University of Amsterdam

MSc Mathematics

Master Thesis

Heterotic Courant algebroids and T-duality

Author:
Mária Anna Sisák

Supervisor:
prof. dr. G.R. Cavalcanti
prof. dr. R.R.J. Bocklandt

Examination date:
October 24, 2019

Korteweg-de Vries Institute for
Mathematics

Abstract

In this thesis we investigate the recent developments of generalized geometry on transitive Courant algebroids in relation with the Strominger System. First we define heterotic Courant algebroids, show that they naturally arise as reductions of exact Courant algebroids via extended actions and classify them via the cohomological notion of string class. Then we study geometric structures and operators on them such as generalized metrics and connections. After that we describe an interpretation of T-duality as an isomorphism of Courant algebroids. Finally we present a geometric formulation of the Strominger System in terms of spinorial equations on a heterotic Courant algebroid and discuss its behaviour under T-duality.

Title: Heterotic Courant algebroids and T-duality

Author: Mária Anna Sisák, m.anna.sisak@gmail.com, 11662891
Supervisor: prof. dr. G.R. Cavalcanti (UU), prof. dr. R.R.J. Bocklandt
Second Examiner: dr. H.B. Posthuma

Examination date: October 24, 2019

Korteweg-de Vries Institute for Mathematics
University of Amsterdam
Science Park 105-107, 1098 XG Amsterdam
http://kdvi.uva.nl

Contents

1 Introduction **5**

2 Linear algebra **9**
2.1 The double of a vector space . . . 9
2.2 The extended double . . . 10
2.3 Clifford algebras and spinors . . . 11

3 Courant algebroids **14**
3.1 Exact Courant algebroids . . . 16
3.2 The double of a Lie algebroid . . . 20
3.3 Heterotic Courant algebroids . . . 22

4 Reduction **30**
4.1 Simple reduction . . . 30
4.2 Reduction via extended actions . . . 31
4.3 Commuting actions . . . 39

5 Heterotic Courant algebroids by reduction **41**
5.1 Extended actions and string classes . . . 41
5.2 Reduction and heterotic Courant algebroids . . . 46

6 Generalized metrics **50**
6.1 Exact case . . . 51
6.2 Heterotic case . . . 52
6.3 Generalized metrics and reduction . . . 54

7 Generalized connections **56**
7.1 Metric compatible generalized connections . . . 58
7.2 Dirac operators . . . 60
7.3 The canonical Levi-Civita connection . . . 62

8 T-duality **65**
8.1 T-duality of torus bundles . . . 65
8.2 T-duality as a map of exact Courant algebroids . . . 72
8.3 Heterotic T-duality . . . 75

9 The Strominger System **82**
9.1 Killing spinor equations . . . 84

9.2 T-duality and and the Strominger System 88

10 Appendix **94**
10.1 Cohomology of compact Lie groups 94
10.2 Exact basic forms 96
10.3 On the second cohomology 97

1 Introduction

Generalized geometry has been introduced by Hitchin [26] in the early 2000s as the study of geometric structures on exact Courant algebroids. These objects are essentially the the double of the tangent bundle, that is the direct sum of the tangent and cotangent bundles, but there is more structure on them. These structures are related to a closed 3-form H on the underlying manifold M that is

$$dH = 0.$$

This motivated the relation to Type II string theory where the bosonic field called the NS-flux is a closed 3-form. Generalized complex geometry was further explored by Gualtieri in his DPhil thesis [22] where he characterized solutions to Type II string theory with extended $\mathcal{N} = 2$ supersymmetry as generalized Kähler manifolds. Later T-duality was also rephrased as an isomorphism of exact Courant algebroids in [7] by Cavalcanti and Gualtieri.

The success of generalized geometry in Type II string theory inspired mathematicians to try to find relations to heterotic string theory as well. In this case the NS-flux is still a 3-form but it is not necessarily closed anymore, instead it is subject to the condition known as the Bianchi identity

$$dH - \alpha(\mathrm{tr}(R_\nabla \wedge R_\nabla) - \mathrm{tr}(F \wedge F)) = 0$$

where R_∇ is the curvature of a metric connection ∇, F is the curvature of a principal connection and α is a constant parameter. In 2009 Chen, Stienon and Xu characterized a large class of Courant algebroids [8] and recovered the equation of the NS-flux as the defining property of some Courant algebroids. These Courant algebroids are the extensions of the double of the tangent bundle by some other bundles. In his DPhil thesis Rubio described the geometry of a Courant algebroid which is the double of the tangent bundle extended by a trivial line bundle [35]. Later Baraglia and Hekmati described extensions by a more general class of vector bundles which they called heterotic Courant algebroids [2]. They also extended the notion of T-duality to these objects corresponding to the duality of heterotic strings. Most recently, Garcia-Fernandez, Rubio and Tipler reformulated the defining equations of $\mathcal{N} = 1$ supersymmetric heterotic string theory to the language of generalized geometry [18] and showed that T-duals to solutions with torus symmetry naturally arise in this framework. In this thesis we explore these recent advancements towards understanding heterotic string theory in terms of generalized geometry.

The thesis is structured in the following way.

Sections 2 and 3 serve as an introduction to Courant algebroids over a manifold M, which are vector bundles together with an inner product, a bilinear bracket and a bundle map to the tangent bundle of M. The bracket is thought of as an nonskew-symmetric analogue to the Lie bracket of vector fields. We present three examples: exact Courat algebroids which are, as vector bundles, the double of the tangent bundle $TM \oplus T^*M$, the double of a Lie algebroid L which is analogously $L \oplus L^*$ and finally heterotic Courant algebroids. These are related to principal G-bundles P over M and as vector bundles are given by

$$\mathcal{H} \cong TM \oplus \mathfrak{g}_P \oplus T^*M$$

where $\mathfrak{g}_P$ is the adjoint vector bundle of P and $\mathcal{H}$ denotes the heterotic Courant algebroid. The Courant algebroid structure on these objects is caracterized by a pair (H, A) of a 3-form H on M and a principal connection A on P subject to the equation

$$dH - c(F \wedge F) = 0 \tag{1.0.1}$$

where c is a non-degenerate symmetric bilinear pairing on the Lie algebra $\mathfrak{g}$ of G and F is the curvature of A. These objects are going to be the main focus point of this thesis. The equation 1.0.1 is basically a reformulation of the Bianchi identity which suggests strong connection to heterotic string theory.

In Section 4 we describe a reduction procedure of exact Courant algebroids over a principal G-bundle $P \to M$ called reduction via extended actions. Essentially we describe a non-trivial way to lift the infinitesimal action of the Lie algebra $\mathfrak{g}$ from the tangent bundle TP to the exact Courant algebroid on P. This was first introduced in [5] as a way to produce exact Courant algebroids on the base M from an exact Courant algebroid on the total space P. Later in [2] it was realized that the non exact Courant algebroids obtained via this reduction procedure are precisely the heterotic Courant algebroids giving the following theorem.

Theorem 1.0.1. [2] *Every heteoric Courant algebroid on a smooth manifold M is given by reduction of an exact Courant algebroid $E \cong TP \oplus T^*P$ over a principal G-bundle $\sigma : P \to M$.*

The proof of this statement is the topic of Section 5. First we show that specific types of extended actions are classified by the same equation 1.0.1 as heterotic Courant algebroids. Then we construct an explicit isomorphism between the reduced space and the corresponding Courant algebroid. Furthermore, we classify the solutions of equation 1.0.1 as degree three cohomology classes called *string classes* and with this we have the following theorem.

Theorem 1.0.2. *Given a principal bundle $P \to M$ with compact, connected semisimple structure group G there is a one-to-one correspondence between real string classes on P and isomorphism classes of heterotic Courant algebroids on M.*

In Section 6 and 7 we build the analogue of Riemannian geomerty on Courant algebroids. First we define generalized metrics and classify them on exact and heterotic Courant algebroids. We also investigate how these structures behave under the reduction defined in Section 4. Then we define generalized connections, torsion-free generalized connections and also connections that are compatible with a generalized metric. We introduce divergence operators which measure the trace of a generalized connection and used to restrict the class of connections compatible with a metric. We will see that unfortunately there is no analogue of the Levi-Civita connection of a generalized metric as there are many torsion-free metric compatible generalized connections even after fixing the divergence. Nevertheless, we construct a natural connection, which we call the canonical Levi-Civita connection, that can be used as a reference point in the affine space of Levi-Civita connections. Furthermore, we show that there are still operators that only depend on a metric and a divergence operator. These are Dirac operators acting on spinor bundles corresponding to the generalized metric.

Section 8 is about T-duality. We first describe the original formulation by Bouwknegt, Evslin, Mathai and Hannabuss [3, 4] which relates to each other two principal torus bundles over a common base endowed with degree 3 cohomology classes. Then we present the results of Cavalcanti and Gualtieri [7] who reformulated T-duality as an isomorphism between invariant sections of exact Courant algebroids. In [2] this construction was extended to heterotic Courant algebroids via the reduction procedure of Section 4. We present this construction and prove the following theorem.

Theorem 1.0.3. *Let (P, h) be a pair of a principal $G \times T^k$-bundle over M and a T-dualisable string class, and let $(\tilde{P}, \tilde{h})$ be a T-dual pair of a $G \times \tilde{T}^k$-bundle over M and a degree 3 cohomology class. Then the T-dual class $\tilde{h}$ is again a string class and there is an isomorphism of the corresponding simply-reduced heterotic Courant algebroids on M*

$$\varphi : \mathcal{H}/T^k \;\cong\; \tilde{\mathcal{H}}/\tilde{T}^k.$$

In Section 9 we finally apply the theory we built in the previous sections to heterotic string theory. We introduce the Strominger System which is the system of partial differential equations describing $\mathcal{N} = 1$ supersymmetric solutions of the theory. We present the result of Garcia-Fernandez, Rubio and Tipler who in [18] proved the following remarkable theorem.

Theorem 1.0.4. *A solution of the Strominger System in 6 dimensions is equivalent to a heterotic Courant algebroid $\mathcal{H}$ together with a generalized metric, an element $e \in \mathcal{H}$ given by an exact 1-form and a nowhere vanishing right-handed spinor η satisfying the Killing spinor equations*

$$\begin{aligned} D^e_+ \eta &= 0, \\ \not{D}^e_- \eta &= 0. \end{aligned}$$

The operators in this equations are the unique operators that we have constructed in Section 7 on the spinor bundle corresponding to the generalized metric and a divergence operator. The divergence operator that is used here is defined by the element $e \in \mathcal{H}$ and the canonical Levi-Civita connection of the metric.

This theorem enables us to view the Strominger System as a natural structure on a heterotic Courant algebroid. With this formulation it is clear that Courant algebroid isomorphisms should take solutions to solutions. As in Section 8 we reformulated T-duality as a Courant algebroid isomorphim it becomes clear that T-duality exchanges solutions. On the other hand there are slight problems with this construction for example the heterotic Buscher rules have not yet been fully reproduced from generalized geometry. We discuss these problems and the proposed solutions at the end.

2 Linear algebra

In this chapter we take a look at vector spaces that will become the local models of the vector bundles we consider later. See [22] and [35] for more details.

2.1 The double of a vector space

Let V be a real vector space of dimension n. Exact generalized geometry is concerned with the study of bundles that are locally $V \oplus V^*$ the *double* of V.

The vector space $V \oplus V^*$ has a natural inner product

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\iota_X \eta + \iota_Y \xi) = \frac{1}{2}(\eta(X) + \xi(Y)) \qquad X, Y \in V,\ \xi, \eta \in V^*$$

of signature (n, n). There is also a natural orientation as the highest exterior power can be decomposed as

$$\wedge^{2m}(V \oplus V^*) = \wedge^m V \otimes \wedge^m (V^*).$$

Then one can define the mapping

$$\wedge^m V \otimes \wedge^m (V^*) \to \mathbb{R}$$
$$(u_1 \wedge ... \wedge u_n) \otimes (u^1 \wedge ... \wedge u^n) \mapsto det(u^i(u_j))$$

and the natural orientation is given by bases that map to $\mathbb{R}^+$.

Equipped with these structures the double of V has symmetry group $\mathrm{SO}(n, n)$. The Lie algebra of this group is given by

$$\mathfrak{so}(n, n) = \{T \in \mathrm{End}(V \oplus V^*) |\ \langle Tx, y \rangle + \langle x, Ty \rangle = 0\ \forall x, y \in V \oplus V^*\}.$$

Any $T \in \mathrm{End}(V \oplus V^*)$ can be written in a block matrix form

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A : V \to V$, $B : V^* \to V$, $C : V \to V^*$ and $D : V^* \to V^*$. Imposing the condition $T \in \mathfrak{so}(n, n)$ yields the restrictions

$$A^* = -D \qquad B + B^* = 0 \qquad C + C^* = 0.$$

Therefore we see that

$$\begin{aligned}\mathfrak{so}(n,n) =& \mathrm{End}(V) \oplus \wedge^2 V \oplus \wedge^2 V^* \\ =& \Big\{ \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix} \Big| A \in \mathrm{End}(V), B \in \wedge^2 V^*, \beta \in \wedge^2 V \Big\}.\end{aligned}$$

By exponentiating we can find some orthogonal symmetries of $V \oplus V^*$ that will be important later.

- The *B-transform* for some $B \in \wedge^2 V^*$

$$\exp(B) = 1 + B + \frac{1}{2}B^2 + ... = 1 + B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}.$$

- Its dual notion the *β-transform* for some $\beta \in \wedge^2 V$

$$\exp(\beta) = 1 + \beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.$$

- For some $A \in \mathrm{End}(V)$ we have that

$$\exp(A) = \begin{pmatrix} \exp(A) & 0 \\ 0 & \exp(A^*)^{-1} \end{pmatrix}$$

  which defines a map from $GL(V)^+$ into the identity component of $\mathrm{SO}(V \oplus V^*)$.

2.2 The extended double

Now we turn to the vector space $\mathcal{H} = V \oplus \mathfrak{g} \oplus V^*$ where $\mathfrak{g} \cong \mathbb{R}^m$ endowed with a non-degenerate inner product c of any signature. Of course these contain $V \oplus V^*$ when $m = 0$ but considering non-zero $\mathfrak{g}$ will be relevant to heterotic generalized geometry. There is a natural non-degenerate pairing on H given by

$$\langle X + s + \xi, Y + t + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y)) + c(s,t)$$

For $X, Y \in V$, $\xi, \eta \in V^*$ and $s, t \in \mathfrak{g}$. There is again a possibility to orient $\mathcal{H}$ if an orientation on $\mathfrak{g}$ is chosen by orienting $V \oplus V^* \subset \mathcal{H}$ as before.

The Lie algebra of orthogonal transformations is then given by

$$\mathfrak{so}(\mathcal{H}) = \{T \in \mathrm{End}(\mathcal{H}) | \ \langle Th_1, h_2 \rangle + \langle h_1, Th_2 \rangle = 0 \ \ \forall h_1, h_2 \in \mathcal{H}\}.$$

One can write any element as

$$T = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

with $A : V \to V$, $B : \mathfrak{g} \to V$, $C : V^* \to V$, $D : V \to \mathfrak{g}$ and so on.

Since $T|_{V \oplus V^*} \in \mathfrak{so}(n,n)$ just as in the exact case we find

$$I = -A^*, \quad C \in \wedge^2 V, \quad \mathfrak{g} \in \wedge^2 V^*.$$

Then taking $s, t \in G$, $X \in V$, $\xi \in V^*$ we have

- $\langle Ts, t\rangle + \langle s, Tt\rangle = 0 \Rightarrow E \in \mathfrak{so}(\mathfrak{g})$,
- $\langle Ts, X\rangle + \langle s, TX\rangle = 0 \Rightarrow Hs = -2c(D., s)$,
- $\langle Ts, \xi\rangle + \langle s, T\xi\rangle = 0 \Rightarrow Bs = -c(F., s)$.

Therefore T is of the form

$$T = \begin{pmatrix} A & -2c(F., .) & C \\ D & E & F \\ G & -2c(D., .) & -A^* \end{pmatrix}$$

and we have the decomposition

$$\mathfrak{so}(\mathcal{H}) = \mathrm{End}(V) \oplus \wedge^2 V \oplus \wedge^2 V^* \oplus \mathfrak{so}(\mathfrak{g}) \oplus (V^* \otimes \mathfrak{g}) \oplus (V \otimes \mathfrak{g}).$$

By exponentiation we obtain a new type of transformation unique to this case which we call A-*transform* for some $A : V \to \mathfrak{g}$

$$\exp(A) = \begin{pmatrix} 1 & 0 & 0 \\ A & 1 & 0 \\ -c(A., A.) & -2c(A., .) & 1 \end{pmatrix}.$$

Together with the B-transform from the exact case we have the (B, A)-*transform* acting via

$$(B, A) = \exp(B + A) = e^B e^A = \begin{pmatrix} 1 & 0 & 0 \\ A & 1 & 0 \\ B - c(A., A.) & -2c(A., .) & 1 \end{pmatrix}.$$

These form a subgroup of $\mathrm{SO}(V \oplus G \oplus V^*)$ with composition given by

$$(B, A) \circ (B', A') = (B + B' - c(A \wedge A'), A + A').$$

2.3 Clifford algebras and spinors

So far we have considered vector spaces together with non-degenerate inner products of various signatures. In this section we recall some classical fats on Clifford algebras, spin groups and the spin representation of the Lie algebra of the spin group.

Let E be now any vector space with nondegenerate inner product $\langle \cdot, \cdot \rangle$ of dimension n. Using the inner product we can define the Clifford algebra $\mathcal{C}\ell(E)$ of E as

$$\mathcal{C}\ell(E) = \otimes^\bullet E / (u^2 - \langle u, u\rangle),$$

i.e. the free tensor algebra of E factored by the relations $uv + vu = 2\langle u, v\rangle$. The spin group sits inside $\mathcal{C}\ell(E)$ as

$$Spin(E) = \{v_1 \cdot ... \cdot v_{2k} |\ k \geq 0,\ v_i \in E :\ \langle v_i, v_i\rangle = \pm 1\},$$

and it is the double cover of $SO(E)$ via the map

$$\begin{aligned} \rho :\ & Spin(E) \to SO(E) \\ & \rho(x)(v) = x \cdot v \cdot x^{-1} \end{aligned}$$

where $x \in Spin(E)$, $v \in E$ and " $\cdot$ " is the Clifford multiplication. Since ρ is a covering map its differential is an isomorphism

$$\begin{aligned} d\rho :\ & \mathfrak{spin}(E) \to \mathfrak{so}(E) \\ & d\rho_x(v) = x \cdot v - v \cdot x. \end{aligned}$$

This isomorphism allows us to lift elements of $\mathfrak{so}(E) \cong \wedge^2(E)$ (via the inner product) to the Clifford algebra. Let $e_1, ..., e_n$ be a basis of E and $\tilde{e}_1..., \tilde{e}_n$ a dual basis in E in the sense that

$$\langle e_i, \tilde{e}_j\rangle = \delta_{ij}.$$

Then an element $\tilde{e}_i \wedge e_j$ acts on E by

$$\tilde{e}_i \wedge e_j :\quad e_i \mapsto e_j,$$

and zero otherwise. This element lifts to $\mathcal{C}\ell(E)$ as

$$\tilde{e}_i \wedge e_j \in \mathfrak{so}(E) \quad \mapsto \quad \frac{1}{2} e_j \tilde{e}_i \in \mathcal{C}\ell(E). \tag{2.3.1}$$

Indeed, using the Clifford relations

$$\begin{aligned} \frac{1}{2}(e_j\tilde{e}_i).e_k &= \frac{1}{2}(e_j\tilde{e}_i e_k - e_k e_j \tilde{e}_i) \\ &= \frac{1}{2}(-e_j e_k \tilde{e}_i + e_k \delta_{ik} + e_j e_k \tilde{e}_i + e_j \delta_{ik}) \\ &= \delta_{ik} e_j. \end{aligned}$$

The spin group has a unique representation S which is of dimension 2^m if the dimension of E is $n = 2m$ or $n = 2m+1$. By differentiation we obtain the *spin representation* of $\mathfrak{so}(E)$. This representation is irreducible when n is odd and decomposes into two irreducible components $S = S^+ \oplus S^-$ when n is even which we call *positive and negative chirality half-spin representations.*

Example. Let $E = V \oplus V^*$ with the natural inner product as in Section 2.1. In this case $V \oplus V^*$ acts naturally on the exterior algebra $S = \wedge^\bullet V^*$ of forms via

$$(X + \xi)\varphi = \iota_X \varphi + \xi \wedge \varphi$$

for $X + \xi \in V \oplus V^*$ and $\varphi \in \wedge^\bullet V^*$. This action is compatible with the Clifford multiplication. i.e.

$$(X + \xi)^2 \varphi = \xi(X) \cdot \varphi$$

so it makes S into a Clifford-module. Restricting to $\mathfrak{so}(V \oplus V^*)$ we find the spin representation which decomposes into half-spin representations characterised by the parity of degree

$$S = \wedge^{even} V^* \oplus \wedge^{odd} V^*.$$

3 Courant algebroids

In the previous section we have considered the double of a vector space V and described some of its properties. Now we would like to consider the of the tangent bundle of a manifold. This leads to the notion of exact Courant algebroids which were first defined in [32].

We have extended the construction of a double vector space further by adding an orthogonal $\mathbb{R}^m$ with its own non-degenerate pairing. We will define the global version of this construction as well which leads us to heterotic Courant algebroids introduced by Baraglia and Hekmati in [2]. Structures in the case of $m = 1$ were further explored by Rubio in his DPhil thesis [35] which also provided guidelines to the calculations.

Let M be a smooth real manifold of dimension n and TM its tangent bundle. To motivate Courant algebroids let us consider the bundle $TM \oplus T^*M$ on M. Similarly to the linear case we have a non-degenerate inner product of split (n, n) signature

$$\langle X + \xi, Y + \eta \rangle = \xi(Y) + \eta(X),$$

where now $X, Y \in \Gamma(TM)$ are vector fields and $\xi, \eta \in \Omega^1(M)$ are 1-forms.

We also have a natural projection $\pi : TM \oplus T^*M \to TM$ to the first coordinate which is a smooth bundle-map and we call it the *anchor*.

In the case of the tangent bundle sections act on themselves via the Lie bracket of vector fields. It turns out that there is a generalization of this bracket on the double bundle $TM \oplus T^*M$ which restricts to the Lie bracket on vector fields via the anchor map π. An skew-symmetric bracket was first defined by Courant in [9], then a nonskew-symmetric version by Dorfman [10]. The two brackets encode the same data and called either Courant or Dorfman brackets. In this thesis we are always considering the nonskew-symmetric form and refer to it as the Dorfman bracket but inconsistencies may occur. In defining the bracket we are following the treatment in [30] where the not nonskew-symmetric bracket is derived in the same manner as the Lie bracket of vector fields utilising actions on the differential forms.

On a smooth manifold M vector fields act on the graded exterior algebra of differential forms via the interior product

$$X.\varphi = \iota_X \varphi \quad X \in \Gamma(TM),\ \varphi \in \Omega^\bullet(M).$$

The interior product is a degree -1 operator and we also have the exterior differential d acting on $\Omega^\bullet(M)$ which is of degree +1. The Lie bracket of two vector fields X, Y then defined as the unique section of TM satisfying

$$\iota_{[X,Y]}.\varphi = [\mathcal{L}_X, \iota_Y].\varphi = [[\iota_X, d], \iota_Y].\varphi \quad \forall \varphi \in \Omega^\bullet(M).$$

The commutators are meant to be supercommutators of operators

$$[A, B] = A \circ B - (-1)^{|A| \cdot |B|} B \circ A,$$

where $|A|$ denotes the degree of the operator A.

To generalize the bracket notice that $TM \oplus T^*M$ also acts on forms via the Clifford action

$$(X + \xi).\varphi = \iota_X \varphi + \xi \wedge \varphi,$$

which is the smooth version of the action we saw in Section 2.3. One can then define the Dorfman bracket of two sections $e_1, e_2 \in \Gamma(TM \oplus T^*M)$ as the unique section satisfying

$$[e_1, e_2].\varphi = [e_1, [d, e_2]].\varphi.$$

Here, although the action of $TM \oplus T^*M$ has mixed degree, both parts are of odd degree therefore $[d, e_1] = d \circ e_1 + e_1 \circ d$. Writing out the action one finds

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi$$

where $X + \xi, Y + \eta \in \Gamma(TM \oplus T^*M)$.

The 4-tuple $(T \oplus T^*, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$ is the first example of a *Courant algebroid.* These objects were axiomatized by Liu, Weinstein and Xu in [32] for the skew-symmetrized version of the bracket. The following definition has been reformulated in terms of the nonskew-symmetric Dorfman bracket.

Definition 3.0.1. A *Courant algebroid* $\mathcal{E}$ over a manifold M is a 4-tuple $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$, where $E \to M$ is a vector bunlde, $\langle \cdot, \cdot \rangle$ is a fibrewise non-degenerate symmetric bilinear pairing, $[\cdot, \cdot]$ is a bilinear bracket on the smooth sections $\Gamma(E)$ and $\pi : E \to TM$ is a smooth bundle map called the *anchor* such that the following axioms hold for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$:
C1) $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$
C2) $\pi([e_1, e_2]) = [\pi(e_1), \pi(e_2)]$
C3) $[e_1, f e_2] = \pi(e_1)(f) e_2 + f[e_1, e_2]$
C4) $\pi(e_1)\langle e_2, e_3 \rangle = \langle [e_1, e_2], e_3 \rangle + \langle e_2, [e_1, e_3] \rangle$
C5) $[e_1, e_1] = \mathcal{D}\langle e_1, e_1 \rangle$
where the bracket in C3) on the right hand side is the Lie bracket of vector fields and in C5) $\mathcal{D} = \frac{1}{2}\pi^* \circ d : \mathcal{C}^\infty(M) \to \Gamma(E)$ (using $\langle \cdot, \cdot \rangle$ to identify E and E^*).

We will sometimes denote a courant algebroid $\mathcal{E}$ simply by E when it does not cause confusion. We say that a Courant algebroid $\mathcal{E}$ is *transitive* if the anchor $\pi : E \to TM$ is surjective and that it is *exact* if it fits into the short exact sequence of vector bundles

$$0 \longrightarrow T^*M \xrightarrow{\frac{1}{2}\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0.$$

Clearly $TM \oplus T^*M$ is an exact Courant algebroid over M.

To understand these objects more one can try to describe the inner morphisms (symmetries) that preserve the structures.

Definition 3.0.2. An *automorphism of $\mathcal{E}$ covering the diffeomorphism* $\varphi : M \to M$ is a bundle automorphism $F : E \to E$ covering the diffeomorphism φ such that

1. $\varphi^*\langle F., F.\rangle = \langle ., .\rangle$ i.e. F is orthogonal,
2. $[F., F.] = F[., .]$ i.e. F is bracket preserving,
3. $\varphi_* \circ \pi = \pi \circ F$ i.e. F is compatible with the anchor.

The group of automorphisms of $\mathcal{E}$ is denoted by $Aut(\mathcal{E})$.

Note that the first two confitions already imply compatibility with the anchor via axiom $C4$).Clearly,

$$\begin{aligned}
\pi(Fe_1)\langle Fe_2, Fe_3\rangle &= \langle [Fe_1, Fe_2], Fe_3\rangle + \langle Fe_2, [Fe_1, Fe_3]\rangle && (C4)\\
&= \langle F[e_1, e_2], e_3\rangle + \langle Fe_2, F[e_1, e_3]\rangle && (2.)\\
&= (\varphi^*)^{-1}(\langle [e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle) && (1.)\\
&= (\varphi^*)^{-1}(\pi(e_1)\langle e_2, e_3\rangle) && (C4)\\
&= (\varphi_*\pi e_1)(\varphi^*)^{-1}\langle e_2, e_3\rangle\\
&= (\varphi_*\pi e_1)(\langle Fe_2, Fe_3\rangle). && (1.)
\end{aligned}$$

The infinitesimal symmetries of a Courant algebroid are given by considering the tangent space of $Aut(\mathcal{E})$ at the identity yielding the following definition.

Definition 3.0.3. The Lie algebra of derivations of $\mathcal{E}$ is given by linear first order diffenential operators D_X acting on $\Gamma(E)$ covering vector fields $X \in \Gamma(TM)$ such that

1. $X\langle ., .\rangle = \langle D_X ., .\rangle + \langle ., D_X .\rangle$,
2. $D_X[., .] = [D_X ., .] + [., D_X .]$.

We denote the derivations of $\mathcal{E}$ by $Der(\mathcal{E})$, which is a Lie algebra by the usual commutator of differential operators.

3.1 Exact Courant algebroids

Recall, that an exact Courant algebroid on a manifold M is a Courant algebroid $\mathcal{E}$ fitting into the short exact sequence

$$0 \longrightarrow T^*M \xrightarrow{\frac{1}{2}\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0.$$

and the most straightforward example was given by the direct sum $TM \oplus T^*M$ together with the natural pairing and the Dorfman bracket

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X\eta - \iota_Y d\xi$$

derived in the previous section.

Exact Courant algebroids are just a slight generalization of $TM \oplus T^*M$ as an isotropic splitting $s : TM \to E$ of the sequence

$$0 \longrightarrow T^*M \xrightarrow{\frac{1}{2}\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0$$

renders E isomorphic to $TM \oplus T^*M$ and the pairing becomes the natural pairing. Whenever we talk about a *splitting* of E we mean an isotropic splitting s of the above sequence. Splittings always exist as the inner product has split signature since the image of T^*M is isotropic.

Such a splitting $s : TM \to E$ not only defines an isomorphism $E \cong TM \oplus T^*M$ but also a closed 3-form $H \in \Omega^3(M)$ via

$$H(X, Y, Z) = \langle [s(X), s(Y)], s(Z)\rangle \quad \forall X, Y, Z \in \Gamma(TM).$$

Different isotropic splittings of E are globally related by 2-forms $B \in \Omega^2(M)$ which change H by an exact 3-form. Therefore the cohomology class of H is independent of the splitting and it also characterises exact Courant algebroids. The class of H in $H^3(M, \mathbb{R})$ is called the *Ševera class* of $\mathcal{E}$ [37].

In the split description of $E \cong TM \oplus T^*M$ the bracket is twisted by the 3-form H and takes the form

$$[X + \xi, Y + \eta]_H = [X, Y] + \mathcal{L}_X\eta - \iota_Y d\xi + \iota_Y \iota_X H.$$

From the split point of view the H-twisted Dorfman bracket can be derived the same way as the untwisted bracket by swiching the exterior derivative to the H-twisted version $d_H = d - H\wedge$ [30].

The automorphism group of an exact Courant algebroid $\mathcal{E}$ is captured in the following two propositions which follow the treatment of [22].

Proposition 3.1.1. *Let $F : E \to E$ be a vector bundle isomorphism covering the identity on M that is orthogonal with respect to the inner product and preserves the anchor, i.e. $\forall e_1, e_2 \in \Gamma(E)$*

1. $\langle e_1, e_2\rangle = \langle Fe_1, Fe_2\rangle$
2. $\pi(e_1) = \pi \circ F(e_1)$.

Then F is a B-transform for some $B \in \Omega^2(M)$.

Proof. Consider a splitting $E \cong TM \oplus T^*M$. Then F can be written in a block matrix form

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

In the splitting the anchor is just projection to the first component. Therefore considering $e_1 = X$ for some $X \in \Gamma(TM)$ and $e_1 = \xi$ for $\xi \in \Gamma(T^*M)$ by (2.) above we find that F is of the form

$$F = \begin{pmatrix} 1 & 0 \\ C & D \end{pmatrix}.$$

Where 1 denotes $id : TM \to TM$. Enforcing orthogonality first for $e_1 = X$ and $e_2 = Y$ for all $X, Y \in \Gamma(TM)$ yields $C + C^* = 0$ i.e. $C \in \Omega^2(M)$. Then taking $e_1 = e_2 = X + \xi$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(T^*M)$ we find that $D = id : T^*M \to T^*M$. Consequently, $F = e^C$ for $C \in \Omega^2(M)$. ∎

On the other hand, such a B-field transform does not necessarily preserve the Dorfman bracket. If $E \cong TM \oplus T^*M$ is equipped with the H-twisted bracket we have

$$\begin{aligned}[e^B(X+\xi), e^B(Y+\eta)]_H =& [X + \xi + \iota_X B, Y + \eta + \iota_Y B]_H \\ =& [X+\xi, Y+\eta]_H + d\iota_X\iota_Y B - \iota_Y d\iota_X B + \iota_X d\iota_Y B \\ =& [X+\xi, Y+\eta]_H + \iota_Y\iota_X B + \iota_{[X,Y]}B \\ =& e^B[X+\xi, Y+\eta]_{H+dB}.\end{aligned}$$

Therefore e^B is an automorphism of the exact Courant algebroid E if and only if B is closed. With this we are ready to describe $Aut(\mathcal{E})$.

Proposition 3.1.2. *The automorphism group $Aut(\mathcal{E})$ of a Courant algebroid is a semidirect product fitting into the following short exact sequence*

$$1 \to \Omega^2_{cl}(M) \to Aut(\mathcal{E}) \to Diff_{[H]}(M) \to 1.$$

Where $Diff_{[H]}(M)$ is the subgroup of diffeomorphisms of M preserving the Severa class of $\mathcal{E}$ and $\Omega^2_{cl}(M)$ is the space of closed 2-forms on M.

