm\tilde{w}_{12} - \tau.$$

We notice that w_{12} and $\tilde{w}_{12}$ are different. Luckily $\mathrm{d}w_{12}$ is the same as $\mathrm{d}\tilde{w}_{12}$ up to orientation. Indeed, this follows from remark 2.2 from [14]: Calculating $\mathrm{d}w_{12}$ gives

$$\mathrm{d}w_{12} = \pm\mathrm{d}\tilde{w}_{12} - \mathrm{d}\tau.$$

Saying $\mathrm{d}w_{12}$ is the same as $\mathrm{d}\tilde{w}_{12}$ up to orientation, is the same as saying that $\mathrm{d}\tau = 0$. Calculating $\mathrm{d}\tau$ yields

$$\mathrm{d}\tau = \mathrm{d}f \wedge \mathrm{d}g - \mathrm{d}g \wedge \mathrm{d}f = 2\mathrm{d}f \wedge \mathrm{d}g.$$

Let $p \in U \subseteq \Sigma$ such that $f(p) \neq 0$. Wedging $f\mathrm{d}f + g\mathrm{d}g$ with $\mathrm{d}g$ gives $f\mathrm{d}f \wedge \mathrm{d}g$. Notice that $f\mathrm{d}f + g\mathrm{d}g$ is the derivative of $f^2 + g^2 = 1$. Therefore, $\mathrm{d}f \wedge \mathrm{d}g = 0$. Suppose $f(p) = 0$. Then $g(p) \neq 0$. Wedging $f\mathrm{d}f + g\mathrm{d}g$ with $\mathrm{d}f$ gives $g\mathrm{d}f \wedge \mathrm{d}g = 0$. In all cases $\mathrm{d}\tau = 0$.

We can finally conclude the following theorem:

Theorem 2.24 (do Carmo [5], Proposition 2, pg. 92)**.** *The Gaussian curvature, defined by the relation* $\mathrm{d}w_{12} = -Kw_1 \wedge w_2$ *is independent of the choice of frame.*

Proof: Suppose $\{e_1, e_2\}$ and $\{\tilde{e}_1, \tilde{e}_2\}$ are two orthonormal frames. Let $\{w_1, w_2\}$ and $\{\tilde{w}_1, \tilde{w}_2\}$ ne the corresponding dual frames and let $w_{12} = \langle \nabla e_1, e_2 \rangle$ and $\tilde{w}_{12} = \langle \widetilde{\nabla} \tilde{e}_1, \tilde{e}_2 \rangle$ be the corresponding connection forms. Then

$$-Kw_1 \wedge w_2 = \mathrm{d}w_{12} = \pm\mathrm{d}\tilde{w}_{12} = \mp\tilde{K}\tilde{w}_1 \wedge \tilde{w}_2 = -\tilde{K}w_1 \wedge w_2.$$

Therefore we can conclude that $K = \tilde{K}$. □

Corollary 2.25 (Hairy ball theorem, proof from Simic [14], pg. 6-7)**.** *There exists no non-vanishing smooth vector field on* $S^2 \subseteq \mathbb{R}^3$

Proof: Suppose that there exists a non-vanishing vector field X. Let $e_1 = \frac{X}{|X|}$ and let e_2 be such that $\{e_1, e_2\}$ is an orthonormal frame. Let $\{w_1, w_2\}$ be de corresponding coframe. Because X is non-vanishing, $\{e_1, e_2\}$ is a global frame on S^2. Recall that the Gaussian curvature of S^2 is 1 at every point. Using that $w_1 \wedge w_2$ is the volume-form we can calculate the area

$$\mathrm{Area}(S^2) = \int_{S^2} w_1 \wedge w_2$$

Using the relation $\mathrm{dw}_{12} = -Kw_1 \wedge w_2$ we get

$$\mathrm{Area}(S^2) = -\int_{S^2} \mathrm{d}w_{12}$$

Now we can apply stokes. Because S^2 has no boundary, we get that the integral will be zero:

$$\text{Area}(S^2) = -\int_{\partial S^2} w_{12} = 0$$

This is a contradiction, because the surface of a unit sphere has an area of 4π. □

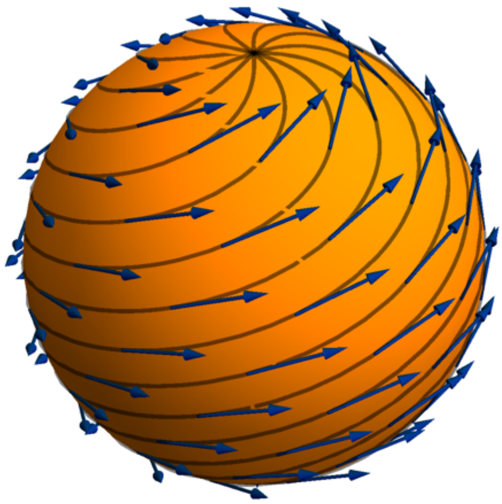


Figure 7: *An example of a vector field on S^2. Notice that on the top there is a singularity. According to the hairy ball theorem, it is not possible to find a vector field without singularities.*

2.5 The general notion of curvature

In calculus on $\mathbb{R}^n$, the partial derivatives of a multi-variable function will commute. Directional derivatives of a real valued function only commute when the Lie-bracket of vector fields is equal to zero. Because connections are a generalization of directional derivatives, we might wonder if there is an equation to measure the interchangeability like that for directional derivatives. We will investigate in this section.

Let $M = \mathbb{R}^n$ and let $\widetilde{\nabla}$ be the Levi-Civita connection of M. We want to calculate what happens when we commute two connections. Let X,Y and Z be vector fields. Let $\{e_i\}$ be the default coordinate basis and denote Z in this basis by:

$$Z = \sum_{i=1}^{n} Z_i e_i$$

Because $\widetilde{\nabla}$ is equal to the directional derivative in $\mathbb{R}^n$, is $\widetilde{\nabla}_X Z$ equal to $\sum_{i=1}^n X(Z_i)e_i$. To check that $\widetilde{\nabla}_X$ and $\widetilde{\nabla}_Y$ commute we calculate the following:

$$\begin{aligned}
\widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z &= \sum_{i=1}^{n}(\widetilde{\nabla}_X(Y(Z_i)) - \widetilde{\nabla}_Y(X(Z_i)))e_i \\
&= \sum_{i=1}^{n}(X(Y(Z_i)) - Y(X(Z_i)))e_i \\
&= \sum_{i=1}^{n}([X,Y](Z_i))e_i \\
&= \widetilde{\nabla}_{[X,Y]}Z
\end{aligned}$$

We can conclude from this that for $\mathbb{R}^n$, we have that $\widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z = 0$. For general Riemannian manifolds, it is not necessarily equal to zero. The rest of this section we will investigate how $\widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z$ will relate to the Gaussian curvature.

Definition 2.26. *Let (M,g) be a Riemannian manifold and let ∇ be the Levi-Civita connection. We define the* **Riemann curvature endomorphism** *to be the map $R\colon \Gamma(TM)\times\Gamma(TM)\times\Gamma(TM)\to\Gamma(TM)$ defined by*

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Lemma 2.27 (Lee [7], Prop. 7.1, pg. 117-118)**.** *The curvature endomorphism is a tensor field.*

Proof: Notice that R is multi-linear over $\mathbb{R}$. We only have to show that R is multi-linear over $C^\infty(M)$. Let f be a real valued function over M. From the properties of connections, it follows that

$$\begin{aligned}
R(X, fY)Z =& \nabla_X \nabla_{fY} Z - \nabla_{fY} \nabla_X Z - \nabla_{[X,fY]} Z \\
=& \nabla_X (f \nabla_Y Z) - f \nabla_Y \nabla_X Z - \nabla_{X(f)Y + f[X,Y]} Z \\
=& X(f) \nabla_Y Z + f \nabla_X (\nabla_Y Z) - f \nabla_Y \nabla_X Z - f \nabla_{[X,Y]} Z - X(f) \nabla_Y Z \\
=& f(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \\
=& fR(X, Y)Z.
\end{aligned}$$

