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The Yang-Mills Moduli Space and The Nahm Transform

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Abstract

In this thesis we will discuss gauge theories and instantons. We will start with discussing the necessary analysis to define the Yang-Mills moduli space. This includes Sobolev spaces, Fredholm transversality and elliptic differential operators. The Nahm transform over the four-torus is defined and proved to be an involution. Finally we will discuss the ADHM construction.

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Chapter 1

Introduction

In gauge theory one studies bundles with connections and the curvature of these connections. Of particular importance are the (anti)-self-dual connections, called instantons. These were introduced by physicists, as they are classical solutions of the equations of motion of the classic field theory on Euclidean spacetime. In the late 1970s, instantons on smooth four-manifolds were more rigorously studied by geometers.

The first major result about instantons was the ADHM construction of $SU(n)$-instantons on $\mathbb{R}^4$. In [2] Atiyah, Drinfeld, Hitchin and Manin discovered a way to construct all $SU(n)$ anti-self-dual connections on S^4. They discovered algebraic equations, called the ADHM equations, whose solutions are in 1-1 correspondence between gauge equivalence classes of irreducible anti-self-dual connections over S^4 for the structure group $SU(n)$. This construction was extended in [23] by Nahm to time-invariant instantons, called monopoles. Nahm discovered a correspondence between instantons on $\mathbb{R}^4$ which are translation invariant in one direction and instantons which are translation invariant in three directions. In [4] Braam and van Baal applied this idea to study $U(n)$ anti-self-dual connections on flat four-tori. They found a transformation which maps anti-self-dual connections on a flat four-torus to anti-self-dual connections on the dual torus. Applying the same transformation gives the identity. For $SU(n)$-connections this transformation interchanges the second Chern class and rank of the bundle.

Later it became clear that these constructions are special examples of a wider framework. One has a Nahm transformation for instantons on $\mathbb{R}^4$ which are invariant under some translation group $\Lambda \subset \mathbb{R}^4$. The simplest case is probably when the translation group is a maximal lattice. This gives the Nahm transform over the torus. If the manifold $\mathbb{R}^4/\Lambda$ is non-compact then it is in general a lot harder to find the Nahm transformed data and proving that the Nahm transform has an inverse. Other cases for which the Nahm transform is well understood are: $\Lambda = \mathbb{Z}$ (calorons), $\Lambda = \mathbb{Z}^2$ (doubly periodic instantons), $\Lambda = \mathbb{Z}^4$ (flat four-torus). A complete description of $\Lambda = \mathbb{Z}^3$ (spatially periodic instantons) is still missing. The proof for the calorons was only completed in 2007 in [5]. A neat exposition of this and more can be found in [13]. The Nahm transform for periodic instantons can be generalized to hyperkähler manifolds, this is done in [3]. The Nahm transform also generalizes to the much larger class of spin four-manifolds of non-negative scalar curvature.

Often there is more geometric structure present on the moduli spaces. A natural question to ask is whether the Nahm transform preserves this extra structure. For example, if the original manifold X is hyperkähler then the moduli space of anti-self-dual connections on X is also hyperkähler. It is known for the ADHM construction and the Nahm transform over the flat four-torus that the Nahm transform is a hyperkähler isomorphism, see [4, 17].

In this thesis we will study the local structure of the Yang-Mills moduli space and the Nahm transform. These require to first give an introduction to Banach manifolds, Fredholm transversality theory, infinite-dimensional vector bundles and anti-self-dual connections. We will prove a slice theorem for the moduli space of gauge equivalence classes of connections and show how the Yang-Mills moduli space is generically a smooth finite-dimensional manifold. Then we introduce the Nahm transform over a flat Riemannian four-torus. The Nahm transform is then described in terms of holomorphic geometry. A proof that this transformation is an involution is included. Finally we will discuss the ADHM construction and show how to construct from the algebraic ADHM data an instanton and vice versa.

This is only a tip of a multifaceted mathematical theory which changed the way people approach smooth four-manifold topology. It was started by Donaldson who proved Donaldson's theorem, which states that a positive definite intersection form of a simply connected smooth four-manifold is diagonalizable over the integers to the identity matrix. It is a difficult result by Freedman that for every even form there is exactly one topological manifold having it as intersection form, while for every odd form there are exactly two. It follows that a non-diagonalizable intersection form gives rise to a four-manifold which does not allow a differentiable structure.

This led to the existence of exotic $\mathbb{R}^4$'s. In fact there is a remarkable result by Taubes and Gompf: There is an uncountable collection of exotic $\mathbb{R}^4$'s. For closed manifolds there can only be countably-many distinct smooth structures. There are no smooth four-manifolds known to have only finitely-many differentiable structures.

All this is much in contrast to dimensions unequal to four. For simply connected manifolds of dimension at least five, the homotopy type and characteristic classes are enough to determine the manifold up to finitely many diffeomorphism possibilities.

Gauge theory can also be used to produce new algebraic invariants of smooth four-manifolds, e.g., the Donaldson invariants and the Seiberg-Witten invariants. These invariants have no analogue in dimensions other then four.

Chapter 2

Analysis of the Yang-Mills moduli space

In this chapter we want to give a local description of the Yang-Mills moduli space. While we will see that for generic choices this space is a smooth finite-dimensional manifold, it arises as the reduction of an infinite-dimensional manifold (the space of all connections on a vector bundle) by the action of an infinite-dimensional group (the gauge group). Therefore in order to describe this space we first must delve into the theory of infinite-dimensional manifolds, which in our context will be Banach manifolds. The fact that the Yang-Mills moduli space is finite-dimensional, will follow from elliptic theory.

Good references for the first section are [15, 14].

2.1 Banach Manifolds

Banach manifolds are a generalization of manifolds to allow for infinite dimensions. To define differentiable Banach manifolds we first need a notion of differentiation of a map between Banach spaces.

Definition 2.1. Let E and F be Banach spaces, and $U \subset E$ an open subset. A function $f : U \to F$ is *differentiable* at $x \in U$ if there exists a bounded linear operator $A_x : E \to F$ such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - A_x(h)\|_F}{\|h\|_E} = 0.$$

If f is differentiable at x then we define its derivative as $(Df)_x := A_x$. The function f is said to be differentiable if it is differentiable at every point

$x \in U$. The derivative is a function

$$Df : U \to B(E,F);\ x \mapsto (Df)_x.$$

The function is C^1 if it is differentiable and if Df is continuous, where $B(E,F)$ is the Banach space of bounded linear maps with the operator norm.

Now we can inductively define r-times differentiable maps and also smooth maps. We will denote the set of r-times continuously differentiable maps on U by $C^r(U)$ and smooth maps by $C^\infty(U)$. For the remainder of this thesis we will mostly work with smooth maps. It is an easy exercise to show that the composition of two C^r maps is again a C^r map.

If the Banach spaces E and F are complex then a smooth map f is holomorphic if its derivative $(Df)_x$ is complex linear for every x.

Definition 2.2. A family of pairs (called charts) $\{U_i, \varphi_i\}_{i\in I}$ is a $C^{r\geq 0}$ *atlas* of X if

(i) $U_i \subset X$ and the U_i cover X.

(ii) each φ_i is a bijection from U_i to an open subset of some separable Banach space E_i, and for any i and j, $\varphi_i(U_i \cap U_j)$ is open in E_i.

(iii) the transition functions

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are r-times continously differentiable.

A new chart (U, φ) is called compatible with an atlas $\{U_i, \varphi_i\}_{i\in I}$ if for every $i \in I$ the transition function

$$\varphi_i \circ \varphi^{-1} : \varphi(U \cap U_i) \to \varphi_i(U \cap U_i)$$

is C^r. Two atlases are called compatible if for every chart of one atlas is compatible with all charts from the other atlas. This defines an equivalence relation on the set of all atlases on X.

Definition 2.3. A C^r*-Banach manifold* X is a second-countable Hausdorff space with an equivalence class of C^r atlases on X.

Remark 2.4. Over a connected component of X the Banach spaces E_i are isomorphic to some fixed E. If X is connected then we will say that X

is modeled on E. If in the above definition all the Banach spaces E_i are Hilbert spaces then X is called a Hilbert manifold. If the Banach spaces E_i are complex and all the transition functions are holomorphic then X is a complex manifold.

We will only be dealing with smooth Banach manifolds, so when we say that X is a Banach manifold we mean that X is a smooth Banach manifold. The following two propositions are straightforward generalizations from the finite-dimensional setting.

Proposition 2.5. (Inverse function theorem) *Let E, F be Banach spaces and $U \subset E$ an open subset. Suppose that $f : U \to F$ is a smooth map and at a point $x_0 \in U$ the derivative $(Df)_{x_0} : E \to F$ is an isomorphism. Then there is an open neighbourhood V of x_0, which is contained in U, such that the restriction of f to V is a diffeomorphism on its image.*

Proposition 2.6. (Implicit function theorem) *Let U, V be open sets of Banach spaces E_1 and E_2 respectively, and let $f : U \times V \to F$ be a smooth map of Banach spaces. Suppose the partial derivative with respect to the second variable at a point $(a, b) \in U \times V$,*

$$(D_2 f)(a, b) : E_2 \to F,$$

is an isomorphism. Then there is a smooth map

$$h : U_0 \to V,$$

defined on an open neighbourhood $U_0 \subset U$ of a such that for all $x \in U_0$

$$f(x, h(x)) = f(a, b)$$

and $h(a) = b$. Moreover if (x_1, x_2) is a solution close enough to (a, b) of $f(x_1, x_2) = f(a, b)$ then $x_2 = h(x_1)$.

Let $f : X \to Y$ be a map between Banach manifolds. The map is called smooth if for all $x \in X$ there is a chart (U, φ) with $x \in U$ and a chart (V, ψ) with $f(x) \in V$ such that $f(U) \subset V$ and $\psi \circ f \circ \varphi^{-1}$ is smooth.

A smooth map $f : X \to Y$ is said to be Fredholm if for all $x \in X$ the derivative $(Df)_x$ is a Fredholm map. The index is a deformation invariant, hence if X is connected then the index of $(Df)_x$ is independent of x and is called the index of f.

The following proposition decomposes a Fredholm map in a linear part and a finite-dimensional non-linear part. This form allows us to generalize some transversality theory to infinite dimensions.