Proof. Let F be an automorphism of E covering the diffeomorphism φ. Consider a splitting $E \cong TM \oplus T^*M$ with the bracket twisted by $H \in \Omega^3(M)$. There is a canonical lift of φ to $TM \oplus T^*M$ via $\tilde{\varphi} = \varphi_* + (\varphi^*)^{-1}$ which is not an automorphism but preserves the inner product, is compatible with the anchor and changes the bracket via

$$\begin{aligned}[\tilde{\varphi}(X+\xi), \tilde{\varphi}(Y+\eta)]_H &= [\varphi_*X, \varphi_*Y] + \mathcal{L}_{\varphi_*X}(\varphi^{-1})^*\eta - \iota_{\varphi_*Y} d(\varphi^{-1})^*\xi + \iota_{\varphi_*Y}\iota_{\varphi_*X}H \\ &= \varphi_*[X,Y] + (\varphi^{-1})^*\mathcal{L}_X\eta - (\varphi^{-1})^*\iota_Y d\xi + (\varphi^{-1})^*\iota_Y\iota_X\varphi^*H \\ &= \varphi[X+\xi, Y+\eta]_{\varphi^*H}.\end{aligned}$$

Then $\tilde{\varphi}^{-1}F$ is a fibre preserving orthogonal transformation of E that also preserves the anchor. Therefore by Proposition 3.1.1 it must be a B-transform for some $B \in \Omega^2(M)$ changing the bracket via

$$[e^B e_1, e^B e_2]_H = e^B[e_1, e_2]_{H+dB}.$$

Finally as $F = \tilde{\varphi}e^B$ we must have $dB = H - \varphi^*H$, and hence

$$Aut(\mathcal{E}) = \{(\varphi, B) \in Diff_{[H]}(M) \times \Omega^2(M) |\ dB = H - \varphi^*H\}.$$

With this result one can describe $Aut(\mathcal{E})$ in a splitting independent way as in the proposition. ∎

The Lie algebra $Der(\mathcal{E})$ of infinitesimal symmetries is obtained by differentiating one parameter families of automorphisms around the identity. Since these are by definition homotopic to the identity they always preserve the cohomology class $[H]$. Let $\{F_t\} = \{e^{tB}\tilde{\varphi}_t\}$ be such a family with $X \in \Gamma(TM)$ the vector field associated to $\{\varphi_t\} \subset Diff(M)$. Consider a splitting of E with the Dorfman bracket twisted by H. Then

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} e^{tB}\tilde{\varphi}_t(Y+\eta) =& \frac{d}{dt}\Big|_{t=0}(\varphi_t)_*Y + (\varphi_{-t})^*\eta + \iota_{(\varphi_t)_*Y}tB \\ =& \mathcal{L}_X(Y+\eta) + \iota_Y B.\end{aligned}$$

Moreover, the constraint $\varphi_t^*H - H = t\cdot dB$ becomes $\mathcal{L}_X H = dB$ and we have

$$Der(\mathcal{E}) = \{(X,B) \in \Gamma(TM)\times\Omega^2(M)|\ \mathcal{L}_X H = dB\}.$$

Consequently we obtain the following splitting independent description of $Der(\mathcal{E})$.

Proposition 3.1.3. *The Lie algebra of derivations $Der(\mathcal{E})$ is an abelian extension of the Lie algebra $Der(TM)$ fitting into the short exact sequence*

$$0 \to \Omega^2_{cl}(M) \to Der(\mathcal{E}) \to Der(TM) \to 0$$

where $Der(TM)_{[H]}$ is the space of vector fields corresponding to 1-parameter groups of diffeomorphisms preserving the cohomology class $[H] \in H^3(M,\mathbb{R})$.

Similarly to the case of vector fields $\Gamma(E)$ acts on itself via the adjoint action as a derivation. On the other hand, in this case the map $\Gamma(E) \to Der(\mathcal{E})$ is neither injective nor surjective. Instead its image is described by the following proposition.

Proposition 3.1.4. *The adjoint action of $\Gamma(E)$ via the Dorfman bracket fits into the following exact sequence*

$$0 \longrightarrow \Omega^1_{cl}(M) \xrightarrow{\frac{1}{2}\pi^*} \Gamma(E) \xrightarrow{ad} Der(\mathcal{E}) \xrightarrow{\chi} H^2(M,\mathbb{R}) \longrightarrow 0.$$

Proof. Recall that in a certain splitting

$$[X+\xi, Y+\eta]_H = [X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi + \iota_Y\iota_X H.$$

Clearly, $ad(X+\xi) = 0$ if and only if $X = 0$ and $d\xi = 0$. In the previous proposition we saw that $Der(\mathcal{E})$ consists of ordered pairs (X,B) acting via the Lie derivative and contraction satisfying $dB = \mathcal{L}_X H$. Therefore we may define

$$\chi(X,B) = [\iota_X H - B] \in H^2(M,\mathbb{R}).$$

The map is well defined since H is closed therefore $d(\iota_X H - B) = \mathcal{L}_X H - dB = 0$ and surjective since we may chose $B \in \Omega^2(M)$ arbitrarily. The kernel of χ then consists of (X,B) such that $\iota_X H - B = d\xi$ which act via

$$\begin{aligned}(X,B).(Y+\eta) &= \mathcal{L}_X(Y+\eta) + \iota_Y B \\ &= [X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi + \iota_Y\iota_X H \\ &= [X+\xi, Y+\eta],\end{aligned}$$

which proves the exactness of the sequence. ∎

3.2 The double of a Lie algebroid

Another important type of Courant algebroids generalize further the notion of an exact Courant algebroid by replacing TM with a more general structure, a Lie algebroid.

Definition 3.2.1. A *Lie algebroid* over a smooth manifold M is a vector bundle $L \to M$ together with a skew-symmetric bilinear bracket $[\cdot,\cdot]$ acting on the smooth sections of L and a smooth bundle map $\pi : L \to TM$ called the *anchor* so that for all $e_1, e_2, e_3 \in \Gamma(L)$ and $f \in \mathcal{C}^\infty(M)$:
L1) $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$
L2) $\pi([e_1, e_2]) = [\pi(e_1), \pi(e_2)]$
L3) $[e_1, fe_2] = \pi(e_1)(f)e_2 + f[e_1, e_2]$
A Lie algebroid is called *regular* if the anchor has constant rank and it is called *transitive* if the anchor is surjective.

Definition 3.2.2. Let $(L, [\cdot,\cdot], \pi)$ be a Lie algebroid. Then the automorphism group $Aut(L)$ of L consists of vector bundle automorphisms $F : L \to L$ covering diffeomorphisms $\varphi : M \to M$ such that for all $e_1, e_2 \in \Gamma(L)$
i.) $[Fe_1, Fe_2] = F[e_1, e_2]$
ii.) $\pi \circ F = \varphi_* \circ \pi$.

Example. The tangent bundle TM of a smooth manifold M is a Lie algebroid together with the Lie bracket and the identity map as the anchor. Its automorphism group is *Diff*(M), where $\varphi \in$ *Diff*(M) acts on the fibres via the pushforward φ_*.

In this thesis we assume every Lie algebroid to be transitive as we only work with transitive Courant algebroids, although most of the results hold for more general classes of Lie algebroids as well, e.g. for regular ones.

On a Lie algebroid $(L, [\cdot,\cdot], \pi)$ one can build differential calculus of tensors in parallel to the one on the tangent and cotangent bundle of a smooth manifold. Here we follow closely [34] but we omit the proofs. There is a natural notion of a Lie derivative $\mathcal{L}^L_X$ with respect to a section X of L. On smooth functions $f \in \mathcal{C}^\infty(M)$ it is defined via the anchor as

$$\mathcal{L}^L_X f = \iota_{\pi(X)} df \in \mathcal{C}^\infty(M),$$

and on sections Y of L it is defined using the bracket

$$\mathcal{L}^L_X(Y) = [X, Y].$$

These definitions then define the action of $\mathcal{L}^L_X$ on any mixed degree tensor $T \in L^{\otimes k} \otimes (L^*)^{\otimes l}$ via the Leibniz rule.

One can also consider Lie algebroid differential forms which are just smooth sections of the graded exterior algebra of L^* which we denote by $\Omega^\bullet(M, L)$. The Lie derivative then defines a Lie algebroid exterior derivative $d_L : \Omega^k(M, L) \to \Omega^{k+1}(M, L)$ via the usual definition, i.e. if $\omega \in \Omega^k(M, L)$ and $X_0, ..., X_k \in \Gamma(L)$ then

$$d_L\omega(X_0, ..., X_k) = \sum_{i=0}^{k}(-1)^i \mathcal{L}^L_{X_i}\omega(X_0, ..., \hat{X}_i, ..., X_k) + \sum_{0\leq i<j\leq k}(-1)^{i+j}\omega([X_i, X_j], X_0, ..., \hat{X}_i, .., \hat{X}_j, .., X_k),$$

where the hat means that we omit that vector field.

The new differential d_L and Lie derivative $\mathcal{L}^L$ satisfy the usual Cartan formula when acting on Lie algebroid differential forms

$$\mathcal{L}^L_X = \iota_X d_L + d_L \iota_X.$$

It can be shown that $d_L \circ d_L = 0$, hence one can build the Lie algebroid cohomology complex $H^\bullet(L)$ which is the analogue of the de Rham cohomology of a manifold.

Clearly for any Lie algebroid the bundle $L \oplus L^*$ carries a natural non-degenerate pairing of split signature. It also carries a bracket derived using the Lie-algebroid differential d_L and the action of $L \oplus L^*$ on $\Omega^\bullet(M, L)$ analogously to the exact case [30]. The bracket takes the familiar form

$$[X + \xi, Y + \eta] = [X, Y]_L + \mathcal{L}^L_X \eta - \iota_Y d_L \xi.$$

Together with the anchor map $\pi : L \oplus L^* \to L \to TM$ induced by the anchor π of L the vector bundle $L \oplus L^*$ becomes a Courant algebroid.

Similarly to the exact case, one can then define twisted versions of $L \oplus L^*$ by the Courant algebroid E that fits into the short exact sequence

$$0 \longrightarrow L^* \xrightarrow{\frac{1}{2}\rho^*} E \xrightarrow{\rho} L \longrightarrow 0.$$

We call such Courant algebroids *double Lie algebroids*. By an *isotropic splitting* of E we mean an isotropic splitting of the above sequence.

By the same arguments as in the exact case such splittings give rise to closed Lie algberoid 3-forms $H \in \Omega^3(M, L)$ and the space of two forms $\Omega^2(M, L)$ acts on the splittings tranditively changing H to $H + dB$. Therefore, these Courant algebroids are characterized by degree 3 cohomology classes $h \in H^3(L)$ which we also call the Ševera class of E [37].

The automorphism group of such a Courant algebroid is not as easy to describe as in the exact case. The Lie algebroid L in general has a large automorphism group which could be hard to describe and relate to the diffeomorphisms of M. Moreover, the anchor has kernel in L there could be fibre preserving automorphisms of E that are more than just B-transforms for some d_L-closed $B \in \Omega^2(M, L)$.

Nevertheless, automorphisms of L still lift to orthogonal vector bundle automorphism of $E \cong L \oplus L^*$ preserving the anchor. If $\varphi \in Aut(L)$ then $\tilde{\varphi} = \varphi + (\varphi^{-1})^*$ changes the Dorfman bracket in the usual way

$$[\tilde{\varphi}\ , \tilde{\varphi}\ ]_H = \tilde{\varphi}[\ ,\ ]_{\varphi^* H}.$$

The B-transforms generalize to this setting as well, so we have the following propositions whose proofs are analogous to the exact case.

Proposition 3.2.1. *The automorphism group of the Courant algebroid E corresponding to a transitive Lie algebroid L contains a subgroup of $Aut_0(E)$ that fits into the short exact sequence*

$$0 \longrightarrow \Omega^2_{cl}(M, L) \longrightarrow Aut_0(E) \longrightarrow Aut(L) \longrightarrow 0.$$

The group $Aut_0(E)$ contains the automorphisms that preserve the projection to L in the short exact sequence $0 \to L^* \to E \to L \to 0$.

Sections of E still act as derivations via the adjoint action, and the kernel of the map still consists of the closed Lie-algebroid 1-forms. We find the following generalization of the exact statement.

Proposition 3.2.2. *The Lie subalgebra $Der_0(E)$ of $Der(\mathcal{E})$ generated by infinitesimal automorphisms in $Aut_0(E)$ fits into the exact sequence*

$$0 \longrightarrow \Omega^1_{cl}(M, L) \xrightarrow{\frac{1}{2}\pi^*} \Gamma(E) \xrightarrow{ad} Der_0(E) \longrightarrow H^2(L) \longrightarrow 0.$$

3.3 Heterotic Courant algebroids

A third example of transitive Courant algebroids called *heterotic Courant algebroids* are related to principal bundles and were defined by Baraglia and Hekmati in [2]. As all of our examples before these are a special case of so called *regular Courant algebroids* classified by Chen, Stienon and Xu [8].

Principal bundles and the Atiyah algebroid

First, let us recall some background knowledge on principal bundles and its Atiyah algebroid. Most of these facts can be found in classic books on the topic, we refer to Kobayashi and Nomizu [29] for basics of principal bundles and Mackenzie [33] for statements regarding Lie algebroids.

In this thesis principal bundles have a *right* action. We assume that the principal bundles have compact connected semisimple structure groups G and that there is a non-degenerate G-invariant symmetric bilinear pairing c on $\mathfrak{g}$ (e.g. the Killing-form). The Lie bracket on the Lie algebra $\mathfrak{g}$ is defined as as the Lie bracket of *left invariant* vector fields on G for sign convention.

Let $\pi : P \to M$ be a principal G-bundle. The fundamental vector fields on P are defined as

$$\psi : \mathfrak{g} \to \Gamma(TP) = Vect(TP)$$
$$\psi(x)|_p = \frac{d}{dt}\Big|_{t=0} p.\exp(tx).$$

The map ψ is called the *infinitesimal action* of $\mathfrak{g}$. With our sign convention ψ is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields $Vect(TP)$, i.e. we have

$$\psi([x,y]) = [\psi(x), \psi(y)] = \mathcal{L}_{\psi(x)}\psi(y).$$

The fundamental vector fields span the *vertical* subbundle of TP which consists of vectors that are in the kernel of π_*.

Atiyah algebroids. The action of G on P lifts naturally to the tangent bundle $TP \to P$ by differentiation. For $X \in TP$ and $g \in G$

$$(g.X)|_{pg} = (R_g)_*(X|_p)$$

where R_g denotes the right action of g on P. A vector field is called *invariant* whenever

$$(R_g)_* X = X.$$

Definition 3.3.1. Let $\pi : P \to M$ be a principal G bundle. Then the *Atiyah algebroid* corresponting to P is the vector bundle $\mathcal{A} = TP/G$ over $M = P/G$.

Proposition 3.3.1. [21] *The Atiyah algebroid of P is a Lie algebroid over M with surjective anchor induced by $\pi_* : TP \to TM$. Moreover $\mathcal{A}$ fits into the short exact sequence of Lie algebroids*

$$0 \longrightarrow \mathfrak{g}_P \xrightarrow{\ j\ } \mathcal{A} \xrightarrow{\ \pi\ } TM \longrightarrow 0$$

where $\mathfrak{g}_P$ is the vector bundle associated to the adjoint representation of G.The sequence is called the Atiyah sequence *corresponding to the principal bundle P.*

Proof. Firstly, sections of $\mathcal{A} = TP/G$ can be identified with G-invariant sections of TP. The Lie bracket of G-invariant vector fields is again G-invariant hence it induces a well defined skew-symmetric bracket satisfying the Jacobi identity on sections of $\mathcal{A}$. The bracket is also compatible with π_*, therefore $(\mathcal{A}, [\cdot,\cdot], \rho)$ becomes a Lie algebroid with the anchor induced by π_*.

Clearly the anchor is surjective, so it only remains to show that $\ker(\rho) \cong \mathfrak{g}_P$. At a point $p \in P$ the subspace $ker(\pi_*)|_p$ is isomorphic to $\mathfrak{g}$ via the map that sends $x \in \mathfrak{g}$ to the fundamental vector $\psi(x)|_p$ corresponding to x at the point p.

The fundamental vector fields are not G-invariant for the action of G on TP but as we saw satisfy $\psi([x,y]) = [\psi(x), \psi(y)]$ which if we integrate for y becomes

$$(R_g)_*\psi(x)|_p = \psi(Ad(g^{-1})x)|_{g.p}.$$

Therefore, using the isomorphism $ker(\pi_*) \cong \mathfrak{g}$, taking the quotient by the action of G accounts for the identification $(p,x) \sim (g.p, Ad(g^{-1})x)$ which is precisely the definiton of $\mathfrak{g}_P$.

The statement that the map $\mathcal{A} \to TM$ is compatible with the bracket follows from the fact that every invariant vector field X on P induces a vector field $\pi_* X$ on M and hence $\pi_*[X,Y] = [\pi_* X, \pi_* Y]$. ∎

A differential form ω on P is called *invariant* if $R_g^*\omega = \omega$ for all $g \in G$. The space of invariant differential forms is denoted by $\Omega^\bullet(P)^G$. Sections of the Atiyah algebroid are identified with G-invariant sections of TP and hence the Atiyah sequence induces a filtration of $\Omega^\bullet(P)^G$

$$\Omega^\bullet(M) = \mathcal{F}^0 \subset \mathcal{F}^1 \subset ... \subset \mathcal{F}^m = \Omega^\bullet(P)^G \tag{3.3.1}$$

where $\mathcal{F}^i = Ann(\wedge^{i+1}\mathfrak{g})$ and m is the dimension of $\mathfrak{g}$.

Connections. A *connection 1-form* on the principal bundle P is an *equivariant* Lie-algebra valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ meaning that

$$R_g^* A = Ad(g^{-1})A \quad \forall g \in G,$$

where Ad is the adjoint action of G on the Lie algebra, such that

$$\iota_{\psi(x)} A = x \quad \forall x \in \mathfrak{g}.$$

By equivariance a connection 1-form descends to a right splitting of the Atiyah sequence. In particular $A(j(x)) = x$ holds for all $x \in \mathfrak{g}$ as well.

A connection on P can also be thought of as a choice of G-invariant *horizontal distribution* H of TP. More precisely for all $g \in G$ and $p \in P$

$$(R_g)_* H_p = H_{gp}$$

and for all $X \in \Gamma(TM)$ there is a unique G-invariant $X^H \in \Gamma(H)$ such that

$$\iota_{X^H} A = 0 \ \text{ and } \ \pi_* X^H = X.$$

This viewpoint gives the right splitting of the Atiyah sequence via $X \mapsto X^H$ corresponding to the left splitting given by A.

Given a connection $A \in \Omega^1(P, \mathfrak{g})$ the Atiyah sequence splits $\mathcal{A} \cong \mathfrak{g}_P \oplus TM$. Then we can identify sections of $\mathcal{A}$ with G-invariant sections of TP which can be written as

$$\begin{array}{rcl} TM \oplus \mathfrak{g}_P & \to & TP \\ X + s & \mapsto & X^H + j(s) = X^H + s \end{array}$$

where X^H is the horizontal lift and j is the map from the Atiyah sequence. We omit j in the notation later, although one has to be careful with signs which we will point out later.

A connection form A on P turns the filtration 3.3.1 of invariant differential forms into a decomposition. At degree k it is given by

$$\Omega^k(P)^G = \bigoplus_{i=0}^{k} \Omega^i(M, \wedge^{k-i}\mathfrak{g}).$$

Therefore, if $A = A^i e_i$ in some basis $\{e_i\}$ of $\mathfrak{g}$, any invariant differential k-form ω can be written as

$$\omega = \omega^0 + \omega_i^1 A^i + \frac{1}{2}\omega_{ij}^2 A^{ij} + ... + \frac{1}{k!}\omega_{i_1..i_k}^k A^{i_1...i_k}$$

where $A^{i_1...i_k} = A^{i_1} \wedge ... \wedge A^{i_k}$, and $\omega_{i_1..i_j}^j$ are basic forms of degree $k-j$, i.e. pullbacks of forms on M via π. The numerical factors are just a case of convention, sometimes they make calculations cleaner.

Curvature. The curvature 2-form of a connection A is given by

$$F = dA + \frac{1}{2}[A, A],$$

where $[A, A]$ is a Lie algebra valued 2-form obtained by wedging the 1-form part of A and taking the Lie-bracket of the components in $\mathfrak{g}$. The curvature is a basic form and it measures the failure of the horizontal distribution to be involutive for the Lie bracket, i.e. for $X, Y \in \Gamma(TM)$

$$F(X, Y) = [X, Y]^H - [X^H, Y^H].$$

Induced connection and bracket. For any transitive (or regular) Lie algebroid $(L, [\cdot,\cdot], \rho)$ sections of the kernel of the anchor $K = ker(\rho)$ carry a $\Gamma(L)$-module structure. Given $a \in \Gamma(L)$ and $b \in \Gamma(K)$ we have

$$a.b = [a, b].$$

Clearly $[a, b] \in \Gamma(K)$ as the bracket is compatible with the anchor.

The action of the Atiyah algebroid $\mathcal{A}$ on the kernel $\mathfrak{g}_P$ is related to G-connections on $\mathfrak{g}_P$ and principal connections on P. If A is a principal connection on P, we may define a G-connection on $\mathfrak{g}_P$ via

$$\nabla_X s = [X^H, s]_{\mathcal{A}},$$

for $X \in TM$ and $s \in \mathfrak{g}_P$ and X^H is the horizontal lift of X. It turns out that the connection defined this way is precisely the connection on $\mathfrak{g}_P$ associated to A via the usual construction.

Moreover, the bracket $[\cdot,\cdot]$ restricted to $ker(\pi_*) \cong \mathfrak{g}_P$ is related to the Lie bracket of $\mathfrak{g}$. We defined the Lie bracket of $\mathfrak{g}$ via left invariant vector fields on G, so the infinitesimal action ψ would become a Lie algebra homomorphism. On the other hand, the map $j : \mathfrak{g} \to TP$ mapping to invariant vector fields is a Lie algebra homomorphism if one defines the bracket of $\mathfrak{g}$ via right invariant vector fields, so in our convention we have

$$j[x, y] = -[jx, jy].$$

We will try to keep track of this sign by omitting j but always writing out ψ.

Consequently, in the splitting induced by the connection A, the bracket on the Atiyah algebroid then becomes

$$\begin{aligned}[X+s,Y+t]_{\mathcal{A}} &= [X^H+s,Y^H+t]_P \\ &= [X^H,Y^H]+[X^H,t]-[Y^H,s]-[s,t] \\ &= [X,Y]^H - F(X,Y)+\nabla_X t-\nabla_Y s-[s,t].\end{aligned}$$

The invariant pairing c on $\mathfrak{g}$ defines an invariant bilinear pairing on the fibres of $\mathfrak{g}_P$ as well. This extra structure is captured in the following definition.

Definition 3.3.2. A *quadratic Lie algebroid* over a manifold M is a transitive (or regular) Lie algebroid $(L,[\cdot,\cdot],\rho)$ together with a non-degenerate symmetric bilinear pairing $\langle\cdot,\cdot\rangle$ on the kernel of the anchor $K=ker(\rho)$ which respects the module structure on K, i.e. for all $a\in\Gamma(L)$ and $b,c\in\Gamma(K)$ we have

$$\rho(a)\langle b,c\rangle = \langle [a,b],c\rangle + \langle b,[a,c]\rangle.$$

It is clear now that the G-invariant pairing c on $\mathfrak{g}$ makes $\mathcal{A}$ into a quadratic Lie algebroid.

Heterotic Courant algebroids

With the above preparation, we are ready to define our third example of a Courant algebroid which will become the main focus of this thesis. In the discussion we follow [2] and [15]. Let $(\mathcal{H},[\cdot,\cdot],\langle\cdot,\cdot\rangle,\pi)$ be a transitive Courant algebroid over the smooth manifold M. As the anchor $\pi:\mathcal{H}\to TM$ is surjective, dualising it we obtain an injection $\pi^*:T^*M\to\mathcal{H}$. Then $\mathcal{H}/T^*M$ becomes a Lie algebroid over TM with the Dorfman bracket. The pairing induces a non-degenerate pairing on the kernel of the induced anchor which turns $\mathcal{H}/T^*M$ into a quadratic Lie algebroid.

Definition 3.3.3. [2] A transitive Courant algebroid $(\mathcal{H},[\cdot,\cdot],\langle\cdot,\cdot\rangle,\pi)$ is called a *heterotic Courant algebroid* if $\mathcal{H}/T^*M$ is isomorphic as a quadratic Lie algebroid to the Atiyah algebroid correspondig to a principal G-bundle P with a symmetric non-degenerate pairing c on its Lie algebra $\mathfrak{g}$.

If $\mathcal{H}$ is a heterotic Courant algebroid, then let $\mathcal{K}$ be the kernel of the anchor π which is a smooth subbundle of $\mathcal{H}$ as π is of constant rank. Given that $\mathcal{H}/T^*M\cong\mathcal{A}$ for some principal bundle P we have the following two short exact sequences of vector bundles

$$0\longrightarrow\mathcal{K}\longrightarrow\mathcal{H}\xrightarrow{\ \pi\ }TM\longrightarrow 0$$

$$0\longrightarrow T^*M\xrightarrow{\ \pi^*\ }\mathcal{K}\longrightarrow\mathfrak{g}_P\longrightarrow 0.$$

A *splitting* of $\mathcal{H}$ is an isotropic right splitting $\lambda:TM\to\mathcal{H}$ of the first sequence above. A splitting λ then splits the second sequence as well via the restriction of the projection $\sigma:\mathcal{H}\to\mathcal{H}/T^*M\cong\mathcal{A}$ to the subspace $\lambda(TM)^\perp$

$$\sigma|_{\lambda^\perp}^{-1}:\mathfrak{g}_P\cong\mathcal{K}\cap\lambda(TM)^\perp.$$

Note that a splitting also induces a splitting of the Atiyah algebroid and therefore defines a connection on the principal bundle P.

Isotropic right splittings of the first sequence always exist, therefore heterotic Courant algebroids are of the form

$$\begin{aligned}\Psi_s : TM \oplus \mathfrak{g}_P \oplus T^*M \;&\cong\; \mathcal{H}\\ X + s + \xi \;&\mapsto\; \lambda(X) + \sigma|_{\lambda^\perp}^{-1} s + \pi^*\xi.\end{aligned}$$

Then the anchor and pairing is given by

$$\pi(X + s + \xi) = X$$

$$\langle X + s + \xi, Y + t + \eta\rangle = \frac{1}{2}(\xi(Y) + \eta(X)) + c(s, t).$$

Different isotropic splitting induce different isomorphisms. Let λ' be another isotropic splitting, then clearly

$$\lambda' - \lambda :\; TM \to \mathcal{K} \cong T^*M \oplus \mathfrak{g}_P$$

so let $A : TM \to \mathfrak{g}_P$ and $B : TM \to T^*M$ and write λ' as

$$\lambda'(X) = \lambda(X) + \sigma|_{\lambda^\perp}^{-1} A(X) + \pi^* B(X).$$

Since the image of λ' is isotropic we find that for all $X, Y \in TM$

$$c(A(X), A(Y)) + \iota_X B(Y) + \iota_Y B(X) = 0.$$

Therefore $B' = B - c(A., A.)$ is skew symmetric in X, Y and we can write λ' for a suitable $A \in T^*M \oplus \mathfrak{g}_P$ and $B \in \Omega^2(M)$ as

$$\lambda'(X) = \lambda(X) + \sigma|_{\lambda^\perp}^{-1} A(X) + \pi^*(\iota_X B - c(A(X), A.)).$$

From this description we see that the change of a splitting amounts to a global (B, A)-transform of $TM \oplus \mathfrak{g}_P \oplus T^*M$ i.e. that isomorphic splittings are an affine space modelled on the vector space

$$\Omega^2(M) \oplus \Omega^1(M, \mathfrak{g}).$$

It is clear that the Dorfman bracket restricted to $\mathcal{A}$ is the bracket of the Atiyah algebroid. The remaining parts of it were characterized in [8] which we summarize in the following proposition.

Proposition 3.3.2. [8] *Given a heterotic Courant algebroid $\mathcal{H}$ on M and a splitting of it $\mathcal{H} \cong TM \oplus \mathfrak{g}_P \oplus T^*M$ determined by a principal connection A on P, there exist a 3-form $H \in \Omega^3(M)$ such that the following hold.*

1. *The Dorfman bracket in the splitting is given by*

$$\begin{aligned}[X + s + \xi, Y + t + \eta]_{\mathcal{H}} = &[X, Y] - F(X, Y) + \nabla_X t - \nabla_Y s - [s, t]\\ &+ \mathcal{L}_X \eta - \iota_Y d\xi + \iota_Y \iota_X H\\ &+ 2c(t, \iota_X F) - 2c(s, \iota_Y F) + 2c(\nabla s, t).\end{aligned}$$

where $X, Y \in \Gamma(TM)$, $s, t \in \Gamma(\mathfrak{g}_P)$, $\xi, \eta \in \Gamma(T^*M)$ and F is the curvature of A.

2. The 3-form H satisfies

$$dH = c(F, F) \in \Omega^4(M).$$

(Here we wedge the 2-form part of F and evaluate the Lie algebra part via c.)

Conversely, given a principal G-bundle P with a non-degenerate symmetric bilinear pairing c on $\mathfrak{g}$ and a pair (A, H) of a connection 1-form and a basic 3-form such that $dH = c(F, F)$, there exist a heterotic Courant algebroid corresponding to the data, given by the above construction.

Remark 3.3.1. Notice that the above proposition carries a topological obstruction to what principal bundles admit heterotic Courant algebroids. The 4-form $c(F, F)$ is closed and defines a real cohomology class called the *first Pontryagin class* of P

$$p_1(P) = [c(F, F)] \in H^4(M, \mathbb{R})$$

corresponding to the bilinear pairing c on $\mathfrak{g}$. This class is topological, i.e. it does not depend on the connection chosen but it does depend on c. Therefore, given a smooth manifold M, there exist a heterotic Courant algebroid corresponding to a certain principal bundle P on M with bilinear form c if and only if

$$p_1(P) = [c(F, F)] = [dH] = 0 \in H^4(M, \mathbb{R})$$

the first Pontryagin class of P vanishes.

Different splittings of $\mathcal{H}$ induce different brackets. We know that splittings are related by (B, A)-transforms and in any splitting the bracket is as in Theorem 3.3.2. We denote this bracket by $[\ ,\ ]_{H,\nabla}$ since it is classified by the pair (H, ∇) where $H \in \Omega^3(M)$ and ∇ is a G-connection on $\mathfrak{g}_P$. This allow us to find how the bracket changes from splitting to splitting.

Proposition 3.3.3. *For $B \in \Omega^2(M)$ and $A \in \Omega^1(M, \mathfrak{g}_P)$ the (B, A)-transform changes the (H, ∇) heterotic Dorfman bracket as follows*

$$e^{(B,A)}[\ ,\ ]_{H,\nabla} = [e^{(B,A)}\ , e^{(B,A)}\ ]_{H',\nabla'}$$

where

$$\begin{aligned} H' &= H + dB + 2c(A, F) + c(A, \nabla A) + \frac{1}{3}c(A, [A, A]) \\ \nabla' &= \nabla + A. \end{aligned}$$

Proof. The proof is a straightforward but cumbersome calculation which we will avoid and come back to the proof later with a much more elegant solution. To begin, note that since we know that the bracket is of the same form in a different splitting, we can simplify the process a bit (e.g. by only considering sections $X, Y \in TM$ under the transformations). We know that for some H' and ∇' we have

$$e^{(B,A)}[X, Y]_{H,\nabla} = [e^{(B,A)}X, e^{(B,A)}Y]_{H',\nabla'},$$

and we can continue from here later. ∎

Remark 3.3.2. In the two previous cases we have seen that the Dorfman bracket can be *derived* from a spin representation of the Clifford algebra. It is natural to ask if this construction generalizes to the heterotic setting as well. It has been shown in [1] that whenever $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ admits a spin bundle S there exist a *Dirac generating operator* $\not{d}$ with $\not{d}^2 = 0$ that induces the Courant bracket via the action on a twisted spin bundle $\mathcal{S}$. The bracket is then obtained by the usual formula

$$[[e_1, \not{d}], e_2].\alpha = [e_1, e_2]_{\mathcal{H}}.\alpha$$

for $e_1, e_2 \in \Gamma(\mathcal{H})$ and $\alpha \in \Gamma(\mathcal{S})$[1]. The existence of such a spinor bundle S depends on the *second Stiefel-Whitney class* $w_2(\mathcal{H})$ of $\mathcal{H}$. By the properties of characteristic classes we have

$$w_2(\mathcal{H}) = w_2(TM) + w_2(\mathfrak{g}_P) + w_2(T^*M) = w_2(\mathfrak{g}_P)$$

so it depends on the class of $\mathfrak{g}_P$ whether the bracket is derived or not. Nevertheless, locally spin bundles always exist so the bracket is always derived on trivial patches.

[1] The discussion in [1] is for general Courant algebroids and in the exact and double Lie algebroid case arrives to the same construction presented before. In particular the Dirac generator operators are d_H and d_H^L.

4 Reduction

In the previous section we have encountered three types of Courant agebroids. It might seem somewhat arbitrary why we picked out these examples from all the possible structures allowed by the definition. Turns out that there is a connection between them via a reduction procedure which was developed in [5] by Bursztyn, Cavalcanti and Gualtieri.