Notice from the definition that $R(X, Y)Z = -R(Y, X)Z$. Therefore,

$$R(fX, Y)Z = -R(Y, fX)Z = -fR(Y, X)Z = fR(X, Y)Z.$$

At last we will check if R is $C^\infty(M)$-linear in Z:

$$\begin{aligned}
R(X, Y)(fZ) =& \nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{[X,Y]} (fZ) \\
\\
=& \nabla_X (Y(f)Z + f \nabla_Y Z) - \nabla_Y (X(f)Z + f \nabla_X Z) - \\
& -([X, Y](f))Z - f \nabla_{[X,Y]} Z \\
\\
=& X(Y(f))Z + Y(f) \nabla_X (Z) + X(f) \nabla_Y Z + f \nabla_X (\nabla_Y Z) - \\
& -Y(X(f))Z - X(f) \nabla_Y Z - Y(f) \nabla_X Z - f \nabla_Y \nabla_X Z - \\
& -([X, Y](f))Z - f \nabla_{[X,Y]} Z \\
\\
=& f(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) + \\
& +(X(Y(f)) - Y(X(f)) - [X, Y](f))Z \\
\\
=& fR(X, Y)Z
\end{aligned}$$

Therefore R is a tensor field. □

Definition 2.28. *Let (M, g) be a Riemannian manifold and let ∇ be the Levi-Civita connection. The* **Riemann curvature tensor***, Rm, is a 4-tensor defined by*

$$Rm(X, Y, Z, W) = g(R(X, Y, Z), W).$$

Let (M, g) be a Riemannian manifold that is isometrically immersed into $(\tilde{M}, \tilde{g})$. We will investigate how the curvature tensor Rm of M is related to the curvature tensor $\widetilde{Rm}$ of $\tilde{M}$.

Let $^\top$ denote the projection to the tangent bundle TM and $^\perp$ denote the projection to the normal bundle NM. Then for vector fields X and Y on M, the Levi-Civita connection $\widetilde{\nabla}$ for $\tilde{M}$ can be written as

$$\widetilde{\nabla}_X Y = (\widetilde{\nabla}_X Y)^\top + (\widetilde{\nabla}_X Y)^\perp,$$

because $T\tilde{M}|_M = TM \oplus NM$. From Lemma 2.22 follows that $(\widetilde{\nabla}_X Y)^\top = \nabla_X Y$, where ∇ is the Levi-civita connection of M. In this section we will focus more on the term $(\widetilde{\nabla}_X Y)^\perp$:

Definition 2.29. *Let (M, g) be a Riemannian manifold that is isometrically immersed into $(\tilde{M}, \tilde{g})$. The* **second fundamental form** *of M is the map $II : \Gamma(TM) \times \Gamma(TM) \to \Gamma(NM)$ and is defined by*

$$II(X, Y) = (\widetilde{\nabla}_X Y)^\perp.$$

Lemma 2.30 (Lee [7], Lm. 8,1, pg. 134)**.** *The second fundamental form of M is symmetric.*

Proof: Because the Levi-Civita connection is torsion-free, it follows that

$$II(X, Y) - II(Y, X) = (\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X)^\perp = [X, Y]^\perp.$$

We assumed that X and Y are sections of the tangent bundle. Therefore, the Lie bracket of X and Y will lie in the tangent bundle. Therefore, the projection to the normal bundle will yield zero. Thus, we can conclude that fundamental form is symmetric. □

Lemma 2.31 (Weingarter equation, Lee [7], Lm. 8.3, pg. 136)**.** *Let $X, Y \in \Gamma(TM)$ and suppose that $N \in \Gamma(NM)$. Then*

$$\langle \widetilde{\nabla}_X N, Y \rangle = -\langle N, II(X, Y) \rangle.$$

Proof: This proof will use the same tricks used in Lemma 2.4. Because N is orthonormal to Y, it follows that $\langle N, Y \rangle = 0$ and so will the derivative in the direction of X be zero. Using the compatibility of the metric it follows that:

$$\begin{aligned} 0 =& X\langle N, Y \rangle \\ =& \langle \widetilde{\nabla}_X N, Y \rangle + \langle N, \widetilde{\nabla}_X Y \rangle. \end{aligned}$$

Next we split $\widetilde{\nabla}$ into its normal and tangent components. The tangent components will disappear against the inner-product with the normal vector.

$$\begin{aligned} 0 =& \langle \widetilde{\nabla}_X N, Y \rangle + \langle N, \nabla_X Y + II(X,Y) \rangle \\ =& \langle \widetilde{\nabla}_X N, Y \rangle + \langle N, II(X,Y) \rangle \end{aligned}$$

From this we conclude the lemma. □

Lemma 2.32 (The Gauss Equation, Lee [7], Thm. 8.4, pg. 136)**.** *For any $X, Y, Z, W \in T_pM$, the following holds:*

$$\widetilde{Rm}(X,Y,Z,W) = Rm(X,Y,Z,W) - \langle II(X,W), II(Y,Z) \rangle + \langle II(X,Z), II(Y,W) \rangle$$

Proof: Recall the definition of the Riemann curvature:

$$\widetilde{Rm}(X,Y,Z,W) = \langle \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z, W \rangle$$

We will expand $\widetilde{\nabla}$ into its normal and tangent components:

$$\begin{aligned} \widetilde{Rm}(X,Y,Z,W) =& \langle \widetilde{\nabla}_X (\nabla_Y Z + II(Y,Z)), W \rangle - \langle \widetilde{\nabla}_Y (\nabla_X Z + II(X,Z)), W \rangle - \\ & - \langle \nabla_{[X,Y]} Z + II([X,Y],Z), W \rangle \end{aligned}$$

Because II lives in the normal bundle, the term $II([X,Y],Z)$ will cancel against the inner product with W. For the same reason, we can apply Lemma 2.31 to the equation:

$$\begin{aligned} \widetilde{Rm}(X,Y,Z,W) =& \langle \widetilde{\nabla}_X \nabla_Y Z, W \rangle - \langle \widetilde{\nabla}_X W, II(Y,Z) \rangle - \\ & - \langle \widetilde{\nabla}_Y \nabla_X Z, W \rangle + \langle \widetilde{\nabla}_Y W, II(X,Z) \rangle - \\ & - \langle \nabla_{[X,Y]} Z, W \rangle \end{aligned}$$

We again expand $\widetilde{\nabla}$ into its normal and tangent components. Notice that half of the terms will fall away due to the fact that the tangent bundle is normal to the normal bundle.

$$\begin{aligned} \widetilde{Rm}(X,Y,Z,W) =& \langle \nabla_X \nabla_Y Z + II(X, \nabla_Y Z), W \rangle - \langle \nabla_X W + II(X,W), II(Y,Z) \rangle - \\ & - \langle \nabla_Y \nabla_X Z + II(Y, \nabla_X Z), W \rangle + \langle \nabla_Y W + II(Y,W), II(X,Z) \rangle - \\ & - \langle \nabla_{[X,Y]} Z, W \rangle \\ \\ =& \langle \nabla_X \nabla_Y Z, W \rangle - \langle II(X,W), II(Y,Z) \rangle - \\ & - \langle \nabla_Y \nabla_X Z), W \rangle + \langle II(Y,W), II(X,Z) \rangle - \\ & - \langle \nabla_{[X,Y]} Z, W \rangle \end{aligned}$$

By reordering we will obtain te required result

$$\begin{aligned}\widetilde{Rm}(X,Y,Z,W) =&\langle \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z) - \nabla_{[X,Y]}Z, W\rangle - \\ &-\langle II(X,W), II(Y,Z)\rangle - \langle II(Y,W), II(X,Z)\rangle \\ \\ =&Rm(X,Y,Z,W) - \langle II(X,W), II(Y,Z)\rangle + \langle II(Y,W), II(X,Z)\rangle\end{aligned}$$

from which we conclude the proof. □

Using the previous lemma we will relate the Riemann curvature to the Gaussian curvature. Let (M, g) be a Riemannian manifold. According to Nash embedding theorem [12] there is a positive integer n such that M can be isometrically immersed into $\mathbb{R}^n$. Recall that the Riemann curvature of the Euclidean space is zero. Lemma 2.32 can thus be reduced to

$$Rm(X,Y,Z,W) = \langle II(X,W), II(Y,Z)\rangle - \langle II(Y,W), II(X,Z)\rangle.$$

We will relate the Gaussian curvature to this formula.