Proposition 2.7. *Let X and Y be Banach manifolds modeled on E and F, respectively. Let $f : X \to Y$ be a smooth Fredholm map. There exist charts centered at $x \in U$ and $f(x) \in V$ such that*

$$\begin{array}{rcl} U & \hookrightarrow & E_0 \oplus K, \\ V & \hookrightarrow & E_0 \oplus C, \end{array}$$

where K and C are finite-dimensional and in which f takes the form

$$f(e,k) = (e, l(e,k)).$$

Moreover, $\dim(K) - \dim(C) = \operatorname{ind}(f)$ *and the derivative of l vanishes at* 0*.*

Proof. Pick charts centered at $x \in U$ and $f(x) \in V$. The derivative of f at 0 gives us decompositions $E = E_0 \oplus K$ and $F = E_0 \oplus C$, with $K \cong \ker((Df)_0)$ and $C \cong \operatorname{coker}((Df)_0)$. We know that

$$(Df)_0 = \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix},$$

for some linear isomorphism L. Now we write $f(e,k) = (\alpha(e,k), \beta(e,k))$, for some $\alpha : E_0 \oplus K \to E_0$ and $\beta : E_0 \oplus K \to C$. We define the map $h : E_0 \oplus K \to E_0 \oplus K$ by $h(e,k) = (\alpha(e,k), k)$. By the inverse function theorem we know that h is invertible in a neighbourhood around 0 and that h^{-1} is of the form $h^{-1}(e,k) = (\gamma(e,k), k)$. So we have that

$$(f \circ h^{-1})(e,k) = (e, l(e,k)),$$

where $l(e,k) = \beta(\gamma(e,k), k)$. The chain rule tells us that $D(f \circ h^{-1})_0 = (Df)_0 \circ (Dh^{-1})_0$, hence the derivative of l has to vanish as 0. □

2.1.1 Transversality theory

We begin with an extension of Sard's theorem, due to Smale. A second category set of a Banach manifold X is an analogue for a generic set. A set $S \subset X$ is a second category set if it contains a countable intersection of open dense sets. The Baire category theorem states that every second category subset of a complete metric space is dense. Since Banach manifolds are regular spaces and by definition they are second countable we can apply Urysohn's lemma to see that Banach manifolds are complete metric spaces. Remember that the regular values of a function $f : X \to Y$ are the points $y \in Y$ such that $(Df)_x$ is surjective for all $x \in f^{-1}(y)$. If $(Df)_x$ is surjective then x is called a regular point.

Proposition 2.8. *Let $f : X \to Y$ be a smooth Fredholm map between Banach manifolds. The regular values of f are a second category set.*

Proof. First note that the function f is locally closed. Fix charts as in Proposition 2.7. If given a bounded sequence (e_n, k_n) in $E_0 \oplus K$ such that $f(e_n, k_n) \to (e, c)$. Then $e_n \to e$ and k_n is a bounded sequence in a finite-dimensional vector space and thus has a convergent subsequence. This implies that $f(e, k) = (e, c)$ and thus that f is locally closed. We pick a countable subcover of charts as in Proposition 2.7. This is possible because X is second countable and thus in particular Lindelöf. Denote the regular values of $f|_U$ by $R(U)$. It is enough to show that the sets $R(U)$ are open and dense. Since the regular values of f are the points in the countable intersection $\bigcap_U R(U)$. The regular points of $f|_U$ form an open set. Since f is locally closed we get that $R(U)$ is an open set. From Proposition 2.7 we get that

$$(Df)_{(e,k)} = \begin{pmatrix} I & 0 \\ (D_1 l)_{(e,k)} & (D_2 l)_{(e,k)} \end{pmatrix}.$$

Hence $(Df)_{(e,k)}$ is surjective if and only if $(D_2 l)_{(e,k)}$ is surjective. The map $l(e, -) : K \to C$ is a map between finite-dimensional vector spaces and $(D_2 l)_{(e,k)} = \mathrm{d}_k l(e, -)$. The ordinary Sard's theorem tells us that the regular values of $l(e, k)$ are dense. This means that $R(U) \cap \{e\} \times C \subset \{e\} \times C$ is dense. So $R(U)$ is dense in $E_0 \oplus C$. □

For a regular value y of f we have that $f^{-1}(y)$ is a smooth submanifold of dimension $\mathrm{ind}(f)$.

Now we will state a Fredholm transversality theorem which will be used later on to show that the moduli space of irreducible ASD connections is generically a smooth manifold.

Definition 2.9. Let $f : M \to Y$ be a smooth map between Banach manifolds and $Q \subset Y$ a submanifold. The map f is *transverse* to Q if

$$\mathrm{im}(Df)_x + T_q Q = T_q Y, \qquad \forall x \in f^{-1}(q),\ \forall q \in Q,$$

where $T_q Q$ is the tangent space to Q at q and $T_q Y$ is the tangent space of Y at q. We will write this as $f \pitchfork Q$.

If $f \pitchfork Q$ then $f^{-1}(Q)$ is a submanifold of M. In the following $f : X \times S \to Y$ will be a smooth map between Banach manifolds, which is transverse to some submanifold $Q \subset Y$. We have that $f^{-1}(Q)$ is a submanifold of $X \times S$ with a natural projection map $\pi : f^{-1}(Q) \to S$.

If $Q \subset Y$ is a submanifold then we will denote by $\nu_Q := TY/TQ$ the normal bundle.

Theorem 2.10. *Let $f : X \times S \to Y$ be a smooth map between Banach manifolds. Let $Q \subset Y$ be a submanifold such that $f \pitchfork Q$ and $(D_1 f)_{(x,s)} : T_x X \to (\nu_Q)_{f(x,s)}$ is Fredholm. Then for generic values of s we have that $f_s = f(\cdot, s) \pitchfork Q$.*

A proof of this theorem follows from the following

Proposition 2.11. *Let f be as in Theorem 2.10. A point $s \in S$ is a regular value for π if and only if $f_s \pitchfork Q$.*

2.1.2 Vector bundles

Let $p : E \to M$ be a smooth map between manifolds. By a (vector bundle) trivialization of E over U we mean a pair (U, ψ), where $U \subset M$ is an open subset and ψ is a diffeomorphism

$$\psi : p^{-1}(U) \to U \times V,$$

which respects the fibers, i.e. the diagram

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\psi} & U \times V \\ & \searrow^{p} & \downarrow \mathrm{pr}_1 \\ & & U \end{array}$$

commutes. Here V is a Banach space, called the standard fiber.

Two trivializations (U_α, ψ_α) and (U_β, ψ_β) are called compatible if for $x \in U_\alpha \cap U_\beta$ we have that

$$(\psi_\beta \circ \psi_\alpha^{-1})(x, v) = (x, \psi_{\beta\alpha}(x) v),$$

where $\psi_{\beta\alpha}(x)$ is a linear isomorphism and

$$\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(V)$$

is smooth. A vector bundle atlas of the map p is a family of trivializations $\{(U_\alpha, \psi_\alpha)\}_\alpha$ such that the U_α cover M and such that for any pair of trivializations (U_α, ψ_α), (U_β, ψ_β) are compatible. Two vector bundle atlases are called equivalent if their union is again an atlas.

Definition 2.12. A *vector bundle* is a smooth map $p : E \to M$ together with an equivalence class of vector bundle atlases for the map p.

Remark 2.13. The condition that $\psi_{\beta\alpha}$ is smooth is implicit in the finite-dimensional case. This is not the case in the infinite dimensional case.

If the standard fiber V is a complex vector space then we can define holomorphic vector bundles over a complex base manifold by replacing all the smooth maps by holomorphic maps.

We would like to generalize the usual constructions we have for finite-dimensional vector bundles (e.g. direct sums, duals, tensor products, pull-backs). However we immediately see that in general this is quite subtle, since there is no preferred topology on the dual or tensor product for Banach spaces. Direct sums and pullbacks pose a problem. The dual of a Hilbert space and tensor products of Hilbert spaces do have preferred topologies.

It often is the case that there is more structure on the standard fiber. If this structure is preserved by some subgroup $G \subset GL(V)$ which is also a submanifold then we would like this extra structure to be present in the vector bundle. For this one needs that the transition functions $\psi_{\beta\alpha}$ takes values in the group G. The group G is called the structure group of the vector bundle. An important example of such a reduction occurs when the standard fiber is a Hilbert space. When V is a Hilbert space we have the subgroup $\mathrm{Hilb}(V) \subset GL(V)$ of topological linear automorphisms which preserve the inner product.

Proposition 2.14. *The group* $\mathrm{Hilb}(V)$ *is a closed submanifold of* $GL(V)$*.*

For a proof we refer the reader to [15].

Definition 2.15. A *Hilbert bundle* is a vector bundle where the standard fiber V is a Hilbert space and the transition functions take values in $\mathrm{Hilb}(V)$.

On a Hilbert bundle we define a fiberwise inner product for every $x \in M$ by

$$h_x(v, w) = \langle \psi_{\alpha,x} v, \psi_{\alpha,x} w \rangle \,, \qquad v, w \in E_x,$$

where $\langle , \rangle$ is the inner product on V. Since the transition functions take values in $\mathrm{Hilb}(V)$ the definition does not depend on α.

2.1.3 Bundle maps and exact sequences

Definition 2.16. Given two smooth vector bundle $p : E \to M$ and $p' : E' \to M'$. A *smooth bundle* map from E to E' is a pair of smooth functions

$f : E \to E'$ and $f_0 : M \to M'$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ M & \xrightarrow[f_0]{} & M' \end{array}$$

commutes, and for each $x \in M$ the restriction to the fiber $f_x : E_x \to E'_{f(x)}$ is continuous linear, and for each $x_0 \in M$ there exists neighbourhoods U of x_0 and U' of $f(x_0)$ with trivializations

$$\begin{aligned} \psi : p^{-1}(U) &\to U \times V \\ \psi' : p'^{-1}(U') &\to U' \times V', \end{aligned}$$

such that the map

$$U \ni x \mapsto \psi'_{f_0(x_0)} \circ f_x \circ \psi_x^{-1} \in \mathrm{Lin}(V, V')$$

is smooth, where $\psi_x = \psi|_{\{x\} \times V} : \{x\} \times V \to E_x$, and similarly for ψ'. We will denote this bundle map usually simply by f.

A smooth family of bundle maps $\{P^t\}_{t \in T}$ is a family of operators $P^t : E \to E'$ parametrized by a smooth manifold T, such that it is given by a smooth bundle map P :

$$\begin{array}{ccc} \mathrm{pr}_2^* E & \xrightarrow{P} & E' \\ \downarrow & & \downarrow \\ T \times M & \longrightarrow & M', \end{array}$$

such that the restriction of P to the slice $\{t\} \times M$ is given by P^t.

It is often the case that the base manifold is fixed. From now on E, E', E'' will be vector bundles over the same base manifold M.

Suppose we are given a sequence of vector bundles

$$E \xrightarrow{f} E'' \to 0.$$

We say that the sequence is exact at E'' if f is fiberwise surjective and if there exists a trivializing cover $\{(U_\alpha, \psi_\alpha)\}$ such that f corresponds to a projection

$$U_\alpha \times W \oplus V'' \to U_\alpha \times V''; \qquad (x, (w, v'')) \mapsto (x, v'').$$

The kernel of f is the bundle with fibers $\ker(f_x)$. It is easy to see that this defines a bundle, since a transition function $h(x) : W \oplus V'' \to W \oplus V''$ of E has to preserve the space $W \times \{0\}$, hence it is given by a matrix of the form

$$h(x) = \begin{pmatrix} h_{11}(x) & h_{12}(x) \\ 0 & h_{22}(x) \end{pmatrix}.$$

This implies that $h_{11}(x)$ is invertible and thus this defines the transition function for the bundle $\ker(f)$.

Remark 2.17. This results still holds in the holomorphic setting. The only extra thing one has to check is that the transition function h_{11} is holomorphic, but this follows immediately from the fact that h is holomorphic and that the map Dh and the almost complex structure preserve W.