We start with an exact Courant algebroid E over the manifold P and suppose there is a Lie group G acting on P via diffeomorphisms on the right. In this section we will assume P to be a principal bundle with compact connected structure group G so the reduced space $P/G = M$ is a smooth manifold. We want to consider an action of G on the total space of E via automorphisms of E and reduce it to a new non-exact Courant algebroid on the base M.

Coming from the action of G we have the Lie-algebra homomorphism

$$\begin{aligned} \psi : \mathfrak{g} &\to \Gamma(TP) \\ x &\mapsto \psi(x)|_p = \frac{d}{dt}\Big|_{t=0} p.\exp(tx) \end{aligned}$$

which defines the infinitesimal action of $\mathfrak{g}$ on $\Gamma(TP)$ via the adjoint action (Lie bracket) of vector fields.

Definition 4.0.1. A *lifted action* of G on E is a right action of G on E via automorphisms covering the action of G on P. A *lifted infinitesimal action* of $\mathfrak{g}$ on E is a Lie algebra homomorphism $\alpha : \mathfrak{g} \to Der(E)$ covering the infinitesimal action ψ of $\mathfrak{g}$ on TP.

Given a lifted action of G on E by differentiation we obtain a lifted infinitesimal action of $\mathfrak{g}$ on E. Conversely if a lifted infinitesimal action is obtained via differentiating a lifted action of G we say that the lifted infinitesimal action *integrates to an action of* G on E.

4.1 Simple reduction

First, we want to see whether under a group action the invariant sections of an exact Courant algebroid could become a Courant algebroid itself.

Let $\sigma : P \to M$ be a principal G-bundle and $(E, [\cdot,\cdot], \langle\cdot,\cdot\rangle, \pi)$ an exact Courant algebroid on P so that G acts via diffeomorphisms preserving the Ševera class of E. As G is compact by averaging we may chose a splitting $E \cong TP \oplus T^*P$ such that the bracket is twisted by a G-invariant 3-form $H \in \Omega^3(M)^G$.

Then there is a natural extended action of G on $TP \oplus T^*P$ via

$$\varphi(X+\xi) = \begin{pmatrix} \varphi_* & 0 \\ 0 & (\varphi^{-1})^* \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = \varphi_* X + (\varphi^{-1})^* \xi.$$

By Proposition 3.1.2 this action is an automorphism of the H-twisted Courant algebroid if and only if $\varphi^* H = H$ which is satisfied as we chose H to be G-invariant.

Consequently if P is a principal G-bundle with compact connected structure group, the group action naturally lifts to an action on E via Courant automorphisms.

Now we can define a new Courant algebroid over the base $M = P/G$ manifold of P. Firstly, E/G is a vector bundle over M as the action is free and proper on P and acts via automorphisms on E. The sections of E/G naturally identify with the G-invatiant sections of E. These sections are closed under the Courant bracket and their inner product is a G-invariant function on P. Hence $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$ descend to well defined operations on $E/G \to P/G$. Finally, to define the anchor for our new Courant algebroid notice that the anchor π of E sends G-invariant sections of E to G-invariant sections of TP which project to sections of $T(P/G)$. Therefore we can use π as the anchor for E/G

$$\pi^G : E/G \to TP/G \to T(P/G).$$

It is easy to check that $(M, E/G, \pi^G, [\cdot,\cdot], \langle\cdot,\cdot\rangle)$ satisfies the axioms of a Courant algebroid over M. The reduced Courant algebroid is not exact, as the rank of E/G is too large but still transitive. Moreover, considering the G-invariant splitting $E \cong TP \oplus T^*P$ it is clear that TP/G becomes a Lie algebroid over M and $(TP/G)^* \cong T^*P/G$. Hence E/G fits into the short exact sequence

$$0 \longrightarrow (TP/G)^* \xrightarrow{\frac{1}{2}\pi^*} E/G \xrightarrow{\pi} TP/G \longrightarrow 0.$$

Therefore, the resulting Courant algebroid is the double of the Lie algebroid TP/G. The above process is what we call *simple reduction.*

Remark 4.1.1. The above construction easily generalizes to any Courant algebroid E over a principal G-bundle $P \to M$. Whenever the action of G lifts to E via Courant algebroid automorphisms one can define a new Courant algebroid E/G over the base M. We also call this the simple reduction of E.

4.2 Reduction via extended actions

There are other ways to reduce exact Courant algebroids over principal G-bundles. The procedure was described in generality in [5] but for the purpose of this thesis we only need to consider the simplest case which we will call reduction via *extended actions.*[1]

[1]In the reference [5] there is a more general definition of an extended action and the case that we considered is called reduction via trivially extended actions.

Extended actions

Let again $(E, [\cdot,\cdot], \langle\cdot,\cdot\rangle, \pi)$ be an exact Courant algebroid on the principal G-bundle $\sigma : P \to M$. Suppose we have a lifted infinitesimal action $\alpha_0 : \mathfrak{g} \to Der(E)$ so that the image lies in the image of the adjoint action (see Proposition 3.1.4). One then can consider lifting α_0 to a morphism $\alpha : \mathfrak{g} \to \Gamma(E)$ that still extends the infinitesimal action ψ of $\mathfrak{g}$ on TP.

To consider such an extension one has to describe the algebraic structure of $\Gamma(E)$ which is what we call a *Courant algebra* over the Lie algebra $(\Gamma(TP), [.,.])$. The structure is formalized in [5] but in this thesis we only need the most trivial case so we omit the precise definition. For our purposes it is enough to consider extensions which define bracket preserving maps from $\mathfrak{g}$ to $\Gamma(E)$.

Definition 4.2.1. [5] Let G be a connected Lie group and P a principal G-bundle with infinitesimal action $\psi : \mathfrak{g} \to \Gamma(TP)$. An *extended action* of the Lie algebra $\mathfrak{g}$ on E is a linear map $\alpha : \mathfrak{g} \to \Gamma(E)$ such that

1. $\alpha([x, y]) = [\alpha(x), \alpha(y)]$,
2. the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{id} & \mathfrak{g} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\psi} \\ \Gamma(E) & \xrightarrow{\rho} & \Gamma(TP) \end{array}$$

   commutes,
3. and the action of $\mathfrak{g}$ via α integrates to a G-action on the total space of E.

The conditions 1.) and 2.) are that α is a morphism of Courant algebras covering ψ.

Suppose now that we have an extended action α and consider a splitting of $E \cong TP \oplus T^*P$. By the definition α in this splitting decomposes as

$$\begin{aligned} \alpha : \mathfrak{g} &\to \Gamma(TP \oplus T^*P) \\ x &\mapsto \psi(x) + \xi(x) \end{aligned}$$

where ψ is the infinitesimal action on TP.

Recall that the adjoint action of E is given by the Dorfman-bracket

$$\begin{aligned} ad(\psi(x) + \xi(x))(Y + \eta) &= [\psi(x) + \xi(x), Y + \eta]_H \\ &= [\psi(x), Y] + \mathcal{L}_{\psi(x)}\eta - \iota_Y d\xi(x) + \iota_Y \iota_{\psi(x)} H \\ &= \begin{pmatrix} \mathcal{L}_{\psi(x)} & 0 \\ \iota_\psi(x) H - d\xi(x) & \mathcal{L}_\psi(x) \end{pmatrix} \begin{pmatrix} Y \\ \eta \end{pmatrix}. \end{aligned}$$

Clearly α integrates to an action of G if the action of $\mathfrak{g}$ respects the splitting $E \cong TP \oplus T^*P$. Moreover if α integrates to an action of G on E by averaging we can always find a G-invariant splitting of E which is then also a $\mathfrak{g}$-invariant splitting.

Therefore to find an extension α that integrates to an action of G we only have to find solutions where

$$\iota_{\psi(x)} H = d\xi(x) \quad \forall x \in \mathfrak{g}. \tag{4.2.1}$$

We also require α to preserve the bracket, i.e.

$$\alpha([x,y]) = [\alpha(x), \alpha(y)]_H \quad \forall x, y \in \mathfrak{g}.$$

Expanding the above formula we find that for ψ

$$\psi([x,y]) = [\psi(x), \psi(y)],$$

which is satisfied since ψ is a Lie-algebra homomorphism and for ξ

$$\xi([x,y]) = \mathcal{L}_{\psi(x)}\xi(y) - \iota_{\psi(y)} d\xi(x) + \iota_{\psi(y)}\iota_{\psi(x)} H.$$

Using condition 4.2.1 the above constraint reduces to

$$\xi([x,y]) = \mathcal{L}_{\psi(x)}\xi(y) \quad \forall x, y \in \mathfrak{g}. \tag{4.2.2}$$

Together the conditions 4.2.1 and 4.2.2 can be expressed using the Cartan complex of equivariant cohomology associated to the principal bundle P [20]. In this complex the degree-k forms are given by

$$\Omega^k_G(P) = \bigoplus_{2p+q=k} \Omega^q(P, S^p(\mathfrak{g}^*))$$

where $S^p(\mathfrak{g}^*)$ is the space of degree-p symmetric products of $\mathfrak{g}^*$. The differential is

$$d_G\omega = d\omega - e^i \iota_{\psi(e_i)}\omega$$

where $\{e_i\}_i$ is a basis for the Lie algebra $\mathfrak{g}$ and $\{e^i\}$ is the dual basis of $\mathfrak{g}^*$. We also take the symmetrization of the second term over the tensor components in $\mathfrak{g}^*$.

Remember that ξ is a linear map from $\mathfrak{g}$ to the sections of T^*P. Therefore one can think of ξ as a section of T^*P with values in the dual Lie algebra, i.e.

$$\xi \in \Omega^1(P, \mathfrak{g}^*).$$

Moreover ξ is G-equivariant, i.e.

$$\xi(Ad(g)x) = g\xi(x)$$

because the equivariance condition is the integrated form of 4.2.2. To see what 4.2.1 means in this setting note that H is an equivariant 3-form

$$H \in \Omega^3(P)^G$$

therefore together $\Phi = H + \xi$ form an element of $\Omega^3_G(P)$. The condition 4.2.1 then implies that $d_G\Phi$ defines a symmetric bilinear form on $\mathfrak{g}$ as

$$\begin{aligned} d_G\Phi = d_G(H+\xi) &= dH + d\xi - e^i\iota_{\psi(e_i)}H - e^i\iota_{\psi(e_i)}\xi \\ &= -e^i\iota_{\psi(e_i)}\xi. \end{aligned}$$

To have a better understanding of this formula write ξ in local coordinates as

$$\xi = \xi(e_j)_i dx^i \otimes e^j$$

so we have

$$d_G\Phi = -\xi(e_{(j})_i\psi(e_{k)})^i e^j e^k = -\frac{1}{2}\Big(\xi(e_j)_i\psi(e_k)^i + \xi(e_k)_i\psi(e^k)^i\Big)e^j e^k.$$

Therefore if $x = x^i e_i$ and $y = y^i e_i$ are elements of the Lie algebra $\mathfrak{g}$ the bilinear form $d_G\Phi$ evaluates to

$$\begin{aligned} d_G\Phi(x,y) =& -\frac{1}{2}\Big(\xi(e_j)_i\psi(e_k)^i + \xi(e_k)_i\psi(e^k)^i\Big)x^j y^k \\ =& -\frac{1}{2}\Big(x^j y^k\iota_{\psi(e_j)}\xi(e_k) + y^j x^k\iota_{\psi(e_j)}\xi(e_k)\Big) \\ =& -\langle\alpha(x),\alpha(y)\rangle. \end{aligned}$$

Here we used that we can write the $\alpha(x)$ using the basis $\{e_i\}$ as

$$\begin{aligned} \alpha(x) &= \psi(x) + \xi(x) \\ &= x^i(\psi(e_i) + \xi(e_i)) \end{aligned}$$

and $\langle\cdot,\cdot\rangle$ is the natural inner product of sections of E.

Consequently suitable extensions of a Lie algebra action on E give rise to solutions of the equation

$$d_G\Phi = -\alpha^*\langle\ ,\ \rangle. \tag{4.2.3}$$

In the following we will see that equivalence classes of solutions are in one-to one correspondence with equivalent extended actions.

Reduction procedure

Now given an extended action we can reduce the exact Courant algebroid E from the principal bundle P to a not necessarily exact one on the base of M. We follow the treatment of [5] closely but the proofs were re-written to the specific case that we consider.

An extended action defines two natural distributions of E, the image of $\mathfrak{g}$ under the action $K = \alpha(\mathfrak{g})$ and its orthogonal complement $K^\perp$. Firstly, $x \in \mathfrak{g}$ acts on K via the Dorfman bracket and

$$x.\alpha(y) = [\alpha(x),\alpha(y)] = \alpha([x,y])$$

therefore K is G-invariant. As derivations of E integrate to orthogonal transformations, the G-action preserves the pairing $\langle \cdot, \cdot \rangle$ hence $K^\perp$ is also G-invariant. We want to consider G-invariant sections of K or $K^\perp$ and pass to their equivalence classes over $M = P/G$. For this we need the reducible distribution to be a genuine subbundle. In the general case there are obstructions to this but in our case we have the following proposition.

Lemma 4.2.1. *The distributions K, $K^\perp$ and $K \cap K^\perp$ have constant rank.*

Proof. Since G acts freely on P $\psi(\mathfrak{g})$ has constant rank in TP as fibrewise $\mathfrak{g} \cong \psi(\mathfrak{g})|_p$ for any $p \in P$. Therefore $K = \{\psi(x) + \xi(x) | x \in \mathfrak{g}\} \subset TP \oplus T^*P$ has at least rank $dim(\mathfrak{g})$ but also it cannot have any higher therefore K is of constant rank. Consequently $K^\perp$ is of constant rank as well.

Moreover we have seen that the quadratic form c on $\mathfrak{g}$ is the pullback of the natural inner product on E. Hence the inner product of sections that are in the image of α is constant along P. As these sections generate K fibrewise $K \cap K^\perp$ is isomorphic to the nullspace $Ann(\mathfrak{g})$ of c and hence is of constant rank. ∎

Lemma 4.2.2. *The map $\pi : K^\perp \to TP$ is surjective.*

Proof. By definition $K^\perp = \{Y + \eta | \ \langle Y + \eta, \psi(x) + \xi(x) \rangle = 0 \ \forall x \in \mathfrak{g}\}$. If $Y \in \Gamma(TP)$ then $Y \in \pi(K^\perp)$ if and only if there exists $\eta \in \Gamma(T^*P)$ so that

$$\xi(e_i)(Y) + \eta(\psi(e_i)) = 0 \ \ \forall i = 1, ...m$$

where $\{e_i\}$ is a basis for $\mathfrak{g}$. This system of equations can be solved for any Y since $\psi(e_i)$ are nowhere vanishing vector fields and $dim(P) > dim(\mathfrak{g})$. ∎

To reduce the Courant structure we need to check the integrability of K and $K^\perp$. The subbundle K is integrable with respect to the Dorfman bracket as α is an algebra morphism. For the invariant sections $e_1, e_2 \in \Gamma(K^\perp)^G$ we can use axiom C4) to find

$$\langle \alpha(x), [e_1, e_2] \rangle = -\langle [e_1, \alpha(x)], e_2 \rangle + \pi(e_1) \langle \alpha(x), e_2 \rangle.$$

The second term vanishes as $e_2 \in \Gamma(K^\perp)$. For the first term by C5) we have

$$[e_1, \alpha(x)] = -[\alpha(x), e_1] + \mathcal{D}\langle \alpha(x), e_1 \rangle = -[\alpha(x), e_1] = -x.e_1 = 0$$

because $e_1 \in \Gamma(K^\perp)$ and G-invariant.

Therefore, the sections of $K^\perp$ are closed under the inner product and the bracket. Moreover, $\pi : K^\perp \to TP$ is surjective and π is G-equivariant so $\Gamma(K^\perp)^G$ also surjects to $\Gamma(TP)^G$. Hence the only obstruction to cosidering $K^\perp/G$ as a Courant algebroid over P/G is the possible degeneracy of the inner product. But by Lemma 4.2.1 $K^\perp$ and $K \cap K^\perp$ are subbundles of E so we may consider

$$E_{red} = \frac{K^\perp}{K \cap K^\perp} \Big/ G.$$

Then E_{red} is a vector bundle on $M = P/G$ which becomes a Courant algebroid with the induced inner product and bracket. Moreover the induced anchor is clearly surjective by Lemma 4.2.2 therefore E_{red} is a transitive Courant algebroid over M.

We will consider the cases when c is a non-degenerate symmetric pairing on $\mathfrak{g}$. Then $K \cap K^{\perp} = 0$ and

$$E_{red} = K^{\perp}/G.$$

Isomorphic reductions

Suppose now that α and α' are extended actions inducing isomorphic reductions. This happens precisely when they induce the same G-action on $E \cong TP \oplus T^*P$ and the image bundles $K = im(\alpha)$ and $K' = im(\alpha')$ are isomorphic, more precisely related by a B-transform for some $B \in \Omega^2(P)$. In a $\mathfrak{g}$-invariant splitting for α given $x \in \mathfrak{g}$ we have $\alpha(x) = \psi(x) + \xi(x)$ acting via the H-twisted Dorfman bracket where H is G-invariant.

Via the B-transform the equivalent extended action is of the form $\alpha'(x) = \psi(x) + \xi(x) + \iota_{\psi(x)}B$ acting via the $H - dB$-twisted bracket. Here $\xi \in \Omega^1(P, \mathfrak{g}^*)^G$. As $\alpha'(x)$ is also an extended action integrating to the same G-action on E it must satisfy 4.2.1 and 4.2.2 which yield:

$$4.2.1: \; d\iota_{\psi(x)}B = -\iota_{\psi(x)}dB, \quad 4.2.2: \; \mathcal{L}_{\psi(x)}B = 0.$$

These conditions are equivalent and show that B must be a G-invariant 2-form. Therefore $H - dB$ is also induced by a G-invariant splitting.

The above considerations give the following equivalence relation of extended actions.

Definition 4.2.2. Two extended actions α and α' are called *equivalent* if their images are related by B-transforms as above.

We have seen in Section 4.2 that extended actions on a certain exact Courant algebroid E on a principal G-bundle P give rise to solutions to the equation

$$d_G\Phi = c. \tag{4.2.4}$$

Here $\Phi = H + \xi \in \Omega^3_G(P)$ with H representinf the Ševera class of E and c a quadratic form on $\mathfrak{g}$ the Lie algebra corresponding to P. Given any solution Φ coming from an extended action and an equivariant 2-form $\beta \in \Omega^2_G(P)$, $\Phi' = \Phi + d_G(\beta)$ again solves 4.2.4. On the other hand, it is not immediately clear if it again defines an extended action.

The following theorem describes the correspondence between extended actions and solutions to 4.2.4.

Theorem 4.2.1. [5] *Let G be a compact connected semi-simple Lie group and c a quadratic form on its Lie algebra $\mathfrak{g}$. Let E be an exact Courant algebroid on a principal G-bundle P. Then extended actions $\alpha : \mathfrak{g} \to \Gamma(E)$, such that $c(x) = -\langle \alpha(x), \alpha(x) \rangle$, up to equivalence are in one-to-one correspondence with the solutions to the equation*

$$d_G\Phi = c$$

up to d_G-exact forms, where $\Phi = H + \xi$ is a G-equivariant 3-form and H represents the Ševera class of E.

Proof. Let $\{e_i\}$ be a basis for the Lie algebra $\mathfrak{g}$ and $\{e^i\}$ the dual basis for $\mathfrak{g}^*$. Pick a connection one form $A \in \Omega^1(P, \mathfrak{g})^G$ which is written as $A = A^i e_i$ in the basis.

Suppose α' is a trivially extended action equivalent to α, therefore their images are related by a B-transform such that B is a G-invariant 2-form. Via the B-transform we have

$$\xi'(x) = \xi(x) + \iota_{\psi(x)}B, \quad H' = H - dB$$

therefore Φ changes to $\Phi' = \Phi - dB + \iota B = \Phi - d_G(B)$. Consequently equivalent extended actions induce equivalent solutions of 4.2.4.

Now we will show that changing a solution $\Phi = H + \xi$ by an exact equivariant 2-form produces equivalent actions. The space of equivariant 2-forms is $\Omega^2(P)^G \oplus \Omega^0(P, \mathfrak{g}^*)^G$.

Suppose $\Phi' - \Phi = d_G(f)$ for some $f \in \Omega^0(P, \mathfrak{g})$. Then we can write $f = f_i e^i$ and use the connection A to produce an invariant 1-form $\omega = f_i A^i$. Now as $\iota A^i = e^i$ we find that $f = -d_G(\omega) + d\omega$. Therefore the change in the action is actually given by

$$\Phi' - \Phi = d_G(f) = d_G(d\omega)$$

where $d\omega \in \Omega^2_{cl}(P)^G$, therefore $\alpha' = \alpha - \iota(d\omega)$ is an equivalent extended action with the B-transform given by $B = d\omega$.

Suppose now that $\Phi' - \Phi = d_G(B)$ for some $B \in \Omega^2(P)^G$. Then we are already in the setting of a B-transform with $H' = H - dB$ and $\xi' = \xi + \iota B$ giving the equivalent extended action defining Φ'. ∎

Generalization to the double of a Lie algebroid

The above construction may be generalized to Courant algebroids related to Lie algebroids [2]. Let $(L, [\cdot,\cdot], \pi)$ be a transitive Lie algebroid on the principal G-bundle $P \to M$ and $E \cong L \oplus L^*$ be the Courant algebroid described in Section 3.2 fitting into the short exact sequence

$$0 \longrightarrow L^* \longrightarrow E \stackrel{\rho}{\longrightarrow} L \longrightarrow 0.$$

Assume that G is connected, semisimple, compact and that there is a symmetric G-invariant bilinear pairing c on its Lie algbera $\mathfrak{g}$.

Let $\psi_0 : \mathfrak{g} \to \Gamma(TP)$ be the infinitesimal action of G on TP. To extend the action to $\alpha : \mathfrak{g} \to \Gamma(L \oplus L^*)$ we first have to assume that ψ_0 lifts to an infinitesimal action ψ of $\mathfrak{g}$ on L that integrates to a G-action on L covering the action on P. With this assumption we have the commuting diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\psi} & \Gamma(L) \\ & \searrow^{\psi_0} & \downarrow \pi \\ & & \Gamma(TP) \end{array}$$

where $\psi(x)$ acts via the adjoint action of the Lie algebroid L.

Given the infinitesimal action ψ on L, the exterior algebra of forms $\Omega^\bullet(P, L)$ becomes a *G-differential algebra* (defined in [20]) via the contraction operator $\iota_{\psi(x)} : \Omega^k(P, L) \to \Omega^{k-1}(P, L)$. This allows us to generalize the Cartan complex construction of equivariant cohomology.

Let $\Omega^p(L, S^q\mathfrak{g}^*)^G$ be the space of G-equivariant p-forms with values in the q-th symmetric power of $\mathfrak{g}^*$. Equivariance for a given $\omega \in \Omega^p(L, S^q\mathfrak{g}^*)^\mathfrak{g}$ is meant as

$$\mathcal{L}_{\psi(x)}\omega(y) = \omega([x, y]) \quad \forall x \in \mathfrak{g}$$

where $\mathcal{L} = \iota \circ d_L + d_L \circ \iota$, which is the integrated version of G-equivariance.

Then the Cartan complex is again given by

$$\Omega^k_G(L) = \bigoplus_{p+2q=k} \Omega^p(L, S^q\mathfrak{g}^*)^G$$

and the equivatiant differential is

$$d_G = d - e^i\iota_{\psi(e_i)}$$

given a basis $\{e_i\}$ of $\mathfrak{g}$.

Now just as in the exact case we may look for extended actions which take the form $\alpha = \psi + \xi : \mathfrak{g} \to \Gamma(L \oplus L^*)$ in a G-invariant splitting characterized by a 3-form $H \in \Omega^3(L)^G$. Then $\xi \in \Omega^1(L, \mathfrak{g}^*)^G$ and H have to satisfy the straightforward generalizations of 4.2.1 and 4.2.2.

Setting $c(x) = -\langle\alpha(x), \alpha(x)\rangle$ we obtain $c \in \Omega^0(L, S^2\mathfrak{g}^*)^G$ and extended actions correspond to solutions of $d_G\Phi = c$ with $\Phi = H + \xi$.

The reduction procedure also generalizes to this case. The distributions $K = im(\alpha)$, $K^\perp$ and $K \cap K^\perp$ remain genuine subbundles of constant rank as $\psi(x)$ covers the infinitesimal action of $\mathfrak{g}$ on P. A slight difference is that not only $\pi : K^\perp \to TM$ but also $\rho : K^\perp \to L$ is surjective. Integrability of K and $K^\perp$ follows as before.

Consequently, we can also define the reduced Courant algebroid as

$$E_{red} = \frac{K^\perp}{K \cap K^\perp}\Big/G$$

which is just $E_{red} = K^\perp/G$ whenever c is non-degenerate.

Moreover, as we have been looking for extended actions that are of the form $\alpha = \psi + \xi$ in a certain isotropic splitting equivalence of extended actions generalizes too. If α and α' have isomorphic images and they induce the same G-action on $L \oplus L^*$ they must be related by a map preserving $\rho : E \to L$, i.e. by a B-transform for some $B \in \Omega^2(L)^G$ (see Proposition 3.2.1). Consequently we have the following generalization of Theorem 4.2.1.

Theorem 4.2.2. [2] *Let P be a principal bundle with compact structure group G and L a transitive Lie algebroid on P. Let E be the Courant algebroid on P with Ševera class*

$h \in H^3(L)$ and suppose that there is an infinitesimal action $\psi : \mathfrak{g} \to \Gamma(L)$ that integrates to a G-action on L covering the action on P. Given $c \in \Omega^0(L, S^2\mathfrak{g}^*)^G$ there is a one to one correspondence between

1. *solutions to the equation*

$$d_G\Phi = c \tag{4.2.5}$$

   up to d_G-exact forms where $\Phi = H + \xi$ is an equivariant 3-form with $[H] = h$,

2. *and extended actions $\alpha : \mathfrak{g} \to \Gamma(E)$ up to equivalence such that there is a G-invariant isotropic splitting $E \cong L \oplus L^*$ so that α is of the form $\psi + \xi$ for some $\xi \in \Omega^1(P, \mathfrak{g}^*)^G$ and $c = -\alpha^*\langle\ ,\ \rangle$.*

Proof. The proof is straightforward generalization of the proof of Theorem 4.2.1 except for one detail. If two solutions Φ and Φ' of 4.2.5 differ by a d_G-exact equivariant function $f \in \Omega^0(L, \mathfrak{g}^*)^G$ then let A be a connection one-form on P and consider its pullback via the dual of the anchor to $B = \pi^*(A) \in \Omega^1(L, \mathfrak{g})^G$. Given a basis $\{e_i\}$ of $\mathfrak{g}$ one can write $B = B^i e_i = \pi^*(A^i)e_i$ with $\iota_{\psi(e_i)}B^j = \delta_i^j$. Then if $f = f_i e^i$ then as in the exact case we have

$$\Phi' - \Phi = d_G(d\omega)$$

for $\omega = f_i B^i \in \Omega^1(L)^G$. ∎

4.3 Commuting actions

We saw two different ways to reduce exact Courant algebroids over a principal bundle. In this section we will explore what constraints are necessary for the two types of reductions to be interchangeable following [2].

Let $\sigma : P \to X$ be a principal G-bundle and $\pi_0 : X \to M$ a principal T-bundle where G and T are compact connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{t}$. Suppose the action of T lifts to P and the lifted action commutes with the action of G so P becomes a $G \times T$ principal bundle over M. Set P_0 to be P/T a principal G-bundle over M with projection σ_0. Moreover P is a principal T-bundle over P_0 with projection map π and we have the following commuting diagram.

$$\begin{array}{ccccc} & & G & & G \\ & & \vdots & & \vdots \\ T & \dashrightarrow & P & \xrightarrow{\pi} & P_0 \\ & & \downarrow \sigma & & \downarrow \sigma_0 \\ T & \dashrightarrow & X & \xrightarrow{\pi_0} & M \end{array}$$

Suppose now that we have an exact Courant algebroid E over P and that $G \times T$ preserves the Ševera class of E. We then have an infinitesimal action

$$\psi_{\mathfrak{g}} \oplus \psi_{\mathfrak{t}} : \mathfrak{g} \oplus \mathfrak{t} \to \Gamma(TP).$$

Since $G \times T$ is compact, by averaging, we can find a $G \times T$-invariant representative $H \in \Omega^3(P)$ of the Ševera class E so we can extend the action of $G \times T$ to E. Suppose also that the action of G is coming from a trivially extended action $\alpha_{\mathfrak{g}} : \mathfrak{g} \to \Gamma(E)$ corresponding to a nondegenerate bilinear form c on $\mathfrak{g}$.

We want to reduce E via the trivially extended action of G as in section 4.2 and then via simple reduction as in section 4.1 for the T action. To be able to do that we must require $\alpha_{\mathfrak{g}}$ to map into T-invariant sections of E. The following theorem then holds.

Theorem 4.3.1. [2] *Let E be an exact Courant algebroid over the principal $G \times T$ bundle P with an extended $G \times T$ action satisfying all the conditions above. Then one can first reduce E by T then by G or the other way around and the reduced transitive Courant algebroid over M satisfies*

$$E_{red}/T \cong (E/T)_{red}.$$

Here $(\ )_{red}$ denotes reduction via the trivially extended action $\alpha_{\mathfrak{g}}$ and $/T$ denotes simple reduction by the group T.

Proof. Firstly, it is clear that since $\alpha_{\mathfrak{g}}$ maps into T-invariant sections of E, the image of $\alpha_{\mathfrak{g}}$, K and its orthogonal complement $K^{\perp}$ are both invariant under the action of T. As the action of T and G commute the T-action on $E_{red} = K^{\perp}/G$ is well-defined and we can identify sections of E_{red}/T with $G \times T$-invariant sections of $K^{\perp}$.

Secondly, we want to first reduce E by T. As $\alpha_{\mathfrak{g}}$ maps into T-invariant sections of E the reduced bundle E/T carries an induced extended action of $\mathfrak{g}$. The induced map

$$\alpha' : \mathfrak{g} \to \Gamma(E/T)$$

is an extended action on the Courant algebroid associated to the Lie algebroid TP/G over $X = P/G$. Then the image of α' is the bundle K/T which is still of constant rank and its orthogonal complement is $(K/T)^{\perp} \cong K^{\perp}/T$. Consequently, sections of $(E/T)_{red}$ can also be identified with $G \times T$-invariant sections of $K^{\perp}$ yielding the desired isomorphism. ∎

5 Heterotic Courant algebroids by reduction

In this section we will show that heterotic Courant algebroids are always obtained by reduction of an exact Courant algebroid via an extended action from the corresponding principal bundle. This construction is due Baraglia and Hekmati [2].

Let $P \to M$ be a principal bundle with compact connected semi-simple structure group G acting on the right. Let $\psi : \mathfrak{g} \to \Gamma(TP)$ be the infinitesimal action of G. It is assumed that $\mathfrak{g}$ is endowed with a G-invariant nondegenerate symmetric bilinear form c. We have seen in Theorem 4.2.1 that given an exact Courant algebroid E on P, up to equivalence, extended actions $\alpha : \mathfrak{g} \to \Gamma(E)$ are in one-to one correspondence with solutions to

$$d_G\Phi = c$$

up to d_G-exact forms. Here $\Phi = H + \xi$ with $H \in \Omega^3(P)^G$ representing the Ševera class of E in a G-invariant splitting, $\xi \in \Omega^1(P, \mathfrak{g}^*)^G$ such that $\alpha = \psi + \xi$ in the splitting and $c = -\alpha^*\langle\cdot,\cdot\rangle$.

5.1 Extended actions and string classes

The following proposition classifies extended actions in yet another way.

Proposition 5.1.1. [2] *Given a principal bundle $\sigma : P \to M$ as above, equivalence classes of solutions $\Phi = H + \xi$ to the equation $d_G\Phi = c$ are represented by pairs (H_0, A) where $H_0 \in \Omega^3(M)$ and $A \in \Omega^1(P, \mathfrak{g})$ is a connection one form, such that*

$$H = \sigma^*(H_0) - CS_3(A)$$

$$\xi = -cA.$$

Here $CS_3(A) \in \Omega^3(P)$ is the Chern-Simons 3-form corresponding to A given by

$$CS_3(A) = c(F, A) - \frac{1}{3!}c(A, [A, A])$$

where $F = dA + \frac{1}{2}[A, A]$ is the curvature of A. The Chern-Simons 3-form satisfies $dCS_3(A) = c(F, F)$.

Proof. Let $\Phi = H + \xi$ be a solution to $d_G\Phi = c$. As c is nondegenerate there exists $A_0 \in \Omega^1(P, \mathfrak{g})$ such that $\xi = -cA_0$. Since ξ and c are G-invariant, so is A_0 as for any $g \in G$ using the right action R_g on P we have

$$R_g^*\xi(x) = \xi(Ad(g)x) = -c(A_0, Ad(g)x) = -c(Ad(g^{-1})A_0, x)$$

therefore $R_g^*A_0 = Ad(g^{-1})A_0$.

Let now $A_1 \in \Omega^1(P, \mathfrak{g})$ be a connection 1-form on P. If $e_1, ..., e_m$ is a basis for $\mathfrak{g}$ and $e^1, ..., e^m$ is the dual basis for $\mathfrak{g}^*$ we can decompose A_1 as $A_1 = A_1^i e_i$ where A_1^i are G-equivariant 1-forms on P such that $\iota_{\psi(e_i)}A_1^j = \delta_i^j$. Then A_0 is

$$A_0 = a_j^i A_1^j e_i + B^i e_i \tag{5.1.1}$$

where $B^i \in \Omega^1(M)$ are basic forms (omitting pullback notation), i.e. $\iota_{\psi(x)}B^i = 0$ for all $x \in \mathfrak{g}$ and $i = 1, .., m$, and a_j^i are G-invariant functions.