Theorem 2.33. *Let Σ be a Riemannian 2-manifold and let e_1 and e_2 be a local orthonormal frame on Σ. Let w_1 and w_2 be the corresponding coframe. Then the Riemann curvature of Σ and the Gaussian curvature of Σ are the same, that is:*

$$K = Rm(e_1, e_2, e_2, e_1)$$

Proof: According to Nash [12], we can assume that Σ is isometrically immersed into $\mathbb{R}^n$ for some large enough n. Extend the given frame to an orthonormal frame $\{e_1, e_2, e_3, \ldots e_n\}$ in $\mathbb{R}^n$. Also extend the coframe.

Let $\widetilde{\nabla}$ be the Levi-Civita connection on $\mathbb{R}^n$. Recall that the second fundamental form is the projection of $\widetilde{\nabla}$ to the normal bundle. In the given frame, the second fundamental form is equal to

$$II(X,Y) = \sum_{k=3}^{n} \langle \widetilde{\nabla}_X Y, e_k\rangle e_k.$$

Using this we will calculate the Riemann curvature.

$$
\begin{aligned}
&Rm(e_1, e_2, e_2, e_1) = \langle II(X, W), II(Y, Z)\rangle - \langle II(Y, W), II(X, Z)\rangle \\
&= \left\langle \sum_{k=3}^{n} \langle \widetilde{\nabla}_{e_1} e_1, e_k\rangle e_k, \sum_{k=3}^{n} \langle \widetilde{\nabla}_{e_2} e_2, e_k\rangle e_k \right\rangle - \left\langle \sum_{k=3}^{n} \langle \widetilde{\nabla}_{e_2} e_1, e_k\rangle e_k, \sum_{k=3}^{n} \langle \widetilde{\nabla}_{e_1} e_2, e_k\rangle e_k \right\rangle \\
&= \sum_{k=3}^{n} \langle \widetilde{\nabla}_{e_1} e_1, e_k\rangle \langle \widetilde{\nabla}_{e_2} e_2, e_k\rangle - \langle \widetilde{\nabla}_{e_2} e_1, e_k\rangle \langle \widetilde{\nabla}_{e_1} e_2, e_k\rangle
\end{aligned}
$$

Recall that w_{ij} is an 1-form defined as $w_{ij}(X) = \langle \widetilde{\nabla}_X e_i, e_j\rangle$. Using this we can reformulate the Riemann curvature as the sum of the wedges of 1-forms. We will use the anti-symmetry of w_{ij}:

$$
\begin{aligned}
Rm(e_1, e_2, e_2, e_1) &= \sum_{k=3}^{n} \langle \widetilde{\nabla}_{e_1} e_1, e_k\rangle \langle \widetilde{\nabla}_{e_2} e_2, e_k\rangle - \langle \widetilde{\nabla}_{e_2} e_1, e_k\rangle \langle \widetilde{\nabla}_{e_1} e_2, e_k\rangle \\
&= \sum_{k=3}^{n} w_{1k}(e_1) w_{2k}(e_2) - w_{1k}(e_2) w_{2k}(e_1) \\
&= \sum_{k=3}^{n} -w_{1k}(e_1) w_{k2}(e_2) - w_{1k}(e_2) w_{k2}(e_1) \\
&= \sum_{k=3}^{n} w_{1k} \wedge w_{k2}(e_2, e_1)
\end{aligned}
$$

Recall from the structure relations of Theorem 2.19 that $\mathrm{d}w_{ij} = \sum_k w_{ik} \wedge w_{kj}$. Using the definition of the Gaussian curvature we then get that

$$
\begin{aligned}
Rm(e_1, e_2, e_2, e_1) &= \mathrm{d}w_{12}(e_2, e_1) \\
&= -K w_1 \wedge w_2(e_2, e_1) \\
&= K
\end{aligned}
$$

from which we conclude the proof. □

3 Singularities of vector fields

To prove the Gauss-Bonnet theorem we will need the notion of an index of vector fields. In this chapter we will define what an index is and how we can calculate it. Next we will see how the index relates with the method of moving frames. Here we notice that we made a lot of choices. We will show that the index only depends on the vector field and not on the choices of path, frame and metric. At last we will give a relationship between indices and the Euler characteristic. The source of this chapter is do Carmo [5, Lemma 5 of chapter 5 and chapter 6].

3.1 An intuitive definition of the index

In this chapter we use the following definitions: Let Σ be a compact, oriented Riemannian 2-manifold and let $\{e_1, e_2\}$ be an orthonormal frame defined on a simply connected neighborhood of $p \in \Sigma$. Let $\{w_1, w_2\}$ be the corresponding dual frame and let w_{12} be the corresponding connection form. Let X be a smooth vector field.

Using X we can define an orthonormal frame. Let $\tilde{e}_1 = \frac{X}{|X|}$ and let e_2 such that $\{\tilde{e}_1, \tilde{e}_2\}$ is an orthonormal frame for which the orientation coincides with Σ. From this we can define the corresponding $\tilde{w}_1, \tilde{w}_2$ and $\tilde{w}_{12}$. Notice that this frame is well-defined on all points $p \in \Sigma$ where $X(p) \neq 0$. If $X(p) = 0$, then we call p a **singular** point. We would like to know the *number of "turns" given by the vector field as we go around an isolated singularity.* [5, pg. 100]. We define this number as follows:

Let $U \in \Sigma$ be a neighborhood such that it only contains one singularity p. If such U exists, we call p **isolated**. We assume that X has only isolated singularities. Since Σ is compact, X has finitely many isolated singularities. Pick in U a neighborhood V of p such that V is homeomorphic to a disk. Suppose that $\{e_1, e_2\}$ is defined on V. Let γ be a path which is the parameterization of ∂V. Let $\phi\colon I \to \mathbb{R}$ be a continuous function which is the 'angle'[3] between e_1 and $\tilde{e}_1$ along γ. Notice that ϕ can be made continuous by adding of subtracting multiples of 2π. By continuity also follows that $\phi(t_{start}) \equiv \phi(t_{end}) \mod 2\pi$. We define the **index** of X at p as $\frac{1}{2\pi}(\phi(t_{end}) - \phi(t_{start}))$.

For example, take a look at Figure 8(this is also example 4.3 in [14]). This vector field has a singularity at the origin. Let U and V be the closed disk at the origin with radius one. Now γ is the clockwise parameterization of the circle. At positions 1 to 5 the angle is measured and is visualized in the circles next to the vector field. It looks

[3]Later in this chapter we will define ϕ more formally.

like that the angle rotates one time clockwise. Because the index is the number of times the vector field turns *counterclockwise*, is the index in this example equal to -1.