Suppose we are given a sequence of vector bundles

$$0 \to E' \xrightarrow{f} E.$$

The sequence is called exact at E' if f is injective and if there exists a trivializing cover $\{(U_\alpha, \psi_\alpha)\}$ such that $V = V' \oplus W$ and f corresponds to the natural injection

$$U_\alpha \times V' \to U_\alpha \times V' \oplus W; \qquad (x, v') \mapsto (x, (v', 0)).$$

Proposition 2.18. *Let E, E'' be Banach vector bundles over M and $f : E \to E''$ a bundle map. Suppose that for all $x \in M$ the map*

$$f_x : E_x \to E''_x$$

is surjective and admits a splitting $P_x : E''_x \to E_x$, i.e. $f_x \circ P_x = \mathrm{id}_{E''_x}$. Then

$$E \to E'' \to 0$$

is exact and $\ker(f)$ *is a subbundle of E.*

Proof. Fix a point $x_0 \in M$. Let U be a neighbourhood of x_0 such that E and E'' are trivializable over U, i.e.

$$\begin{array}{ccc} E_U & \longrightarrow & E''_U \\ \downarrow \psi & & \downarrow \psi'' \\ U \times V & \longrightarrow & U \times V''. \end{array}$$

So we have that $\tilde{f}(x_0) \equiv \psi''_{x_0} \circ f_{x_0} \circ \psi_{x_0}^{-1} : V \to V''$ admits a splitting $\tilde{P}(x_0) \equiv \psi_{x_0} \circ P_{x_0} \circ \psi''^{-1}_{x_0}$. This gives us a decomposition of Banach spaces $V = W \oplus V''$, where $W = \ker \tilde{f}(x_0)$. This decomposition allows us to write

$$\tilde{f}(x) = \begin{pmatrix} h_1(x) & h_2(x) \end{pmatrix} : W \oplus V'' \to V''.$$

Let $h(x) : W \oplus V'' \to W \oplus V''$ be the map given by

$$h(x) := \begin{pmatrix} 1 & 0 \\ h_1(x) & h_2(x) \end{pmatrix}.$$

Note that $h(x_0)$ is the identity, hence for U small enough we have that $h(x)$ is invertible for all $x \in U$. The inverse is given by

$$h^{-1}(x) = \begin{pmatrix} 1 & 0 \\ g_1(x) & g_2(x) \end{pmatrix},$$

for some $g_1,\ g_2$ which satisfy $h_1(x) + h_2(x)g_1(x) = 0$ and $h_2(x)g_2(x) = 1$. The composition $\tilde{f}(x) \circ h^{-1}(x)$ is given by

$$\begin{pmatrix} h_1(x) + h_2(x)g_1(x) & h_2(x)g_2(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix},$$

which is the projection on the factor V''. So if we compose the trivialization ψ with h we see that we get an exact sequence

$$E \to E'' \to 0.$$

Hence $\ker(f)$ is a subbundle of E. □

Dually we have the following proposition.

Proposition 2.19. *Let E, E' be Banach vector bundles over M and $f : E' \to E$ a bundle map. Suppose that for all $x \in M$ the map*

$$f_x : E'_x \to E_x$$

is injective and admits a splitting $P_x : E_x \to E'_x$, i.e. $P_x \circ f_x = id_{E'_x}$. Then

$$0 \to E' \to E$$

is exact and $\mathrm{im}(f)$ *is a subbundle of E.*

Proposition 2.20. *For an exact sequence*

$$0 \to E' \to E$$

we can form the factor bundle E/E', which is the union of E_x/E'_x.

Proof. In the notation from above we have local trivializations for E/E' given by

$$(E/E')_{U_\alpha} \to U_\alpha \times W.$$

The coordinate transformations for E are represented by matrices

$$\begin{pmatrix} h_{11}(x) & h_{12}(x) \\ h_{21}(x) & h_{22}(x) \end{pmatrix} : V' \oplus W \to V' \oplus W,$$

which have to preserve the subspace $V' \times \{0\}$ in $V' \oplus W$. This implies that $h_{21}(x) = 0$ and thus $h_{22}(x)$ has to be invertible. This gives us that the transition function for E/E' are given by $h_{22}(x)$ and are smooth. Hence E/E' is a vector bundle. □

Remark 2.21. If the bundles E and E' are Hermitian vector bundles with a compatible holomorphic structure then for an exact sequence $0 \to E' \to E$ of holomorphic maps the factor bundle E/E' is also holomorphic.

2.2 ASD connections and holomorphic geometry

In this section we will introduce the Yang-Mills moduli space. We will prove that the Yang-Mills moduli space is generically a smooth finite-dimensional manifold. We will start discussing some basic properties of connections. We will introduce ASD connections and explain how they interact with holomorphic structures. Most of the theory about the Yang-Mills moduli space can be found in [8].

2.2.1 Connections on vector bundles

In this subsection we let $E \to X$ be a finite-dimensional vector bundle. A connection is a partial differential operator of order 1 which is defined as follows.

Definition 2.22. A *connection* is a linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*X \otimes E),$$

which satisfies the Leibniz rule

$$\nabla(fs) = f\nabla(s) + \mathrm{d}f \otimes s,$$

for all $f \in C^\infty(X)$ and $s \in \Gamma(E)$.

We introduce the notation $\Omega^k_X(E) := \Gamma(\Lambda^k_X \otimes E)$, where $\Lambda^k_X = \Lambda^k T^*X$. Note that $\Omega_X(E) := \bigoplus_k \Omega^k_X(E)$ is a graded bimodule over $\Omega(X)$ by taking the wedge product on the form part. Let $\mathrm{d} : \Omega^k(X) \to \Omega^{k+1}(X)$ be the de Rham operator.

A connection induces a linear operator

$$\nabla : \Omega^k_X(E) \to \Omega^{k+1}_X(E).$$

This operator is uniquely determined by the Leibniz rule

$$\nabla(\alpha \wedge \theta) = \mathrm{d}\alpha \wedge \theta + (-1)^{|\alpha|}\alpha \wedge \nabla\theta,$$

for all $\alpha \in \Omega^k(X)$ and $\theta \in \Omega^l_X(E)$.

Definition 2.23. The *curvature* of a connection ∇ is the operator

$$F_\nabla := \nabla \circ \nabla : \Omega^k_X(E) \to \Omega^{k+2}_X(E).$$

Alternatively we can view the curvature as an element of $\Omega^2_X(\mathrm{End}(E))$.[1]

Suppose that $\{e_1, \dots, e_n\}$ is a local frame of the bundle E over U then

$$\nabla e_i = \sum_{j=1}^n A^j_i \otimes e_j,$$

for certain $A^j_i \in \Omega^1(U)$. We define an $n \times n$ matrix of 1-forms A by $A = (A^j_i)_{ij}$, so $A \in \Omega^1_X(\mathrm{End}(E))$. This matrix depends on the local framing. For any section $s \in \Gamma(E_U)$ we now have that

$$\nabla s = \mathrm{d}s + As,$$

where d is the derivative on functions, so

$$\sum_{i=1}^n \mathrm{d}(s^i e_i) = \sum_{i=1}^n \mathrm{d}s^i \wedge e_i, \qquad s^i \in C^\infty(U).$$

In this local framing we can also express the curvature

$$\nabla(\nabla e_i) = \nabla(\sum_j A^j_i \wedge e_j) = \sum_j \mathrm{d}A^j_i \wedge e_j + \sum_{j,k} A^k_j \wedge A^j_i \wedge e_k.$$

[1]There is a bijection between the space of all endomorphisms of the graded $\Omega(X)$-module $\Omega_X(E)$ which increases the degree by p and elements of $\Omega^p(\mathrm{End}(E))$.

We introduce the following shorthand notation for this

$$F_\nabla = \mathrm{d}A + A \wedge A.$$

We want to know how the connection matrix and the curvature change when we change the local framing. The next lemma tells us precisely this. The proof is left as an exercise.

Lemma 2.24. *Let $g \in \mathrm{End}(E_U)$ be a local gauge transformation. Let $\{e'_1, \ldots, e'_n\}$ be the framing defined by*

$$e'_i := \sum_{j=1}^{n} g_i^j e_j.$$

Then the connection matrix A' with respect to the framing $\{e'_1, \ldots, e'_n\}$ is given by

$$A' = gAg^{-1} + (\mathrm{d}g)g^{-1}.$$

And the curvature is given by

$$F_{\nabla'} = gF_\nabla g^{-1}.$$

This lemma tells us that the curvature transforms as a tensor under bundle automorphisms. An other thing we obtain from the lemma is that we can define a connection by the following data. The bundle $E \to X$ can be defined by an open cover $\{U_\alpha\}_\alpha$ of X and transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n),$$

such that $g_{\alpha\beta}g_{\beta\alpha} = 1$ on $U_\alpha \cap U_\beta$ and $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$. The connection is now defined by a family $\{A_\alpha\}_\alpha$ of matrix 1-forms which satisfy

$$A_\beta = g_{\alpha\beta}A_\alpha g_{\alpha\beta}^{-1} + (\mathrm{d}g_{\alpha\beta})g_{\alpha\beta}^{-1}.$$

Example 2.25. A connection ∇ on a bundle E induces a connection on the endomorphism bundle $\mathrm{End}(E)$. It is the unique connection which satisfies the Leibniz rule

$$\nabla(gs) = (\nabla g)s + g\nabla s, \qquad \text{for } g \in \mathrm{End}(E),\ s \in \Gamma(E).$$

Let ∇_1 and ∇_2 be connections on E_1 and E_2, respectively. There is a natural connection ∇ on $E_1 \otimes E_2$ which is defined on sections of the form $s_1 \otimes s_2$ as

$$\nabla(s_1 \otimes s_2) = \nabla_1 s_1 \otimes s_2 + s_1 \otimes \nabla_2 s_2.$$

It is easy to check that if we extend this linearly to all sections of $E_1 \otimes E_2$ then this defines a connection.

Let $E \to X$ be a vector bundle with connection ∇ and a map $f : Y \to X$. The pullback connection $f^*\nabla$ is a connection on f^*E which is determined uniquely by

$$(f^*\nabla_X)(f^*s) := f^*(\nabla_{f_*X}(s)).$$

Just as before the connection ∇ on $\mathrm{End}(E)$ induces an operator $\nabla : \Omega^k_X(\mathrm{End}(E)) \to \Omega^{k+1}_X(\mathrm{End}(E))$. This allows us to state the well know Bianchi identity

Proposition 2.26. *(Bianchi identity)* $\nabla(F_\nabla) = 0$.

Proof. For any section s of E we have

$$\begin{aligned} \nabla^3 s &= \nabla(\nabla^2 s) = \nabla(F_\nabla s) = (\nabla F_\nabla)s + F_\nabla(\nabla s) \\ &= \nabla F_\nabla)s + \nabla^2(\nabla s) = (\nabla F_\nabla)s + \nabla^3 s. \end{aligned}$$

Since this holds for every section s the result follows. $\square$

If we have a Hermitian vector bundle, i.e. a finite-dimensional complex Hilbert bundle, we would like our connection to be compatible with the Hermitian structure in the following sense.

Definition 2.27. A connection ∇ on a Hermitian bundle (E, h) is a *Hermitian connection* if

$$\mathrm{d}h(s_1, s_2) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$$

for any sections $s_1,\ s_2 \in \Gamma(E)$.