The constraint $c(x, y) = -\langle \alpha(x), \alpha(y)\rangle$ together with $\alpha = \psi - cA_0$ yields that $2c(x, y) = c(\iota_{\psi(x)}A_0, y) + c(\iota_{\psi(y)}A_0, x)$ which in the basis gives

$$c_{ij} = \frac{1}{2}(a_i^r c_{jr} + a_j^r c_{ir}) = a_{(i}^r c_{j)r}.$$

Therefore $a_i^r c_{jr} = a_{(i}^r c_{j)r} + a_{[i}^r c_{j]r} = c_{ij} + \beta_{ij}$ where $\beta_{ij} = -\beta_{ji}$. Substituting then $a_i^k = \delta_i^k + c^{kr}\beta_{ri}$ into 5.1.1 gives

$$A_0 = A_1 + B^i e_i + c^{kr}\beta_{ri}A_1^i e_k.$$

here $A = A_1 + B^i e_i$ is also a connection 1-form on P, and we will show that $\xi' = -cA$ with an equivalent H' gives an equivalent solution to $d_G\Phi = c$. Change Φ by the exact equivariant 3-form $d_G(\beta_{ij}A_1^i A_1^j)$ to get an equivalent solution Φ'. Then if $\Phi' = \xi' + H'$ we find that $H' = H + d(\beta_{ij}A_1^i A_1^j)$ which amounts of changing the splitting on E. Therefore we find that any solution is equivalent to $\Phi = \xi + H$ with $\xi = -cA$ for a connection 1-form A on P.

Given now a solution (H, A) such that $\Phi = H - cA$ we can write the equivariant 3-form $H \in \Omega(P)$ using the connection $A = A^i e_i$ as

$$H = H_0 + H_i^1 A^i + \frac{1}{2}H_{ij}^2 A^{ij} + \frac{1}{3!}H_{ijk}^3 A^{ijk},$$

where $A^{i_1 \dots i_k} = A^{i_1} \wedge ... \wedge A^{i_k}$ and $H_0, H_i^1, H_{ij}^2, H_{ijk}^3$ are basic forms (omitting pullback notation). The equation $d_G\Psi = c$ then becomes

$$dH - e^i\iota_{\psi(e_i)}H = -d(cA) = -c_{ij}dA^i e^j.$$

This shows that H must satisfy separately $dH = 0$ and $e^i\iota_{\psi(e_i)}H = c_{ij}dA^i e^j$. Let f_{jk}^i be the structure constants of the Lie algebra $\mathfrak{g}$, then we can express the curvature $F = F^i e_i$

as $F^i = dA^i + \frac{1}{2}f^i_{jk}A^{jk}$ which gives a formula for dA^i. Using this, and the fact that F^i are basic from the second equation we find

$$H^1_i = -c_ijF^j \quad H^2_{ij} = 0 \quad H^3_{ijk} = -c_{ir}f^r_{jk},$$

therefore H can be rewritten as

$$H = H_0 - CS_3(A).$$

Finally, $dH = 0$ is satisfied if and only $H_0 \in \Omega^3(M)$ is such that $dH_0 = c(F, F)$ which proves the proposition. ∎

Remark 5.1.1. The above theorem gives the same topological constraint to the existence of an extended action on a principal G-bundle P as there is for the existence of a heterotic Courant algebroid corresponding to P. We find again that the first Pontryagin class $p_1(P) = [c(F, F)] \in H^4(M, \mathbb{R})$ of P must vanish for an extended action to exist.

We have seen before that the equivalent solutions $\Phi = H + \xi$ to $d_G\Phi = c$ induce the same cohomology class $[H]$. In the following proposition we will see that more is true. The equivalence class of a solution only depends on the cohomology class of H. A stronger version of the following theorem can be found in [2] but it has been relaxed and the proof changed to avoid the usage of spectral sequences as long as we can.

Proposition 5.1.2. *Any two solutions $\Phi = H + \xi$ and $\Phi' = H' + \xi'$ to $d_G\Phi = c$ such that*

$$[H] = [H'] \in H^3(P, \mathbb{R})$$

are equivalent i.e. differ by a d_G-exact equivariant 2-form.

Proof. By Proposition 5.1.1 there exist connection 1-forms $A, A' \in \Omega^1(P, \mathfrak{g})$ such that (after equivalence)

$$\xi = -cA, \quad H = H_0 - CS_3(A), \quad \xi' = -cA' \quad \text{and} \quad H' = H'_0 - CS_3(A').$$

Since A and A' are connection 1-forms their difference is a basic Lie algebra valued 1-form $a = A' - A \in \Omega^1(P, \mathfrak{g})_{bas} \cong \Omega^1(M, \mathfrak{g}_P)$. Then the curvature 2-form F of A is related to the curvature 2-form F' of A' as

$$\begin{aligned} F' &= d(A + a) + \frac{1}{2}[A + a, A + a] \\ &= F + da + [A, a] + \frac{1}{2}[a, a] \\ &= F + \nabla a + \frac{1}{2}[a, a] \end{aligned}$$

where ∇ is the covariant derivative on the adjoint vector bundle $\mathfrak{g}_P$ induced by A.

The Chern-Simons 3-form changes as follows

$$\begin{aligned}
CS_3(A') &= c(F', A') - \frac{1}{3!}c(A, [A, A]) \\
&= c(F, A) + c(\nabla a + \frac{1}{2}[a, a], A) + c(F, a) + c(\nabla a + \frac{1}{2}[a, a], a) \\
&- \frac{1}{3!}c(A, [A, A]) - \frac{1}{2}c(a, [A, A]) - \frac{1}{2}c(a, [A, A]) - \frac{1}{3!}c(a, [a, a]) \\
&= CS_3(A) + c(F, a) + c(\nabla a, a) + \frac{1}{3}c(a, [a, a]) \\
&+ c(\nabla a, A) - c(a, \frac{1}{2}[A, A]).
\end{aligned}$$

The last two terms can be rewritten as

$$\begin{aligned}
c(\nabla a, A) - c(a, \frac{1}{2}[A, A]) &= c(da, A) + c([A, a], A) - c(a, \frac{1}{2}[A, A]) \\
&= d(c(a, A)) + c(a, dA) + c(a, [A, A]) - c(a\frac{1}{2}[A, A]) \\
&= c(a, F) + d(c(a, A)).
\end{aligned}$$

Finally, we have found

$$CS_3(A') = CS_3(A) + 2c(F, a) + c(\nabla a, a) + \frac{1}{3}c(a, [a, a]) + d(c(a, A)).$$

Substituting back to H and H' yields

$$H' - H = H_0' - H_0 - 2c(F, a) - c(\nabla a, a) - \frac{1}{3}c(a, [a, a]) - d(c(a, A))$$

as $[H] = [H']$ this difference is exact and equivariant. Moreover, since the first five terms are basic we find (see Appendix 10.2) that there exist $B \in \Omega^2(P)_{bas}$ such that

$$H - H' = dB - d(c(a, A)).$$

Consequently, we have

$$\begin{aligned}
\Phi' - \Phi &= H_0' - CS_3(A') - H_0 + CS_3(A) - cA' + cA \\
&= dB - d(c(a, A)) - cA - ca + cA \\
&= d_G(B - c(a, A))
\end{aligned}$$

as $\iota c(a, A) = -ca$, which proves the proposition.

To summarize our results we have found

$$\begin{aligned}
A' &= A + a \\
H_0' &= H_0 + dB + 2c(F, a) + c(\nabla a, a) + \frac{1}{3}c(a, [a, a]) \\
H' &= H + d(B + c(a, A)).
\end{aligned}$$

∎

The above propositions show that the equivalence classes of extended actions are in one-to one correspondence with classes $h \in H^3(P, \mathbb{R})$ that can be represented by a 3-form H such that

$$H = H_0 - CS_3(A)$$

with H_0 basic and A a connection 1-form on P. Moreover, if a class can be represented as above for a specific principal connection A it can be represented for *any* connection. Indeed, in Proposition 5.1.2 we have seen that if $A' = A + a$ an other connection, then $CS_3(A) = CS_3(A') - (\text{basic forms}) - dc(a, A)$, therefore

$$H' = H - dc(A, a) = H^0 + (\text{basic forms}) - CS_3(A') = H^{0\prime} - CS_3(A').$$

Turns out that this condition is topological. Given a connection 1-form A, its restriction to any fibre G is also a connection on the principal bundle G over the one-point space. Such principal bundles have a unique connection 1-form ω_1 given by the *Maurer-Cartan 1-form* which is flat. Since we assumed G to be compact, connected and semisimple its cohomology classes are $H^0(G, \mathbb{R}) = \mathbb{R}$, $H^1(G, \mathbb{R}) = H^2(G, \mathbb{R}) = 0$ and $H^3(G, \mathbb{R}) \neq 0$ with one of the generators being the class of ω_3, the *Maurer-Cartan 3-form*

$$\omega_3 = \frac{1}{3!} c(\omega_1, [\omega_1, \omega_1])$$

for a choice of non-degenerate pairing c on the Lie algebra (see Appendix 10.1).

Clearly, the 3-forms arising from extended actions restrict to ω_3 on any fibre, and this property only depends on the cohomology class not the specific form (see [2]). This motivates the following definition.

Definition 5.1.1. A *real string class* on a principal G-bundle P is a degree 3 real cohomology class $h \in H^3(P, \mathbb{R})$ that restricted to any fibre G agrees with the class of ω_3 for some choice of c. The set of string classes is denoted by $H^3_{str}(P, \mathbb{R})$.

Remark 5.1.2. When the group G is simple the third cohomology group is one-dimensional $H^3(G, \mathbb{R}) = \mathbb{R}$ generated by $[\omega_3]$. In this case if a class $h \in H^3(P, \mathbb{R})$ restricts to the same number on each fibre one can scale c so h becomes a string class. When G is semisimple its third cohomology is a direct sum of the third cohomologies of its simple components $H^3(G, \mathbb{R}) = \mathbb{R}^d$. In this case $h \in H^3(P, \mathbb{R})$ is a string class for a properly scaled c whenever it restricts to the same vector with nonzero components to each fibre.

The existence of such classes requires $p_1(P) = 0$ but the converse is also true, whenever $p_1(P) = 0$ the set of string classes is non-empty. These classes on P are precisely the classes that admit representatives of the form $H^0 - CS_3(A)$ which proves the following theorem.

Theorem 5.1.1. *On a principal bundle P with compact, connected semisimple structure group G there is a one-to-one correspondence between real string classes and equivalence classes of extended actions.*

5.2 Reduction and heterotic Courant algebroids

In the previous section we have found a certain representation for extended actions which is reminiscent of what we have seen in Section 3.3 for heterotic Courant algebroids. We also have discovered the same topological condition for the existence of both. Moreover, we found in Proposition 5.1.2 a kind of transformation law for extended actions that looks like (B, A)-transforms of heterotic Courant algebroids. All of these similarities are not accidental as we will see in this section.

Suppose that E is an exact Courant algebroid on the principal bundle P as above with $p_1(P) = 0$ and an extended action $\alpha : \mathfrak{g} \to \Gamma(E)$. In Section 4.2 we considered the Courant subbundle $K^\perp = im(\alpha)^\perp$ and defined the reduced Courant algebroid

$$E_{red} \cong K^\perp / G$$

whose sections are identified with G-invariant sections of $K^\perp$. The following theorem defines the structure of E_{red} through heterotic Courant algebroids.

Proposition 5.2.1. [2] *Let $\sigma : P \to M$ be a principal bundle as above, E an exact Courant algebroid on P and (H_0, A) a pair giving an extended action on E as in Proposition 5.1.1. Then if $\mathcal{H} = T^*M \oplus \mathfrak{g}_P \oplus TM$ is the heterotic Courant algebroid associated to (P, H_0, A) as in Theorem 3.3.2, there is an isomorphism of Courant algebroids $f : \mathcal{H} \to E_{red}$. It is given by*

$$f : \; X + s + \xi \to X^H + s + c(A, s) + \sigma^*(\xi)$$

where X^H is the unique horizontal lift of X defined by the principal connection A and s is thought of as a G-invariant vertical vector field.

Proof. First, we will show that f is a well defined isomorphism of vector bundles. Sections of E_{red} are identified with G-invariant sections of the bundle $K^\perp$. Given the principal connection A, G-invariant sections of TP decompose as

$$\Gamma(TP)^G \cong \Gamma(\mathfrak{g} \oplus TM).$$

Splitting E so that the Courant bracket is twisted by $H = H_0 - CS_3(A)$ the extended action is $\alpha = \psi - cA$, therefore $K = \{\psi(x) - c(A, x)| \; x \in \mathfrak{g}\}$. One can also write K using the map j which identifies $\mathfrak{g}$ with invariant sections of TP, then we have $K = \{x - c(A, x)| \; x \in \mathfrak{g}\}$. It is clear then that f maps into $K^\perp$ as for any $x, s \in \mathfrak{g}$ $X \in TM$ and $\xi \in T^*M$ we have

$$\begin{aligned} 2\langle x - c(A, x) \;, \; X^H + s + c(A, s) + \sigma^*(\xi)\rangle = & \\ =& c(A(x), s) + \sigma^*(\xi)(x) - c(A(X^H), x) - c(A(s), x) \\ =& c(x, s) - c(s, x) = 0. \end{aligned}$$

Moreover f is injective and the rank of $\mathcal{H}$ is the same as the rank of E_{red} therefore f is an isomorphism of vector bundles.

It is clear then that $X = \pi_{\mathcal{H}} = \pi_{E_{red}} \circ f$ and that f preserves the fibrewise inner product, which is given by

$$\langle X+s+\xi, X+s+\xi\rangle = \xi(X) + c(s,s)$$

on $\mathcal{H}$. The only thing remains to show is that f preserves the Courant bracket which in the heterotic case was given in Theorem 3.3.2. For E_{red} it is given by the usual exact bracket, with the Lie bracket taken as the bracket of invariant vector fields on the Atiyah algebroid. Omiting pullback notation, we have

$$\begin{aligned}
[f(X+s+\xi), f(Y+t+\eta)]_E =&[X^H + s + c(A,s) + \xi, Y^H + t + c(A,t) + \eta] \\
=&[X^H + s, Y^H + t] + \mathcal{L}_{X^H+s}(c(A,t)+\eta) \\
&- \iota_{Y^H+t} d(c(A,s)+\xi) \\
&+ \iota_{Y^H+t}\iota_{X^H+s}(H_0 - CS_3(A)).
\end{aligned}$$

The first term is the projection of $[f(\cdot), f(\cdot)]$ to $TP \cap K^\perp$ which clearly matches the projection of $f[\cdot,\cdot]$. It remains to show that $[f(\cdot), f(\cdot)] = f[\cdot,\cdot]$ on $T^*P \cap K^\perp$. Using the Cartan formula $\mathcal{L}_X = [d, \iota_X]$ we have the following terms

$$d\iota_{X^H+s}(c(A,t)+\xi) = dc(s,t) + d\xi(X)$$

where $dc(s,t) = c(\nabla s, t) + c(s, \nabla t)$ and

$$\iota_{X^H+s}(H_0 - CS_3(A)) = \iota_Y\iota_X H_0 - c(F(X,Y),A) + c(\iota_X F, t) - c(\iota_Y F, s) + c(A,[s,t]).$$

For the last term let $\{e_i\}$ be a basis for $\mathfrak{g}$ then we have

$$\begin{aligned}
\iota_{X^H+s} d(c(A,t)+\xi) =&\iota_{X^H+s}(c(dA,t) + dc(e_i,t)\wedge A^i) + \iota_X d\xi \\
=&\iota_{X^H+s}(c(dA,t) + c(\nabla e_i, t)\wedge A^i \\
&+ c(e_i, \nabla t)\wedge A^i) + \iota_X d\xi \\
=&c(\iota_{X^H}(F - \frac{1}{2}[A,A]), t) + c(\iota_s(F - \frac{1}{2}[A,A]), t) \\
&+ c(A, \nabla_X t) - c(s, \nabla t)
\end{aligned}$$

here we used that $\iota_s dc(e_i,t) = 0$.

$$\begin{aligned}
&=c(\iota_{X^H}F, t) - c([s,A],t) + c(A,\nabla_X t) - c(s,\nabla t) \\
&=c(\iota_{X^H}F, t) - c(A,[s,t]) + c(A,\nabla_X t) - c(s,\nabla t).
\end{aligned}$$

In the last line we used that $-c([s,A],t) = +c(A,[s,t])$ for the right handed bracket on the Lie algebra $\mathfrak{g}$ as we use the map j to identify $\mathfrak{g}$ with invariant sections (cf. Section 3.3). Therefore, we get $-c(A,[s,t])$ in our conventions for the left handed bracket. Putting everything together we find

$$\begin{aligned}
[f(X+s+\xi), f(Y+t+\eta)]_E =&[X,Y]^H + \nabla_X Y - \nabla_Y X - [s,t] - F(X,Y) \\
&+ c(\nabla_X t, A) - c(\nabla_Y s, A) - c([s,t],A) \\
&- c(F(X,Y),A) - 2c(\iota_Y F, s) + 2c(\iota_X F, t) \\
&+ c(\nabla s, t) = \mathcal{L}_X\eta - \iota_Y d\xi + \iota_Y\iota_X H,
\end{aligned}$$

which equal to $f([X+s+\xi, Y+t+\eta]_{\mathcal{H}})$. ∎

The above two statements together yield the following classification of heterotic Courant algebroids.

Theorem 5.2.1. [2] *Every heteoric Courant algebroid on a smooth manifold M is given by reduction of an exact Courant algebroid $E \cong TP \oplus T^*P$ over a principal G-bundle $\sigma : P \to M$ with a real string class Ševera class $h \in H^3_{str}(P, \mathbb{R})$ corresponding to a non-degenerate bilinear pairing c on the Lie algebra. Furthermore, h is represented by $H = \sigma^*(H_0) - CS_3(A)$ for a 3-form $H_0 \in \Omega^3(M)$ and a principal connection A on P and the extended action in the corresponding splitting is given by $\xi \in \Omega^1(P, \mathfrak{g}^*)^G$ of the form $\xi = -cA$.*

Clearly, equivalent actions give rise to isomorphic reductions, and hence to isomorphic heterotic Courant algebroids, that is they are related by (B, A)-transforms. Recall Proposition 3.3.3 where we looked at how (B, A)-transforms change the Dorfman bracket. Applying our findings in Proposition 5.1.2 which relates two extended actions we can finally finish the proof without the cumbersome calculation as follows.

Proposition (Proposition 3.3.3)**.** *Let $\mathcal{H}$ be a split heterotic Courant algebroid on the principal bundle P represented by the pair (H_0, ∇) with $H_0 \in \Omega^3(M)$ and ∇ a G-connection on $\mathfrak{g}_P$. For $B \in \Omega^2(M)$ and $A \in \Omega^1(M, \mathfrak{g}_P)$ the (B, A)-transform changes the (H_0, ∇) heterotic Dorfman bracket as follows*

$$e^{(B,A)}[\ ,\ ]_{H_0,\nabla} = [e^{(B,A)}\ , e^{(B,A)}\ ]_{H_0',\nabla'}$$

where

$$\begin{aligned} H_0' &= H_0 + dB + 2c(A, F) + c(A, \nabla A) + \frac{1}{3}c(A, [A, A]) \\ \nabla' &= \nabla + A. \end{aligned}$$

Proof. First, note that $\Omega^1(M, \mathfrak{g}_P) \cong \Omega^1(P, \mathfrak{g})_{bas}$ and that we have

$$e^{(B,A)}[X, Y]_{H_0,\nabla} = [e^{(B,A)}X, e^{(B,A)}Y]_{H_0',\nabla'},$$

for some (H_0', ∇') representing an other splitting of $\mathcal{H}$. By Theorem 5.2.1 we have the following commutative diagram.

$$\begin{array}{ccc} & (E, h) & \\ \swarrow^{\alpha'} & & \searrow^{\alpha} \\ (\mathcal{H}, H_0, \nabla) & \xrightarrow[(B,A)]{\cong} & (\mathcal{H}, H_0', \nabla') \end{array}$$

Here (E, h) is the exact Courant algebroid with Ševera class h that we reduce via the extended actions α and α' (we denote the reduction process also by α and α'). Now

Proposition 5.1.2 relates $\alpha = H_0 - CS_3(A) - cA$ to $\alpha' = H_0' - CS_3(A') - cA'$ and we find

$$\begin{aligned} H_0' &= H_0 + dB + 2c(A, F) + c(A, \nabla A) + \frac{1}{3}c(A, [A, A]) \\ \nabla' &= \nabla + A. \end{aligned}$$

Note that the two representatives of h, $H = H_0 - CS_3(A)$ and $H' = H_0' - CS_3(A')$ are related by a B-transform on E

$$H' = H + d(B + c(A_0, A)),$$

where A_0 is the principal connection on P associated to ∇. ∎

6 Generalized metrics

Courant algebroids were proposed as a generalization of the tangent bundle of a manifold, so naturally generalized geometry aims to define the analogues of the geometric structures and operators on the tangent bundle as well. These have been found useful in geometrizing solutions to partial differential equations that arose in string theory. In the next chapter we introduce the generalization of Riemannian metrics on transitive Courant algebroids and see how they behave under reduction. For the exact case the the calculations and definitions can be found in [22] or [23] and for the heterotic case we refer to [2], [15] and [35].

Let $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ be a Courant algebroid on a manifold M of dimension n.

Definition 6.0.1. A *generalized metric* on E is a smooth self-adjoint, orthogonal bundle automorphism $\mathcal{G} : E \to E$ which is positive definite in the sense that

$$\langle \mathcal{G}e, e\rangle > 0,$$

for all non-zero sections $e \in \Gamma(E)$.

Orthogonality implies that $\mathcal{G}\mathcal{G}^* = Id$, and together with the self-adjoint property we find that

$$\mathcal{G}^2 = Id.$$

Therefore, a generalized metric also defines a decomposition of E into its $+1$ and -1 eigenspaces which we denote by

$$E_- = \ker(Id + \mathcal{G}) \quad \text{and} \quad E_+ = \ker(Id - \mathcal{G}).$$

As $\mathcal{G}$ is positive definite the restriction of the inner product $\langle\cdot,\cdot\rangle$ to E_- is negative definite and to E_+ is positive definite. Denote the projections to the subbundles $E_\pm$ as

$$\Pi_\pm = \frac{1}{2}(Id \mp \mathcal{G}) : E \to E_\pm.$$

From this description it is clear that E_+ and E_- are orthogonal with respect to the inner product.

Therefore, if the inner product on a Courant algebroid has signature (p, q) then a generalized metric is equivalent to a decomposition of E into two orthogonal subbundles $E = E_+ \oplus E_-$ of rank p and q respectively, such that the restriction of the inner product to E_+ (E_-) is positive (negative) definite. Clearly, defining one of the bundles determines the other one immediately because $E_+ = E_-^\perp$. One can then recover the bundle morphism $\mathcal{G}$ by defining it as the identity on E_+ and minus the identity on E_-. In the following we will mostly work with this equivalent description of a generalized metric.

When E is a transitive Courant algebroid, i.e. the anchor $\pi : E \to TM$ is surjective, the cotangent bundle T^*M injects into E via π^* and its image is isotropic. Consequently, if the signature of the inner product on E is (p, q) transitivity implies that $p \geq n$ and $q \geq n$ where $n = \dim(M)$.

Lemma 6.0.1. *Given a generalized metric on a transitive Courant algebroid, the anchor restricted to the subbundles E_- and E_+*

$$\pi_\pm = \pi|_{E_\pm} : E_\pm \to TM$$

is surjective.

Proof. If no element $e \in E_-$ projects to some vector $X \in TM$ then there exist $\xi \in T^*M$ such that $\xi(X) \neq 0$ but $\xi(Y) = 0$ for any $Y \in TM$ outside of the linear span of X. Then ξ as an element of E is orthogonal to E_- and therefore an element of E_+ of zero length which contradicts the positive definite property of the inner product on E_+. The same argument shows that π_+ is surjective as well. ∎

Note that when the signature of the inner product is $(n+m, n)$, restricting the anchor to E_- yields an isomorphism $\pi_- : E_- \to TM$. Therefore its inverse

$$\lambda = (\pi_-)^{-1} : TM \to E_- \subset E$$

is an injective bundle-map with negative definite image, which induces a Riemannian metric g on M via

$$g(X, Y) = -\langle \lambda(X), \lambda(Y) \rangle \quad X, Y \in TM.$$

6.1 Exact case

For an exact Courant algebroid $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$ on a manifold M of dimension n the inner product has signature (n, n). Therefore a generalized metric is equivalent to defining a rank n negative definite subbundle $E_- \subset E$ and we have the decomposition $E = E_+ \oplus E_-$ for $E_+ = E_-^\perp$.

Generalized metrics can also be described from the point of view of the underlying manifold utilising the bundle map $\lambda = (\pi_-)^{-1} : TM \to E$ defined in the previous section. We have the following proposition.

Proposition 6.1.1. *A generalized metric E_- on an exact Courant algebroid E over the manifold M is equivalent to a Riemannian metric g on M and an isotropic splitting $E \cong TM \oplus T^*M$ such that E_- and E_+ are of the form*

$$E_- = \{X - gX \mid X \in TM\}$$
$$E_+ = \{X + gX \mid X \in TM\}.$$

Proof. Let g be the induced Riemannian metric $g(X,Y) = -\langle \lambda(X), \lambda(Y) \rangle$ for $X, Y \in TM$. Then we can define an isotropic splitting $\lambda' : TM \to E$ via

$$\lambda'(X) = \lambda(X) + g(X).$$

Indeed, the inner product on the image of λ' vanishes as

$$\begin{aligned}\langle \lambda(X) + g(X), \lambda(Y) + g(Y) \rangle &= \langle \lambda(X), \lambda(Y) \rangle + \frac{1}{2}\iota_{\lambda(X)} g(Y) + \frac{1}{2}\iota_{\lambda(Y)} g(X) \\ &= -g(X,Y) + \frac{1}{2} g(X,Y) + \frac{1}{2} g(Y,X) \\ &= 0.\end{aligned}$$

Using λ' we have an isomorphism $E \cong TM \oplus T^*M$ and the subbundle E_- is given by

$$E_- = im(\lambda) = im(\lambda' - g)$$

from which we have the required form. Calculating the orthogonal complement of E_- in this splitting yields the form of E_+. ∎

Remark 6.1.1. By The proposition above a generalized metric on E is equivalent to the pair (g, H) where g is the Riemannian metric on M and H is the representative of the Ševera class of E defined by the splitting.

Two-forms act transitively on isotropic splittings of E via the B-transform. Therefore, in an arbitrary splitting $E \cong TM \oplus T^*M$ the two subbundles defined by a generalized metric have the following form:

$$\begin{aligned}E_- &= \{X - g(X) + B(X) \mid X \in TM\}, \\ E_+ &= \{X + g(X) + B(X) \mid X \in TM\}.\end{aligned}$$

Here g is the Riemannian metric defined by $\mathcal{G}$ metric and B is a two-form.

6.2 Heterotic case

Let $P \to M$ be a principal bundle with compact connected semi-simple structure group G and c a non-degenerate invariant bilinear pairing on the Lie algebra $\mathfrak{g}$ so that the first Pontryagin class of P vanishes with respect to c. So far we have not specified the signature of c but from here we require it to be *positive definite.*

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$ be a heterotic Courant algebroid over M associated to the principal bundle P. Then we have the two short exact sequences

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{H} \xrightarrow{\pi} TM \longrightarrow 0 \tag{6.2.1}$$

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} \mathcal{K} \longrightarrow \mathfrak{g}_P \longrightarrow 0. \tag{6.2.2}$$

In this case the inner product has signature $(n+m,n)$ where m is the dimension of the Lie algebra $\mathfrak{g}$. Therefore, a generalized metric is again equivalent to a negative definite subbundle $\mathcal{H}_- \subset \mathcal{H}$ whose orthogonal complement we again denote by $\mathcal{H}_+$. We also have the injective bundle morphims

$$\lambda = (\pi_-)^{-1} : TM \to \mathcal{H}_- \subset \mathcal{H}$$

which is an isomorphism between TM and $\mathcal{H}_-$ and has negative definite image.

Then we have the following generalization of Proposition 6.1.1.

Proposition 6.2.1. *A generalized metric $\mathcal{H}_-$ on the heterotic Courant algebroid $\mathcal{H}$ is equivalent to a Riemannian metric g on M and an isotropic splitting such that the generalized metric in the splitting is given by*

$$\begin{aligned}\mathcal{H}_- =& \{X - gX \mid X \in T\},\\ \mathcal{H}_+ =& \{X + s + gX \mid X \in T,\ s \in \mathfrak{g}_P\}.\end{aligned}$$

Proof. The proof is analogous to the exact case. Let g be again the induced Riemannian metric $g(X,Y) = -\langle \lambda(X), \lambda(Y)\rangle$ for $X, Y \in TM$. Then we have the isotropic splitting $\lambda' : TM \to \mathcal{H}$ defined as before

$$\lambda'(X) = \lambda(X) + g(X).$$

As we have seen in Section 3.3, an isotropic splitting define an isomorphism

$$\begin{aligned}\Psi_{\lambda'} : T \oplus \mathfrak{g}_P \oplus T^* &\to \mathcal{H}\\ X + s + \xi &\mapsto \lambda'(X) + \sigma|_{\lambda'^\perp}^{-1} s + \xi\end{aligned}$$

where $\sigma : \mathcal{H} \to \mathcal{H}/T^*$ is the natural projection and its restriction $\sigma|_{\lambda'^\perp}$ to $\mathcal{K} \cap \lambda'(TM)^\perp$ induces an isomorphism to $\mathfrak{g}_P$.

Notice, that $\lambda'(TM) = \{h + g(\pi(h)) \mid h \in \mathcal{H}_-\}$ and hence $\lambda'(TM)^\perp = \{h - g(\pi(h)) \mid h \in \mathcal{H}_+\}$. Consequently,

$$\lambda'(TM)^\perp \cap \mathcal{K} = \mathcal{H}_+ \cap \mathcal{K}$$

and the isomorphism $\Psi_{\lambda'}$ can be written as $X + s + \xi \to \lambda'(X) + \sigma|_{\mathcal{H}_+}^{-1} s + \xi$.

Finally, the subspace $\mathcal{H}_-$ is given by $\{\lambda(X) = \lambda'(X) - g(X) \mid X \in TM\}$ and hence has the desired form under the isomorphism $\Psi_{\lambda'}$. The form of $\mathcal{H}_+$ follows from orthogonality. ∎

Remark 6.2.1. By the proposition above we see that a generalized metric on the heterotic Courant algebroid $\mathcal{H}$ is equivalent to a triple (g, ∇, H) where g is a Riemannian metric on M and (∇, H) are the G-connection on $\mathfrak{g}_P$ and $H \in \Omega^3(M)$ associated to the isotropic splitting λ'.

In the heterotic case isotropic splittings differ by (B, A)-transforms, therefore the defining bundles $\mathcal{H}_-$ and $\mathcal{H}_+$ of a generalized metric have the following form in an arbitrary splitting $\mathcal{H} + TM \oplus \mathfrak{g}_P \oplus T^*M$:

$$\mathcal{H}_- = \{X + A(X) - g(X) - c(A(X), A.) + B(X) \mid X \in TM\}$$
$$\mathcal{H}_+ = \{X + A(X) + s + B(X) - c(A(X), A.) - 2c(s, A) + g(X) \mid X \in TM \ s \in \mathfrak{g}_P\}.$$

6.3 Generalized metrics and reduction

In this section we investigate how generalized metrics behave under reduction.

Simple reduction. Let E be a transitive Courant algebroid over a principal bundle $X \to M$ with compact connected structure group T. Assuming that the group action lifts to E via Courant automorphisms we defined a new Courant agebroid

$$E/T$$

over the base M which we called the simple reduction of E.

Sections of E/T correspond to T-invariant sections of E, therefore the inner product on E/T has the same signature as on E. Hence a T-invariant generalized metric on E defined via the splitting $E = E_+ \oplus E_-$ reduces to a decomposition

$$E/T = E_+/T \oplus E_-/T,$$

as T-invariance of the generalzied metric is equivalent to the T-invariance of its defining subbundlles. Clearly, the inner product restricted to the reduced bundles is still positive and negetaive definite respectively.

On the other hand, if we have a generalized metric on E/T defined by the decomposition $E/T = (E/T)_+ \oplus (E/T)_-$ over M, pulling back to X yields a decomposition of E into the T-invariant preimages of $(E/T)_+$ and $(E/T)_-$. Moreover, the signature of inner product on the pullback bundles agrees with the signature on the original bundle. Consequently we have the following proposition.

Proposition 6.3.1. *Generalized metrics on the simple reduction E/T of E are in one to one correspondence with T-invariant generalized metrics on E.*

Reduction via extended actions. We have seen that heterotic Courant algebroids arise as reductions of exact Courant algebroids. In this case we first restrict to a subbundle of the original Courant algebroid which we then simply reduce. In this section we investigate which generalized metrics are compatible with this restriction and which metrics can be extended back from a subbundle to the total space.