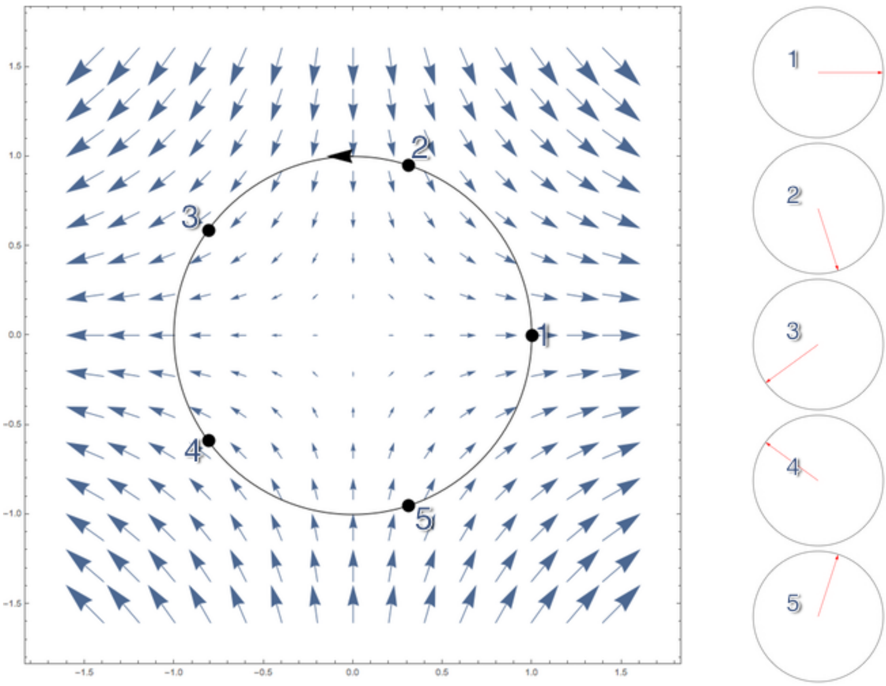


Figure 8: *The vector field in* $\mathbb{R}^2$ *of* $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. *The circle drawn is the image of the path* $\gamma(t) = \{\cos(t), \sin(t)\}$. *At the points 1 to 5 the angle is measured between the vector field and the default* $\frac{\partial}{\partial x}$ *vector of* $\mathbb{R}^2$. *These angles are shown next to the vector field. At the origin there is a singularity, which has an index of -1.*

We can calculate the index of this example explicitly. The path $\gamma\colon [0, 2\pi] \to \mathbb{R}^2$ is defined as $\gamma(t) = (\cos t, \sin t)$. In this example the angle function is continuous when it is defined as:

$$\phi(t) = \begin{cases} \arctan\frac{-\sin t}{\cos t} & \text{if} \quad 0 \le t < \frac{\pi}{2} \\ -\frac{\pi}{2} & \text{if} \quad t = \frac{\pi}{2} \\ \arctan\frac{-\sin t}{\cos t} - \pi & \text{if} \quad \frac{\pi}{2} < t < \frac{3\pi}{2} \\ -\frac{3\pi}{2} & \text{if} \quad t = \frac{3\pi}{2} \\ \arctan\frac{-\sin t}{\cos t} - 2\pi & \text{if} \quad \frac{3\pi}{2} < t < 2\pi \end{cases}$$

Notice that this function is equal to the function $\phi(t) = -t$. Therefore the index is equal to $\frac{1}{2\pi}(-2\pi - 0) = -1$.

3.2 Using the method of moving frames to calculate the index

In chapter 2.4 we found some relations when we compared two different orthonormal frames. From the change of coordinates we found the functions $f, g\colon V - \{p\} \to \mathbb{R}$ such that:

$$\begin{aligned}\tilde{e}_1 =& f e_1 + g e_2 \\ \tilde{e}_2 =& - g e_1 + f e_2\end{aligned}$$

We also defined $\tau = f\mathrm{d}g - g\mathrm{d}f$ where we found that $\mathrm{d}\tau = 0$ and

$$w_{12} = \tilde{w}_{12} - \tau.$$

We might wonder how we can interpret this 1-form τ. It turns out that τ will give the rate of change of the angle between both frames. Next lemma will state this more formally. This is lemma 5 in do Carmo [5]

Lemma 3.1 (do Carmo [5], lemma 5, pg. 91)**.** *Let $p \in U \subseteq \Sigma$ be a point and let $\gamma\colon I \to U$ be a curve such that $\gamma(t_0) = p$. Let ϕ_0 be the angle between e_1 and $\tilde{e}_1$. Then*

$$\phi(t) = \phi_0 + \int_{t_0}^{t} \left(f(\gamma(t)) \frac{\mathrm{d}g(\gamma(t))}{\mathrm{d}t} - g(\gamma(t)) \frac{\mathrm{d}f(\gamma(t))}{\mathrm{d}t} \right) \mathrm{d}t$$

is a differentiable function such that

$$\cos\phi(t) = f(\gamma(t)), \quad \sin\phi(t) = g(\gamma(t)), \quad \phi(t_0) = \phi_0 \ \textit{ and } \ \mathrm{d}\phi = \gamma^*\tau.$$

Proof: First we prove the simple statements. Recall that that integral of a continuous function is differentiable. Therefore, ϕ is differentiable. Also $\phi(t_0) = \phi_0 + \int_{t_0}^{t_0} \ldots \mathrm{d}t = \phi_0$. Lastly note that the part in the brackets is equal to $\gamma^*\tau$. Therefore, ϕ can be written as

$$\phi(t) = \phi_0 + \int_{t_0}^{t} \gamma^*\tau \mathrm{d}t.$$

Using the fundamental theorem of calculus we get that

$$\mathrm{d}\phi = \gamma^*\tau.$$

We still have to prove that $\cos\phi(t) = f(\gamma(t))$ and $\sin\phi(t) = g(\gamma(t))$. First we show that $f(\gamma(t)) \cdot \cos\phi(t) + g(\gamma(t)) \cdot \sin\phi(t) = 1$. For brevity let $\hat{f} = f \circ \gamma$ and $\hat{g} = g \circ \gamma$. Differentiating yields

$$(\hat{f}\cos\phi + \hat{g}\sin\phi)' = \hat{f}'\cos\phi - \hat{f}\sin\phi\phi' + \hat{g}'\sin\phi + \hat{g}\cos\phi\phi'.$$

Using the fact that $\phi' = \hat{f}\hat{g}' - \hat{g}\hat{f}'$ we get

$$\begin{aligned}(\hat{f}\cos\phi + \hat{g}\sin\phi)' =& \hat{f}'\cos\phi - \hat{f}(\hat{f}\hat{g}' - \hat{g}\hat{f}')\sin\phi + \hat{g}'\sin\phi + \hat{g}(\hat{f}\hat{g}' - \hat{g}\hat{f}')\cos\phi \\ =& (\hat{f}' + \hat{f}\hat{g}\hat{g}' - \hat{g}^2\hat{f}')\cos\phi + (-\hat{f}^2\hat{g}' + \hat{f}\hat{g}\hat{f}' + \hat{g}')\sin\phi.\end{aligned}$$

We can add the relation $\hat{f}^2 + \hat{g}^2 = 1$ anywhere. If we place this at the right spots we get:

$$\begin{aligned}(\hat{f}\cos\phi + \hat{g}\sin\phi)' =& ((\hat{f}^2 + \hat{g}^2)\hat{f}' + \hat{f}\hat{g}\hat{g}' - \hat{g}^2\hat{f}')\cos\phi + \\ &+ (-\hat{f}^2\hat{g}' + \hat{f}\hat{g}\hat{f}' + (\hat{f}^2 + \hat{g}^2)\hat{g}')\sin\phi \\ =& \hat{f}(\hat{f}\hat{f}' + \hat{g}\hat{g}')\cos\phi + \hat{g}(\hat{f}\hat{f}' + \hat{g}\hat{g}')\sin\phi \\ =& (\hat{f}\hat{f}' + \hat{g}\hat{g}')(\hat{f}\cos\phi + \hat{g}\sin\phi)\end{aligned}$$

Notice from $\hat{f}^2 + \hat{g}^2 = 1$ follows that $\hat{f}\hat{f}' + \hat{g}\hat{g}' = 0$. Thus the above equation is equal to zero. Therefore, $\hat{f}\cos\phi + \hat{g}\sin\phi$ is constant. By hypothesis, ϕ_0 is the angle between e_1 and $\tilde{e}_1$. This is equivalent by saying that $\cos\phi_0 = f(p)$ and $\sin\phi_0 = g(p)$. Using this fact it follows that

$$\hat{f}(t_0)\cos\phi(t_0) + \hat{g}(t_0)\sin\phi(t_0) = f(p)\cos\phi_0 + g(p)\sin\phi_0 = f^2(p) + g^2(p) = 1.$$

Using this it follows that

$$(\hat{f} - \cos\phi)^2 - (\hat{g} - \sin\phi)^2 = \hat{f}^2 + \hat{g}^2 - 2(\hat{f}\cos\phi + \hat{g}\sin\phi) + sin^2\phi + cos^2\phi = \quad 0.$$