For a Hermitian bundle we can choose orthogonal local trivializations. For orthogonal trivializations we have that the transition functions $g_{\alpha\beta}$ have values in the unitary group $U(n)$. In an orthogonal framing we have that the connection matrix of a Hermitian connection takes values in the Lie algebra of $U(n)$. To see this let $\{e_1, \dots, e_n\}$ be an orthogonal framing. Then for any two sections $s_1,\ s_2$ we have that

$$\begin{aligned} \mathrm{d}\left(\sum_{i=1}^n s_1^i s_2^i\right) &= h(\sum_{i=1}^n \mathrm{d}s_1^i e_i + As_1, \sum_{i=1}^n s_2^i e_i) + h(\sum_{i=1}^n s_1^i e_i, \sum_{i=1}^n \mathrm{d}s_2^i e_i + As_2) \\ &= \mathrm{d}\left(\sum_{i=1}^n s_1^i s_2^i\right) + h(As_1, s_2) + h(s_1, As_2). \end{aligned}$$

From which we see that $A^* = -A$, which means that $A \in \mathfrak{u}(n)$. We get similar results by replacing complex numbers by real numbers and $U(n)$ by $O(n)$.

2.2.2 Parallel transport

Parallel transport is a geometric way to compare different fibers of a vector bundle $E \to X$.

Let $\gamma : [0,1] \to X$ be a smooth path. Parallel transport along γ is a linear map

$$h_\gamma : E_{\gamma(0)} \to E_{\gamma(1)},$$

defined as follows. A section s is parallel along γ if

$$(\gamma^* \nabla_{\partial/\partial t})(\gamma^* s) = 0.$$

Parallel sections are locally solutions of an ODE, hence for every $e \in E_{\gamma(0)}$ there exists a unique parallel section s_e along γ with $s_e(\gamma(0)) = e$. The map h_γ is defined by

$$h_\gamma(e) = s_e(\gamma(1)).$$

This is clearly linear. We can extend the definition along piecewise smooth paths by defining

$$h_{\gamma_1 * \gamma_2} := h_{\gamma_1} \circ h_{\gamma_2},$$

where $\gamma_1 * \gamma_2$ is the concatenation of paths. It is easy to see that for the reversed path γ^{-1} we have that $h_{\gamma^{-1}} = (h_\gamma)^{-1}$. In particular parallel transport is a linear isomorphism between the fibers. Lastly we remark that if the connection is Hermitian then parallel transport is a unitary map.

2.2.3 ASD connections

Let V be an n-dimensional Hermitian vector space with θ a preferred unit n-vector. Remember that the Hodge star is an operator $* : \Lambda^k V \to \Lambda^{n-k} V$, which can be defined by

$$\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle \, \theta,$$

where $\langle , \rangle$ is the usual inner product on $\Lambda^k V$.

In case $n = 4$ we have that $* : \Lambda^2 V \to \Lambda^2 V$ and $*^2 = 1$. This means that we can decompose $\Lambda^2 V$ in the $+1$ and (-1)-eigenspace of $*$. Vectors in the $+1$ eigenspace are called self-dual and vectors in the -1 eigenspace are called anti-self-dual.

One can repeat this construction for the cotangent spaces of an oriented Riemannian manifold X. This gives an operator $* : \Omega^k(X) \to \Omega^{n-k}(X)$, where $n = \dim(X)$. For a four-manifold we have that the two-forms split in self-dual (SD) and anti-self-dual (ASD) forms. The same construction works for forms with values in vector bundles, by letting the Hodge act on the form part. Note that the notion of self-dual and anti-self-dual are conformally invariant. We will denote the self-dual two-forms with values in E by $\Omega^+_X(E)$ and the anti-self-dual two forms with values in E by $\Omega^-_X(E)$.

Remember that the curvature of a connection ∇ is a matrix valued 2-form $F_\nabla \in \Omega^2_X(\mathrm{End}(E))$.

Definition 2.28. A connection ∇ is *SD* or *ASD* if the curvature F_∇ is an element of $\Omega^+_X(\mathrm{End}(E))$ or $\Omega^-_X(\mathrm{End}(E))$, respectively.

We will now give a different characterization of the ASD condition. The Yang-Mills functional for $SU(n)$-bundles $E \to X$ is the function which sends a unitary connection to the L^2-norm of its curvature, i.e.

$$\|F_\nabla\|^2 = \int_X |F_\nabla|^2 \, \mathrm{d}\mu.$$

On $\Lambda^2_X \otimes \mathrm{End}(E)$ we use the norm which is determined by

$$|\alpha \otimes A|^2 = g(\alpha, \alpha)\mathrm{Tr}(AA^*) = -g(\alpha, \alpha)\mathrm{Tr}(A^2),$$

where g is the Riemannian metric on two-forms. For a conformal rescaling $\lambda : X \to (0, \infty)$ we have that $\mathrm{d}\mu_{\lambda g} = \lambda^{-m/2}\mathrm{d}\mu_g$ and $|\cdot|^2_{\lambda g} = \lambda^2|\cdot|^2_g$, where $m = \dim(X)$. Hence if X is four-dimensional then the Yang-Mills functional is conformally invariant. Another important feature of the Yang-Mills functional in four dimensions is its interaction with the self-dual and anti-self-dual forms. We introduce the notation $\gamma^2 := \gamma \wedge \gamma$. If $\alpha \in \Omega^+(\mathfrak{g}_E)$ and $\beta \in \Omega^-(\mathfrak{g}_E)$ then $\alpha \wedge \beta = 0$. This gives us that for any two-form γ we have that

$$\gamma^2 = \left(\gamma^+\right)^2 + \left(\gamma^-\right)^2 = (|\gamma^+|^2 - |\gamma^-|^2) \, \mathrm{d}\mu,$$

where γ^+ is the self-dual part of γ and γ^- the anti-self-dual part. We also have that

$$|\gamma|^2 = |\gamma^+|^2 + |\gamma^-|^2.$$

Hence we can conclude that

$$\mathrm{Tr}(F^2_\nabla) = |F_\nabla|^2 \, \mathrm{d}\mu \iff \nabla \text{ is ASD}.$$

So we have an inequality

$$\|F_\nabla\|^2 \geq \int_X \mathrm{Tr}(F^2_\nabla) \, \mathrm{d}\mu,$$

and this is an equality if and only if ∇ is ASD. The Chern-Weil representation of characteristic classes tells us that the right-hand-side of this inequality is independent of the connection and equal to $8\pi^2\left(c_2(E) - \frac{1}{2}c_1(E)^2\right)$. Remember that for an $SU(n)$-bundle E the first Chern class is always zero. For an $SU(n)$-bundle E the second Chern class $c_2(E)$ is called the charge of the bundle. A connection of charge k is a connection on a bundle of charge k. We have obtained the following.

Theorem 2.29. *An ASD connection is a global minimum of the Yang-Mills functional.*

The Euler-Lagrange equation for the Yang-Mills functional is $\nabla^*(F_\nabla) = 0$, where ∇^* is the adjoint of the connection ∇ on $\mathrm{End}(E)$.

For an $SU(2)$-bundle the second Chern class completely classifies the topological type of the bundle.

2.2.4 Holomorphic bundles

If E and X are complex manifolds then a vector bundle $\pi : E \to X$ is a holomorphic vector bundle if the fibers are complex vector spaces and the bundle map π is holomorphic. An equivalent definition is that the transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{C})$$

are holomorphic. For a holomorphic vector bundle $E \to X$ there is a linear operator

$$\overline{\partial}_E : \Omega_X^{0,k}(E) \to \Omega_X^{0,k+1}(E)$$

which is uniquely determined by:

1. $\overline{\partial}_E(fs) = (\overline{\partial} f)s + f\overline{\partial}_E s$.
2. $(\overline{\partial}_E s)(x) = 0$ for all x in an open subset $U \subset X$ if and only if s is holomorphic over U.

We can construct this operator by taking it to be the ordinary $\overline{\partial}$-operator in holomorphic trivializations, i.e. $\overline{\partial}_E s = \psi_\alpha^{-1}\overline{\partial}(\psi_\alpha \circ s)$. This is well-defined since for any other holomorphic trivialization ψ_β we have

$$\begin{aligned}
\overline{\partial}_E s &= \psi_\alpha^{-1}\overline{\partial}(\psi_\alpha \circ s) \\
&= \psi_\alpha^{-1}\overline{\partial}(\psi_\alpha \circ \psi_\beta^{-1} \circ \psi_\beta \circ s) \\
&= \psi_\alpha^{-1}\overline{\partial}(g_{\alpha\beta} \circ \psi_\beta \circ s) \\
&= \psi_\alpha^{-1} g_{\alpha\beta}\overline{\partial}(\psi_\beta \circ s) \\
&= \psi_\beta^{-1}\overline{\partial}(\psi_\beta \circ s),
\end{aligned}$$

where we used that $\overline{\partial} g_{\alpha\beta} = 0$.

A connection d_A over a complex base manifold splits into two operators

$$\mathrm{d}_A = \partial_A \oplus \overline{\partial}_A : \Omega^0_X(E) \to \Omega^{1,0}_X(E) \oplus \Omega^{0,1}_X(E).$$

The connection is said to be compatible with the holomorphic structure if $\overline{\partial}_A = \overline{\partial}_E$. The following integrability theorem tells precisely when the $\overline{\partial}_A$ operator is compatible with a holomorphic structure.

Theorem 2.30. *The* $(0,1)$*-part of a connection on a complex vector bundle over a complex manifold is compatible with a holomorphic structure if and only if* $\overline{\partial}_A^2 = 0$.

A proof can be found in [8, p.50]. For a given holomorphic structure there are many compatible connections. However the following easy result tells us that there can be only one Hermitian connection compatible with a holomorphic structure.

Proposition 2.31. *A Hermitian connection on a Hermitian vector bundle* $E \to X$ *is compatible with a holomorphic structure if and only if it has curvature of type* $(1,1)$*, and in this case the connection is uniquely determined by the metric and holomorphic structure.*

It is easy to check that if the connection is ASD then it is of type $(1,1)$. This implies that if d_A is an ASD connection then $\overline{\partial}_A^2 = 0$ and thus an ASD connection on a complex vector bundle over a complex base manifold defines a holomorphic structure.

We now come to an alternative characterization of ASD connections. For this we need the notion of a hyperkähler manifold .

Definition 2.32. A *hyperkähler manifold* X is a Riemannian manifold together with three covariantly constant endomorphisms $I, J, K : TX \to TX$ which satisfy the quaternionic relations $I^2 = J^2 = K^2 = IJK = -1$.

The following lemma says that on a hyperkähler manifold the ASD connections are precisely the connections compatible with all almost complex structures.