Let $P \to M$ be a principal bundle with compact, connected, semisimple structure group G and c a non-degenerate, positive definite bilinear pairing on its Lie algebra $\mathfrak{g}$. Suppose $p_1(P) = 0$, let $h \in H^3(P, \mathbb{R})$ be a string class and consider the exact Courant algebroid E on P corresponding to h. Let n be the dimension of the base manifold M and m the dimension of the Lie group G.

Recall that in the reduction procedure we specify a splitting $E \cong TP \oplus T^*P$ and an extended action $\alpha : \mathfrak{g} \to \Gamma(TP \oplus T^*P)$. We then consider the image of α which is a genuine subbundle of E

$$im(\alpha) = K \subset E.$$

Moreover, we have seen that the inner product restricted to the sections of K agrees with $-c$ and therefore it is negative definite. Consequently, the inner product has signature $(n+m, n)$ on $K^\perp$.

The reduced Courant algebroid $E_{red} \cong \mathcal{H}$ is defined as the simple reduction of $K^\perp$, i.e.

$$\mathcal{H} \cong E_{red} = K^\perp / G.$$

Therefore, a generalized metric $E = E_- \oplus E_+$ can be reduced to a generalized metric on E_{red} if it is G-invariant and respects the decomposition $E = K \oplus K^\perp$ or equivalently $\mathcal{G}(K) \subset K$. This in particular implies that K is a subbundle of E_- and that $K^\perp \cap E_-$ is of rank $n = \dim(M)$. Then we can decompose $K^\perp$ into positive and negative definite parts as

$$(K^\perp)_+ = E_+ \quad \text{and} \quad (K^\perp)_- = E_- \cap K^\perp.$$

as these subbundles are G-invariant they define a generalized metric on $\mathcal{H} = K^\perp/G$.

On the other hand a generalized metric on $\mathcal{H} = E_{red}$ can be defined as a decomposition

$$K^\perp/G = \mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$$

where $\mathcal{H}_-$ is of rank n and $\mathcal{H}_+$ is of rank $n+m$. Such a decomposition pulls back to P and we get a G-invariant decomposition of $K^\perp$

$$K^\perp = \hat{\mathcal{H}}_- \oplus \hat{\mathcal{H}}_+$$

into negative and positive definite subbundles (here the hat denotes pullback). Then $\hat{\mathcal{H}}_+$ is a rank $n + d = \dim(P)$ positive definite G-invariant subbundle of E and therefore it determines a generalized metric. This generalized metric is then given by its defining subbundles as

$$E_- = K \oplus \hat{\mathcal{H}}_- \quad \text{and} \quad E_+ = \hat{\mathcal{H}}_+.$$

In conclusion we have the following proposition.

Proposition 6.3.2. *Let $\mathcal{H}$ be a heterotic Courant algebroid coming from a reduction of an exact Courant algebroid E over the principal bundle P. Then if the bilinear pairing c on the Lie algebra $\mathfrak{g}$ of the structure group G is positive definite, any generalized metric on $\mathcal{H}$ uniquely lifts to a generalized metric on E.*

Remark 6.3.1. In this section we always assumed that c is positive definite. For indefinite c the picture is a bit more complicated. Generalized metrics on a heterotic Courant algebroid still lift to the unreduced space but this lift is not unique anymore as one still has to make a choice of splitting K into positive and negative definite orthogonal subspaces.

7 Generalized connections

We have seen in the previous section that metrics have an analogue on Courant algebroids which motivates the search for the analogue of connections and metric compatible connections especially for a unique Levi-Civita connection. We will see that the framework in this case is not so clean but there are still some interesting invariants related to generalized metrics when they are paired together with certain differential operators. These extra operators seem arbitrary from the mathematical point of view as they are motivated by the existence of the *dilaton field* arising in string theory so we postpone their justification to a later point.

Let $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$ be a transitive Courant algebroid over a manifold M. Our discussion in this section is going to be general and we apply our findings to specific cases later.

Generalized connections were introduced by Gualtieri in [24] for exact Courant algebroids and later adapted to the transitive setting in [15] by Garcia-Fernandez.

Definition 7.0.1. A *generalized connection* D on E is a first order differential operator

$$D : \Gamma(E) \to \Gamma(E^* \otimes E),$$

which satisfies the *π-Leibniz rule*

$$D_{e_1}(fe_2) = \pi(e_1)(f)e_2 + fD_{e_1}e_3 \quad \forall f \in C^\infty(M), e_1, e_2 \in \Gamma(E),$$

and compatible with the inner product

$$\pi(e_1)\langle(e_2, e_3) = \langle D_{e_1}e_2, e_3\rangle + \langle e_2, D_{e_1}e_3\rangle \quad \forall e_1, e_2, e_3 \in \Gamma(E).$$

That is, D is an *orthogonal E-connection*[1]. We denote the set of generalized connections on E by $\mathcal{D}$.

Remark 7.0.1. Note that $\mathcal{D}$ is non-empty, as given any standard orthogonal connection ∇ on the vector bundle $(E, \langle \cdot, \cdot \rangle)$ we can defined a generalized connection D via

$$D_{e_1}e_2 = \nabla_{\pi(e_1)}e_2,$$

for $e_1, e_2 \in \Gamma(E)$.

[1]In this thesis we only consider generalized connections that are compatible with the inner product but clearly one could define them without this restriction as well. This has been done in [1] where these operators are called E-connections.

As standard orthogonal connections form an affine space modelled on $\Gamma(T^*M \otimes \mathfrak{so}(E))$, the space of generalized connections is an affine space modelled on the vector space

$$\Gamma(E^* \otimes \mathfrak{so}(E)).$$

Definition 7.0.2. The *torsion* $T_D \in \Gamma(\wedge^3 E^*)$ of a generalized connection D is defined by

$$T_D(e_1, e_2, e_3) = \langle D_{e_1} e_2 - D_{e_2} e_1, e_3 \rangle + \langle D_{e_3} e_1, e_2 \rangle.$$

Remark 7.0.2. The torsion formula above applies only to generalized metrics compatible with the inner product. A more general form was proposed in [1] using the skew-symmetric Courant bracket $[[e_1, e_2]] = \frac{1}{2}([e_1, e_2] - [e_2, e_1])$:

$$T_D(e_1, e_2, e_3) = \Big(\frac{1}{2}\langle D_{e_1} e_2 - D_{e_2} e_1, e_3 \rangle - \frac{1}{3}\langle [[e_1, e_2]], e_3 \rangle\Big) + c.p.,$$

where *c.p* denotes cyclic permutations. This is the straightforward generalization of the standard definition of the torsion of a connection and it applies to non-orthogonal E-connections as well. By a lengthy calculation one can show that the two formulas are equivalent for orthogonal E-connections.

Via the inner product we have an isomorphism $\Gamma(E^* \otimes \mathfrak{so}(E)) \cong \Gamma(E \otimes \wedge^2 E)$ of the vector space on which $\mathcal{D}$ is modelled. Fixing the torsion restricts $\mathcal{D}$ to a subset which we denote by

$$\mathcal{D}(T) = \{ D \in \mathcal{D} \mid T_D = T \in \Gamma(\wedge^3 E^*) \ \}.$$

It is an affine space modelled on

$$\Sigma = \{ \sigma \in \Gamma(E^{\otimes 3}) | \ \sigma_{abc} = -\sigma_{acb}, \ \sigma_{abc} + c.p. = 0 \},$$

where *c.p.* denotes cyclic permutation. Indeed, the first condition is that σ is in $\Gamma(E \otimes \wedge^2 E)$ and the second is that it does not affect the torsion.

Similarly to fixing the torsion, fixing the trace of a generalized connection is a reasonable step. In [17] this process was called *Weyl gauge fixing* as it has physical motivations which we will see later. Here, we introduce *divergence operators* which provide the right mathematical framework for this process.

Definition 7.0.3. [1] The *divergence* of an element $e \in \Gamma(E)$ corresponding to a generalized connection D is

$$div_D(e) = tr \ De = \sum_{i=1}^{r_E} \langle D_{e_i} e, \tilde{e}_i \rangle \in C^\infty(M),$$

where $r_E = \text{rank}(E)$, $\{e_i\}$ is a basis of E and $\{\tilde{e}_i\}$ is the dual basis (i.e. $\langle e_i, \tilde{e_j} \rangle = \delta_{ij}$).

Clearly, div_D defines a first order differential operator $\Gamma(E) \to C^\infty(M)$ which also satisfies the π-Leibniz rule

$$div_D(fe) = \pi(X)f + f div_D(e), \tag{7.0.1}$$

since the trace is $C^\infty(M)$-linear. The above construction can be made independent of the connection D as well.

Definition 7.0.4. [1] A *divergence operator* on E is a first order differential operator $div : \Gamma(E) \to C^\infty(M)$ satisfying 7.0.1.

Taking the trace defines a map on the vector space $\Gamma(E \otimes \wedge^2 E)$, on which the affine space $\mathcal{D}$ is modelled

$$\mathrm{tr} : \Gamma(E \otimes \wedge^2 E) \to \Gamma(E).$$

Restricted to Σ it admits a right inverse via the map

$$e \mapsto \sigma^e$$

where

$$\sigma^e(e_1, e_2, e_3) = \langle e_1, e_2 \rangle \langle e, e_3 \rangle - \langle e, e_2 \rangle \langle e_1, e_3 \rangle.$$

Therefore the vector space Σ decomposes as

$$\Sigma = \Sigma_0 \oplus \Gamma(E)$$

Here Σ_0 denotes the kernel of tr which is then explicitely given by

$$\Sigma_0 = \Big\{ \sigma \in \Sigma \;\Big|\; \sum_{i=1}^{r_E} \sigma(e_i, \tilde{e}_i, .) = 0 \Big\}.$$

We can now define yet another subset of generalized connections with fixed torsion and divergence operator

$$\mathcal{D}(T, div) = \{ D \in \mathcal{D} |\; T_D = T \text{ and } div_D = div \}.$$

As div fixes precisely the trace of a connection $\mathcal{D}(T, div)$ is clearly an affine space modelled on Σ_0.

7.1 Metric compatible generalized connections

Recall that a generalized metric on a transitive Courant algebroid E over a manifold M of dimension is given by a decomposition

$$E = E_- \oplus E_+$$

into positive and negative definite subbundles. Alternatively, a generalized metric is an orthogonal bundle morphism $\mathcal{G} : E \to E$ such that $\mathcal{G}^2 = Id$. We define compatible connections analogously to the standard case of Riemannian geometry.

Definition 7.1.1. A generalized connection $D : \Gamma(E) \to \Gamma(E^* \otimes E)$ is said to be *compatible with the generalized metric* E_- if it respects the decomposition $E = E_- \oplus E_+$, i.e. whenever

$$D(\Gamma(E_\pm)) \subset \Gamma(E^* \otimes E_\pm).$$

We also call these connections *metric connections* and denote their space by $\mathcal{D}(E_-)$.

A metric connection defines four first order differential operators via the splitting, namely

$$D_+^+ : \Gamma(V_+) \to \Gamma(V_+^* \oplus V_+) \qquad D_-^+ : \Gamma(V_+) \to \Gamma(V_-^* \oplus V_+)$$
$$D_+^- : \Gamma(V_-) \to \Gamma(V_+^* \oplus V_-) \qquad D_-^- : \Gamma(V_-) \to \Gamma(V_-^* \oplus V_-).$$

These operators satisfy the $\pi_\pm$-Leibniz rules where $\pi_\pm = \pi|_{E_\pm}$. Metric compatible connections also form an affine space, modelled on

$$\Gamma(E^* \otimes \mathfrak{so}(E_-)) \oplus \Gamma(E^* \otimes \mathfrak{so}(E_+))$$

reflecting the fact that these connections respect the splitting induced by E_-.

Initially it seems like specifying a metric connection amounts to specifying its four different component, but it turns out that the *mixed-type operators* D_-^+ and D_+^- are fixed by the torsion. Moreover, whenever the torsion T_D is of *pure type*, i.e.

$$T_D \in \Gamma(\wedge^3 E_-^*) \oplus \Gamma(\wedge^3 E_+^*)$$

they are uniquely determined by the metric E_-. We call the operators D_+^+ and D_-^- *pure-type operators* as well.

For $e \in \Gamma(E)$ let us denote by $e^\pm$ its projections to $E_\pm$ which can be written as

$$e^\pm = \frac{1}{2}(Id - \mathcal{G})e = \Pi_\pm e.$$

Lemma 7.1.1. [17] *For $D \in \mathcal{D}(E_-)$ with torsion $T \in \Gamma(\wedge^3 E^*)$ the following holds.*

1. *For $D' \in \mathcal{D}(E_-)$ if $T_{D'} = T$ then the mixed type operators agree $D'^{\pm}_{\mp} = D^\pm_\mp$.*
2. *If T is of pure type then for $e_1, e_2 \in \Gamma(E)$ the mixed-type operators are given by*

$$D_{e_1^-} e_2^+ = [e_1^-, e_2^+]^+ \quad D_{e_1^+} e_2^- = [e_1^+, e_2^-]^-.$$

Proof. For any $e_1, e_2, e_3 \in \Gamma(E)$ the mixed-type torsion is given by

$$T_D(e_1^-, e_2^+, e_3^+) = \langle D_{e_1^-} e_2^+ - [e_1^-, e_2^+]^+, e_3^+ \rangle,$$

since $D_{e_1} e_2^\pm \subset E_\pm$. This formula fully determines $D^\pm_\mp$ from the mixed-type part of T_D and yields (2.) when the mixed-type torsion vanishes. ∎

From here we turn our attention to torsion-free metric compatible connections whose space we denote by

$$\mathcal{D}(0, E_-) = \mathcal{D}(E_-) \cap \mathcal{D}(0).$$

It is clear from our previous discussion that $\mathcal{D}(0, E_-)$ is an affine space modelled on

$$\Sigma^+ \oplus \Sigma^-,$$

where $\Sigma^\pm = \Gamma(\wedge^3 E_\pm) \cap \Sigma$. Moreover, the decomposition into trace and traceless part carries over as well

$$\Sigma^\pm = \Sigma_0^\pm \oplus \Gamma(E_\pm).$$

Remark 7.1.1. It is important to note that metric connections and even torsion free metric connections always exist for any generalized metric E_-. As the restriction of the inner product $\langle\cdot,\cdot\rangle$ to the subbundles E_- and E_+ is non-degenerate there exist standard metric connections ∇^- and ∇^+ on them. Then for $e_1, e_2 \in \Gamma(E)$ let

$$D_{e_1^\pm} e_2^\pm = \nabla^\pm_{\pi_\pm(e_1^\pm)} e_2^\pm$$

and define the mixed-type operators as in Lemma 7.1.1. Then D is a metric connection with pure-type torsion T_D, therefore

$$D^0 = D - \frac{1}{3} T_D$$

is a torsion-free metric connection since T_D can be regarded as an element in $\Gamma(E^* \otimes \mathfrak{so}(E_-)) \oplus \Gamma(E^* \otimes \mathfrak{so}(E_+))$.

In the presence of a divergence operator *div* we will also consider torsion-fee metric connections with specific trace which we denote by

$$\mathcal{D}(E_-, div) = \mathcal{D}(0, E_-) \cap \mathcal{D}(0, div).$$

Clearly, $\mathcal{D}(E_-, div)$ is an affine space modelled on $\Sigma_0^+ \oplus \Sigma_0^-$.

7.2 Dirac operators

We can see that our goal of finding a unique "generalized Levi-Civita connection" is not achievable in general. Nevertheless, there have been attempts to single out a canonical natural connection ([18]) from the elements of $\mathcal{D}(E_-, div)$ specifically for the case of heterotic Courant algebroids but for now we are staying in the general setting. Remarkably, even though, there is a large family of "Levi-Civita connections" corresponding to a generalized metric there are still geometric invariants that only depend on $\mathcal{D}(E_-, div)$.

First let us state some general facts about connections on bundles of Clifford algebras and the corresponding spin bundles (see [1]). Let V be a vector bundle over M with a non-degenerate inner product $\langle\cdot,\cdot\rangle$ and suppose M admits a spin bundle S corresponding to the bundle of Clifford algebras $\mathcal{C}\ell(V)$. This holds whenever $w_2(V)$ the second Stiefel-Whitney class of V vanishes. An orthogonal V-connection ∇ on V generates a unique V-connection $\nabla^{\mathcal{C}\ell}$ on $\mathcal{C}\ell(V)$ via

$$\nabla^{\mathcal{C}\ell}_e(uv) = (\nabla^{\mathcal{C}\ell}_e u)v + u(\nabla^{\mathcal{C}\ell}_e v) \quad e \in \Gamma(V),\ u, v \in \mathcal{C}\ell(V).$$

The affine space of V-connections on V is modelled on $\Gamma(V^* \otimes \mathfrak{so}(V))$, therefore the difference in two Clifford V-connection for any $e \in \Gamma(V)$ is

$$\nabla^{\mathcal{C}\ell}_e - \tilde{\nabla}^{\mathcal{C}\ell}_e = A_e \in \mathfrak{so}(V).$$

Here $A_e \in \mathfrak{so}(V)$ is lifted to the Clifford algebra as it was described in Section 2.3.

The spin bundle S also admits V-connections ∇^S and we call them *compatible with the Clifford connection* $\nabla^{\mathcal{C}\ell}$ if

$$\nabla^S_e(u \cdot \alpha) = (\nabla^{\mathcal{C}\ell}_e u) \cdot \alpha = u \cdot (\nabla^S_e \alpha)$$

holds for all $e \in V$, $u \in \mathcal{C}\ell(V)$ and $\alpha \in S$. Spin connections in general may differ by an element in $\Gamma(V^* \otimes End(S)) \cong \Gamma(V^* \otimes \mathcal{C}\ell(V))$ but compatibility with a Clifford connection requires the part in $\mathcal{C}\ell(V)$ to be central. In conclusion, for any $e \in \Gamma(V)$ two spin connections compatible with the same Clifford connection differ by a function

$$\nabla^S_e - \tilde{\nabla}^S_e = f \ \in C^\infty(M).$$

To associate a spin connection to a Clifford connection $\nabla^{\mathcal{C}\ell}$ which is independent of choices assume that the line bundle $(detS^*)^{1/r_S}$ exist where $r_S = \operatorname{rank} S$ and consider the bundle

$$\mathcal{S} = S \otimes (detS^*)^{1/r_S}.$$

Any spin connection ∇^S on S determines a connection on $(detS^*)^{1/r_S}$ and therefore on $\mathcal{S}$ by taking the product connection. This is now independent of the choice of ∇^S as the change of connection on $(detS^*)^{1/r_S}$ amounts to $-f \in C^\infty(M)$. Denote the resulting V-connection on $\mathcal{S}$ by $D^{\mathcal{S}}$ which now only depends on the choice of Clifford connection $\nabla^{\mathcal{C}\ell}$.

Definition 7.2.1. The *Dirac operator* associated to a spin V-connection $D^{\mathcal{S}}$ on $\mathcal{S}$ is a first order differential operator

$$\not{D}^{\mathcal{S}} : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$$

which is defined as follows. For $r = \operatorname{rank} V$ let $e_1, .., e_r$ be a local orthogonal frame of V. Then for $\alpha \in \mathcal{S}$

$$\not{D}\alpha = \frac{1}{2}\sum_{i=1}^{r} e_i \cdot D^{\mathcal{S}}_{e_i}\alpha,$$

where the multiplication denotes the Clifford action.

Note that when r is even the spin bundle S decomposes into the two irreducible half-spin representations

$$S = S^R \oplus S^L.$$

Let $\mathcal{S}^R = S^R \oplus (detS^*)^{1/r_S}$ (and $\mathcal{S}^L$ analogously), then the Diarc operator $\not{D}$ maps $\mathcal{S}^R$ to $\mathcal{S}^L$ and $\mathcal{S}^L$ to $\mathcal{S}^R$. The $R - L$ notation seems somewhat arbitrary here, it reflects that in physics elements of the half-spin representations are thought of as "left and right moving" particles.

Coming back to the case of metric compatible connections, let E_- be a generalized metric on the Courant algebroid E. Since the inner product of E restricts to non-degenerate inner products on the bundles E_- and E_+ we can consider the bundle of

Clifford algebras $\mathcal{C}\ell(E_\pm)$. Suppose that $E_\pm$ both admit a spinor bundle $S_\pm$ and that the line bundles $(detS^*_\pm)^{1/rs_\pm}$ exist. We have seen that a generalized connection D compatible with the metric E_- induces $E_\pm$-connections on $E_\pm$ via the pure-type operators operators $D^\pm_\pm$. Moreover, the operators $D^\pm_\pm$ are also compatible with the restricted inner product, i.e. they define orthogonal $E_\pm$-connections on the subbundles $E_\pm$. Consider now the associated spin connections on

$$\mathcal{S}_\pm = S_\pm \otimes (detS^*_\pm)^{1/rs_\pm}$$

and the induced dirac operators

$$\not{D}_\pm : \Gamma(\mathcal{S}_\pm) \to \Gamma(\mathcal{S}_\pm).$$

Lemma 7.2.1. *The operators $\not{D}_\pm$ do not depend on the choice of generalized connection $D \in \mathcal{D}(E_-, div)$.*

Proof. Two generalized connections $D, D' \in \mathcal{D}(E_-, div)$ differ by elements in $\Sigma_0^+ \oplus \Sigma_0^-$. Therefore, there exist a traceless element $\sigma \in \Sigma_0^- \subset \Gamma(E_- \otimes \wedge^2 E_-)$ such that

$$\langle (D_-^-)_{e_1} e_2 - (D'^-_-)_{e_1} e_2, e_3 \rangle = \sigma(e_1, e_2, e_3),$$

and $\sigma(e_1, e_2, e_3) + c.p. = 0$ for all $e_1, e_2, e_3 \in \Gamma(E_-)$. The Clifford connection also changes by σ which in turn changes the spin connection on $(detS^*_-)^{1/rs_-}$ by a factor (which we can ignore now) and by σ on S acting via the formula defined in Section 2.3.

More precisely, if $r = \operatorname{rank} E_-$ and $e_1, ..., e_r$ is an orthogonal basis of E_- then σ acts via the Clifford action as the element

$$\sigma = \sum_{i,j,k} \sigma^{kji} e_k e_j e_i \ \in \ \mathcal{C}\ell(E)$$

with the property that $\sigma^{kji} + c.p. = 0$, therefore $\sigma = 0$ in $\mathcal{C}\ell(E)$, so we have that

$$\not{D}'_- = \not{D}_-.$$

The same calculation holds for $\not{D}_+$. ∎

7.3 The canonical Levi-Civita connection

We have seen that torsion free metric compatible generalized connections are far from unique. Nevertheless, in [18] a natural connection was constructed for transitive Courant algebroids which can be regarded as a canonical Levi-Civita connection. The definition extends the construction of *Gualtieri-Bismut connection* which was defined on exact Courant algebroids (see [24]) to the transitive case.

Let $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ be a transitive Courant algebroid and $E_- \subset E$ a generalized metric, and $\mathcal{G}$ the corresponding bundle map. Define the subbundle $C^+ = E_-^* \subset E_+$ and a bundle map $C : E \to E$

$$C = \pi|_{E_-}^{-1} \circ \pi \circ \Pi_+ + \pi|_{C_+}^{-1} \circ \pi \circ \Pi_-$$

where $\Pi_\pm = \frac{1}{2}(Id \pm \mathcal{G})$ the projections to $E_\pm$. Clearly, C sends E_- to C^+ and E_+ to E_-.

Example. When E is exact a generalized metric induces a splitting in which $E_- = \{X - gX | X \in TM\}$ and $E_+ = \{X + gX | X \in TM\}$ and the map C is

$$C(X \pm gX) = X \mp gX.$$

In the heterotic case the induced splitting is $E_- = \{X - gX | X \in TM\}$ and $E_+ = \{X + s + gX | X \in TM, s \in \mathfrak{g}_P\}$ and C is

$$C(X - gX) = X + gX \quad \text{and} \quad C(X + s + gX) = X - gX.$$

Definition 7.3.1. [18][24] The *Gualtieri-Bismut connection* D^B on E is defined by

$$D^B_{e_1} e_2 = [e_1^-, e_2^+]^+ + [e_1^+, e_2 -]^- + [Ce_1^+, e_2^+]^+ + [Ce_1^-, e_2^-]^-,$$

where $e^\pm = \Pi_\pm e$.

Remark 7.3.1. The Gualtieri-Bismut connection is well-defined. Bilinearity is clear as the Dorfman bracket is bilinear, the Leibniz-rule follows from axiom C3 (see Definition 3.0.1), compatibility with the metric from C4 and finally for $C^\infty(M)$-linearity in the first entry one can again use C3 and the fact that the Dorfman bracket is skew-symmetric when evaluated on orthogonal elements.

When E is exact this connection restricts to the original definition of the Gualtieri-Bismut connection from [24]. The name reflects the fact that when explicitly calculated D^B contains the *Bismut connection* on the tangent bundle which is defined for a Riemannian metric g and a 3-form H as

$$\nabla^- = \nabla^g - \frac{1}{3} g^{-1} H.$$

Here ∇^g is the Levi-Civita connection corresponding to g. An important feature of ∇^- that it has totally skew-symmetric torsion $-H$.

From Lemma 7.1.1 we readily see that the torsion of D^B is of pure-type. Therefore, we can construct a torsion-free connection out of it which is still compatible with the metric.

Definition 7.3.2. [18] The *canonical Levi-Civita connection* of E_- is defined by

$$D^{LC} = D^B - \frac{1}{3} T_{D^B}.$$

The most important feature of this construction is naturality in the sense that D^{LC} is interchanged by generalized diffeomorphisms between Courant-algebroids. More precisely if $f : E \to E'$ is an morphism of Courant algebroids then

$$D^{LC}(E_-) = f^* D^{LC}(f(E_-)).$$

Remark 7.3.2. Note that the canonical Levi-Civita connection is not divergence-free in any sense. It is instead used as a reference point for the affine space of torsion-free metric connections and the Weyl gauge fixing process of choosing a divergence is done with respect to the Levi-Civita connection. This simplifies calculations as compared to a singled out connection fixing the divergence simply amounts to choosing an element $e \in \Gamma(E)$.

Example. In the case of both exact and heterotic Courant algebroids the divergence of the canonical Levi-Civita connection can be computed fairly easily using normal coordinates $\{x_1, .., x_n\}$ for the induced Riemannian metric g. We then find for an element $e \in E$

$$div_{D^{LC}} e = \sum_{i=1}^{n} \frac{\partial \pi(e)^i}{\partial x^i},$$

which is the usual notion of divergence in multivariable calculus. This also supports our choice of taking D^{LC} to be our reference connection.

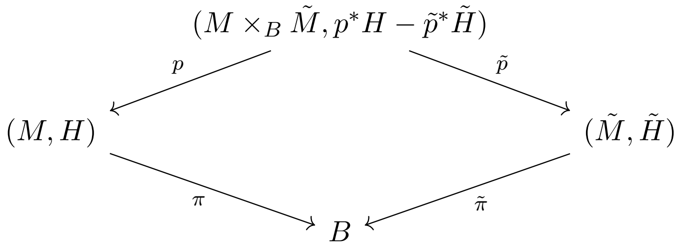
8 T-duality

The notion of mathematical T-duality has been introduced by Bouwknegt, Evslin, Mathai and Hannabuss [3, 4] as a relation between principal torus bundles. Later it has been reinterpreted by Cavalcanti and Gualtieri [7] as an isomorphism of exact Courant algebroids over the principal torus bundles. In this section we review these constructions then we generalize it to the heterotic case following [2].

8.1 T-duality of torus bundles

Let M and $\tilde{M}$ be principal torus bundles of the same dimension over the common base B. We write T^k and $\tilde{T}^k$ for the structure groups of M and $\tilde{M}$ respectively. Let H and $\tilde{H}$ be invariant closed 3-forms over M and $\tilde{M}$ respectively. Consider the fibered product $C = M \times_B \tilde{M}$ with projections p and $\tilde{p}$ to M and $\tilde{M}$. The space C, called the *correspondence space*, is a principal T^{2k} bundle over B and we have the closed invariant 3-form $p^*H - \tilde{p}^*\tilde{H}$ on it.

We summarize the above data in the following commutative diagram.



Definition 8.1.1. (Differential T-duality) The torus bundles together with the closed invariant 3-forms (M, H) and $(\tilde{M}, \tilde{H})$ are said to be T-dual if there exist an invariant 2-form F on $M \times_B \tilde{M}$ such that

$$dF = p^*H - \tilde{p}^*\tilde{H}$$

and such that $F : \mathfrak{t}^k_M \otimes \mathfrak{t}^k_{\tilde{M}} \to \mathbb{R}$ is non-degenerate where $\mathfrak{t}^k$ is tangent to the fibre of $M \times_B \tilde{M} \to \tilde{M}$ and $\tilde{\mathfrak{t}}^k$ is tangent to the fibre of $M \times_B \tilde{M} \to M$.

More precisely, given a basis of periodic elements $\{\partial_{\theta_i}\}$ and $\{\partial_{\tilde{\theta}_j}\}$ of the fibres of M and $\tilde{M}$ respectively we have

$$(F(\partial_{\theta_i}, \partial_{\tilde{\theta}_j})) \in GL(k, \mathbb{R}).$$

The condition above is topological in the sense that if (M, H) and $(\tilde{M}, \tilde{H})$ are T-dual then so are any invariant representatives of the classes $[H] \in H^3(M, \mathbb{R})$ and $[\tilde{H}] \in H^3(\tilde{M}, \mathbb{R})$. Indeed, if $B \in \Omega^2(M)$ and $\tilde{B} \in \Omega^2(\tilde{M})$ are invariant 2-forms then

$$p^*(H + dB) - \tilde{p}^*(\tilde{H} + d\tilde{B}) = d(F + p^*B - \tilde{p}^*\tilde{B})$$

and $F + p^*B - \tilde{p}^*\tilde{B}$ restricted to $\mathfrak{t}^k \otimes \tilde{\mathfrak{t}}^k$ agrees with F, therefore still non-degenerate. Consequently, we may redefine T-duality in the following way.

Definition 8.1.2. (Topological T-duality) The T^k torus bundles together with the degree 3 real cohomology classes (M, h) and $(\tilde{M}, \tilde{h})$ are said to be T-dual if

$$p^*h = \tilde{p}^*\tilde{h} \ \in H^3(M \times_B \tilde{M}, \mathbb{R}),$$

and for any invariant representatives $H \in \Omega^3(M)$ and $H' \in \Omega^3(\tilde{M})$ such that $p^*H - \tilde{p}^*\tilde{H} = dF$ for $F \in \Omega^2(M \times_B \tilde{M})$ the restriction of F to $\mathfrak{t}^k \otimes \tilde{\mathfrak{t}}^k$ is non-degenerate.

Remark 8.1.1. The above notions of T-duality are less restrictive than what is used in physics literature. One usually asks for H and $\tilde{H}$ to represent integral cohomology classes and F to be unimodular in the sense that

$$F(\partial_{\theta_i}, \partial_{\tilde{\theta}_j}) \in GL(k, \mathbb{Z})$$

where $\{\partial_{\theta_i}\}$ and $\{\partial_{\tilde{\theta}_j}\}$ are bases of invariant period 1 elements of the fibres of M and $\tilde{M}$ respectively. This restricted version is used to interpret T^k and $\tilde{T}^k$ as dual tori.

In case of integral cohomology classes the T-duality condition can be further restricted to $p^*h = \tilde{p}^*\tilde{h}$ as integral cohomology classes in $H^3(M \times_B \tilde{M}, \mathbb{Z})$. The existence and uniqueness of T-duals in this case was explored further in [3] and [4] with similar, although more nuanced results as presented below.

Suppose now we have a T-dual pair of T^k torus bundles (M, h) and $(\tilde{M}, \tilde{h})$ in the sense of Definition 8.1.2. Let $\theta \in \Omega^1(M, \mathfrak{t}^k)$ and $\tilde{\theta} \in \Omega^1(\tilde{M}, \tilde{\mathfrak{t}}^k)$ be principal connections on M and $\tilde{M}$ respectively. Pulling them back to the correspondence space they form a principal T^{2k} connection. Chose a basis $t_1, ..., t_k$ for $\mathfrak{t}^k$ and $\tilde{t}_1, ..., \tilde{t}_k$ for $\tilde{\mathfrak{t}}^k$. Then we can write $\theta = \theta^i t_i$ and $\tilde{\theta} = \tilde{\theta}^i \tilde{t}_i$. We denote the dual bases by $t^1, ..., t^k$ and $\tilde{t}^1, ..., \tilde{t}^k$.

Let H and $\tilde{H}$ be invariant representatives of h and $\tilde{h}$ such that, omitting pullback notation, $H - H' = dF$ for some invariant 2-form F on the correspondence space. Using the connections θ and $\tilde{\theta}$ we can write in the usual notation

$$\begin{aligned}
H &= H^0 + H^1_i \theta^i + H^2_{ij} \theta^{ij} + H^3_{ijk} \theta^{ijk}, \\
\tilde{H} &= \tilde{H}^0 + \tilde{H}^1_i \tilde{\theta}^i + \tilde{H}^2_{ij} \tilde{\theta}^{ij} + \tilde{H}^3_{ijk} \tilde{\theta}^{ijk}, \\
F &= F_0 + F^1_i \theta^i + \tilde{F}^1_i \tilde{\theta}^i + F^2_{ij} \theta^{ij} + \tilde{F}^2_{ij} \tilde{\theta}^{ij} + F^3_{ij} \theta^i \tilde{\theta}^j,
\end{aligned}$$

where $H^0, H^1_i, ..., \tilde{H}^0, \tilde{H}^1_i, ..., F^0, F^1_i, \tilde{F}^1_i ...$ are all basic forms, i.e. pullbacks from B.