Therefore we can conclude that $f \circ \gamma = \cos\phi$ and $g \circ \gamma = \sin\phi$. □

Using this lemma we can relate τ to the index. Let C be the boundary of the closure of V and let $\gamma\colon I \to C$ be a parameterization. Recall that the index I of X at p is $2\pi I = \phi(t_{end}) - \phi(t_{start})$. From the fundamental theorem of calculus follows that $2\pi I = \int_{t_{start}}^{t_{end}} \mathrm{d}\phi$. Lemma 3.1 tells us that $\mathrm{d}\phi = \gamma^*\tau$. We can conclude that

$$2\pi I = \int_C \tau.$$

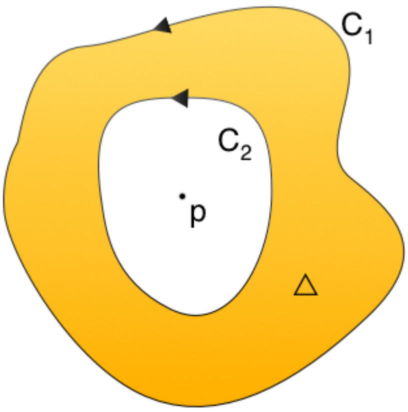
We can use this to formally define the index of X at p:

Definition 3.2. *Using the definitions of the variables above. The* **index** I *of* X *at* p *is defined as*

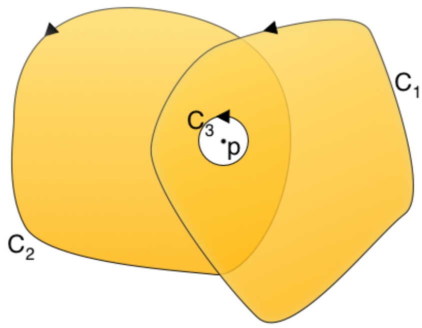
$$I = \frac{1}{2\pi}\int_\gamma \tau.$$

We made a lot of choices in this definition. We chose a curve γ, an orthonormal frame $\{e_1, e_2\}$ and we chose a Riemannian metric. We will prove that the index is independent of any of this.

Remark: There is also a topological definition of the index of a vector field. (People not familiar to algebraic topology can skip this remark and continue reading) Recall that $\mathbb{R}^n - \{p\}$ *is homotopic to* S^{n-1}*, a loop without self intersections is homotopic to a circle and that the homology group of* $H_n(S^n)$ *is* $\mathbb{Z}$*. Then, when we take a small enough loop* γ *homotopic to a circle around the singularity* p*, the map* $X|_\gamma \colon S^1 \to S^1$ *induces a map in homology* $H_1(S^1) \to H_1(S^1)$*. Because* $H_1(S^1) \simeq \mathbb{Z}$*, an element* α *will be mapped to* $I\alpha$*. This number* I *is the degree of the map* $X|_\gamma$ *and we define the index of* X *at* p *as this degree. This definition is equivalent as above, but does not depend on the choice of path, because homotopic paths induce the same homomorphism [6, Prop. 2.10, pg. 111]. It also does not depend on the metric, because there is always an homotopy between metrics. Finally, the frame* $\{e_1, e_2\}$ *doesn't matter either, because this frame is defined on a null-homotopic neighborhood* V*. So changing the frame will not change the induced map* $H_1(S^1) \to H_1(S^1)$*. With this in mind, the next three lemmas are trivial and can be skipped.*



(a) *In the case two closed simple paths don't intersect, they enclose an area that looks like an annulus.*



(b) *In the case two closed simple paths* C_1 *and* C_2 *do intersect, we can always pick a simple closed path* C_3 *such that* C_3 *is in the interior of the intersection between* C_1 *and* C_2*.*

Figure 9: *The index* I *does not depend on the choice of path.*

Lemma 3.3 (do Carmo [5], Lemma 1, pg. 100)**.** *The index* I *does not depend on* *the choice of path.*

Proof: Let C_1 and C_2 be two closed curves around p such that the only singularity they enclose is p. Suppose that C_1 and C_2 don't intersect. This is shown in Figure 9a. The paths C_1 and C_2 are the boundary of an annulus like surface Δ. Using Stokes' theorem it follows that the difference between the corresponding indices I_1 and I_2 is

$$I_1 - I_2 = \frac{1}{2\pi}\int_{C_1}\tau - \frac{1}{2\pi}\int_{C_2}\tau = \frac{1}{2\pi}\int_{\Delta}\mathrm{d}\tau = 0.$$

We also used the fact that $\mathrm{d}\tau = 0$.

Suppose that the paths C_1 and C_2 do intersect. Take a look at Figure 9b. We pick a path C_3 in the interior of the intersection of C_1 and C_2, such that C_3 circles around p. Using that $I_3 = I_1$ and $I_3 = I_2$ we can conclude that

$$I_1 = I_3 = I_2.$$

□

Lemma 3.4 (Simic [5], Lemma 4.6)**.** *The index I does not depend on the choice of frame $\{e_1, e_2\}$. More precisely, let S_r be the boundary of the disk of radius r and center p. Then the limit*

$$\lim_{r\to 0}\frac{1}{2\pi}\int_{S_r}\tilde{w}_{12} = \tilde{I}$$

exists and $\tilde{I} = I$.

Proof: First we prove that the limit exists. We will use the Cauchy criterion. This means we have to show that

$$\lim_{r_1,r_2\to 0}\left|\int_{S_{r_1}}\tilde{w}_{12} - \int_{S_{r_2}}\tilde{w}_{12}\right| = 0.$$

Suppose w.l.o.g. $r_1 > r_2$. As we can see in Figure 9a, the paths S_{r_1} and S_{r_2} enclose an annulus Δ. Using Stokes theorem we get that

$$\int_{S_{r_1}}\tilde{w}_{12} - \int_{S_{r_2}}\tilde{w}_{12} = \int_{\partial\Delta}\tilde{w}_{12} = \int_{\Delta}\mathrm{d}\tilde{w}_{12}.$$

Notice that when r_1 and r_2 goes to zero, the area of Δ will also go to zero. Since $\mathrm{d}w_{12}$ is bounded, the limit $\lim\limits_{r_1,r_2\to 0}\int_{\Delta}\mathrm{d}\tilde{w}_{12}$ is zero. Therefore, the limit exists.

Now we will show that this limit is equal to the index I at p. Let r_1 be arbitrary small but fixed and let $r_2 < r_1$. We take the limit when r_2 goes to 0. Then by Stokes we have that

$$\begin{aligned}\int_{S_{r_1}} \tilde{w}_{12} &= \lim_{r_2 \to 0} \int_{S_{r_1} - S_{r_2}} \tilde{w}_{12} + \lim_{r_2 \to 0} \int_{S_{r_2}} \tilde{w}_{12} \\ &= \int_{B_{r_1}} \mathrm{d}\tilde{w}_{12} + 2\pi \tilde{I} = - \int_{B_{r_1}} K \tilde{w}_1 \wedge \tilde{w}_2 + 2\pi \tilde{I}.\end{aligned}$$

We may notice that $\mathrm{d}\tilde{w}_{12}$ is not everywhere defined on B_{r_1}. But because $\mathrm{d}\tilde{w}_{12} = -K\tilde{w}_1 \wedge \tilde{w}_2 = -K w_1 \wedge w_2$ we can extend it.