Lemma 2.33. *Let* X *be a four-dimensional hyperkähler manifold. Then*

$$\Lambda^- = \bigcap_I \Lambda^{1,1}_I,$$

where the intersection runs over all complex structures I *which are compatible with the metric and orientation.*

Proof. It is enough to check it for $X = \mathbb{R}^4$. For the standard metric and complex structure I on $\mathbb{R}^4$ the ASD two-forms are

$$\mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}x_3 \wedge \mathrm{d}x_4,\ \mathrm{d}x_1 \wedge \mathrm{d}x_4 - \mathrm{d}x_2 \wedge \mathrm{d}x_3,\ \mathrm{d}x_1 \wedge \mathrm{d}x_3 - \mathrm{d}x_4 \wedge \mathrm{d}x_2$$

and the forms of type $(1,1)$ are

$$\begin{aligned}
\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} &= -2i\mathrm{d}x_1 \wedge \mathrm{d}x_2, \\
\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_2} &= \mathrm{d}x_1 \wedge \mathrm{d}x_3 + \mathrm{d}x_2 \wedge \mathrm{d}x_4 + i(\mathrm{d}x_2 \wedge \mathrm{d}x_3 - \mathrm{d}x_1 \wedge \mathrm{d}x_4), \\
\mathrm{d}z_2 \wedge \mathrm{d}\overline{z_1} &= \mathrm{d}x_3 \wedge \mathrm{d}x_1 + \mathrm{d}x_4 \wedge \mathrm{d}x_2 + i(\mathrm{d}x_4 \wedge \mathrm{d}x_1 - \mathrm{d}x_3 \wedge \mathrm{d}x_2), \\
\mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2} &= -2i\mathrm{d}x_3 \wedge \mathrm{d}x_4.
\end{aligned}$$

So we see that $\Lambda^- \subset \Lambda^{1,1}_I$. Any other complex structure which is compatible with the metric and orientation is given by $I_A := A \circ I \circ A^{-1}$ for some A in $SO(4)$. The matrix A preserves Λ^- and $\Lambda^{1,1}_{I_A} = A(\Lambda^{1,1}_I)$, hence $\Lambda^- \subset \Lambda^{1,1}_{I_A}$. The complex dimension on Λ^- is 3 and the complex dimension of $\Lambda^{1,1}_I$ is 4. Clearly we can find a matrix A such that $\Lambda^{1,1}_I \neq \Lambda^{1,1}_{I_A}$. Hence we conclude that $3 \leq \dim_{\mathbb{C}}(\bigcap_I \Lambda^{1,1}_I) < 4$ and thus that $\Lambda^- = \bigcap_I \Lambda^{1,1}_I$. □

The alternative characterization also allows for a generalization of ASD connections to hyperkähler manifolds of dimension greater then 4.

Definition 2.34. If X is a hyperkähler manifold then the *twistor space* is a sphere bundle $p : Z \to X$ with fiber $p^{-1}(x)$ all complex structures of the form $\{aI + bJ + cK : a^2 + b^2 + c^2 = 1\}$.

A quaternionic instanton is a pair (E, ∇), where E is a complex vector bundle and ∇ is a connection whose curvature at a point $x \in X$ takes values in

$$\bigcap_{u \in p^{-1}(x)} \Lambda^{1,1}_u T^* X \otimes \mathrm{End}(E_x),$$

where $p : Z \to X$ is the twistor space. When $\dim(M) = 4$ this agrees with the definition of an instanton since ASD connections are precisely the 2-forms of type $(1,1)$ with respect to all complex structures compatible with the hyperkähler metric.

2.3 The Yang-Mills moduli space

In this section X will be a smooth connected compact oriented Riemannian four-manifold and $E \to X$ a smooth vector bundle with a certain structure group G. We will denote the space of all connections on E by $\mathscr{A}$. Remember that this is an affine space modeled on $\Omega^1(\mathfrak{g}_E)$, where $\mathfrak{g}_E$ is the subbundle of $\mathrm{End}(E)$ such that for each fiber $(\mathfrak{g}_E)_x = \mathfrak{g}$ is the Lie algebra of G. The group of all gauge transformations will be denoted by $\mathscr{G} \subset \Gamma(\mathrm{End}(E))$, it is defined as all bundle automorphisms which respect the structure on the fibers. For some results it will be important that the structure group G is compact. We will always assume that G is compact.

We want to construct a space $\mathscr{B}$ of all gauge equivalence classes of connections, i.e. $\mathscr{B} = \mathscr{A}/\mathscr{G}$. Remember that the action of $\mathscr{G}$ on $\mathscr{A}$ is given by

$$(g, \mathrm{d}_A) \mapsto g\mathrm{d}_A g^{-1} = \mathrm{d}_A - (\mathrm{d}_A g)g^{-1}.$$

In order to describe this space it is convenient to work with suitable completions of the spaces $\mathscr{A}$ and $\mathscr{G}$. The completions will be the Sobolev spaces $H_k(T^*X \otimes \mathfrak{g}_E) \equiv \Omega^1_k(\mathfrak{g}_E)$ and $\mathscr{G}_{k+1} \subset H_{k+1}(\mathrm{End}(E))$ for some $k \in \mathbb{N}$ (See Definition A.5). Remember that for a Hermitian vector bundle $F \to X$ over a compact manifold X, the Sobolev spaces on X carry a Hilbert topology. The bundle T^*X has an inner product from the Riemannian structure. On the fibers $(\mathfrak{g}_E)_x$ we have the Hilbert-Schmidt inner product $\langle A, B\rangle = \mathrm{Tr}(A^*B)$. Together with the Riemannian metric on forms this gives us a Hilbert structure on $\Omega^1_k(\mathfrak{g}_E)$. Note that by the Sobolev embedding theorems we know that for $k > 2$ these Sobolev spaces embed in the continuous sections.

For now we will not write the subscripts and it will turn out that the resulting moduli space does not depend on this index.

We will assume that E is an Hermitian vector bundle and thus that $\mathscr{G}$ is a subgroup of the Hermitian bundle automorphisms.

We give the space $\mathscr{B}$ the quotient topology. We denote by $[\mathrm{d}_A]$ the class of $\mathrm{d}_A \in \mathscr{A}$ in $\mathscr{B}$. We can define a distance function on $\mathscr{B}$ by

$$d([\mathrm{d}_A], [\mathrm{d}_B]) = \inf_{g \in \mathscr{G}} \|\mathrm{d}_A - g(\mathrm{d}_B)\|,$$

where $\|\cdot\|$ is the norm from the Hilbert structure on the Sobolev space. This is well-defined on the equivalence classes because

$$\|g(\mathrm{d}_A) - g(\mathrm{d}_B)\| = \|g(\mathrm{d}_A - \mathrm{d}_B)\| = \|g(A-B)\| = \|g(A-B)g^{-1}\| = \|A-B\|,$$

where the last equality follows from the invariance of the trace under conjugation, i.e.

$$\mathrm{Tr}\left((gCg^{-1})^* gCg^{-1}\right) = \mathrm{Tr}\left(gC^*Cg^{-1}\right) = \mathrm{Tr}(C^*C).$$

Lemma 2.35. *If the structure group G is compact then the distance function d defines a metric on $\mathscr{B}$.*

The non-trivial thing to check is that $d([\mathrm{d}_A],[\mathrm{d}_B]) = 0$ implies that $[\mathrm{d}_A] = [\mathrm{d}_B]$. This follows directly from the following.

Lemma 2.36. *Let $(d_{A_n})_n$ and $(d_{B_n})_n$ be sequences in $\mathscr{A}$ which converge to d_A and d_B in H_3, respectively. Suppose that there is a sequence $(g_n)_n$ in $\mathscr{G}_4$ such that*

$$g_n(d_{A_n}) = d_{B_n}.$$

Then there is a subsequence of $(g_n)_n$ which converges to some $g \in \mathscr{G}_4$. This limit satisfies $g(d_A) = d_B$.

Proof. We will write $\mathrm{d}_{A_n} = \mathrm{d}_A + a_n$ and $\mathrm{d}_{B_n} = \mathrm{d}_A + b_n$. The equation $g_n(\mathrm{d}_{A_n}) = \mathrm{d}_{B_n}$ can in a local trivialization be rewritten as

$$\nabla_A(g_n) = b_n g_n - g_n a_n,$$

where ∇_A is the connection on $\mathrm{End}(E)$ induced by the connection d_A on E. By the Sobolev embedding theorem we know that the g_n are continuous and since the gauge group G is compact we know that g_n is bounded in H_0, independent of n. This gives us that the right-hand-side of this equation is bounded in H_0. Since the base manifold is compact we can use Lemma A.4 to pick a subsequence of $(g_n)_n$ which converges to g in H_0 and also converges weakly in H_1 to some g. Let ϕ be any smooth test section of $\mathrm{End}(E)$. We have

$$\langle \nabla_A g, \phi \rangle = \lim_{n\to\infty} \langle \nabla_A g_n, \phi \rangle = \lim_{n\to\infty} \langle b_n g_n - g_n a_n, \phi \rangle = \langle (\mathrm{d}_B - \mathrm{d}_A) g, \phi \rangle \,,$$

hence g satisfies the equation $\nabla_A g = (\mathrm{d}_B - \mathrm{d}_A)g$. By the Sobolev multiplication theorem (See Lemma A.3) we know that the right-hand-side is in H_1. By bootstrapping we get that g is in H_k for every k. Hence we have that $g \in \mathscr{G}_4$ and $g(\mathrm{d}_A) = \mathrm{d}_B$. □

We want to further investigate the structure on $\mathscr{B}$. The space of connections $\mathscr{A}$ is an affine Banach space. The group of gauge transformations $\mathscr{G}$ is a Banach manifold. A chart around the identity is obtained from the exponential map $\exp : \Omega^0_{k+1}(\mathfrak{g}_E) \to \mathscr{G}_{k+1}$ for $k > 2$. This can easily be seen, since for $k > 2$ the Sobolev spaces embed into the continuous sections and thus it reduced pointwise to the fact that $\exp : \mathfrak{g} \to G$ is a chart around the identity.

The action $\mathscr{G} \times \mathscr{A} \to \mathscr{A}$ given by

$$(g, \mathrm{d}_A) \mapsto g\mathrm{d}_A g^{-1} = \mathrm{d}_A - (\mathrm{d}_A g)g^{-1}$$

is a smooth map between Banach manifolds. The partial derivative to the first coordinate of this map at a point $(1, \mathrm{d}_A)$ can easily be computed by

$$\frac{d}{dt}\bigg|_{t=0} \left(\mathrm{d}_A - (\mathrm{d}_A \exp(tX)) \exp(tX)^{-1}\right) = -\mathrm{d}_A X,$$

where we use the logarithm to write a gauge transformation g near 1 as $\exp(tX)$ for some $X \in \Omega^0(\mathfrak{g}_E)$. So the partial derivative is given by

$$-\mathrm{d}_A : \Omega^0(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E).$$

Remember that elliptic theory gives us that

$$\Omega^1(\mathfrak{g}_E) = \mathrm{im}(\mathrm{d}_A) \oplus \ker(\mathrm{d}_A^*), \qquad \Omega^0(\mathfrak{g}_E) = \ker(\mathrm{d}_A) \oplus \mathrm{im}(\mathrm{d}_A^*).$$

These are orthogonal sums in the Sobolev spaces. The tangent space of $\mathscr{A}$ is $\Omega^1(\mathfrak{g}_E)$ and the tangent space of the orbit through d_A is $\mathrm{im}(\mathrm{d}_A)$. We now want to construct local models for the space $\mathscr{B}$.