Differentiating F, omitting the wedge in the products, we obtain

$$\begin{aligned} dF = & dF_0 + dF_i^1\theta^i - F_i^1 d\theta^i + d\tilde{F}_i^1\tilde{\theta}^i - \tilde{F}_i^1 d\tilde{\theta}^i \\ & + dF_{ij}^2\theta^{ij} + 2F_{ij}^2(d\theta^i)\theta^j + d\tilde{F}_{ij}^2\tilde{\theta}^{ij} + 2\tilde{F}_{ij}^2(d\tilde{\theta}^i)\tilde{\theta}^j \\ & + dF_{ij}^3\theta^i\tilde{\theta}^j + F_{ij}^3(d\theta^i)\tilde{\theta}^j - F_{ij}^3\theta^i(d\tilde{\theta}^j). \end{aligned}$$

Note that $d\theta$ and $d\tilde{\theta}$ are the curvature 2-forms of θ and $\tilde{\theta}$ and hence basic.

The T-duality condition $dF = H - \tilde{H}$ then amounts to

$$H_{ijk}^3 = \tilde{H}_{ijk}^3 = 0, \quad H_{ij}^2 = dF_{ij}^2, \quad \tilde{H}_{ij}^2 = -d\tilde{F}_{ij}^2, \quad dF_{ij}^3 = 0,$$

$$H_i^1 = dF_i^1 + 2F_{ji}^2 d\theta^j - F_{ij}^3 d\tilde{\theta}^j, \quad \tilde{H}_i^1 = -d\tilde{F}_i^1 - 2\tilde{F}_{ji}^2 d\tilde{\theta}^j - F_{ji}^3 d\theta^j$$

$$H^0 - \tilde{H}^0 = dF^0 - F_i^1 d\theta^i - \tilde{F}_i^1 d\tilde{\theta}^i.$$

Notice now that changing the representatives $H \to H - d(F_{ij}^2\theta^{ij} + F_i^1\theta^i - F_0)$ and $\tilde{H} \to \tilde{H} - d(\tilde{F}_{ij}^2\tilde{\theta}^{ij} + \tilde{F}_i^1\tilde{\theta}^i)$ changes F to only $F_{ij}^3\theta^i\tilde{\theta}^j$ and we reach the following forms

$$H = H^0 - F_{ij}^3 d\tilde{\theta}^i\theta^j, \quad \tilde{H} = H^0 - F_{ij}^3\tilde{\theta}^i d\theta^j,$$

where now the basic parts of H and $\tilde{H}$ agree and

$$H - \tilde{H} = d(F_{ij}^3\theta^i \wedge \tilde{\theta}^j).$$

In conclusion we have the following theorem.

Theorem 8.1.1. *Let (M, h) and $(M, \tilde{h})$ be T-dual torus bundles. Then for any pair of principal connections θ and $\tilde{\theta}$ there exist a 3-form $H^0 \in \Omega^3(B)$ on the common base such that h and $\tilde{h}$ are represented by*

$$H = \pi^* H^0 - F_{ij}\theta^i \wedge d\tilde{\theta}^j \quad \text{and} \quad \tilde{H} = \tilde{\pi}^* H^0 - F_{ij} d\theta^i \wedge \tilde{\theta}^j$$

respectively, and

$$F = F_{ij}\theta^i \wedge \tilde{\theta}^j.$$

Whenever F is unimodular on $\mathfrak{t}^k \otimes \tilde{\mathfrak{t}}^k$ we may regard it as the natural pairing between $\mathfrak{t}^k$ and $\mathfrak{t}^{k*}$, in general we obtain an isomorphism $F : \mathfrak{t}^{k*} \to \tilde{\mathfrak{t}}^k$.

As an abuse of notation we regard F as a pairing on $\mathfrak{t}^k \otimes \tilde{\mathfrak{t}}^k$ we will write $F(A, B)$ for A a form with values in $\mathfrak{t}^k$ and B a form with values in $\tilde{\mathfrak{t}}^k$. More precisely, given a basis in which $A = A^i t_i$ and $B = B^i \tilde{t}_i$ we have

$$F(A, B) = F_{ij} A^i \wedge B^j.$$

In this notation the two-form F of Theorem 8.1.1 is written as $F(\theta, \tilde{\theta})$.

The above theorem shows that not every degree 3 cohomology class has T-duals. Recall the filtration of invariant forms on a principal bundle

$$\Omega^\bullet(B) = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \dots \subset \mathcal{F}^k = \Omega^\bullet(M)$$

with $\mathcal{F}^i = Ann(\wedge^{i+1}\mathfrak{t}^k)$. By the previous theorem if $h \in H^3(M,\mathbb{R})$ has a T-dual, then it has an invariant repesentative that is in $\mathcal{F}^1$. Turns out this is the only obstruction to having a T-dual whenever we have a class representing an integral cohomology class.

Definition 8.1.3. Let $M \to B$ be a principal torus bundle and $h \in H^3(M,\mathbb{R})$. We call h a *T-dualisable class* if it represents an integral cohomology class and it has a representative H that lies in $\mathcal{F}^1$.

Theorem 8.1.2. *Given a principal T^k-bundle $M \to B$ and a T-dualisable class $h \in H^3(M,\mathbb{R})$, there exist a principal torus bundle $\tilde{M}$ and a T-dualisable class $\tilde{h}$ such that (M,h) and $(\tilde{M},\tilde{h})$ are T-dual.*

Proof. Let $\theta \in \Omega^1(M,\mathfrak{t}^k)$ be a connection on M. Then as h is T-dualisable it has a representative $H \in \mathcal{F}^1$ and there are forms $\tilde{c} \in \Omega^2(B,\mathfrak{t}^{k*})$ and $H_0 \in \Omega^3(B)$ such that

$$H = \langle \tilde{c}, \theta \rangle + H_0$$

where $\langle \cdot, \cdot \rangle$ represents the natural pairing between $\mathfrak{t}^k$ and its dual Lie algebra $\mathfrak{t}^{k*}$.

As H is closed so is $\tilde{c}$ since

$$dH = \langle d\tilde{c}, \theta \rangle + \langle \tilde{c}, d\theta \rangle + dH_0 = 0$$

and $d\theta = d\theta + \frac{1}{2}[\theta,\theta] = F_\theta$ is basic. Moreover, as H represents an integral cohomology class, so does $\tilde{c}$ in $H^2(B,\mathfrak{t}^{*k})$.

Therefore, we can interpret $\tilde{c}$ as the real image of the Chern class of a principal torus bundle $\tilde{M}$ over B with structure group $\tilde{T}^k$ the torus dual to T^k. Let $\tilde{\theta}$ be a connection on $\tilde{M}$ such that $d\tilde{\theta} = \tilde{c}$ and define the dual 3-form as

$$\tilde{H} = \langle c, \tilde{\theta} \rangle + H_0$$

where $d\theta = c \in \Omega^2(B,\mathfrak{t}^k)$ represents the Chern class of M and H_0 is the basic part of H.

Then, in the correspondence space we have

$$p^*H - \tilde{p}^*\tilde{H} = \langle \tilde{c}, \theta \rangle - \langle c, \tilde{\theta} \rangle = d\langle \theta, \tilde{\theta} \rangle.$$

Clearly with this choice H and $\tilde{H}$ are T-dual in the sense of Definition 8.1.3 and $F = \langle \theta, \tilde{\theta} \rangle$ is unimodular.

Moreover, $\tilde{H}$ is closed as

$$d\tilde{H} = \langle d\theta, d\tilde{\theta} \rangle + dH_0 = dH = 0$$

and represents an integral cohomology class as well. ∎

Remark 8.1.2. The above theorem has some implications in terms of the Chern classes in real cohomology. Let (M, h) and $(\tilde{M}, \tilde{h})$ be T-dual torus bundles and H and $\tilde{H}$ representatives such that $H = H_0 + F(\theta, \tilde{c})$ and $\tilde{H} = H_0 + F(c, \tilde{\theta})$ with $c = d\theta$ and $\tilde{c} = d\tilde{\theta}$. Then the last calculation above yields

$$F(c, \tilde{c}) + dH_0 = 0.$$

Theorem 8.1.2 establishes the existence of a T-dual for all torus bundles with T-dualisable classes h. However, in the construction of a T-dual we made several choices which suggests that T-duals are not unique.

Firstly, we have already chosen a representative H of the T-dualisable class $h \in H^3(M, \mathbb{R})$. This choice determines the real image of the Chern class of the T-dual torus bundle in $H^2(B, \mathfrak{t}^{k*})$. In the following we show that different choices lead to different classes.

Suppose we have an other representative $H' = H^{0\prime} + \langle \tilde{c}', \theta \rangle$. Then

$$H - H' = dC$$

for some invariant 2-form $C \in \Omega^2(M)$. After choosing a basis for $\mathfrak{t}^k$ and $\mathfrak{t}^{k*}$ we can decompose C as

$$C = C^0 + C^1_i \theta^i + C^2_{ij} \theta^{ij}.$$

Substituting into $H - H' = dC$ yields

$$\langle \tilde{c} - \tilde{c}', \theta \rangle = d(C^1_i)\theta^j + d(C^2_{ij}\theta^{ij})$$

with C^2_{ij} constant functions. The first term is exact in $\Omega^2(B, \mathfrak{t}^{k*})$ so only the second term can change the cohomology class of c i.e.

$$[\tilde{c} - \tilde{c}'] \in d\Omega^0(B, \wedge^2 \mathfrak{t}^{k*}).$$

Therefore, the cohomology class $[c]$ is not well defined on B, only on M.

Secondly, even after choosing a representative H the T-dual of a pair (M, H) is not unique, as we interpret $\tilde{c}$ as the image of an integral cohomology class under $H^2(B, \mathbb{Z}) \to H^2(B, \mathbb{R})$. Therefore the torsion part of $[\tilde{c}]$ can be chosen arbitrarily.

Thirdly, in our basis the dual Chern class is $\tilde{c} = \tilde{c}^i t_i$. Given any integers $m_1, ..., m_k \in \mathbb{Z}$ the Torus bundle with real Chern class $\tilde{c}' = \sum_i m_i \tilde{c}^i t_i$ is also T-dual to (M, h) only F is no longer unimodular.

The following examples demonstrate some of the ambiguities described above.

Example 1. The most basic instance of T-duality is the case of circle bundles. For a circle bundle $M \to B$ every cohomology class $h \in H^3(M, \mathbb{R})$ representing an integral class is T-dualisable, as the fibre is one dimensional. Therefore any pair (M, h) has a T-dual. Indeed, after picking a basis $\mathfrak{t}^1 \cong \mathbb{R}$, $H \in \mathcal{F}^1$ a representative of h and $\theta \in \Omega^1(M, \mathbb{R})$ a principal connection we may write $H = \tilde{c} \wedge \theta + H_0$. Then, the T-dual 3-form is

$$\tilde{H} = c \wedge d\tilde{\theta} + H_0$$

with $c = d\theta$. In this case $[\tilde{c}] \in H^2(B, \mathbb{R})$ is uniquely defined by h as $\wedge^2 \mathfrak{t}^{1*} = 0$.

Example 2. A concrete example of a circle bundle is the Hopf fibration which exhibits the 3-sphere S^3 as a circle bundle over S^2. The mapping of this fibration is as follows

$$\begin{aligned} \pi : S^3 \subset \mathbb{C}^2 &\rightarrow S^2 \cong \mathbb{CP}^1 \\ (z_1, z_2) &\mapsto [z_1 : z_2] \end{aligned}$$

and the S^1 fibres are clearly given by $e^{it}(z_1, z_2)$.

Another way to implement the Hopf map is

$$(\eta, \xi_1, \xi_2) \mapsto (2\eta, \xi_1 + \xi_2)$$

where we use polar coordinates (ψ, φ) for S^2 and $(\eta, \xi_1, \xi_2) \mapsto (e^{i\xi_1} \sin \eta, e^{i\xi_2} \cos \eta)$ to parametrise S^3. In this coordinate system $\eta \in [0, \pi/2)$, $\xi_1, \xi_2 \in [0, 2\pi)$ and the round metric is given by

$$ds^2 = d\eta^2 + sin^2 \eta d\xi^2 + cos^2 \eta d\xi^2.$$

Along the Hopf map the volume form $d\psi \wedge d\varphi$ of S^2 pulls back to $2d\eta \wedge (d\xi_1 + d\xi_2)$ on S^3.

The fibres are given by $\pi^{-1}(\psi, \varphi) = (\psi/2, \varphi - t, t)$ where t runs from 0 to 2π, hence the vertical distribution V at a point is given by

$$V|_{(\eta, \xi_1, \xi_2)} = span\{-\partial_{\xi_1} + \partial_{\xi_2}\}.$$

We can use the metric ds^2 to dualise the vector field $-\partial_{\xi_1} + \partial_{\xi_2}$ to find a connection 1-form for the fibration which becomes

$$\theta = -sin^2(\eta) d\xi_1 + cos^2(\eta) d\xi_2.$$

The curvature of this connection is then a volume form for the 2-sphere

$$\sigma = d\theta = sin(2\eta)(d\xi_1 + d\xi_2) \wedge d\eta.$$

Consider now the Hopf fibration S^3 together with the zero cohomology class $h = 0 \in H^3(S^3, \mathbb{R})$. Then (S^3, h) is T-dual to the trivial bundle $S^2 \times S^1$ together with the class represented by $\sigma \wedge \tilde{\theta}$ where $\tilde{\theta}$ is the flat connection on $S^2 \times S^1$.

Considering now a different cohomology class represented by $\sigma \wedge \theta$, the pair $(S^3, [\sigma \wedge \theta])$ is self-T-dual.

Example 3. Let $M \rightarrow B$ be a 2-torus bundle with real Chern class $([c_1], [c_2]) \in H^2(B, \mathbb{R}^2)$ after choosing a basis for $\mathfrak{t}^{2*} \cong \mathbb{R}^2$. Let $\theta = \theta_i t^i$ be a connection on M such that $(d\theta_1, d\theta_2) = (c_1, c_2)$. In this example we construct two topologically different T-duals for M with the T-dualisable class $0 = h \in H^3(M, \mathbb{R})$.

First, take the representative $H = 0$. Then as in the previous example $B \times S^1 \times S^1$ is a T-dual with 3-form

$$\tilde{H} = c_1 \wedge \tilde{\theta^1} + c_2 \wedge \tilde{\theta}^2,$$

where $\tilde{\theta}^i t_i$ is a flat connection for $B \times S^1 \times S^1$.

Another T-dual can be constructed by taking the representative

$$H = d(\theta_1 \wedge \theta_2) = c_1 \wedge \theta_2 - c_2 \wedge \theta_1.$$

Then let $\tilde{M} \to B$ be the torus bundle with the structure group dual to the torus of M and real Chern class $(-[c_2], [c_1])$. Let $\tilde{\theta} = \tilde{\theta}^i t_i$ be a connection on $\tilde{M}$ with $d\tilde{\theta} = -c_2$ and $d\tilde{\theta} = c_1$. Then $\tilde{M}$ is T-dual to M with 3-form

$$\tilde{H} = d(\tilde{\theta}_2 \wedge \tilde{\theta}_1).$$

Indeed, using $d\theta_1 = -d\tilde{\theta}_2$ and $d\theta_2 = d\tilde{\theta}_1$ we have

$$H - \tilde{H} = d\theta_1 \wedge \theta_2 - d\theta_2 \wedge \theta_1 + d\tilde{\theta}_1 \wedge \tilde{\theta}_2 - d\tilde{\theta}_2 \wedge \tilde{\theta}_1 = d(\tilde{\theta}_2 \wedge \theta_2 + \tilde{\theta}_1 \wedge \theta_1).$$

Clearly, the two T-duals $B \times S^1 \times S^1$ and $\tilde{M}$ are topologically distinct whenever $[c_1] \neq 0$ or $[c_2] \neq 0$.

Closed 3-forms on a manifold can be used to twist the exterior derivative d to $d_H = d + H$ as we have seen in the case of exact Courant algebroids. Then, the T-duality relation between two principal torus bundles induces an isomorphism between the twisted differential complexes.

Theorem 8.1.3. *Let (M, H) and $(\tilde{M}, \tilde{H})$ be T-dual principal torus bundles with structure groups T^k and $\tilde{T}^k$. If $p^*H - \tilde{p}^*\tilde{H} = dF$ then the map*

$$\tau : \; \varphi \mapsto \int_{T^k} e^F \wedge p^*\varphi \tag{8.1.1}$$

is an isomorphism of the twisted differential complexes of invariant forms $(\Omega^\bullet(M)^{T^k}, d_H)$ and
$(\Omega^\bullet(\tilde{M})^{\tilde{T}^k}, d_{\tilde{H}})$.

Proof. Showing that τ is an isomorphism of vector spaces is a straightforward calculation of the inverse. The main idea is that if F is of the form $F_{ij}\theta^i\tilde{\theta}^j$ for some principal connections θ and $\tilde{\theta}$ on M and $\tilde{M}$, reversing the process is an almost inverse. More precisely,

$$\int_{\tilde{T}^k} e^{-F} \wedge \int_{\tilde{T}^k} e^F \wedge \varphi = det(F) \cdot \varphi$$

i.e., it accounts to multiplying by the determinant of F. As F is non-degenerate the inverse exists. From this calculation it is easy to generalize to any F by decomposing it to basic parts and parts acting on on $\wedge^2\mathfrak{t}^k$, $\wedge^2\tilde{\mathfrak{t}}^k$ and $\mathfrak{t}^k \otimes \tilde{\mathfrak{t}}^k$.

It remains to show that τ interchanges d_H and $d_{\tilde{H}}$. Indeed,

$$\begin{aligned}
\tau(d_H\varphi) &= \int_{T^k} e^F \wedge d\varphi + e^F \wedge H \wedge \varphi \\
&= \int_{T^k} d(e^F \wedge \varphi) - dF \wedge e^F \wedge \varphi + H \wedge e^F \wedge \varphi \\
&= \int_{T^k} d(e^F \wedge \varphi) + \tilde{H} \wedge \varphi \\
&= d_{\tilde{H}} \tau(\varphi),
\end{aligned}$$

where we used that $dF = H - \tilde{H}$. ∎

8.2 T-duality as a map of exact Courant algebroids

Let (M, h) and $(\tilde{M}, \tilde{h})$ be a T-dual pair and let E and $\tilde{E}$ be the exact Courant algebroids on M and $\tilde{M}$ corresponding to h and $\tilde{h}$ respectively. Given representatives H and $\tilde{H}$ of h and $\tilde{h}$ we have splittings $E \cong TM \oplus T^*M$ and $\tilde{E} \cong T\tilde{M} \oplus T^*\tilde{M}$ with the H and $\tilde{H}$ twisted brackets. Since the 3-forms are invariant we can consider the invariant sections as reduced Courant algebroids over the common base B, as we have seen in Section 4.1. The Courant reduced brackets are derived from the twisted differential and the Clifford action on invariant differential forms.

We have seen in Theorem 8.1.3 that the twisted differential complexes are isomorphic via the T-duality map τ. In [7] it has been shown that there is an isomorphism between the reduced Courant algebroids as well which we will outline here.

The T-duality map τ can be thought of as the composition of a pullback, a B-transform and a pushforward. To find the isomorphism of Courant algebroids it is sufficient to find an isomoprhism of vector bundles

$$\varphi : \quad (TM \oplus T^*M)/T^k \;\rightarrow\; (T\tilde{M} \oplus T^*\tilde{M})/\tilde{T}^k$$

such that it induces an isomorphism of Clifford modules, i.e.

$$\tau(v.\rho) = \varphi(v).\tau(\rho) \tag{8.2.2}$$

for all $v \in \Gamma(TM \oplus T^*M)^{T^k}$, and $\rho \in \Omega^\bullet(M)^{T^k}$.

Indeed, such a map is then immediately an isomorphism of Courant algebroids. It is orthogonal as

$$\langle v, v\rangle \tau(\rho) = \tau(\langle v, v\rangle \rho) = \tau(v.v.\rho) = \varphi(v).\varphi(v).\tau(\rho) = \langle \varphi(v), \varphi(v)\rangle \tau(\rho)$$

and it preserves the Courant bracket as

$$\begin{aligned}
\varphi([v, w]_H).\tau(\rho) &= \tau([[v, d_H], w].\rho) \\
&= [[\varphi(v), d_{\tilde{H}}], \varphi(w)].\tau(\rho) \\
&= [\varphi(v), \varphi(w)]_{\tilde{H}}.\tau(\rho).
\end{aligned}$$

It remains to find such a map φ. We want to generalize the idea of τ and define φ as the composition of a pullback to the correspondence space, a B-transform using F and a pushforward to the T-dual space.

There are some problems with this construction. Firstly, given an invariant section $X + \xi$ of $TM \oplus T^*M$ the pullback of X is not well defined, it depends on a choice of lift $\hat{X}$ to the tangent space of $M \times_B \tilde{M}$. Secondly, given a lift $\hat{X} + p^*\xi$ we can perform the B-transform

$$\hat{X} + p^*\xi \mapsto \hat{X} + p^*\xi - \iota_{\hat{X}} F,$$

but to be able to push $p^*\xi - \iota_{\hat{X}} F$ forward to $\tilde{M}$ it has to be basic.

Fortunately, the ambiguity in the lift of X allows us to solve the problem of the pushforward. As F is invertible there exist precisely one lift $\hat{X}$ of X such that

$$p^*\xi(Y) - F(\hat{X}, Y) = 0 \quad \forall Y \in \mathfrak{t}^k. \tag{8.2.3}$$

Therefore, we may define φ as

$$\varphi(X + \xi) = \tilde{p}_* \circ e^{-F} \circ p^*(X + \xi) = \tilde{p}_*(\hat{X} + p^*\xi - \iota_{\hat{X}} F) \tag{8.2.4}$$

where $\hat{X}$ satisfies 8.2.3.

Proposition 8.2.1. *The map φ defined above is an isomorphism of the reduced vector bundles and satisfy 8.2.2.*

Proof. Clearly, φ is an isomorphism as its inverse is $p_* \circ e^{-F} \circ \tilde{p}^*$. Then for $X + \xi \in \Gamma(TM \oplus T^*M)^{T^k}$ and $\rho \in \Omega^\bullet(M)^{T^k}$ we have (omitting pullback notation)

$$\tau((X + \xi).\rho) = \int_{T^k} e^F \wedge \iota_X \rho + e^F \wedge \xi \wedge \rho$$

and

$$\varphi(X + \xi).\tau(\rho) = \int_{T^k} e^F \wedge \iota_{\hat{X}} \rho + \iota_{\hat{X}} e^F \wedge \rho - \iota_{\hat{X}} F \wedge e^F \wedge \rho + \xi \wedge e^F \wedge \rho.$$

The first two terms agree as ρ is basic for the $\tilde{T}^k$ action so contraction with $\hat{X}$ only depends on X. The last two terms agree as well as e^F has even degree components. Finally the middle two terms in the second expression cancel since $e^F = 1 + F + \frac{1}{2} F \wedge F + ...$ and

$$\iota_{\hat{X}} e^F = \iota_{\hat{X}} F \wedge e^F.$$

∎

The isomorphism φ can be illustrated using connections to split the tangent bundles. Let (M, h) and $(\tilde{M}, \tilde{h})$ be a T-dual pair with corresponding exact Courant algebroids E and $\tilde{E}$.

Given a T^k-connection θ on M the space of T^k-invariant vector fields decomposes as

$$TM/T^k \cong \mathfrak{t}^k \oplus TB.$$

Therefore, if $E \cong TM \oplus T^*M$ with the bracket twisted by $H = H^0 + F(\tilde{c}, \theta)$ representing h, we find

$$E/T^k \cong TB \oplus \mathfrak{t}^k \oplus \mathfrak{t}^{k*} \oplus T^*B.$$

Similarly, $\tilde{E} \cong T\tilde{M} \oplus T^*\tilde{M}$ with $\tilde{H} = H^0 + F(c, \tilde{\theta})$ representing $\tilde{h}$ and we have

$$\tilde{E}/\tilde{T}^k \cong TB \oplus \tilde{\mathfrak{t}}^k \oplus \tilde{\mathfrak{t}}^{k*} \oplus T^*B.$$

The invariant 2-form $F \in \mathfrak{t}^{k*} \otimes \tilde{\mathfrak{t}}^{k*}$ exhibiting the T-duality condition can be interpret as an isomorphism $F : \tilde{\mathfrak{t}}^k \to \mathfrak{t}^{k*}$. In this setting the following proposition holds.

Proposition 8.2.2. *Given the data above, the T-duality map φ on the invariant sections is given by*

$$\varphi : (X, s, t^*, \xi_0) \mapsto (X, -F^{-1}(t^*), -F^*(s), \xi_0)$$

where $X \in TM$, $s \in \mathfrak{t}^k$, $t^* \in \mathfrak{t}^{k*}$ and $\xi_0 \in T^*M$.

Proof. Let θ and $\tilde{\theta}$ be connection 1-forms on M and $\tilde{M}$ as above, such that $F = F(\theta \wedge \tilde{\theta})$. Then any $X + s + t^* + \xi_0 \in (TM \oplus T^*M)/T^k$ can be written as an invariant section of $TM \oplus T^*M$

$$X + s + t^* + \xi_0 \mapsto X^H + s + t^* + \xi_0$$

where X^H is a horizontal lift of X, ξ_0 is thought of as a basic form, $s \in \mathfrak{t}^k$ and $t^* \in \mathfrak{t}^{k*}$ is meant as $t_i^*\theta^i$ in a certain basis. After pullback and B-transform by $-F$ we have

$$X + s + t^* + \xi_0 \quad \mapsto \quad \hat{X} + t^* + \xi_0 - F(\iota_{\hat{X}}\theta, \tilde{\theta}) + F(\theta, \iota_{\hat{X}}\tilde{\theta}),$$

from which we get $\iota_{\hat{X}}\tilde{\theta} = -F^{-1}(t^*)$, i.e. $\hat{X} = X^H + s - F^{-1}(t^*)$. Therefore the T-duality map is

$$\begin{aligned}
&\varphi(X + s + t^* + \xi_0) = \\
&\quad = p_*(X^H + s - F(t^*) + t^* + \xi_0 - F(s, \tilde{\theta}) - t^*) \\
&\quad = X^H - F^{-1}(t^*) - F(s, \tilde{\theta}) + \xi_0 \\
&\quad = X^H - F^{-1}(t^*) - F^*(s) + \xi_0.
\end{aligned}$$

∎

When we take $F = \langle \theta, \tilde{\theta} \rangle$ as in the construction of Theorem 8.1.2 the isomorphism above is simply

$$\varphi : (X, s, t^*, \xi_0) \mapsto (X, -t^*, -s, \xi_0).$$

Remark 8.2.1. Although the construction of the map φ is making use of certain splittings of E but it is actually a splitting-independent statement. Taking a different splitting of E changes the representative of the Ševera class by an exact form $H \mapsto H + dB$. Sections of E change via the B-transform

$$X + \xi \mapsto X + \xi - \iota_X B.$$

The T-duality map φ then acts as

$$X + \xi - \iota_X B \mapsto \tilde{p}_*(\hat{X} + \xi - \iota_X B - \iota_{\hat{X}} F)$$

where $\iota_X B + \iota_{\hat{X}} F = \iota_{\hat{X}}(F + B)$ since B is a pullback from M. This also means that the lift of X in this case is the same as before. Changing F to $F + B$ is also exactly the change that the T-duality condition $H - \tilde{H} = dF$ requires, which shows the independence of φ.

8.3 Heterotic T-duality

In this section we will show that the isomorphism induced by the T-duality relation on exact Courant algebroids descends to the heterotic case via reduction. We follow the treatment of [2] but work in real coefficients for the T-duality relation. This simplifies the construction and we can understand the construction without spectral sequences but we lose uniqueness statements that arise when one uses integer coefficients. Moreover, in our case there is no obstruction to T-duality under reasonable assumptions.

Our framework is the following. Suppose that $P \to M$ is a principal $G \times T^k$-bundle where G is compact, connected, semisimple and T^k is the k-dimensional torus. Then let $X = P/G$ and $P_0 = P/T^k$ be the principal T^k and G bundles induced by factoring out with the other group. Then we have the following commutative diagram.

$$\begin{array}{ccc} P & \xrightarrow{\pi} & P_0 \\ \sigma\downarrow & & \downarrow\sigma_0 \\ X & \xrightarrow{\pi_0} & M \end{array} \tag{8.3.5}$$

We are interested in T-dualizing heterotic Courant algebroids on X. For this let $h \in H^3(P, \mathbb{R})$ be a real string class on P corresponding to a heterotic Courant algebroid $\mathcal{H}$. To interpret T-duality as an isomorphism of heterotic Courant algebroids, we make use of the exact case and T-dualize the exact Courant algebroid E corresponding to h on P.

Firstly, to be able to construct a T-dual for the class h with respect to the T^k-bundle $\pi : P \to P_0$ it has to be a T-dualisable class i.e. satisfy the conditions of Theorem 8.1.2. Secondly, even if we have constructed a T-dual pair $(\tilde{P}, \tilde{h})$ of a $\tilde{T}^k$-bundle and a degree 3 cohomology class, it is not clear whether the G-action on P_0 lifts to an action on $\tilde{P}$ commuting with the $\tilde{T}^k$-action. Such a lift can be obtained whenever $\tilde{P}$ is the pullback of a $\tilde{T}^k$-bundle $\tilde{\pi}_0 : \tilde{X} \to M$ via σ_0. It turns out that this is always the case, which we will explain in detail in the next section.

For now let us assume that h has the right properties and such a T-dual $\tilde{P}$ exists. Then we have the following commutative diagram.

$$\begin{array}{ccccc} P & \xrightarrow{\pi} & P_0 & \xleftarrow{\tilde{\pi}} & \tilde{P} \\ \sigma\downarrow & & \downarrow\sigma_0 & & \downarrow\tilde{\sigma} \\ X & \xrightarrow{\pi_0} & M & \xleftarrow{\tilde{\pi}_0} & \tilde{X} \end{array} \tag{8.3.6}$$

Considering the correspondence space $P \times_{P_0} \tilde{P}$ with projections p and $\tilde{p}$ to P and $\tilde{P}$ respectively, the T-duality condition is $p^*h = \tilde{p}^*\tilde{h}$. We will use this property to show that $\tilde{h}$ is again a string class for the G-bundle $\tilde{P} \to \tilde{X}$.

Now let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be heterotic Courant algebroids over X and $\tilde{X}$ corresponding to the T-dual string classes h and $\tilde{h}$. We consider the exact Courant algebroids E and $\tilde{E}$ on P and $\tilde{P}$ corresponding to the same classes. By the exact case of T-duality we have an isomorphism $\varphi : E/T^k \to \tilde{E}/\tilde{T}^k$. We show that in a certain splitting φ exchanges the extended actions on E/T^k and $\tilde{E}^k/\tilde{T}^k$. Then by our considerations in Section 4.3 after reducing via the extended actions we have the chain of isomorphisms

$$\mathcal{H}/T^k \cong (E/T^k)_{red} \cong (\tilde{E}/\tilde{T}^k)_{red} \cong \tilde{\mathcal{H}}/\tilde{T}^k.$$

T-dualizable string classes

Let $\sigma : P \to M$ a principal $G \times T^k$ bundle fitting into the commutative diagram 8.3.5. Let $h \in H^3(P, \mathbb{R})$ be a class representing an integral cohomology class, so that h is T-dualisable with respect to the torus bundle $P \to P_0$ and a string class with respect to the G-bundle $P \to X$. For this we assume that the bilinear form c on the Lie algebra $\mathfrak{g}$ is normalized so that the Maurer-Cartan 3-form ω represents an integral cohomology class of G.

Let θ be a principal connection for the torus bundle $X \to M$ and pull it back to a G-invariant connection on $P \to P_0$. As h is T-dualisable we may chose a $G \times T^k$-invariant representative H of h that decomposes as

$$H = H^0 + \langle \tilde{c}, \theta \rangle.$$

with $H \in \Omega^3(P_0, \mathbb{R})$ and $\tilde{c} \in \Omega^2(P_0, \mathfrak{t}^{k*})$. The 2-form $\tilde{c}$ is closed and as H represents an integral cohomology class so does $\tilde{c}$. We construct $\tilde{P} \to P_0$ as a torus bundle with real Chern class represented by $\tilde{c}$.

If we consider the Serre spectral sequence for the G-bundle $\sigma_0 : P_0 \to M$ with coefficients in $H^1(T^k, \mathbb{R}) = \mathfrak{t}^{k*}$ (see Appendix 10.3 and [2]), because G is semi-simple (so it has trivial first and second de-Rham cohomology) we find that

$$H^2(P_0, \mathfrak{t}^{k*}) = H^2(M, \mathfrak{t}^{k*}).$$

We see that the class $[\tilde{c}]$ can always represents a torus bundle which is a pullback from M via σ_0, but one has to be careful to chose the torsion part of $[\tilde{c}]$ to be a pullback from M as well.