Recall that $\tilde{w}_{12} = w_{12} + \tau$. Calculating the integral over S_{r_1} gives

$$\int_{S_{r_1}} \tilde{w}_{12} = \int_{S_{r_1}} w_{12} + \int_{S_{r_1}} \tau = \int_{B_{r_1}} \mathrm{d}w_{12} + 2\pi I = - \int_{B_{r_1}} K w_1 \wedge w_2 + 2\pi I.$$

from which we can conclude that $\tilde{I} = I$. □

Lemma 3.5 (do Carmo [5], Lemma 3, pg. 102)**.** *The index I does not depend on the choice of metric*

Proof: Let $\langle \cdot, \cdot \rangle_0$ and $\langle \cdot, \cdot \rangle_1$ be two Riemannian metrics on Σ. Let $t \in [0, 1]$ and define for all t the following function:

$$\langle \cdot, \cdot \rangle_t = t \langle \cdot, \cdot \rangle_1 + (1 - t) \langle \cdot, \cdot \rangle_0$$

Notice that this homotopy between $\langle \cdot, \cdot \rangle_0$ and $\langle \cdot, \cdot \rangle_1$ is a new Riemannian metric on Σ for each $t \in [0, 1]$. Let I_0, I_1 and I_t the corresponding indices. From the Gramm-Schmidt process it follows that the frames and co-frames depend continuous on t. Therefore, the connection forms are also continuous. This means that the difference τ and I_t is also continuous. Recall that the index is integer valued. The only continuous integer valued function is a constant function. We can conclude that $I_0 = I_1$. □

3.3 The relation between the index and the Euler characteristic

We have seen in the previous section how the index relates to the τ. We have seen that the index only depends on the vector field. It turns out that the sum of all indices is independent of the vector field. Therefore is this sum a topological quantity of the surface. We wonder how this relates to another topological quantity, namely the Euler characteristic.

We will sketch the relation between indices and the Euler characteristic. We follow the prove given by do Carmo [5, Remark 1, pg. 103] in chapter 6. Another type of proof can be found in e.g. [11, Chapter 6, pg. 35]. The theorem which relate the Euler characteristic to the sum of indices is called the **Poincare-Hopf** theorem.

Suppose we have a compact 2-manifold Σ. We can describe our surface as a union of triangles in such way that each two triangles have either a common edge, a common vertex or are fully disjoint. We call such a union of triangles a **triangulation**.

Lemma 3.6 (Abate [1],Theorem 6.5.10, pg. 340)**.** *All surfaces admit a triangulation.*

This proof is left as exercise for the reader. For an extended proof see [1, Theorem 6.5.10, pg. 340]. Using triangulation's we will define the Euler characteristic:

Definition 3.7. *Let Σ be a triangulated surface. Define F as the number of faces, E of edges and V of vertices of the triangulation. Then the Euler characteristic $\chi(\Sigma)$ is defined as*

$$\chi(\Sigma) = F - E + V.$$

Remark: If X is a CW complex, then χ is generally defined as the alternating sum of the number of n-cells. These two definitions are the same for surfaces, because triangulation's are CW-complexes.

Theorem 3.8 (Hatcher [6], Th. 2.44, pg. 146)**.** *If two surfaces are homotopy invariant, then they have the same Euler characteristic. Moreover, the Euler characteristic χ of a space X is equal to*

$$\chi(X) = \sum_n (-1)^n \mathrm{rank}\, H_n(X).$$

Proof: *This is an algebraic topological proof. Knowledge of homology is thereby required.* Let

$$0 \to C_k \xrightarrow{d_k} C_{k-1} \xrightarrow{d_{k-1}} C_{k-1} \to \ldots \to C_1 \xrightarrow{d_1} C_0 \to 0$$

be a chain complex of finitely generated abelian groups. Let $Z_n = \ker d_n$, $B_n = \mathrm{Im}\, d_{n+1}$ and $H_n = \frac{Z_n}{B_n}$. From this we can create the following short exact sequences.

$$0 \to Z_n \xrightarrow{id} C_n \xrightarrow{d_n} B_{n-1} \to 0$$

$$0 \to B_n \xrightarrow{d_n} Z_n \xrightarrow{x \mapsto [x]} H_n \to 0$$

From the rank - nullity theorem from linear algebra it follows that $\text{rank} C_n = \text{rank} Z_n + \text{rank } B_{n-1}$. In the same manner it is true that $\text{rank } Z_n = \text{rank } B_n + \text{rank } H_n$. Calculating the alternating sum yields:

$$\begin{aligned}\sum_n (-1)^n \text{rank } C_n &= \sum_n (-1)^n \text{rank } Z_n + \sum_n (-1)^n \text{rank } B_{n-1} \\ &= \sum_n (-1)^n \text{rank } H_n + \sum_n (-1)^n \text{rank } B_n - \sum_n (-1)^n \text{rank } B_n \\ &= \sum_n \text{rank } H_n\end{aligned}$$

Lastly we have to relate the C_n with the sum of n-cells. If we pick $C_n = H_n(X^n, X^{n-1}) = \tilde{H}_n(X^n/X^{n-1})$, we get that C_n is the homology group of the wedge of n-spheres. For each n-cell there is one wedge. Calculating the homology yields that C_n will be equal to $\mathbb{Z}^{\# n-\text{cells}}$. □

Next we will create a vector field such that the sum of the indices is equal to $\chi(\Sigma)$. We will do this by adding singularities on specific places.

Suppose we have a triangulation for our surface, such as that shown in Figure 10. Add a singularity in the center of each face such that the vector field moves away from that point. Such a singularity is called a **source**. Next add a singularity at each vertex such that the vector field moves towards that point. This is called a **sink**. To prevent unwanted singularities we add singularities like those shown in Figure 8 on each edge. Those singularities are called **saddles**. For saddles we know that the index is -1. The same way it can be

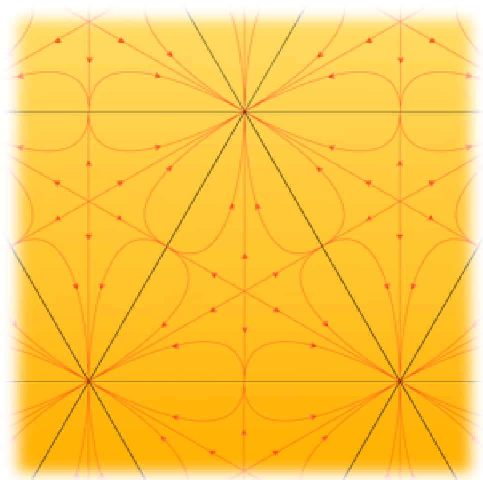


Figure 10: *Example of a vector field where each face, edge and vertex has 1 singularity. The black lines represent the triangulation of the surface. The red line forms the vector field.*

shown that the index of saddles and sources are $+1$. The sum of the indices of this vector field is equal to

$$\sum \text{Index} = \sum_{\text{sources}} 1 - \sum_{\text{saddles}} 1 + \sum_{\text{sinks}} 1 = F - E + V = \chi(\Sigma).$$

Later we will show that the sum of the indices is independent of the vector field. We will follow directly from our proof of the Gauss-Bonnet theorem. Then we can calculate the characteristic purely from the indices.

Example: We know that the sphere is homotopic to a cube. A cube has 6 faces, 8 vertices and 12 edges. Calculating the characteristic using the classical formula gives us that

$$\chi(S^2) = F - E + V = 6 - 12 + 8 = 2$$

Therefore, $\chi(S^2) = 2$.

Example: Look at Figure 7. We will calculate $\chi(S^2)$ using the sum of indices. In Figure 7 there are 2 singularities. One at the top and one at the bottom. At the top there is a sink with index 1 and at the top there is a source with index 1. The sum of indices is thus 2. In this case the sum of the indices agrees with the Euler characteristic.

4 Another type of curvature: Geodesic curvature

We have developed all the tools we need to prove the Gauss-Bonnet theorem for 2-manifolds without boundary. For 2-manifolds with boundary we need to define an extra type of curvature, namely the geodesic curvature. In this chapter we will define this curvature and give a geometric interpretation. We will follow do Carmo [5, Chapter 5, pg 91 to 96]

Let's look back how the normal curvature was defined in $\mathbb{R}^3$. As we may see in Figure 11, We defined the normal curvature by comparing the second derivative of a path to the normal vector. We might wonder what information about curvature we get when we project the second derivative to the tangent plane.