The stabilizer of a connection under the gauge action will be denoted by

$$\Gamma_{\mathrm{d}_A} = \{g \in \mathscr{G} : g(\mathrm{d}_A) = \mathrm{d}_A\} = \{g \in \mathscr{G} : \mathrm{d}_A g = 0\}.$$

We consider Γ_{d_A} as a subgroup of G by the inclusion $(\Gamma_{\mathrm{d}_A})_x \subset \mathrm{Aut}(E_x)$ for some x. Note that for a different choice of base point x we get a conjugate subgroup in G. The following slice theorem will allow us to construct local models for $\mathscr{B}$.

Theorem 2.37. *The action*

$$\mathscr{A} \times \mathscr{G} \to \mathscr{A}$$

has local slices. This means that for every $\mathrm{d}_A \in \mathscr{A}$ *there is a slice* $U \subset \ker(\mathrm{d}_A^*)$ *such that*

$$(U \times \mathscr{G})/\Gamma_{\mathrm{d}_A} \to \mathscr{A}$$

is an equivariant diffeomorphism onto an open subset of $\mathscr{A}$*.*

Proof. Consider the map

$$\varphi : \ker(\mathrm{d}_A^*) \times \mathscr{G} \to \mathscr{A}$$

given by

$$\varphi : (B, g) \mapsto g(\mathrm{d}_A + B).$$

The derivative of this map at the point $(0, 1)$ is given by

$$\begin{pmatrix} 1 & -\mathrm{d}_A \end{pmatrix}.$$

The kernel of this derivative is precisely $\{0\} \times \ker(\mathrm{d}_A)$. Let $g \in \Gamma_{\mathrm{d}_A}$ and $B \in \ker(\mathrm{d}_A^*) \subset \Omega^1_X(\mathfrak{g}_E)$ then

$$\begin{aligned} \mathrm{d}_A^* g(B) &= - * \mathrm{d}_A * (gBg^{-1}) = - * \mathrm{d}_A(g * Bg^{-1}) \\ &= - * \big((\mathrm{d}_A g) * Bg^{-1} + g(\mathrm{d}_A * Bg^{-1})\big) \\ &= g\mathrm{d}_A^* Bg^{-1} = 0. \end{aligned}$$

So Γ_{d_A} leaves $\ker(\mathrm{d}_A^*)$ invariant. We have a natural action of Γ_{d_A} on $\ker(\mathrm{d}_A^*) \times \mathscr{G}$ given by

$$h : (B, g) \mapsto (h(B), gh^{-1}), \qquad h \in \Gamma_{\mathrm{d}_A}.$$

Note that this is a free and properly discontinuous action, so the quotient $(\ker(\mathrm{d}_A^*) \times \mathscr{G})/\Gamma_{\mathrm{d}_A}$ is again a Banach manifold. The map φ descends to the quotient $(\ker(\mathrm{d}_A^*) \times \mathscr{G})/\Gamma_{\mathrm{d}_A}$, since for $h \in \Gamma_{\mathrm{d}_A}$ we have

$$(h(B), gh^{-1}) \mapsto gh^{-1}(\mathrm{d}_A + h(B)) = g(\mathrm{d}_A + B).$$

The orbit of $(0, 1)$ under Γ_{d_A} is $\{0\} \times \Gamma_{\mathrm{d}_A}$ and the tangent space of this orbit at $(0, 1)$ is $\{0\} \times \ker(\mathrm{d}_A)$.

This gives us that the derivative of

$$\varphi : (\ker(\mathrm{d}_A^*) \times \mathscr{G})/\Gamma_{\mathrm{d}_A} \to \mathscr{A}$$

at the orbit of $(0, 1)$ is an isomorphism and thus by the inverse function theorem we know that φ is a local diffeomorphism form an open neighbourhood of the orbit of $(0, 1)$ to an open neighbourhood of d_A.

It remains to show that if we pick a small enough open $U \subset \ker(\mathrm{d}_A^*)$ then restriction of φ is a bijection. Let $U_{\mathrm{d}_{A,\epsilon}}$ be the open set

$$U_{\mathrm{d}_{A,\epsilon}} = \{a \in \Omega^1(\mathfrak{g}_E) : \mathrm{d}_A^* a = 0,\ \|a\| < \epsilon\}.$$

Suppose that φ is not injective for any small ϵ then we find $\epsilon_n \to 0$, $B_n, B_n' \in U_{\mathrm{d}_{A,\epsilon_n}}$, $[(B_n, g_n)] \neq [(B_n', g_n')]$ such that

$$g_n(\mathrm{d}_A + B_n) = g_n'(\mathrm{d}_A + B_n').$$

Define $h_n := g_n'^{-1} g_n$ then we have

$$h_n(\mathrm{d}_A + B_n) = \mathrm{d}_A + B_n'.$$

By Lemma 2.36 we know that we can pick a converging subsequence such that $h_n \to h$ with $h \in \Gamma_A$. So we have that

$$[(B_n, h_n)] \to [(0, h)] = [(0, 1)]$$

and

$$[(B_n', 1)] \to [(0, 1)].$$

By the local diffeomorphism property we can conclude that for $n \gg 0$ we have $[(B_n', 1)] = [(B_n, h_n)]$. This implies that $h_n(B_n) = B_n'$ for $n \gg 0$ and thus that

$$[(B_n, g_n)] = [(h_n(B_n), g_n h_n^{-1})] = [(B_n', g_n')],$$

which is a contradiction.

We conclude that for small enough ϵ the map φ is injective and thus a diffeomorphism. The action of $\mathscr{G}$ on the quotient $(\ker(\mathrm{d}_A^*) \times \mathscr{G})/\Gamma_{\mathrm{d}_A}$ is given by

$$h : [(B, g)] \mapsto [(B, hg)].$$

It is clear that the map φ is equivariant. □

As a corollary of this theorem we obtain a local model for $\mathscr{B}$.

Corollary 2.38. *For small enough ϵ the projection $\mathscr{A} \to \mathscr{B}$ induces a homeomorphism from $U_{\mathrm{d}_{A,\epsilon}}/\Gamma_{\mathrm{d}_A}$ to an open neighbourhood of $[\mathrm{d}_A]$ in $\mathscr{B}$.*

Proof. The action of $\mathscr{G}$ on $(U_{\mathrm{d}_{A,\epsilon}} \times \mathscr{G})/\Gamma_{\mathrm{d}_A}$ acts only on the factor $\mathscr{G}$. The result follows directly from Theorem 2.37. □

Let $Hol_x(\mathrm{d}_A)$ denote the holonomy group of a connection d_A in a point x. This is a closed subgroup $Hol_x \subset \mathrm{Aut}(E_x) \cong G$. So Hol_x can be considered as a Lie subgroup of G.

Definition 2.39. A connection is *reducible* if for each $x \in X$ the holonomy group Hol_x not equal to $\mathrm{Aut}(E_x) \cong G$.

If the base manifold is connected then for any $x, y \in X$, the groups Hol_x and Hol_y are conjugate to each other in G. So in this case one only has to consider the holonomy in one fiber.

Lemma 2.40. *Let* d_A *be any connection over a connected base manifold* X*. The isotropy group* $\Gamma_{\mathrm{d}_A} = \{g \in \mathscr{G} : g(\mathrm{d}_A) = \mathrm{d}_A\}$ *is isomorphic to the centralizer of* Hol_x *in* $\mathrm{Aut}(E_x) \cong G$*.*

Proof. Let $h_\gamma : E_x \to E_y$ be the parallel transport along γ, where $x = \gamma(0)$ and $y = \gamma(1)$. For an element in the centralizer $g_x \in C(Hol_x)$ we define a gauge transformation g by

$$g(e_y) := (h_\gamma \circ g_x \circ h_\gamma^{-1})(e_y), \qquad e_y \in E_y.$$

Since g_x is in the centralizer of Hol_x this does not depend on γ because for a different path α we have that

$$(h_\alpha)^{-1}\, h_\gamma = h_{\alpha^{-1} * \gamma},$$

where $\alpha^{-1} * \gamma$ is the concatenation of paths. From this it follows that

$$\begin{aligned}(h_\alpha \circ g_x \circ h_\alpha^{-1})^{-1}(h_\gamma \circ g_x \circ h_\gamma^{-1}) &= h_\alpha \circ g_x^{-1} \circ h_{\alpha^{-1}*\gamma} \circ g_x \circ h_\gamma^{-1} \\ &= h_\alpha \circ h_{\alpha^{-1}*\gamma} \circ h_\gamma^{-1} = 1.\end{aligned}$$

By construction g is parallel for d_A, hence this gives an injective map $C(Hol_x) \to \Gamma_{\mathrm{d}_A}$. Let $g \in \Gamma_{\mathrm{d}_A}$. Then $\mathrm{d}_A g = 0$, which means that g is parallel and thus that $g_y = h_\gamma g_x h_\gamma^{-1}$. □

Example 2.41. For the structure group $G = SU(2)$ and a simply connected base manifold X there are only two possible reductions. Suppose that d_A is a reducible connection. Then Hol_x is a proper Lie subgroup of $SU(2)$. So Hol_x is contained in some circle $S^1 \subset SU(2)$. Since Hol_x is connected the only possibilities are $Hol_x = S^1$ and $Hol_x = 1$.

If $Hol_x = 1$ then the bundle is trivial and has the product connection. In the other case, $Hol_x = S^1$, there are parallel one-dimensional invariant subspaces L_x and L'_x of E_x. This gives us a splitting $E = L \oplus L'$.

We see that an $SU(2)$-connection d_A is irreducible if and only if $\Gamma_{\mathrm{d}_A} = C(SU(2)) = \{\pm 1\}$.

Consider the open subset $\mathscr{A}^*$ of $\mathscr{A}$ consisting of connections whose isotropy group is minimal, i.e.

$$\mathscr{A}^* = \{\mathrm{d}_A \in \mathscr{A} : \Gamma_{\mathrm{d}_A} = C(G)\}.$$

Let $\mathscr{B}^*$ be the quotient of $\mathscr{A}^*$.

The space $\mathscr{B}$ is a stratified space. The strata are given by

$$\mathscr{B}^{\Gamma} = \{[\mathrm{d}_A] \in \mathscr{B} : \Gamma_{\mathrm{d}_A} \text{ and } \Gamma \text{ are conjugate subgroups of } G\}.$$

The generic stratum is $\mathscr{B}^* = \mathscr{B}^{C(G)}$.

In the case $G = SU(2)$ we have that $C(SU(2)) = \{\pm 1\}$ define global gauge transformations. The action of $\mathscr{G}/\{\pm 1\}$ on $\mathscr{A}^*$ is free. By Corollary 2.38 we know that $\mathscr{B}^*$ is a Hilbert manifold modeled on $U_{\mathrm{d}_{A,\epsilon}}/\{\pm 1\} = U_{\mathrm{d}_{A,\epsilon}}$. This gives us that the projection $\mathscr{A}^* \to \mathscr{B}^*$ is a principal $\mathscr{G}/\{\pm 1\}$-bundle, where the right action is given by $\mathrm{d}_A \cdot g = g^{-1}(\mathrm{d}_A)$. This point of view will turn out to be useful later on.

Now we will have a closer look at the moduli space of ASD connections. We will assume that X is a compact oriented Riemannian manifold and $E \to X$ and $SU(2)$ or $SO(3)$ bundle over X.