We will now show that $\tilde{h}$ is a string class as well. By construction for any connection $\tilde{\theta}$ on $\tilde{P}$, $\tilde{h}$ has a $G \times \tilde{T}^k$-invariant representative $\tilde{H}$ such that

$$\tilde{H} = H^0 + \langle c, \tilde{\theta} \rangle.$$

Here $[c] \in H^2(M, \mathfrak{t}^k) = H^2(P_0, \mathfrak{t}^k)$ represents the Chern class of $X \to M$ as well as of $P \to P_0$, more precisel $c = d\theta$. As P is a pullback of a torus bundle from M we may chose the connection $\tilde{\theta}$ in the above description to be a pullback connection from $\tilde{X} \to M$. Therefore, the restriction of $\tilde{H}$ to any fibre agrees with the restriction of H^0. Moreover, the restriction of H^0 to any fibre is closed as

$$d\tilde{H} = dH = 0 \quad \Rightarrow \quad dH^0 + \langle d\tilde{\theta}, d\theta \rangle = 0,$$

i.e. dH^0 is basic. This also means that in terms of cohomology

$$[dH^0] = -\langle [\tilde{c}], [c] \rangle \in H^4(M, \mathbb{R}).$$

Similarly by the construction of H its restriction to any G-fibre agrees with H^0. As h was a string class this restriction must agree with the Cartan 3-form ω_3 which becomes the restriction of $\tilde{h}$ as well. Therefore, $\tilde{h}$ is a string class as well.

The following theorem summarizes our results.

Theorem 8.3.1. *Let $P \to M$ be a principal $G \times T^k$-bundle fitting into the commutative diagram 8.3.5. Let $h \in H^3_{str}(P, \mathbb{R})$ be a string class representing an integral cohomology class. Then there exist a T-dual $(\tilde{P}, \tilde{h})$ to the torus bundle $P \to P_0$ completing the diagram 8.3.6 if and only if h has a representative H such that*

$$H = H^0 + \langle \tilde{c}, \theta \rangle$$

where H^0 is a basic 3-form for the torus bundle, θ is a principal connection on it and $[\tilde{c}] \in H^2(M, \mathfrak{t}^{k*})$. We call such an h a T-dualisable string class *and whenever the above condition holds the T-dual $\tilde{h}$ is also a T-dualisable string class.*

Reduction and T-duality

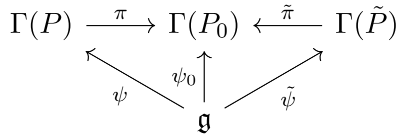
In this section we will show that the T-duality map defined in Section 8.2 interchanges the extended actions on T-dual exact Courant algebroids and hence induces an isomorphism of heterotic Courant algebroids.

By theorem 8.3.1 given a principal $G \times T^k$-bundle fitting into the commutative diagram 8.3.5 and a T-dualisable string class $h \in H^3(P, \mathbb{R})$ there is a T-dual pair $(\tilde{P}, \tilde{h})$ completing the following diagram.

$$\begin{array}{ccccc} P & \xrightarrow{\pi} & P_0 & \xleftarrow{\tilde{\pi}} & \tilde{P} \\ \sigma \downarrow & & \downarrow \sigma_0 & & \downarrow \tilde{\sigma} \\ X & \xrightarrow{\pi_0} & M & \xleftarrow{\tilde{\pi}_0} & \tilde{X} \end{array}$$

Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be the heterotic Courant algebroids on X and $\tilde{X}$ corresponding to the real string classes $h \in H^3(P, \mathbb{R})$ and $\tilde{h} \in H^3(\tilde{P}, \mathbb{R})$. Let E and $\tilde{E}$ be the exact Courant algebroids on P and $\tilde{P}$ corresponding to the same classes.

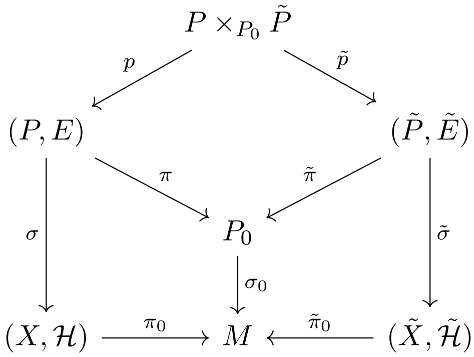
Let ψ and $\tilde{\psi}$ be the infinitesimal actions of G on P and $\tilde{P}$ respectively, and ψ_0 the infinitesimal action on P_0. As the G-action on both P and $\tilde{P}$ covers the action on P_0 we have the following commutative diagram.



As the T-duality map φ is independent of splitting it is sufficient to show the desired property in a certain splitting corresponding to specific extended actions. We will chose to work with pullback connections which will make our statement trivial.

Let $\theta \in \Omega^1(X, \mathfrak{t}^k)$ and $\tilde{\theta} \in \Omega^1(\tilde{X}, \tilde{\mathfrak{t}}^k)$ be torus connections on X and $\tilde{X}$ and pull them back to torus connections on P and $\tilde{P}$. Let $A \in \Omega^1(P_0, \mathfrak{g})$ be a G-connection which we pull back to both P and $\tilde{P}$.

We first want to implement T-duality for the exact Courant algebroids E and $\tilde{E}$ so consider the fibered product $P \times_{P_0} \tilde{P}$ and the resulting commutative diagram.



Notice that the action of G on the correspondence space $P \times_{P_0} \tilde{P}$ pulls back from P_0 via $\pi \circ p = \tilde{\pi} \circ \tilde{p}$ therefore it is a $G \times T^k \times \tilde{T}^k$-bundle.

Using the pullback G-connection A and averaging we can find a $G \times T^k$-invariant representative H of h and $\tilde{H}$ of $\tilde{h}$ which are of the form

$$H = H^0 - CS_3(A) \quad \text{and} \quad \tilde{H} = \tilde{H}^0 - CS_3(A)$$

with $H^0 \in \Omega^3(X)$ and $\tilde{H}^0 \in \Omega^3(\tilde{X})$ basic forms for the G-action. As A is a pullback connection the Chern Simons 3-forms above are basic with respect to the torus actions.

Then we can decompose H^0 and $\tilde{H}^0$ using the pullback torus connections θ and $\tilde{\theta}$. As h and $\tilde{h}$ are T-dual we can find $F \in \Omega^2(P \times_{P_0} \tilde{P})$ such that

$$H - \tilde{H} = H^0 - \tilde{H}^0 = dF$$

with F non-degenerate on $\mathfrak{t}^k \otimes \tilde{\mathfrak{t}}^k$. As H and $\tilde{H}$ are $G \times T^k \times \tilde{T}^k$-invariant so is F. Writing every form in the decomposition using the torus connections and repeating the calculation in Section 8.1 we may find new representatives that are of the form

$$H = H^0 + \langle \tilde{c}, \theta \rangle - CS_3(A) \quad \text{and} \quad \tilde{H} = H^0 + \langle \tilde{\theta}, c \rangle - CS_3(A),$$

where $H^0 \in \Omega^3(M)$, $c = d\theta$, $\tilde{c} = d\tilde{\theta}$ and $F = \langle\theta, \tilde{\theta}\rangle$.

The extended actions corresponding to these representatives of h and $\tilde{h}$ are given by $\xi = -cA$ and $\xi' = -cA$, therefore the corresponding G-equivariant 3-forms are

$$\Phi = H + \xi = H^0 + \langle\tilde{c}, \theta\rangle - CS_3(A) - cA,$$

$$\tilde{\Phi} = \tilde{H} + \tilde{\xi} = H^0 + \langle c, \tilde{\theta}\rangle - CS_3(A) - cA.$$

Note that in particular these forms are now $G \times T^k$ (or $G \times \tilde{T}^k$)-invariant as well. Clearly, on the correspondence space we have $\tilde{\Phi} = \Phi + dF = \Phi + d_G F$ as $\iota F = 0$ since the torus connections are basic for the G-action.

Consider the splittings of E and $\tilde{E}$ corresponding to the representatives H and $\tilde{H}$. Then using the connection 1-forms we have the decompositions

$$\begin{aligned} E/T^k &\cong TP_0 \oplus \mathfrak{t}^k \oplus \mathfrak{t}^{k*} \oplus T^*P_0, \\ \tilde{E}/\tilde{T}^k &\cong TP_0 \oplus \tilde{\mathfrak{t}}^k \oplus \tilde{\mathfrak{t}}^{k*} \oplus T^*P_0 \end{aligned} \tag{8.3.7}$$

In this decomposition the infinitesimal action of G descends to E/T as

$$\begin{aligned} \psi : \mathfrak{g} &\mapsto \Gamma(TP)^{T^k} = \Gamma(TP_0 \oplus \mathfrak{t}^k) \\ x &\mapsto (\psi_0(x), \iota_{\psi(x)}\theta) = (\psi_0(x), 0) \end{aligned}$$

since the torus connection θ is basic for the G-action. Similarly, $\tilde{\psi(x)} = (\psi_0(x), 0)$ in the splitting induced by $\tilde{\theta}$.

Finally, as the G and torus actions commute the extended actions on E and $\tilde{E}$ define extended actions on the reduced Courant algebroids E/T^k and $\tilde{E}/\tilde{T}^k$. In the decomposition 8.3.7 they re given by

$$\begin{aligned} \alpha(x) &= \psi(x) + \xi(x) = (\psi_0(x), 0, 0, -c(A, x)), \\ \tilde{\alpha}(x) &= \tilde{\psi}(x) + \tilde{\xi}(x) = (\psi_0(x), 0, 0, -c(A, x)). \end{aligned}$$

In Section 8.2 we have seen that T-duality map is given by

$$\varphi : (X, s, t^*, \xi) \to (X, -t^*, -s, \xi)$$

in the above setting. It is clear that it interchanges the infinitesimal actions, i.e. $\varphi \circ \alpha = \tilde{\alpha}$. Therefore φ descends to an isomorphism $\varphi : (E/T^k)_{red} \to (\tilde{E}/\tilde{T}^k)_{red}$ and by $\mathcal{H}/T^k \cong (E/T^k)_{red}$ and $\tilde{\mathcal{H}}/\tilde{T}^k \cong (\tilde{E}/\tilde{T}^k)$ we conclude the main theorem.

Theorem 8.3.2. *Let (P, h) be a pair of a principal G-bundle over a torus bundle $X \to M$ together with a T-dualisable string class, and a T-dual pair $(\tilde{P}, \tilde{h})$ completing the diagram 8.3.6. Then the T-dual class $\tilde{h}$ is again a string class and there is an isomorphism of the corresponding simply-reduced heterotic Courant algebroids on M*

$$\varphi : \mathcal{H}/T^k \;\cong\; \tilde{\mathcal{H}}/\tilde{T}^k$$

Example. In general heterotic T-duality requires some understanding of the cohomology of principal bundles. The simplest example is the case of a trivial G-bundle P over a simply connected torus-bundle $X \to M$. Then for $\sigma : X \times G \to X$ by the Künneth theorem (see [25]) the third cohomology group decomposes as

$$H^3(P) = H^3(X) \oplus H^3(G).$$

Therefore h is a string class if and only if it is of the form $h = \sigma^* h_0 + [\omega_3]$, therefore its T-dualisability depends solely on h_0. In this case heterotic T-duality reduces to the T-duality of the classes $h_0 \in H^3(X)$.

Compatible 4-tuples of connections. In [2] the T-duality isomorphism was described for a general class of connections on P and $\tilde{P}$. We now derive the same result from the "other way around" which also motivates their construction without the use of spectral sequences. Suppose we are in the setting of heterotic T-duality as above and let A be any T^k-invariant G-connection on P and $\tilde{A}$ any $\tilde{T}^k$-invariant G-connection on P. To implement T-duality we want to represent the T-dualisable string classes h and $\tilde{h}$ so that the T-duality map is clear and also there is an extended action corresponding to the splitting. More precisely we want to consider 4-tuples of connections $(A, \tilde{A}, \theta, \tilde{\theta})$ so we have representatives

$$\begin{aligned} H &= H^0 - CS_3(A) = K^0 + \langle d\tilde{\theta}, \theta \rangle, \\ \tilde{H} &= \tilde{H}^0 - CS_3(\tilde{A}) = K^0 + \langle \tilde{\theta}, d\theta \rangle, \end{aligned}$$

where $H^0, \tilde{H}^0$ are basic for the G-actions and K^0 is basic for the torus action. We call such 4-tuples of connections *compatible*.

For pullback connections θ_0, $\tilde{\theta}_0$ and A_0 the condition is satisfied, we had $h = [H]$

$$H = H^0 + \langle d\tilde{\theta}_0, \theta_0 \rangle - CS_3(A_0).$$

Taking a different representative $H' = H^{0\prime} + CS_3(A)$ changes the form H basically via a (B, A)-transform as we have seen in Proposition 5.1.2. Therefore if $A = A_0 + a$ for some $a \in \Omega^1(P, \mathfrak{g})_{bas}$ we have

$$H' = H + d(c(a, A_0)) = H^{0\prime} - CS_3(A),$$

where H^0 is basic for the G-action. The form H is also represented as a T-dualisable class $H = K^0 + \langle d\tilde{\theta}, \theta \rangle$ for a 3-form K^0 that is basic for the torus action. We want to bring H' to this form as well. It can be achieved by choosing a basis $t_1, .., t_k$ for $\mathfrak{t}^k$ and writing $\theta_0 = \theta_0^i t_i$, and the dual connection using the dual basis $\tilde{\theta}_0 = \tilde{\theta}_{0i} t^i$. Then $a = a^0 + a_i^1 \theta^i$ where now a^0 and a_i^1 are $\mathfrak{g}$-valued forms on M. Then changing the connection $\tilde{\theta} = \tilde{\theta}_i t^i$ on $\tilde{P}$ from a pullback connection to

$$\tilde{\theta}_{0i} \mapsto \tilde{\theta}_i = \tilde{\theta}_{0i} + c(a_i^1, A),$$

and $\tilde{H}$ to $\tilde{H}' = \tilde{H} + dc(a^0, A_0)$ the right forms for H' and $\tilde{H}'$ using the new connections

$$\begin{aligned} A &= A_0 + a_i^1 \theta_0^i + a^0 \\ \tilde{A} &= A_0 + a^0 \\ \theta &= \theta_0 \\ \tilde{\theta} &= \tilde{\theta}_0 + c(a_i^1, A_0) t^i \end{aligned}$$

One can similarly change the remaining connections as well, which amount to the same changes as above. Finally, any compatible 4-tuple can be obtained via these changes and we get the general formula

$$\begin{aligned}
A &= A_0 + a_i^1\theta_0^i + a^0 \\
\tilde{A} &= A_0 + \tilde{a}^{1i}\tilde{\theta}_{0i} + a^0 \\
\theta &= \theta_0 + c(\tilde{a}^{1i}, A_0)t_i \\
\tilde{\theta} &= \tilde{\theta}_0 + c(a_i^1, A_0)t^i.
\end{aligned}$$

This is equivalent to ([2] 4.6). From this formula it is easy to see that using the induced representatives for the T-dual classes H and $\tilde{H}$ we have on the corresponding space

$$\tilde{H} - H = d(\theta \wedge \tilde{\theta})$$

moreover the extended actions $\Phi = H - cA$ and $\tilde{\Phi} = \tilde{H} - c\tilde{A}$ satisfy

$$\tilde{\Phi} - \Phi = d_G(\theta \wedge \tilde{\theta}).$$

In this setting the extended actions are

$$\begin{aligned}
\alpha(x) &= (\psi_0(x), c(a_i^1, x)t^i, -c(\tilde{a}^{1i}, x)t_i, -cx) \\
\tilde{\alpha}(x) &= (\psi_0(x), c(\tilde{a}^{1i}, x)t_i, -c(a_i^1, x)t^i, -cx)
\end{aligned}$$

and these are again exchanged by the T-duality map.

The upshot of this calculation is that we can chose the representatives of our class h somewhat freely and the T-duality morphism still can be constructed in a way that is useful for calculations. This can come handy when one wants to transfer solutions to the Strominger System where the connection 1-form A plays a crucial role.

9 The Strominger System

The Strominger System of partial differential equations arises in physics when the 10 dimensional $\mathcal{N} = 1$ supersymmetric theory of heterotic strings is compactified to 6 dimensions [27, 38]. It was first considered in the mathematical literature by Li and Yau [31] as a generalization of the Calabi conjecture to non-Kähler complex manifolds of arbitrary dimension. Since then it has been widely studied, and for complex dimensions one and two the existence and uniqueness problem has been fully understood. In three complex dimensions however, which is the most relevant to physics, the question of existence and uniqueness is still unanswered.

In the recent years the system was connected to heterotic generalized geometry [2, 15, 18] which also provided a useful tool to understanding the moduli space of solutions which has been the main interest of research regarding the system lately but here we do not discuss these results.

In this section we first give a short introduction to the Strominger System then we present its reformulation in terms of generalized connections following the work of Garcia-Fernandez, Rubio and Tripler [18, 17]. Later we discuss the latest developments towards understanding how the system behaves under T-duality of Courant algebroids [17, 19].

The Strominger System is formulated on a complex compact manifold X of dimension n with underlying smooth real manifold (X, J) where J is the complex structure. Moreover, X is taken to be a complex Calabi-Yau manifold in the following sense.

Definition 9.0.1. A *complex Calabi-Yau manifold*[1] (X, Ω) is a complex compact manifold X together with a nowhere vanishing holomorphic section Ω of the canonical line bundle $\mathcal{K} = \wedge^n (T^{(1,0)}X)^*$ of X.

In particular this definition does not require X to be Kähler. Given a Riemannian metric g on the complex Calabi-Yau manifold (X, Ω) compatible with the complex structure J one defines the *Kähler form* as

$$\omega = g(., J.).$$

Using the Kähler form the norm of Ω, denoted by $||\Omega||_\omega$, is given by

$$||\Omega||_\omega^2 \frac{\omega^n}{n!} = (-1)^{n(n-1)/2} i^n \Omega \wedge \overline{\Omega}.$$

With this data we can formulate the Strominger System.

[1] Normally a Calabi-Yau manifold is a compact complex manifold X with $SU(3)$ holonomy. This in particular implies that the first Chern class of X vanishes and via Yau's Theorem that the manifold is Kähler and admits a Ricci-flat metric. This particular naming of complex manifolds with trivial canonical line bundle (and hence zero first Chern class) is inspired by the name "symplectic Calabi-Yau" which was used in [12] to denote symplectic manifolds with vanishing first Chern class.

Definition 9.0.2. The *Strominger System* is

1. a complex Calabi-Yau manifold (X, Ω) with a hermitian metric g on (TX, J),
2. a complex vector bundle with a hermitian metric (E, h)
3. and unitary connections ∇ on (TM, J, g) and A on (E, h)

subject to the conditions

$$\begin{aligned} &F_A \wedge \omega^{n-1} = 0, \qquad F_A^{0,2} = 0, \\ &R_\nabla \wedge \omega^{n-1} = 0, \qquad R_\nabla^{0,2} = 0, \\ &d(||\Omega||_\omega \omega^{n-1}) = 0, \\ &dd^c\omega - \alpha(\mathrm{tr}R_\nabla \wedge R_\nabla - \mathrm{tr}F_A \wedge F_A) = 0. \end{aligned} \tag{9.0.1}$$

The first two equations require A and ∇ to be so called *Hermite-Yang Mills* connections which also imposes topological constraints on the vector bundles E and TX. The third equation says that the metric g is conformally balanced, that is conformally equivalent to a balanced metric. It is the case for example when g is a Kähler Ricci-flat (i.e. Calabi-Yau) but these are considered as the trivial examples. The last equation is known as the *Bianchi identity* which couples together the connections and the metric and α denotes the constant parameter of string theory. This equation was studied from topological and geometrical point of view but it stays mysterious analytically.

Example. A trivial solution of this system would be a complex torus as the quotient of the complex plane $\mathbb{C}^n$, with the holomorphic volume form

$$\Omega = dz^1 \wedge ... \wedge dz^n.$$

Taking the Eucledian flat metric g which has closed Kähler form

$$\omega = \sum_{i=1}^{n} dz^i \wedge d\overline{z}^i,$$

and any complex vector bundle with a hermitian metric (E, h) that admits a flat connection A we arrive to a solution of 9.0.1.

This example also follows through when the manifold X is Calabi-Yau with a Kähler Ricci-flat metric g. Then the second and third equations are satisfied and again any flat complex vector bundle with a hermitian metric will satisfy the first and last equations with a flat unitary connection.

These are the first and simplest examples one can produce. Finding non-Kähler examples is not an easy task and it is currently a research question, recent examples can be found in e.g. [19, 13]. We refer to [16] for a thorough review on the known results regarding the Strominger System.

Strominger System and generalized geometry. It is the Bianchi identity that motivates strongly the connection to heterotic generalized geometry. From here we assume that (X, Ω) is a complex Calabi-Yau manifold with a hermitian metric g and unitary connection ∇, together with a hermitian vector bundle (E, h) and a unitary connection A as in the definition of the Strominger System. We will show that the Bianchi identity can be used to define a heterotic Courant algebroid. To make this clearer let P_h be the principal bundle of unitary frames of (E, h) on X and P_g the principal bundle of unitary frames of (TX, g, J). Then consider the fibered product

$$P = P_g \times P_h$$

which is now a $SU(n) \times SU(r)$ principal bundle with semisimple Lie algebra $\mathfrak{g} = \mathfrak{su}(r) \oplus \mathfrak{su}(3)$. The Killing form on these Lie algebras is given by the trace, therefore

$$c = \alpha(\mathrm{tr}_{-\mathfrak{su}(n)} - \mathrm{tr}_{\mathfrak{su}(r)})$$

is a positive definite non-degenerate inner product on $\mathfrak{g}$.

Then take the product connection $\nabla_0 = \nabla \times A$ on P with ∇ and A connections on P_g and P_h respectively, associated to the linear connections on (TM, J, g) and (E, h). With this the Bianchi identity lifts to a condition on the bundle of frames and becomes

$$dd^c\omega - c(F_0 \wedge F_0) = 0$$

where F_0 is the curvature of A_0. This is precisely the condition we have found for a heterotic Courant algebroid to exist on M with real string class

$$h = [d^c\omega - CS_3(A_0)] \in H^3(P, \mathbb{R}).$$

Moreover, together with the metric g on M the triple $(g, d^c\omega, A)$ describes a generalized metric on the Courant algebroid associated to the string class h.

However, a heterotic Courant algebroid $\mathcal{H}$ together with a generalized metric $\mathcal{H}_-$ does not fully describe a solution of the Strominger System. For example, there is no Calabi-Yau structure initially, the metric and $SU(n)$-connection only defines a complex structure on M.

9.1 Killing spinor equations

The full characterization of the Strominger System in terms of heterotic generaized geometry was solved in [18] by Garcia-Fernandez, Rubio and Tripler for six real dimensions. They used a different characterisation of the Strominger System in terms of non-vanishing spinors on X which was based on earlier work of Strominger and Hull.

For this setup let X be a smooth compact six dimensional real spin manifold with a Riemannian metric g. Consider the Clifford algebra bundle generated by the cotangent bundle $\mathcal{C}\ell(T^*X)$ and an associated spin bundle S. Consider the twisted spin bundle $\mathcal{S} = S \otimes (detS^*)^{1/rs}$ where Clifford connections induce unique spin connections. As X is even dimensional the spin bundle splits into half-spin representations

$$\mathcal{S} = \mathcal{S}^L \oplus \mathcal{S}^R,$$

as we have seen in Section 7.2. The space of differential forms $\wedge^\bullet T^*X$ sits naturally inside $\mathcal{C}\ell(T^*X)$ therefore the forms act on spinors.

Let P_K be a principal bundle on X with structure group $K = SU(r)$. Let $\phi \in C^\infty(X)$ be a smooth function on X, $H \in \Omega^3(X)$ a 3-form, ∇ a metric connection on X, A a principal connection on P_K and $\eta \in \mathcal{S}^R$ a nowhere vanishing right-moving spinor. Recall, that given a three-form H on X the Bismut connection is given by

$$\nabla^- = \nabla^g - \frac{1}{3}g^{-1}H$$

where ∇^g is the Levi-Civita connection. For the data $(g, \phi, H, \nabla, A, \eta)$ we have the following theorem.

Theorem 9.1.1. *A solution $(g, \phi, H, \nabla, A, \eta)$ of the system*

$$\begin{aligned} \nabla^-\eta &= 0, \\ (H + 2d\phi)\cdot\eta &= 0, \\ F_A\cdot\eta &= 0, \\ R_\nabla\cdot\eta &= 0, \\ dH - \alpha(trR_\nabla \wedge R_\nabla - trF_A \wedge F_A) &= 0, \end{aligned} \tag{9.1.2}$$

is equivalent to a Calabi-Yau structure Ω on X, with hermitian metric g and connection A on P_K solving the Strominger System 9.0.1 with

$$H = d^c\omega, \quad d\phi = -\frac{1}{2}d\log||\Omega||_\omega.$$

Moreover, any solution of 9.1.2 is a solution to the equations of motion of heterotic supergravity (to first order expansion).

Here to obtain a genuine solution to 9.0.1 one has to take an associated complex hermitian vector bundle (E, h) to the principal bundle P_K and the associated unitary connection.

This theorem was first proved partially by Strominger and Hull [27][38] who did not consider the equation $R_\nabla\cdot\eta = 0$ for the metric connection. In this case the resulting solutions are not necessarily solutions to the equations of motion and the metric connection is not required to be Hermite-Yang Mills. Later Fernández, Ivanov, Ugarte and Villacampa proved [28][11] that the addition of the equation $R_\nabla\cdot\eta = 0$ is necessary and sufficient to find solutions that also solve the equations of motion and fully satisfy 9.0.1.

Generalized Killing spinors. In [18] the system 9.1.2 was reformulated in terms of generalized spinors, metrics and connections on a heterotic Courant algebroid. We present their construction here but omit some calculations.

Let $P \to M$ be a pricipal bundle with compact semisimple structure group G and a positive-definite non-degenerate inner product c on its Lie algebra $\mathfrak{g}$. Let $\mathcal{H}$ be a heterotic Courant algebroid corresponding to a string class $h \in H^3_{str}(P, \mathbb{R})$ and $\mathcal{H}_- \subset \mathcal{H}$ a generalized metric. By 6.2.1 the metric induces a splitting characterized by a basic three-form $H \in \Omega^3(M)$ and a connection A. Then we have $\mathcal{H} \cong TM \oplus \mathfrak{g}_P \oplus T^*M$ and

$$\mathcal{H}_- = \{X - gX \mid X \in TM\}$$
$$\mathcal{H}_+ = \{X + s + gX \mid X \in TM, \ s \in \mathfrak{g}_P\}$$

for a Riemannian metric g on M. Let $\Pi_\pm$ be the projections from $\mathcal{H}$ to $\mathcal{H}_\pm$.

The generalized metric induces an isometry via the anchor

$$\pi|_{\mathcal{H}_-} : (\mathcal{H}_-, -\langle \cdot, \cdot \rangle) \to (TM, g)$$

therefore the second Stiefel-Whitney class of $\mathcal{H}_-$ and TM agrees. As we took M to be a spin manifold, we find that there exist a spin bundle S corresponding to the Clifford algebra bundle $\mathcal{C}\ell(\mathcal{H}_-)$. Here we have to take the fibres at a point x to be $\mathcal{C}\ell(\mathcal{H}_-|_x^*)$ as the Clifford bundle on M is also constructed out of the cotangent bundle.

We take the twisted spinor bundle $\mathcal{S}$ again and since the dimension of M is even the it again splits into half-spin representations

$$\mathcal{S} = \mathcal{S}^R \oplus \mathcal{S}^L.$$

Consider now the canonical Levi-Civita connection D^{LC} on $\mathcal{H}$ corresponding to $\mathcal{H}_-$ as defined in 7.3.2. Recall from Section 7.1 that torsion-free metric compatible generalized connections form an affine space over

$$\Sigma^+ \oplus \Sigma^- = \Sigma_0^+ \oplus \Sigma_0^- \oplus \Gamma(\mathcal{H}_-) \oplus \Gamma(\mathcal{H}_+)$$

where $\Sigma_0^\pm$ denotes the traceless changes and $\Gamma(\mathcal{H}_\pm)$ describe the "Weyl gauge transformations" which we can use to fix the divergence. Therefore, given an element $e \in \mathcal{H}$ we can define $\chi^e \in \Gamma(\mathcal{H}^* \oplus \mathfrak{so}(\mathcal{H}))$ as before

$$\chi^e_{e_1} e_2 = \langle e_1, e_2 \rangle e - \langle e, e_2 \rangle e_1,$$

then

$$(\chi^e)^\pm_{e_1} e_2 = \Pi_\pm \chi^e_{e_1^\pm} e_2^\pm$$

defines the change corresponding to $e^\pm \in \Gamma(\mathcal{H}_\pm)$. Now we can use $(\chi^e)^\pm$ we may define another torsion-free generalized metric

$$D^e = D^{LC} - \frac{1}{3(\operatorname{rk} \mathcal{H}_- - 1)} (\chi^e)^- - \frac{1}{3(\operatorname{rk} \mathcal{H}_+ - 1)} (\chi^e)^- \qquad (9.1.3)$$

whose divergence differs from the divergence of D^{LC} by the element e. We will also denote D^e by $D^e(\mathcal{H}_-)$ when we want to stress the dependence on the metric.

Recall, that a generalized connection defines four first order differential operators on $\mathcal{H}$ corresponding to the decomposition $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$. Here we consider two of them.

Firstly, the negative pure-type operator

$$D_-^{e-} : \Gamma(\mathcal{H}_-) \to \Gamma(\mathcal{H}_-^* \otimes \mathcal{H}_-)$$

defines an orthogonal $\mathcal{H}_-$-connection on $\mathcal{H}_-$ and therefore induces a canonical Dirac-operator $\not{D}_-^{e-}$ on the spin bundle $\mathcal{S}$ which maps the half-spin representations into each other. Denote its restriction to the right-moving sector by

$$\not{D}_-^e : \Gamma(\mathcal{S}^R) \to \Gamma(\mathcal{S}^L). \tag{9.1.4}$$

Secondly, the mixed-type operator

$$D_-^{e+} : \Gamma(\mathcal{H}_-) \to \Gamma(\mathcal{H}_+^* \otimes \mathcal{H}_-)$$

also generates a unique first order differential operator on the twisted spin bundle whose restriction to $\mathcal{S}^R$ we denote by

$$D_+^e : \Gamma(\mathcal{S}^R) \to \Gamma(\mathcal{H}_+^* \otimes \mathcal{S}^R). \tag{9.1.5}$$

Notice, that by Lemmas 7.1.1 and 7.2.1 the two operators 9.1.4 and 9.1.5 are independent from the torsion-free metric connection chosen as long as we fix the divergence. We use these to define the spinorial equations that can be related to the Strominger System.

Definition 9.1.1. Given a generalized metric $\mathcal{H}_-$ and an element $e \in \mathcal{H}$ the *Killing spinor equations* for a nowhere vanishing right-moving spinor $\eta \in \mathcal{S}^R(\mathcal{H}_-)$ are given by

$$\begin{aligned} D_+^e \eta &= 0 \\ \not{D}_-^e \eta &= 0, \end{aligned} \tag{9.1.6}$$

where the operators could also be defined via any generalized connection $D \in \mathcal{D}(\mathcal{H}_-, div_{D^e})$.

Definition 9.1.2. We say that *the pair* $(\mathcal{H}_-, e)$ *is a solution to the Killing spinor equations* if $\mathcal{H}_- \subset \mathcal{H}$ is a generalized metric, e is a section of $\mathcal{H}$ and there exist a spinor bundle $\mathcal{S} = \mathcal{S}^R \oplus \mathcal{S}^L$ and nowhere vanishing right handed spinor $\eta \in \Gamma(\mathcal{S}^R)$ that satisfies 9.1.6 for the generalized connection D^e.

The cotangent bundle T^*M embeds into $\mathcal{H}^*$ so after using the inner product to identify $\mathcal{H}^*$ with $\mathcal{H}$ we can regard 1-forms as sections of $\mathcal{H}$. With this the following theorem holds.

Theorem 9.1.2. [18] *Let $\phi \in C^\infty(M)$ be a smooth function. Then the Killing spinor equations 9.1.6 for $e = d\phi$ are equivalent to the system*

$$\begin{aligned} \nabla^- \eta &= 0 \\ F \cdot \eta &= 0 \\ (H + 2d\phi) \cdot \eta &= 0 \\ dH - c(F \wedge F) &= 0 \end{aligned} \tag{9.1.7}$$

where F is the curvature of the connection A determined by the splitting $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$ and ∇^- is the Bismut connection corresponding to the Riemannian metric defined by $\mathcal{H}_-$.

Sketch proof. The most important observation is that via the isomorphism

$$\pi|_{\mathcal{H}_-} : \mathcal{H}_- \to TM$$

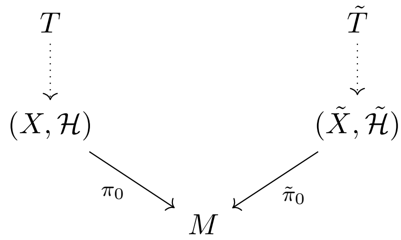
we also get an isomorphism of Clifford bundles $\mathcal{C}\ell(\mathcal{H}_-) \cong \mathcal{C}\ell(T^*M)$ therefore they act on the same twisted spinor bundle. From here the proof is just computing the Levi-Civita connection and the added divergence terms in this specific setting and transfering the equations from one space to the other. ∎

Clearly, plugging in our ansatz $P = P_g \times P_K$ and considering product connections yields the system 9.1.2 which in turn defines a solution of the Strominger System 9.0.1. Although the proof of the theorem is straightforward once one has the right connection to work with, the existence of this connection is truly remarkable. Moreover, we have never lost naturality in the construction, that is if $f : \mathcal{H} \to \mathcal{H}'$ is a Courant isomorphism we have

$$D^e(\mathcal{H}_-) = f^* D^{fe}(f(\mathcal{H}_-)).$$

9.2 T-duality and and the Strominger System

Recall Theorem 8.3.2 that described heterotic T-duality as an isomorphism of simply reduced heterotic Courant algebroids over a common base. More presicely, let $\mathcal{H}$ be a heterotic Courant algebroid over a principal torus bundle $X \to M$ such that the torus action lifts to $\mathcal{H}$ via Courant automorphisms. Let $(\tilde{X}, \tilde{\mathcal{H}})$ be a T-dual $\tilde{T}^k$-torus bundle of $(X, \mathcal{H})$ as in Theorem 8.3.1 that is we have the following diagram:



Then the T-duality map φ is an isomorphism

$$\varphi : \mathcal{H}/T \cong \tilde{\mathcal{H}}/\tilde{T}$$

of the simply reduced Courant algebroids over the common base M.