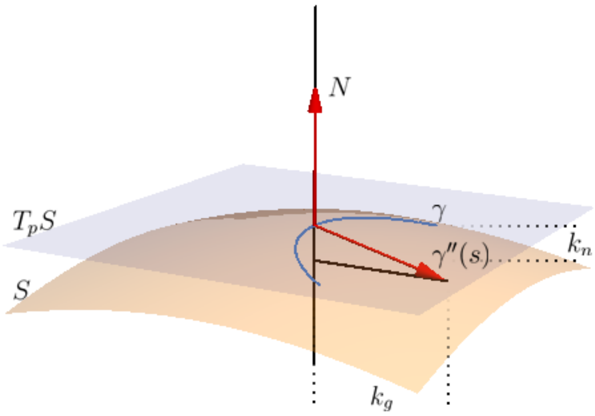


Figure 11: *A graphical illustration of the construction of the geodesic curvature* k_g

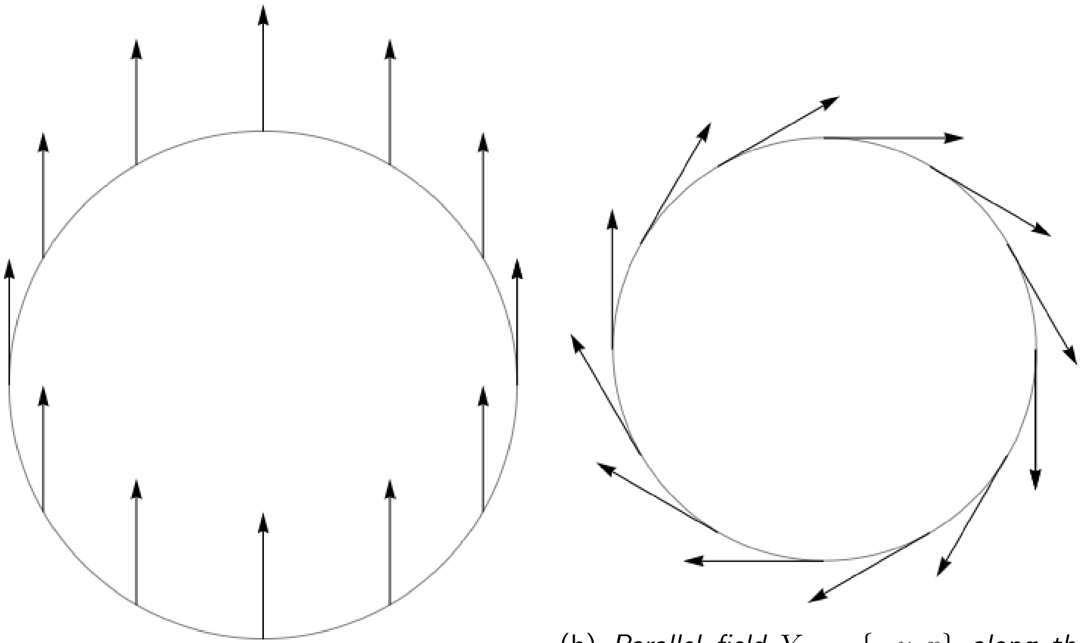
Let $^\top$ denote this projection. Recall from Lemma 2.22 that for an isometric immersion $M \to \tilde{M}$ of Riemannian manifolds we have $(\tilde{\nabla})^\top = \nabla$. We can thus use the covariant derivative of the surface for the projection of the second derivative. We will define the geodesic curvature as follows:

Theorem 4.1 (do Carmo [5], Definition 4, pg. 94)**.** *Let Σ be a Riemannian 2-manifold, with ∇ as its Levi-Civita connection. Let $\gamma\colon I \to \Sigma$ be a regular differentiable curve parametrized by arc length. Pick a local orthonormal frame consistent with the orientation of the surface and such that $e_1 = \gamma'(s)$. Define the geodesic curvature k_g at $\gamma(s)$ as the number*

$$k_g = \langle \nabla_{\gamma'(s)} e_1, e_2 \rangle = w_{12}\left[\gamma'(s)\right] = (\gamma^* w_{12}) \left(\frac{\partial}{\partial s}\right).$$

To give a geometric interpretation we first need to define the following:

Definition 4.2 (do Carmo [5], Definition 2, pg. 94)**.** *A vector field Y is called parallel along a cuve $\gamma(t)$ if the covariant derivative $\nabla_{\gamma'(t)}Y = 0$ for all t.*



(a) *Parallel field $Y = \{0, 1\}$ along the unit circle embedded in $\mathbb{R}^2$.*

(b) *Parallel field $Y = \{-y, x\}$ along the meridian (parameterized by arc length) of S^2.*

Figure 12: *Two examples of parrallel fields along the unit circle.*

Example: In Figure 12 are two examples of parallel fields along the unit circle. In the first example we assume the circle is embedded in $\mathbb{R}^2$. The parallel field in this case is a constant field. Differentiating will yield nothing. Thus, the covariant derivative $\nabla_{\gamma'(t)}Y = 0$.

Example: In Figure 12b we assume that the unit circle is the meridian of S^2. We also assume that the meridian is parameterized by arc length. If we also assume that the circle is parameterized clockwise, we notice that the vector field $Y = \{-y, x\}$ is the first derivative of our path. Recall that the second derivative of a path parameterized by arc length is normal to the tangent space. The covariant derivative must therefore be zero.

It can be shown that for a curve $\gamma\colon I \to M$, $t_0 \in I$ and $v_0 \in T_{\gamma(t_0)}M$ there always exists a unique parallel vector field V along γ such that $V(t_0) = v_0$ [7, Th. 4.11, pg.60]. We will show that the geodesic curvature will be the rate of change of the angle between the tangent vector of the curve and this parallel field:

Theorem 4.3 (do Carmo [5], Proposition 4, pg. 95)**.** *Suppose we have a 2-manifold Σ and a regular curve $\gamma\colon I \to \Sigma$ parametrized by arc length. Suppose we have a parallel vector field Y along γ and let ϕ be the angle between Y and γ'. Then*

$$k_g = \frac{\mathrm{d}\phi}{\mathrm{d}s}.$$

Proof: We pick two local frames $\{e_1, e_2\}$ and $\{\tilde{e}_1, \tilde{e}_2\}$ as follows: Let $e_1 = \frac{Y}{|Y|}$ and $\tilde{e}_1 = \gamma'(s)$. Set e_2 and $\tilde{e}_2$ such that both frames induce the same orientation. Let $w_{12}(X) = \langle \nabla_X e_1, e_2 \rangle$ and $\tilde{w}_{12} = \langle \nabla_X \tilde{e}_1, \tilde{e}_2 \rangle$. From Lemma 3.1 it follows that

$$\gamma^*(\tilde{w}_{12} - w_{12}) = \gamma^*\tau = \mathrm{d}\phi.$$

Because Y is a parallel field along γ it follows that e_1 is a parallel field along γ. Therefore, $\nabla_{\gamma'(s)} e_1 = 0$. Taking the inner product with e_2 yields

$$\gamma^* w_{12} = w_{12}(\gamma'(s)) = \langle \nabla_{\gamma'(s)} e_1, e_2 \rangle = 0.$$

From this it follows that $\mathrm{d}\phi = \gamma^*\tilde{w}_{12}$. The geodesic curvature becomes

$$k_g = (\gamma^*\tilde{w}_{12}) \left(\frac{\partial}{\partial s} \right) = \mathrm{d}\phi \left(\frac{\partial}{\partial s} \right) = \frac{\mathrm{d}\phi}{\mathrm{d}s}$$

and we conclude the proof. □

5 Proof of the Gauss-Bonnet theorem

We have developed all the tools we need to prove the Gauss-bonnet theorem. The theorem states

Theorem 5.1 (do Carmo [5], Theorem 2, pg. 104). *Let Σ be a compact oriented 2-manifold with boundary $\partial\Sigma$. Let X be any smooth vector field on Σ that is nowhere tangent to $\partial\Sigma$. Assume that X has only singularities on its interior and assume all singularities are isolated. Let $p_1, \ldots, p_k$ be the singularities of X and denote with $I_1, \ldots I_k$ the corresponding indices. Then for any Riemannian metric on our surface the following holds:*

$$\int_\Sigma K w_1 \wedge w_2 + \int_{\partial\Sigma} k_g \mathrm{d}s = 2\pi \sum_{i=1}^{k} I_i$$

Proof: Choose a Riemannian metric and define the following orthonormal frames:

- Let $\tilde{e}_1 = \frac{X}{|X|}$ and let $\tilde{e}_2$ orthonormal to $\tilde{e}_1$ such that $\{\tilde{e}_1, \tilde{e}_2\}$ has the same orientation as Σ.
- Pick a neighborhood $V \subseteq M$ of $\partial\Sigma$. Choose an orthonormal frame $\{e_1, e_2\}$ such that when we restrict it to the boundary of Σ we get that e_1 is nowhere tangent to $\partial\Sigma$.