For each $k > 2$ we have a moduli space $M(k) \subset \mathscr{B}(k)$ of k-Sobolev ASD connections modulo $(k+1)$-Sobolev gauge transformations. It turns out that the structure of $M(k)$ does not depend on k.

Proposition. *The natural inclusion of $M(k+1)$ in $M(k)$ is a homeomorphism.*

For a proof we refer the reader to [8, p.134].

The main step in the proof is in showing that this map is surjective, i.e. every k-Sobolev ASD connection can be gauge transformed to a $(k+1)$-Sobolev ASD connection.

The $*$-operator depends on the Riemannian metric on the base manifold X, and hence the moduli space M depends on the metric. The main result of this section is that for most choices of metric g the moduli space $M^*(g)$ of irreducible ASD connection is a smooth finite-dimensional submanifold of $\mathscr{B}^*$.

From now on we will not write the Sobolev index anymore. We obtain local models for M from the local models of $\mathscr{B}$ in the following way.

Let d_A be an ASD connection and define

$$\psi : U_{\mathrm{d}_{A,\epsilon}} \to \Omega^+_X(\mathfrak{g}_E), \qquad \psi(a) = F^+(\mathrm{d}_A + a) = \mathrm{d}^+_A a + (a \wedge a)^+.$$

The zero set of ψ modulo Γ_{d_A} is homeomorphic to a neighbourhood of $[\mathrm{d}_A]$ in M. The derivative of ψ at 0 is given by d^+_A. We will now show that d^+_A is a Fredholm map.

Consider the 'deformation complex' :

$$\Omega^0_X(\mathfrak{g}_E) \xrightarrow{\mathrm{d}_A} \Omega^1_X(\mathfrak{g}_E) \xrightarrow{\mathrm{d}^+_A} \Omega^2_X(\mathfrak{g}_E).$$

This is a complex since d_A is ASD means precisely that $\mathrm{d}_A^+ \circ \mathrm{d}_A = 0$. The symbol complex of this is given by

$$\pi^*(\Omega^0(\mathfrak{g}_E)) \xrightarrow{(x\wedge\cdot)\otimes 1_{\mathfrak{g}_E}} \pi^*(\Omega^1(\mathfrak{g}_E)) \xrightarrow{p^+(x\wedge\cdot)\otimes 1_{\mathfrak{g}_E}} \pi^*(\Omega^2(\mathfrak{g}_E)) \ ,$$

where $\pi : T^*X \to X$ is the projection map and $p^+ : \Omega^2(\mathfrak{g}_E) \to \Omega^+(\mathfrak{g}_E)$ is the projection on $\Omega^+(\mathfrak{g}_E)$. The symbol sequence is easily seen to be exact. This gives us that the operator $\mathrm{d}_A^* \oplus \mathrm{d}_A^+$ is elliptic. Hence the restriction of d_A^+ to $\ker(\mathrm{d}_A^*)$ is a Fredholm map. Elliptic theory tells us that the image of d_A^+ is closed and that the cokernel of this map is finite-dimensional. Moreover we know that the cohomology groups of the deformation complex are finite-dimensional. Hence we have that

$$\ker(\mathrm{d}_A^+|_{\ker(\mathrm{d}_A^*)}) \cong \frac{\ker(\mathrm{d}_A^+)}{\mathrm{im}(\mathrm{d}_A)} = H^1_{\mathrm{d}_A}$$

is finite-dimensional. We conclude that the map ψ is Fredholm. The index of the operator $\mathrm{d}_A^* \oplus \mathrm{d}_A^+$ is given by minus the Euler characteristic of the deformation complex

$$\mathrm{ind}(\mathrm{d}_A^* \oplus \mathrm{d}_A^+) = \dim(H^1_{\mathrm{d}_A}) - \dim(H^0_{\mathrm{d}_A}) - \dim(H^2_{\mathrm{d}_A}).$$

The number $s := \mathrm{ind}(\mathrm{d}_A^* \oplus \mathrm{d}_A^+)$ will be called the virtual dimension of M. Note that

$$\mathrm{ind}(\psi) = s + \dim(\ker(\mathrm{d}_A)) = s + \dim(\Gamma_{\mathrm{d}_A}).$$

The Atiyah-Singer index theorem gives us that

$$s = 8\kappa(E) - \dim(E)(1 - b_1(X) + b_+(X)),$$

where $b_1(X)$ is the first Betti number, and $b_+(X)$ is the dimension of the space of SD harmonic two-forms, and $\kappa(E) = c_2(E)$ for $SU(2)$-bundles and $\kappa(E) = \frac{-1}{4}p_1(E)$ for $SO(3)$-bundles.

Now we give a global description of the moduli space which has a natural extension for variations of the metric. Let ρ be the action of $\mathscr{G}$ on $\Omega^2(\mathfrak{g}_E)$ given by $\rho(g)F = g \cdot F \cdot g^{-1}$. The curvature $F : \mathscr{A} \to \Omega^2(\mathfrak{g}_E)$ is equivariant, i.e. $F(g(\mathrm{d}_A)) = g \cdot F(\mathrm{d}_A) \cdot g^{-1}$. This gives us an equivariant map $F^+ : \mathscr{A}^* \to \Omega^+(\mathfrak{g}_E)$. Hence the map F^+ defines a section of the associated bundle

$$\begin{array}{c} \mathscr{E} := \mathscr{A}^* \times_{\mathscr{G}/\{\pm 1\}} \Omega^+(\mathfrak{g}_E) \ , \\ \downarrow \\ \mathscr{B}^* \end{array}$$

we will denote this section by F^+ as well. The section F^+ is a smooth Fredholm map between Banach manifolds. This can easily be seen by looking at a local slice $U_{\mathrm{d}_{A,\epsilon}}$ of $\mathscr{B}^*$. This gives us a bundle trivialization as explained in appendix Definition B.2 by

$$[\mathrm{d}_A + a, F] \mapsto (a, F).$$

In these trivializations the section F^+ is the map $a \mapsto F^+(\mathrm{d}_A + a)$. This is precisely the map ψ defined above, which we know to be Fredholm. The zero set of this section is precisely the ASD moduli space M.

Let $\mathscr{C}$ be the space of C^k-Riemannian metrics on X. This is a subspace of the Banach space $\Gamma^k(T^*X \otimes T^*X)$ with the C^k-norm.

Proposition 2.42. *The $*$-operators $*_g$ are a smooth family, i.e. the map $* : \mathscr{C} \to End(\Omega^2(\mathfrak{g}_E))$ given by $g \mapsto *_g$ is smooth.*

This gives us a smooth bundle automorphism ϕ of the trivial bundle $(\mathscr{A}^* \times \mathscr{C}) \times \Omega^2(\mathfrak{g}_E) \to (\mathscr{A}^* \times \mathscr{C})$, which in the fiber (d_A, g) is given by $\frac{1}{2}(1 - *_g)$. The kernel of ϕ is a smooth vector bundle $\mathscr{E}$ over $\mathscr{A}^* \times \mathscr{C}$, with the fiber over (d_A, g) the self-dual two forms for the metric g.

The group of gauge transformations $\mathscr{G}$ acts on this bundle and $\{\pm 1\}$ acts trivially. This gives us a quotient bundle

$$\underline{\mathscr{E}} \longrightarrow \mathscr{B}^* \times \mathscr{C}.$$

The section $\psi(\mathrm{d}_A, g) = F_g^+(\mathrm{d}_A)$ of the trivial bundle $(\mathscr{A}^* \times \mathscr{C}) \times \Omega^2(\mathfrak{g}_E)$ descends to a section $\underline{\psi}$ on $\underline{\mathscr{E}}$.

Proposition 2.43. *For an $SU(2)$ or $SO(3)$ bundle $E \to X$, the section $\underline{\psi}$ has zero as a regular value set in $\mathscr{B}^* \times \mathscr{C}$.*

A proof of this proposition can be found in [9].

From Theorem 2.10 we get that for a generic metric g the map ψ_g has zero as a regular value. Hence for a generic metric we have that $M^*(g)$ is a smooth manifold of dimension $\mathrm{ind}(\psi_g) = s$.

Chapter 3

The Nahm transform

In this chapter we want to describe the Nahm transform over a flat four-torus. We will prove that the Nahm transform is an bijection between ASD connections on the four-torus to ASD connections on the dual torus. The Nahm transform makes uses of the Dirac operator. Therefore we will first give an introduction to spin structures and Dirac operators in four dimensions. This theory will allow us to define the Nahm transform for a spin four-manifolds of non-negative scalar curvature. A proof that the Nahm transform is an involution is included. This proof requires some holomorphic geometry and makes use of a spectral sequence.

3.1 Spin structures and Dirac operators

In this section we give some quick working definitions for $Spin(4)$-structures and Dirac operators. Most of this is based on [20]. For more details about spin structures we refer the reader to [16].

3.1.1 Spin structures

We will begin by defining the Clifford algebras and Spin groups. After this we will give more explicit descriptions in four dimensions.

Definition 3.1. Let Q be a quadratic form on a vector space V then the *Clifford algebra* $Cl(Q) \equiv Cl(V)$ is the quotient of the tensor algebra of V by the ideal generated by all elements of the form $v \otimes v - Q(v) \cdot 1$.

If $Q = 0$ then $Cl(Q)$ is the exterior algebra. If $V = \mathbb{R}^n$ and $Q(v) = -\sum_{i=1}^n v_i^2$ then we get the Clifford algebra Cl_n which has generators $e_1, \ldots, e_n$

with the relations

$$e_i^2 = -1, \qquad e_i e_j + e_j e_i = 0 \quad \text{for } i \neq j.$$

The $\mathbb{Z}_{\geq 0}$ grading of the tensor algebra descends to a $\mathbb{Z}/2\mathbb{Z}$ grading of the Clifford algebra.

Let $Cl^{\times}(V)$ be the multiplicative group of units of $Cl(V)$.

Definition 3.2. The *group* $Pin(V)$ is the subgroup of $Cl^{\times}(V)$ generated by all elements $v \in V$ with $\|v\| = 1$. The *group* $Spin(V)$ is the intersection of $Pin(V)$ with the elements of even degree $Cl(V)^{\text{ev}}$.

Since we are mainly interested in the case $n = 4$ we will now reduce to $n =$ even.

Theorem 3.3. *There is a* $\mathbb{Z}_2$*-graded complex vector space* $S_n = S_n^+ \oplus S_n^-$ *and an isomorphism of algebras*

$$\gamma : Cl_n \otimes_{\mathbb{R}} \mathbb{C} \to End(S_n),$$

with $\dim(S_n^+) = \dim(S_n^-) = 2^{n-1}$.

The pair (S_n, γ) is unique up to isomorphism.