In the previous section we have reformulated the Strominger System in terms of natural geometric objects that are interchanged by isomorphisms. Although, T-duality is not a genuine isomorphims of the Courant algebroids $\mathcal{H}$ and $\tilde{\mathcal{H}}$ only of their simple reduction, one can still construct T-dual pairs to the system under reasonable assumptions. In the discussion we follow the treatment of [17].

To see how the system and T-duality relate to each other we should consider first how the structures that characterise the Killing spinor equations behave under simple reduction.

One of the main ingredients of the Killing spinor equations 9.1.6 is a generalized metric on the heterotic Courant algebroid $\mathcal{H}$ on X which we can define by choosing a rank n negative definite subbundle of $\mathcal{H}$, where $n = \dim X$. If $\mathcal{H}_-$ is T-invariant we can reduce it to a generalized metric on $\mathcal{H}/T$ as we have seen in Proposition 6.3.1. Moreover, we still have the decomposition

$$\mathcal{H}/T = \mathcal{H}_-/T \oplus \mathcal{H}_+/T.$$

As the T-duality map is an isomorphism of Courant algebroids it respects the orthogonality of the subbundles $\mathcal{H}_-/T$ and $\mathcal{H}_+/T$ so we get a decomposition

$$\tilde{\mathcal{H}}/\tilde{T} = \varphi(\mathcal{H}_+/T) \oplus \varphi(\mathcal{H}_-/T).$$

As the inner product on $\tilde{\mathcal{H}}/\tilde{T}$ has the same signature this decomposition defines a generalized metric which then lifts to a $\tilde{T}$ generalized metric on $\tilde{\mathcal{H}}$ (see Proposition 6.3.1 again). We will denote the generalized metric on $\tilde{X}$ obtained as above by $\varphi(\mathcal{H}_-)$.

The other building block of the equations 9.1.6 is a generalized connection. A T-invariant generalized connection D on $\mathcal{H}$ is one that sends invariant sections to invariant sections, that is

$$D|_{\Gamma(\mathcal{H})^T} : \Gamma(\mathcal{H})^T \to \Gamma(\mathcal{H}^* \otimes \mathcal{H})^T.$$

For invariant generalized connections we have the following lemma.

Lemma 9.2.1. [17] *There is a natural identification between T-invariant generalized connections $\mathcal{D}_{\mathcal{H}}^T$ on $\mathcal{H}$ and generalized connections $\mathcal{D}_{\mathcal{H}/T}$ on $\mathcal{H}/T$. In particular $\mathcal{D}_{\mathcal{H}}^T$ is non-empty.*

Proof. The sections of $\Gamma(\mathcal{H}/T)$ naturally identify with invariant sections $\Gamma(\mathcal{H})^T$ of $\mathcal{H}$. Via this identification both $\mathcal{D}_{\mathcal{H}}^T$ and $\mathcal{D}_{\mathcal{H}/T}$ are affine spaces modelled on

$$\Gamma(\mathcal{H}^* \oplus \mathfrak{so}(\mathcal{H}))^T.$$

Note that $\mathcal{D}_{\mathcal{H}/T}$ is non-empty as we can construct an element as in Remark 7.0.1. Therefore, it is sufficient to show that every connection in $\mathcal{D}_{\mathcal{H}/T}$ induces a T-invariant connection on $\mathcal{H}$, which also proves that $\mathcal{D}_{\mathcal{H}}^T$ is non-empty.

Let now D be an orthogonal $\mathcal{H}/T$-connection on $\mathcal{H}/T$, which defines an orthogonal connection on the T-invariant sections of $\mathcal{H}$. Locally T-invariant sections span $\mathcal{H}$ which we can use to extend D to a differential operator on all of the sections of $\mathcal{H}$. This initially only satisfies the $\pi_{\mathcal{H}/T}$-Leibniz rule for functions on M but we can enforce the π_E-Leibnicz rule for functions on X as part of the definition. From the existence of local T-invariant frames orthogonality follows as well. ∎

For our purposes we have to consider metric-compatible generalized connection. These were classified as connections respecting the decomposition that defines the generalized metric. From this point of view the following proposition is clear.

Proposition 9.2.1. *A T-invariant generalized connection D_0 on $\mathcal{H}$ compatible with the generalized metric $\mathcal{H}_-$ defines a generalized connection D on $\mathcal{H}/T$ compatible with the generalized metric $\mathcal{H}_-/T$.*

Although we defined the Killing spinor equations via a a specific metric connection, it really only depends on the generalized metric and the divergence operator associated to the connection.

A T-invariant divergence operator *div* on $\mathcal{H}$ is one that maps invariant sections to invariant function, that is

$$div|_{\Gamma(\mathcal{H}^T)} : \Gamma(\mathcal{H})^T \cong \Gamma(\mathcal{H}/T) \to C^\infty(X)^T \cong C^\infty(M).$$

Clearly, it defines a divergence operator on $\mathcal{H}/T$. The converse is also true via the same construction as in Lemma 9.2.1 so we have the following statement.

Lemma 9.2.2. *There is a one-to-one correspondence between T-invariant divergence operators on $\mathcal{H}$ and divergence operators on $\mathcal{H}/T$.*

Recall that we characterised soluions to the Killing spinor equation as pairs $(\mathcal{H}_-, e)$ where $\mathcal{H}_-$ is a generalized metric and $e \in \mathcal{H}$ denotes the gauge-fixing, that is

$$div_{D^e} - div_{D^{LC}} = e.$$

By an *invariant solution* we mean a solution $(\mathcal{H}_-, e)$ such that $\mathcal{H}_-$ is an invariant metric and e is an invariant section of $\mathcal{H}$. Note that the canonical Levi-Civita connection of a T-invariant metric is invariant and therefore so is D^e. With our previous considerations we have the following theorem.

Theorem 9.2.1. [17] *Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be T-dual Courant algebroids over the common base M and suppose we have an invariant solution $(\mathcal{H}_-, e)$ of the Killing spinor equations on $\mathcal{H}$. Then $(\varphi(\mathcal{H}_-), \varphi(e))$ is also an invariant solution of the Killing spinor equation on $\tilde{\mathcal{H}}$. Here we regard $\varphi(e)$ as a $\tilde{T}$ invariant section of $\tilde{\mathcal{H}}$.*

Proof. If $(\mathcal{H}_-, e)$ is an invariant solution of the Killing spinor equation the we can take the twisted spinor bundle $\mathcal{S} = \mathcal{S}^R \oplus \mathcal{S}^L$ to be equivariant and the a nowhere vanishing spinor $\eta \in \Gamma(\mathcal{S}^R)^T$ invariant. As the Levi-Civita connection of an invariant metric is invariant so is the connection $D^e(\mathcal{H}_-)$ and the operators D^e_+ and $\not{D}^e_-$. Therefore there is an induced connection $D^e(\mathcal{H}_-)_T$ on $\mathcal{H}/T$ and operators $(D^e_+)_T$ and $(\not{D}^e_-)_T$ on $\Gamma(\mathcal{S}^R/T)$. Clearly the operators still satisfy 9.1.6 on the reduced bundles for the reduced spinor $\eta_T \in \Gamma(\mathcal{S}^R/T)$.

As under the T-duality isomorphism $\mathcal{H}/T$ is mapped to $\tilde{\mathcal{H}}/\tilde{T}$ one can regard $\Gamma(\mathcal{S}_+/T)$ as a spinor bundle for $\tilde{\mathcal{H}}/\tilde{T}$ as well. The connection $D^e(\mathcal{H}_-)$ is natural hence the induced connection $D^e(\mathcal{H}_-)_T$ as well, which is mapped to

$$\varphi(D^e(\mathcal{H}_-)_T) = D^{\varphi(e)}(\varphi(\mathcal{H}_-))_{\tilde{T}}.$$

Then the corresponding operators acting on the common spinor bundle $\mathcal{S}_+/T$ map to each other as well. Clearly then the operators $(D_+^{\varphi(e)})_{\tilde{T}}$ and $(\not{D}_-^{\varphi(e)})_{\tilde{T}}$ satisfy 9.1.6 for the same spinor $\eta_T \in \Gamma(\mathcal{S}^R/T)$.

Pulling back the spinor bundle $\mathcal{S}/T = \mathcal{S}_+/T \oplus \mathcal{S}_-/T$, the spinor η_T and the operators via the $\tilde{T}$-action to the T-dual space $\tilde{X}$ we get a spinor bundle for $\varphi(\mathcal{H}_-)$ and a solution for the Killing-spinor equation characterized by the pair $(\varphi(\mathcal{H}_-), \varphi(e))$. ∎

This theorem is completely natural, the only reason it takes so many words to prove is that there are a lot of different structures and operators involved and we need to justify that they change into each other under the T-duality map.

With this we can try to prove some T-duality result for the Strominger System but there we are faced with a problem. Recall that from Theorem 9.1.2 that the Strominger System is equivalent to the killing spinor equation when the divergence is taken to be an exact 1-form.

So let $e = d\phi$ for some T-invariant function $\phi \in C^\infty(X)^T$, that is ϕ is a pullback function from the base M and therefore $d\phi$ is also a pullback 1-from from M to X. Suppose that there is an invariant solution $(\mathcal{H}_-, d\phi)$ to the Killing spinor equations on X. Then the T-dual solution on $\tilde{X}$, it is of the form $(\varphi(\mathcal{H}_-), \varphi(d\phi))$. Recall from Section 8. that the T-duality isomorphism roughly exchanges the vectors tangent to the torus-fibres with their dual in the cotangent space. Therefore the T-dual of a basic form $d\phi$, which has no components in the fibres is again just the pullback of $d\phi$ to the dual bundle $\tilde{X}$. Consequently when we transfer solutions of the Strominger System naively as in 9.2.1 we find that

$$\varphi : (\mathcal{H}_-, d\phi) \to (\varphi(\mathcal{H}_-), d\phi).$$

On the other hand, in physics literature the rules of T-duality are fully described and known as the Buscher-rules [6, 36] which includes the transformation of the dilaton field ϕ called the *dilaton shift*. It is given by

$$\tilde{\phi} = \phi - \log\Big(\frac{\det g_T}{\det \tilde{g}_{\tilde{T}}}\Big),$$

where g_T and $\tilde{g}_{\tilde{T}}$ are the T-dual Riemannian metrics restricted to the dual torus fibres. That is, the dilaton shift measures the change in the volume of the fibres.

In our calculations we have found no dilaton shift but there is an explanation to it. Recall that in Theorem 9.1.2 one chooses a spin bundle $\mathcal{S}$ on which both the generalized Killing spinor equations and the spinorial Strominger System is formulated. This spin bundle $\mathcal{S}$ is derived solely from the geometry of the underlying manifold with a specific construction (see the proof of Theorem 9.1.1 in [27, 38]). Then, when one transports the Killing spinor equations under T-duality in Theorem 9.2.1 one uses the same spinor bundle $\mathcal{S}$ to formulate the T-dual generalized Killing spinor equations on $\tilde{X}$. On the other hand however, the spinorial form of the Strominger System is formulated on a different spin bundle $\tilde{\mathcal{S}}$ which is derived from the spin manifold $\tilde{X}$. Therefore to find the proper solution to the Strominger System one has to specify a morphism of spin bundles

$$\tau : \mathcal{S} \to \tilde{\mathcal{S}},$$

that respects the Clifford action. Spin bundles corresponding to the same bundle of Clifford algebras are actually related by tensoring one of them with a line bundle, i.e. there exist a line bundle $\mathcal{L}$ on $\tilde{X}$ such that

$$\tilde{\mathcal{S}} = \mathcal{S} \otimes \mathcal{L}.$$

From this we can see that the isomorphism is only going to affect the action by a scaling factor. This is precisely the missing dilaton shift.

Recall also that in Theorem 8.2.2 we saw the map

$$\tau : (\Omega^\bullet(X), d_H) \to (\Omega^\bullet(\tilde{X}), d_{\tilde{H}})$$

which was an isomorphism of the twisted differential complexes. This map is precisely what one needs to describe the dilaton shift whenever the differential forms can act as a spinor bundle for the generalized metric. This is the case for exact Courant algebroids, therefore the case of the dilaton shift is clear there. Unfortunately so far there has not been a description like that for the heterotic case which leaves the question of the dilatons shift somewhat unexplained.

In [17] Garcia-Fernandez proposed the following explanation. If there is a solution of the Strominger System $(\mathcal{H}_-, d\phi)$ on the heterotic Courant algebroid $\mathcal{H}$ then one can try to first pull it back to the exact Courant algebroid E which it has been reduced from. In Proposition 6.3.2 we saw that $\mathcal{H}_-$ lifts uniquely to a generalized metric in E and $d\phi$ also pulls back. This pulled back system $(\mathcal{H}_-^E, d\phi)$ actually satisfies the analogue of the Killing spinor equations on the exact courant algebroid E (taking the exact canonical Levi-Civita as a reference connection). Then one can T-dualize as in Theorem 9.2.1 but incorporating the isomorphism of the spinor bundles τ. Then he found that the T-dual solution is consistent with the dilaton shift. From here he pushes forward the T-dual metric to the T-dual heterotic Courant algebroid $\tilde{\mathcal{H}}$ together with the T-dual dilaton (which is still basic on the common base). He argues that this process is the same as T-dualizing the heterotic Killing spinor equations as in Theorem 9.2.1 plus twisting the spinor bundle. This point of view is promising but it does not explain how generalized connections can be reduced or lifted therefore the proof seems incomplete.

Nonetheless, when the metrics have constant volume on the fibres the Buscher rules result in $d\tilde{\phi} = d\phi$ and one can freely use the construction in Theorem 9.2.1. This has been used in [19] recently to construct T-dual examples of solutions on non-Kählerian torus bundles over K3 surfaces.

There has been another approach to this problem. In [2] it has been shown that the equations of motion of heterotic string theory are exchanged by T-duality. Solutions are characterized by a generalized metric and a function again but in this case one has to plug in the dilaton shift by hand as it is not part of the geometric formulation. Moreover, this approach does not take supersymmetry into account.

In conclusion, generalized geometry seems like a good tool to mathematically approach heterotic String theory and the Strominger System but there are still unexplored problems that would be interesting to look into.

THE END!

10 Appendix

In this section we include some facts and calculations about the de Rham cohomology of certain Lie groups and principal bundles which we used in Section 8 and Section 5.

10.1 Cohomology of compact Lie groups

Firstly, we define the cohomology of a Lie algebra $\mathfrak{g}$ and the state two important theorems about the de Rham cohomology of compact connected Lie groups. Both of these can be used to get an insight into the cohomology groups especially at lower degrees. These facts are classical, we refer to [14] for the proofs.

Definition 10.1.1. If $\mathfrak{g}$ is a real Lie algebra, its *Lie algebra cohomology ring* is defined as the cohomology of the following complex. The degree k elements are

$$C^k = \wedge^k \mathfrak{g}^*,$$

where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$, and the differential is

$$\begin{aligned} \delta :\ & C^k \to C^{k+1} \\ \delta\alpha(\xi_0, ..., \xi_k) &= \sum_{i<j} \alpha([\xi_i, \xi_j], \xi_0, ..., \hat{\xi_i}, ..., \hat{\xi_j}, ..., \xi_k) \end{aligned}$$

with $\xi_0, ..\xi_k \in \mathfrak{g}$. We denote the complex by $(\wedge^\bullet \mathfrak{g}^*, \delta)$.

Theorem 10.1.1. *Let G be a compact connected Lie group and $\mathfrak{g}$ its Lie algebra. Then the de Rham cohomology of G is isomorphic to the Lie algebra cohomology of $\mathfrak{g}$*

$$H^\bullet(G, \mathbb{R}) \cong H^\bullet(\wedge^\bullet \mathfrak{g}^*, \delta).$$

Rough sketch of proof. By averaging any cohomology class of G can be represented by left invariant elements. Then one can show that the cohomology can be calculated using only left invariant differential forms evaluated on left invariant vector fields. Finally, the exterior differential reduces to δ on these forms. ∎

Theorem 10.1.2. *Let G be a compact connected Lie group and $\mathfrak{g}$ its Lie algebra. Then the de Rham cohomology ring of G is an exterior algebra on odd generators given by*

$$\begin{aligned} H^\bullet(G, \mathbb{R}) &= \Big(\wedge^\bullet \mathfrak{g}^* \Big)^G \\ &= \bigoplus_{k=0}^{m} \Big\{ \alpha \in \wedge^k \mathfrak{g}^* \ \Big|\ \sum_{i=1}^{k} \alpha(\xi_1, ..., [\xi, \xi_i], ..., \xi_k = 0 \ \forall \xi, \xi_1, ..., \xi_k \in \mathfrak{g} \Big\}. \end{aligned}$$

Rough sketch of proof. One can show that the calculation of the cohomology ring can be done restricting to bi-invariant forms as well. These are represented by the above exterior algebra and they are automatically closed which finishes the proof. ∎

Example 1. The cohomology ring of a k-dimensional torus $\mathbb{T}^k$ is just the exterior algebra of the dual $\mathfrak{t}^{k*}$ of its Lie algebra $\mathfrak{t}^k$.

$$H^\bullet(\mathbb{T}^k, \mathbb{R}) = \wedge^\bullet \mathfrak{t}^{k*}$$

It is clear from both of the theorems above as $[\mathfrak{t}, \mathfrak{t}] = 0$, therefore every element in $\wedge^\bullet \mathfrak{t}^{k*}$ is closed and none are exact.

Recall, that a Lie algebra $\mathfrak{g}$ is called *simple*, if it is not abelian and its only ideals are 0 and $\mathfrak{g}$. We call $\mathfrak{g}$ *semisimple* if it is a direct sum of simples. A Lie group is called *simple* (*semisimple*) if its corresponding Lie algebra is simple (semisimple).

Example 2. (The cohomology of compact connected semisimple Lie groups) Let G be a compact, connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Then the first 3 cohomology groups of G are

$$H^0(G, \mathbb{R}) = \mathbb{R}, \quad H^1(G, \mathbb{R}) = H^2(G, \mathbb{R}) = 0, \quad H^3(G, \mathbb{R}) \neq 0.$$

The zeroth cohomology group is clear as G is connected. Suppose now that $\alpha \in (\mathfrak{g}^*)^G$ as in Theorem 10.1.2, therefore

$$\alpha([\xi_1, \xi_2]) = 0 \quad \forall \xi_1, \xi_2 \in \mathfrak{g}$$

but as G is semisimple $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ therefore $\alpha = 0$.
If $\beta \in (\wedge^2 \mathfrak{g}^*)^G$ then

$$\beta([\xi_1, \xi_2], \xi_3) + \beta(\xi_2, [\xi_1, \xi_3]) = 0 \quad \forall \xi_1, \xi_2, \xi_3 \in \mathfrak{g}.$$

and as β is closed

$$-\beta([\xi_1, \xi_2], \xi_3) + \beta([\xi_1, \xi_3], \xi_2) - \beta([\xi_2, \xi_3], \xi_2) = 0 \quad \forall \xi_1, \xi_2, \xi_3 \in \mathfrak{g}.$$

Together these yield

$$\beta([\xi_2, \xi_3], \xi_2) = 0 \quad \forall \xi_1, \xi_2, \xi_3 \in \mathfrak{g}$$

i.e. $\beta = 0$ again.
Finally, for degree 3 we can consider a non-degenerate invariant bilinear form c on $\mathfrak{g}$ (eg. the Killing form), then

$$\gamma(\xi_1, \xi_2, \xi_3) = c([\xi_1, \xi_2], \xi_3) \in (\wedge^3 \mathfrak{g}^*)^G$$

and as c is non-degenerate γ is not zero.

10.2 Exact basic forms

Let $P \to M$ be a principal bundle with compact connected semi-simple structure group G. In general, an invariant exact form $d\beta$ on P may be basic without the form β being basic. An example of this is the Chern-Simons 3-form whose exterior differential is basic but $CS_3(A)$ restricts to a non-trivial cohomology class on the fibres, hence not basic. We will show that this is not the case for 1 and 2-forms, exact elements of $\Omega^k(P,\mathbb{R})_{bas}$ for $k = 1, 2$ are the exterior derivatives of basic forms. This calculation is essentially the same as the computation for the cohomology of Lie groups but it is still useful to point out as it is a unique feature of our setting that we use often (via Proposition 5.1.2).

Proposition 10.2.1. *Let $B \in \Omega^2(P)$ and $C \in \Omega^1(P)$ be invariant forms. If their exterior differential dB and dC are basic forms then so are B and C.*

Proof. Let $A = A^i e_i$ be a principal connection on P written in a basis $e_1, ..., e_m$ of $\mathfrak{g}$, the semisimple Lie algebra corresponding to G. Let f^i_{jk} be the structure constants of $\mathfrak{g}$.

Firstly, write $C = C^0 + C^1_i A^i$ using the connection with C^0 and C^1_i basic. Then the exterior differential is

$$dC = dC^0 + dC^1_i \wedge A^i + C^1_i F^i - C^1_i f^i_{jk} A^{jk},$$

where we used that $F = dA = \frac{1}{2}[A, A]$ and $A^{ij} = \frac{1}{2} A^i \wedge A^j$. Clearly, dC is basic if

$$dC^i = 0 \ \forall i \ \text{ and } \ C^1_i f^i_{jk} = 0 \ \ \forall j, k.$$

Contracting the second condition with any elements in $x, y \in \mathfrak{g}$ we find that $C^1_i [x, y]^i = 0$. As x, y are arbitrary and $\mathfrak{g}$ is semisimple it is equivalent to $C^1_i x^i = 0$ for all $x \in \mathfrak{g}$, i.e. C^1 is essentially an element in $\mathfrak{g}^*$ that annihilates all of $\mathfrak{g}$ hence it must vanish which proves the claim.

Similarly, write $B = B^0 + B^1_i A^i + B^2_{ij} A^{ij}$ (omitting wedge product). The exterior derivative is then

$$dB = dB^0 + dB^1_i A^i - B^1_i F^i + B^1_i f^i_{jk} A^{jk} + dB^2_{ij} A^{ij} + 2B^2_{ij} dF^i A^j - 2B^2_{ij} f^i_{kl} A^{jkl},$$

which is basic whenever

$$dB^1_i = B^2_{ji} F^i, \ \ B^1_i f^i_{jk} = dB^2_{jk}, \ \ B^2_{ij} f^i_{kl} = 0$$

for all i, j, k. The third condition is again for a set j the same as in the case of the one form C which yield $B^2_{ij} = 0$ for all i, j. This turns the second condition into the previous form again, and we find that $B^1_i = 0$ as well for all i. ∎

10.3 On the second cohomology

We want to calculate the the second cohomology group of a principal bundle $G \cdots P \to M$ with coefficients in $H^1(\mathbb{T}^k, \mathbb{R}) = \mathfrak{t}^{k*} \cong \mathbb{R}^k$. The structure group is again compact, connected, semisimple. In case of real coefficients, by the universal coefficient theorem

$$H^1(G, \mathfrak{t}^{k*}) = H^2(G, \mathfrak{t}^{k*}) = 0.$$

Therefore, the second page of the Serre spectral sequence is as follows.

0	0	0	0
0	0	0	0
$H^0(G, \mathfrak{t}^{k*}) = \mathfrak{t}^{k*}$	$\mathfrak{t}^{k*}$	$H^1(M, \mathfrak{t}^{k*})$	$H^2(M, \mathfrak{t}^{k*})$
	$H^0(M, \mathbb{R})$	$H^1(M, \mathbb{R})$	$H^2(M, \mathbb{R})$

In this case all the relevant differentials are already zero and we find that the second cohomology group is just the second cohomology of the base

$$H^2(P, \mathfrak{t}^{k*}) = H^2(M, \mathfrak{t}^{k*}).$$

Popular summary

String theory has been in the focus of theoretical physics research since the 1960s but the precise mathematical formulation of these theories is still not complete. In the early 2000s Nigel Hitchin proposed a new geometric approach called generalized geometry. It formulated Type II string theory on geometric objects called exact Courant algebroids which are a specific type of the wider class of Courant algebroids. Since 2010 generalized geometry has been extended to a different class of Courant algebroids which have found applications to heterotic string theory. In this thesis we explore these new advancements.

We first define Courant agebroids and present three examples including exact Courant algebroids. We then describe a reduction procedure which relates our three examples to each other. This relation brings the first important result. It turns out that a type of Courant algebroids, which were suggestively named heterotic Courant algebroids, can be completely classified as objects that arise via reduction of exact Courant algebroids.

In the coming chapters we investigate the geometry of Courant algebroids. Remarkably, it is possible to describe an analogue of differential calculus on Courant algebroids although it is not as clean as in the standard case. Nevertheless, it provides an useful tool in geometrizing the partial differential equations arising in string theory.

After that we turn our attention to T-duality which is one of the most important dualities in string theory. It relates different Type II theories to each other and heterotic theories to each other. First we explain how this duality is formalized mathematically on torus bundles. Then we present its reformulation into the setting of generalized geometry for exact Courant algebroids which has been applied to Type II string theory. Finally, we describe how T-duality has been extended to the heterotic case as an isomorphism of heterotic Courant algebroids.

We conclude with describing the Strominger System, the defining equations of heterotic string theory. We prove that the system naturally arises as a pair of equations on a heterotic Courant algebroid which is one of the main advancements of research in this area. Not only does this new formulation provide a cleaner framework for the study of the Strominger System but in this setting T-duality is also fairly simple to describe. We present T-duality in this new setup and discuss some unanswered questions that are still under research.

Bibliography

[1] A. Alekseev and P. Xu. *Derived brackets and courant algebroids.* Unpublished, available at http://www.math.psu.edu/ping/anton-final.pdf (2001)

[2] D. Baraglia, P. Hekmati. *Transitive Courant Algebroids, String Structures and T-duality.* Adv. Theor. Math. Phys. Vol 19 No. 3, 613-672 (2015) arXiv:1308.5159v2 [math.DG]

[3] P. Bouwknegt, J. Evslin, V. Mathai. *T -duality: topology change from H-flux.* Comm. Math. Phys., 249(2):383–415, (2004) arXiv:hep-th/0306062.

[4] P. Bouwknegt, K. Hannabuss, V. Mathai. *T-duality for principal torus bundles.* J. High Energy Phys., 3:018, 10 pp. (electronic) (2004) arXiv:hep-th/0312284.

[5] H. Bursztyn, G. R. Cavalcanti, M. Gualtieri. *Reduction of Courant algebroids and generalized complex structures.* Adv. Math., 211 (2), 726-765. (2007) arXiv:math/0509640v3 [math.DG]

[6] T. Buscher. *A symmetry of the string background field equations.* Phys. Lett. B 194 59. (1987) (doi:10.1016/0370-2693(87)90769-6)

[7] G. R. Cavalcanti, M. Gualtieri. *Generalized complex geometry and T-duality.* A Celebration of the Mathematical Legacy of Raoul Bott (CRM Proceedings & Lecture Notes), American Mathematical Society, pp. 341-366. ISBN: 0821847775 (2010) arXiv:1106.1747v1 [math.DG]

[8] Z. Chen, M. Stienon, P. Xu. *On Regular Courant Algebroids.* Journal of Symplectic Geometry 11, no. 1, 1-24. (2013) arXiv:0909.0319 [math.DG]

[9] T. Courant. *Dirac manifolds.* Trans. Amer. Math. Soc. 319, 631–661. (1990)

[10] I.Ya. Dorfman. *Dirac structures of integrable evolution equations.* Phys. Lett. A 125, 240–246. (1987)

[11] M. Fernández, S. Ivanov, L. Ugarte, R. Villacampa. *Non-Kähler heterotic-string compactifications with non-zero fluxes and constant dilaton.* Commun. Math. Phys. 288 677–697. (2009) arXiv:0804.1648v4 [math.DG]

[12] J. Fine, D Panov. *Symplectic Calabi-Yau manifolds, minimal surfaces and the hyperbolic geometry of the conifold.* Journal of Differential Geometry, 82 , no. 1, 155–205. (2009) arXiv:0802.3648v2 [math.SG]

[13] A. Fino, G. Grantcharov, L. Vezzoni. *Solutions to the Hull-Strominger system with torus symmetry.* (2019) arXiv:1901.10322v2 [math.DG]

[14] C.-K. Fok. *Cohomology and K-theory of compact Lie groups.* Unpublished, available at http://pi.math.cornell.edu/ ckfok/Cohomology_Lie_groups.pdf (2010)

[15] M. Garcia-Fernandez. *Torsion-free generalized connections and Heterotic Supergravity.* (2013) arXiv:1304.4294 [math.DG]

[16] M. Garcia-Fernandez. *Lectures on the Strominger System.* Travaux mathématiques, Vol. XXIV 7–61. (2016) arXiv:1609.02615v2 [math.DG]

[17] M. Garcia-Fernandez. *Ricci flow, Killing spinors, and T-duality in generalized geometry.* to appear in Advances in Mathematics. (2016) arXiv:1611.08926v3 [math.DG]

[18] M. Garcia-Fernandez, R. Rubio, C. Tipler. *Infinitesimal moduli for the Strominger System and Killing spinors in generalized geometry.* Math. Ann. 369, no. 1-2, 539-595. (2017) arXiv:1503.07562v2 [math.DG]

[19] M. Garcia-Fernandez. *T-dual solutions of the Hull-Strominger System on non-Kähler threefolds.* to appear in J. Reine Angew. Math. (2018) arXiv:1810.04740v3 [math.DG]

[20] V. L. Ginzburg. *Equivariant Poisson Cohomology and a Spectral Sequence Associated with a Moment Map.* (1996) arXiv:dg-ga/9611002v2

[21] J. Grabowski, A. Kotov, N. Poncin. *Geometric structures encoded in the Lie structure of an Atiyah algebroid.* (2009) arXiv:0905.1226v1 [math.DG]

[22] M. Gualtieri. *Generalized complex geometry.* Oxford University DPhil thesis (2004) arXiv:math/0401221v1 [math.DG]

[23] M. Gualtieri. *Generalized complex geometry.* (2007) arXiv:math/0703298v2 [math.DG]

[24] M. Gualtieri. *Branes on Poisson varieties* The many facets of geometry, 368–394, Oxford Univ. Press, ISBN: 9780199534920 (2010) arXiv:0710.2719v2 [math.DG]

[25] A. Hatcher. *Algebraic topology.* Cambridge University Press, ISBN: 978-0521795401 (2002)

[26] N. Hitchin. *Generalized Calabi-Yau manifolds.* Quart.J.Math.Oxford Ser.54:281-308. (2003) arXiv:math/0209099v1 [math.DG]

[27] C. Hull. *Superstring compactifications with torsion and space-time supersymmetry.* In Turin 1985 Proceedings "Superunification and Extra Dimensions" 347–375. (1986)

[28] S. Ivanov. *Heterotic supersymmetry, anomaly cancellation and equations of motion.* Phys. Lett. B 685 (2-3) 190–196. (2010)

[29] S. Kobayashi, K. Nomizu. *Foundations of Differential geometry Vol II.* Interscience Publishers, ISBN: 0471157325 (1969)

[30] Y. Kosmann-Schwarzbach. *Derived brackets.* Lett. Math. Phys. 69 , 61–87. (2004) arXiv:math/0312524v1 [math.DG]

[31] J. Li, S.-T. Yau. *The existence of supersymmetric string theory with torsion.* J. Diff. Geom. 70 143–181, (2005) arXiv: 0411136 [hep-th].

[32] Z.-J. Liu, A. Weinstein, and P. Xu. *Manin triples for Lie bialgebroids.* J. Differential Geom., 45(3):547–574. (1997) arXiv:dg-ga/9508013v3

[33] K. Mackenzie. *Lie Groupoids and Lie Algebroids in Differential Geometry.* (London Mathematical Society Lecture Note Series) Cambridge University Press, ISBN: 052134882X (1987)

[34] C.-M. Marle. *Differential calculus on a Lie algebroid and Poisson manifolds.* (2008) arXiv:0804.2451v2 [math.DG]

[35] R. Rubio. *Generalized geometry of type B_n.* Oxford University DPhil thesis (2014) https://people.maths.ox.ac.uk/hitchin/files/StudentsTheses/rubio.pdf

[36] M. Serone, M. Trapletti. *A Note on T-duality in Heterotic String Theory.* Phys.Lett.B637:331-337. (2006) arXiv:hep-th/0512272v2

[37] P. Ševera, A. Weinstein. *Poisson geometry with a 3- form background.* Prog.Theor.Phys.Suppl. 144, 145-154. (2001) arXiv:math/0107133v2 [math.SG]

[38] A. Strominger. *Superstrings with torsion.* Nucl. Phys. B 274 (2) 253–284. (1986)