Because all singularities are isolated, we can pick some balls $B_i \subset \Sigma$ with center p_i such that all balls contain only one singularity. From Theorem 2.24 and Stokes theorem follows that

$$\int_{\Sigma - \cup_i B_i} K\tilde{w}_1 \wedge \tilde{w}_2 = -\int_{\Sigma - \cup_i B_i} \mathrm{d}\tilde{w}_{12} = \int_{\cup_i \partial B_i} \tilde{w}_{12} - \int_{\partial\Sigma} \tilde{w}_{12}.$$

Because all balls B_i are disjoint we get that

$$\int_{\Sigma - \cup_i B_i} K\tilde{w}_1 \wedge \tilde{w}_2 + \int_{\partial\Sigma} \tilde{w}_{12} = \sum_{i=1}^{k} \int_{\partial B_i} \tilde{w}_{12}$$

Now we take the limit when the radius of the balls goes to zero. Notice that $\tilde{w}_1 \wedge \tilde{w}_2$ is independent of the choice of frame and represents an 'area element' of the surface. Because the area of the balls tends to zero, the integral $\int_{\Sigma - \cup_i B_i} K\tilde{w}_1 \wedge \tilde{w}_2$ will tend to $\int_\Sigma K\tilde{w}_1 \wedge \tilde{w}_2$. Lemma 3.4 tells us that $\int_{\partial B_i} \tilde{w}_{12}$ goes to $2\pi I_i$ when the radius will tend to zero.

The only thing yet to prove is that $\int_{\partial\Sigma} \tilde{w}_{12} = \int_{\partial\Sigma} k_g \mathrm{d}s$. To show this let $i \colon \partial\Sigma \to \Sigma$ be the inclusion map and let ϕ be the angle between $\tilde{e}_1$ and e_1. From Lemma 3.1 follows that

$$i^*\tilde{w}_{12} = i^*w_{12} + \mathrm{d}\phi.$$

Using this relation in $\int_{\partial\Sigma} \tilde{w}_{12}$ we get that

$$\int_{\partial\Sigma} \tilde{w}_{12} = \int_{\partial\Sigma} i^*\tilde{w}_{12} = \int_{\partial\Sigma} i^*w_{12} + \int_{\partial\Sigma} \mathrm{d}\phi$$

Notice that $\int_{\partial\Sigma} i^*w_{12} = \int_{\partial\Sigma} k_g \mathrm{d}s$ and $\int_{\partial\Sigma} \mathrm{d}\phi$ is integer valued. Because $\tilde{e}_1$ is nowhere tangent to $\partial\Sigma$, the angle $\phi \not\equiv 0 \mod 2\pi$ for all points on the boundary. Thus, the integral $\int_{\partial\Sigma} \mathrm{d}\phi$ is zero. From this we can conclude that

$$\int_{\Sigma} K w_1 \wedge w_2 + \int_{\partial\Sigma} k_g \mathrm{d}s = 2\pi \sum_{i=1}^{k} I_i$$

and we have proven the Gauss-Bonnet theorem. □

Notice that the left side of the equation is not dependent on the vector field and the right hand side does not depend on the metric. Thus, the sum of isolated indices is independent of vector field. In paragraph 3.3 we showed that the sum of indices is equal to the Euler characteristic under the assumption that $\sum_i I_i$ is independent of vector field. Now have showed that this assumption is true and we can conclude that

$$\int_{\Sigma} K w_1 \wedge w_2 + \int_{\partial\Sigma} k_g \mathrm{d}s = 2\pi \sum_{i=1}^{K} I = 2\pi\chi(\Sigma)$$

Example: We will calculate the Euler characteristic for a unit sphere using the curvature. Recall that the curvature of a unit sphere is 1. The area of this surface is equal to 4π. Gauss-Bonnet theorem tells us that $4\pi = 2\pi\chi(S^2)$. From this it follows that the Euler characteristic of a sphere is 2. This is the same for what we have seen in chapter 3.3, where we have calculated the Euler characteristic using the alternating sum of n-cells and using the sum of indices.

Summary and Outlook

Summary

Let us summarize all the topics we have covered in this thesis. We start with surfaces immersed in $\mathbb{R}^3$. Using paths we defined normal curvature as a curvature in a direction. We also defined the Gaussian curvature K as the determinant of the differential of the normal map N. From the properties of N we found that K is equal to the product of the extreme values of the normal curvature.

Secondly we investigated how the Gaussian curvature depends on the ambient space. We introduced the notion of connections and we discovered that there exists a unique linear symmetric connection which is compatible with the metric. Using this connection we defined the 1-form $w_{ij}(X) = \langle \nabla_X e_i, e_j \rangle$. From the structure relations we found that the Gaussian curvature could be written as $\mathrm{d}w_{12}(e_2, e_1)$. From this it follows that K only depends on the induced metric. We had proven Gauss Egregium.

Next we investigated how K depends on choice of frames. The Gaussian curvature is independent of frame, but w_{12} will change. In chapter 3 we found that the difference was related to the index of singularities and the sum of this topological quantity turned out to be the Euler characteristic.

In the mean time we generalized the notion of curvature for Riemannian n-manifolds. It turned out that the Riemann curvature tensor applied to (e_1, e_2, e_2, e_1) was equal to the Gaussian curvature for Riemannian 2-manifolds.

Second to last we defined the geodesic curvature as the projection of the second derivative to the tangent plane. We showed that this curvature measures the change of angle between a path and the parallel field of that path.

Finally, we proved the Gauss-Bonnet theorem. The trick was to do a change of frame. From the extra terms that were introduced we related $\int_\Sigma K \mathrm{d}A$ to the sum of indices, which were then related to the Euler characteristic. We proved that

$$\int_\Sigma K \mathrm{d}A + \int_{\partial\Sigma} k_g \mathrm{d}s = 2\pi\chi(\Sigma).$$

Outlook

In 1944 Chern published a paper showing that the proof of Gauss-Bonnet can be extended to smooth manifold without boundary of even dimension [3]. Given a Riemannian manifold (M, g) we could calculate the curvature tensor and from that the related curvature forms Ω_{ij} using the definition:

$$R^{\nabla}(e_i) = \sum_{j=0}^{n} \Omega_{ij} \otimes e_j$$

From this he defined the top form Ω to be the square root of $\det(\frac{\Omega_{ij}}{2\pi})$[4, Def. 2.9.2, pg. 135]. In explicit formula it is equal to:

$$\Omega = \frac{(-1)^{\frac{n}{2}-1}}{2^n \pi^{\frac{n}{2}} (\frac{n}{2})!} \sum \epsilon_{i_1} \dots \epsilon_{i_n} \Omega_{i_1 i_2} \wedge \Omega_{i_3 i_4} \wedge \dots \wedge \Omega_{i_{n-1} i_n}$$

Using the same tricks as above - pick a vector field and remove small balls around the singularities and let the radius of the balls tend to zero - he proved that

$$\int_M \Omega = \chi(M).$$

Using this we can relate the generalized Gauss-Bonnet theorem to the field of characteristic classes. The idea is that Ω is a representative in $H^{2q}(M, R)$ where H^k is the de Rham cohomology. The class it represents is called the Euler class [4, Def. 2.9.2, pg. 135].

In this thesis we started with the everyday problem of holding a pizza slice. It was a starting point in the study of curvature. We have found results that Gauss, one of the greatest mathematicians, could not describe it in any other way than by calling it surprising! So remember next time when you eat your pizza that what you are eating contains a vast amount of mathematics.

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