In four dimensions the quaternions give a simplification of spinors. We will start by recalling a few basic properties of the quaternions. In the following V will be a four-dimensional real vector space with a quaternionic structure. The quaternions $\mathbb{H}$ are formed from $\mathbb{R}$ by adding three symbols $i,\ j,\ ,k$ which satisfy the identities:

$$i^2 = j^2 = k^2 = ijk = -1.$$

So a general quaternion is of the form

$$x = x_1 + x_2 i + x_3 j + x_4 k,$$

with $x_1, \ldots, x_4 \in \mathbb{R}$. The conjugate of x is defined by

$$\overline{x} = x_1 - x_2 i - x_3 j - x_4 k.$$

One can check that

$$x\overline{x} = \sum_{i=1}^{4} x_i^2,$$

which is equal to the Euclidian norm $|x|^2$. So the multiplicative inverse of x is given by $x^{-1} = \frac{\overline{x}}{|x|^2}$. We see that the elements of $\mathbb{H}$ with norm 1 form a group. Every quaternion x can be written as

$$x = z_1 + jz_2,$$

where $z_1 = x_1 + x_2 i$ and $z_2 = x_3 + x_4 i$. This gives us an identification between $\mathbb{H}$ and $\mathbb{C}^2$. This identification gives us a representation $\mathbb{H}\backslash\{0\} \to GL(\mathbb{C}^2)$ of the left multiplication map $x \mapsto qx$. In this representation we have that

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{3.1.1}$$

and $\mathbb{H}\backslash\{0\} \otimes_{\mathbb{R}} \mathbb{C} \cong GL(\mathbb{C}^4)$. This gives us an identification between the group of units in $\mathbb{H}$ and $SU(2)$. We will identify these two groups from now on. Under this identification the Lie algebra $\mathfrak{su}(2)$ gets identified with the geometric tangent space of the unit elements at the point 1. These are the pure imaginary quaternions, i.e.

$$\mathfrak{su}(2) = \{x \in \mathbb{H} : \overline{x} = -x\}.$$

Definition 3.4. The *Lie group Spin*(4) is the universal cover of $SO(4)$. It can be explicitly given as

$$Spin(4) = SU(2) \times SU(2) \subset GL(4, \mathbb{C}).$$

The covering map $\rho : Spin(4) \to SO(4) \subset GL(V)$ is obtained by $\rho(A_+, A_-)(v) = A_- v A_+^{-1}$, where we use the quaternionic multiplication on V.

Note the the kernel of ρ is given by $\{\pm(1, 1)\}$.

Definition 3.5. The *Lie group Spin*$^c(4)$ is the universal cover of $SO(4) \times U(1)$. Explicitly it can be given as $Spin(4) \times U(1)/\{\pm(1, 1)\}$ or as

$$Spin(4)^c = \left\{ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} : A_+ \in SU_+(2),\ A_- \in SU_-(2),\ \lambda \in U(1) \right\}.$$

The covering map is given by the product of two representations $\rho^c : Spin^c(4) \to SO(4)$ and $\pi : Spin^c(4) \to U(1)$, which are given by

$$\rho^c(\lambda A_+, \lambda A_-)(v) = (\lambda A_+)v(\lambda A_-)^{-1},$$

where we identify $v \in V$ with a 2×2 complex matrix, and

$$\pi\left(\lambda A_+, \lambda A_-\right) = \det(\lambda A_+) = \det(\lambda A_-) = \lambda^2.$$

Note that $Spin(4)$ is a six-dimensional Lie group and it is contained in the seven-dimensional Lie group $Spin^c(4)$.

For $Spin(4)$ we have two natural representations $\rho_\pm : Spin(4) \to GL_{\mathbb{C}}(W_\pm)$, where $W_\pm = \mathbb{C}^2$, defined by left matrix multiplication

$$\begin{aligned} \rho_+ \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} (w_+) &= A_+ w_+, \\ \rho_- \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} (w_-) &= A_- w_-. \end{aligned}$$

In fact since $A_\pm$ are in $SU(2)$ we know that we can identify $\mathbb{C}^2$ with $\mathbb{H}$, so we can also let the above be quaternionic multiplication.

Similarly for $Spin^c(4)$ we have representations $\rho^c_\pm : Spin(4) \to GL_{\mathbb{C}}(W_\pm)$, where $W_\pm = \mathbb{C}^2$, defined by

$$\begin{aligned} \rho^c_+ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} (w_+) &= (\lambda A_+) w_+, \\ \rho^c_- \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} (w_-) &= (\lambda A_-) w_-. \end{aligned}$$

We have an induced representation on $\mathrm{Hom}(W_+, W_-)$. This is given by $\sigma(A_+, A_-)(f) = A_- \circ f \circ A_+^{-1}$, for $f \in \mathrm{Hom}(W_+, W_-)$. Let $\mathrm{Hom}_j(W_+, W_-)$ be the space of homomorphism which intertwine the right multiplication by j. These are precisely the homomorphisms which are given by left multiplication by a quaternion. We obtain that there is an isomorphism of $Spin(4)$-representation spaces

$$\gamma : V \xrightarrow{\cong} \mathrm{Hom}_j(W_+, W_-), \tag{3.1.2}$$

If we complexify we get an isomorphism

$$\gamma : V \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\cong} \mathrm{Hom}(W_+, W_-).$$

Let $W = W_+ \oplus W_-$. Then we have an inclusion $i : \mathrm{Hom}(W_+, W_-) \hookrightarrow \mathrm{End}(W)$, given by

$$i(f) = \begin{pmatrix} 0 & -f^* \\ f & 0 \end{pmatrix},$$

where $f^* : W_- \to W_+$ is the adjoint of f. The space $\mathrm{End}(W)$ is identified with the complexification of the Clifford algebra Cl_4 by

$$
\begin{aligned}
e_1 &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad & e_2 &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \\
e_3 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad & e_4 &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{3.1.3}
$$

These satisfy the relations

$$e_i^2 = -1, \qquad e_i e_j + e_j e_i = 0 \quad \text{for } i \neq j.$$

The map

$$V \otimes_{\mathbb{R}} \mathbb{C} \hookrightarrow \mathrm{End}(W)$$

extends to the exterior algebra

$$\Lambda^{\bullet} V \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{End}(W).$$

This is done by sending the wedge product of e_i with $i \in I$ to the product of e_i with $i \in I$ in $\mathrm{End}(W)$. This is an isomorphism of vector spaces, but not of algebras.

Under this identification the multiplication is given by

$$e_i \cdot \omega = e_i \wedge \omega - e_i \lrcorner \omega,$$

where $\lrcorner$ denotes the internal contraction.

For an oriented Riemannian manifold (M, g) the structure group of TM can be reduced from $GL(4, \mathbb{R})$ to $SO(4)$. In the following we will denote the transition functions of TM by

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(4),$$

where $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M.

Definition 3.6. A *spin structure* on (M, g) is given by an open cover $\{U_\alpha : \alpha \in A\}$ and a collection $\{\bar{g}_{\alpha\beta} : \alpha, \beta \in A\}$ of transition functions

$$\bar{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(4),$$

such that $\rho \circ \bar{g}_{\alpha\beta} = g_{\alpha\beta}$ and the cocycle condition

$$\bar{g}_{\alpha\beta} \bar{g}_{\beta\gamma} \bar{g}_{\gamma\alpha} = \mathrm{Id} \qquad \text{on } U_\alpha \cap U_\beta \cap U_\gamma$$

is satisfied.

The property of admitting a spin structure is a topological property. The obstruction is given by the second Stiefel-Whitney class. More precisely M admits a spin structure if and only if $w_2(TM) = 0$.

Note that if we identify $Spin(4)$ with $SU(2) \times SU(2)$ then a spin structure defines a four-dimensional complex vector bundle S over M. Given a Riemannian manifold (M, g) with a spin structure $\{\bar{g}_{\alpha\beta}\}$ then the transition functions

$$\rho_\pm \circ \bar{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to SU_\pm(2)$$

determine two complex vector bundles of rank two over M, which we will denote by $S_\pm$. The bundles S_+ and S_- are called the positive and negative spinor bundles, respectively. We know that we can also view S_+ and S_- as quaternionic line bundles. Note that $S = S_+ \oplus S_-$. We can extend the isomorphism of (3.1.2) for bundles to get an isomorphism

$$\gamma : TM \xrightarrow{\cong} \mathrm{Hom}_j(S_+, S_-). \tag{3.1.4}$$

We refer to this map as the Clifford multiplication.

Definition 3.7. A *spin^c structure* on (M, g) is given by an open cover $\{U_\alpha : \alpha \in A\}$ and a collection of transition functions

$$\bar{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(4)^c$$

such that $\rho^c \circ \bar{g}_{\alpha\beta} = g_{\alpha\beta}$ and the cocycle condition

$$\bar{g}_{\alpha\beta}\bar{g}_{\beta\gamma}\bar{g}_{\gamma\alpha} = \mathrm{Id} \qquad \text{on } U_\alpha \cap U_\beta \cap U_\gamma$$

is satisfied.

Let (M, g) be spinnable with transition functions $\bar{g}_{\alpha\beta}$ and L is a complex line bundle over M with Hermitian metric and transition functions

$$z_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1).$$

Then we can define a $Spin^c$ structure on M by taking the transition functions to be the product

$$z_{\alpha\beta}\bar{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(4)^c.$$

Given a $Spin^c$ structure defined by $\bar{g}_{\alpha\beta}$ then we get an associated complex line bundle L by taking the transition functions to be

$$\pi \circ \bar{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1).$$

For $Spin^c$ structures we have an isomorphism

$$\gamma : TM \otimes \mathbb{C} \xrightarrow{\cong} \mathrm{Hom}(S_+, S_-).$$

3.1.2 Almost Hermitian manifolds

An almost Hermitian manifold is a Riemannian manifold with an almost complex structure which is preserved by the Riemannian metric.

For an almost Hermitian four-manifold the structure group reduces to $U(2)$. The natural inclusion $U(2) \hookrightarrow Spin^c(4)$ given by

$$A \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \det_{\mathbb{C}} A & 0 & 0 \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & & \multicolumn{2}{c}{A} \end{pmatrix}$$

gives us a $Spin^c$ structure. The associated line bundle has transition functions given by

$$U_\alpha \cap U_\beta \ni x \mapsto \det_{\mathbb{C}}(g_{\alpha\beta}(x)).$$

These transition functions are the same as that of the determinant line bundle $\det_{\mathbb{C}}(TM) = \Lambda^{2,0}TM$.

For a Spin$^c(4)$ structure with transition functions $\bar{g}_{\alpha\beta} : U_{\alpha\beta} \to (Spin(4) \times U(1))/\mathbb{Z}_2$ we have that

$$\bar{g}_{\alpha\beta}(x) = (h_{\alpha\beta}(x), z_{\alpha\beta}(x))\,\mathbb{Z}_2,$$

where

$$\begin{aligned} h_{\alpha\beta} : U_{\alpha\beta} &\to Spin(4), \\ z_{\alpha\beta} : U_{\alpha\beta} &\to U(1), \end{aligned}$$

are smooth functions and $\rho(h_{\alpha\beta}) = g_{\alpha\beta}$. The cocycle condition for $\bar{g}_{\alpha\beta}$ becomes that

$$(\epsilon_{\alpha\beta\gamma}, \zeta_{\alpha\beta\gamma}) = (h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}, z_{\alpha\beta}z_{\beta\gamma}z_{\gamma\alpha}) \in \mathbb{Z}_2.$$

Note that $z_{\alpha\beta}^2 = \det_{\mathbb{C}}(g_{\alpha\beta})$ defines a cocycle and this is the determinant line bundle. The collection $(\epsilon_{\alpha\beta\gamma})$ is a Čech 2-cocyle representing the second Stiefel-Whitney class.

Proposition 3.8. *An almost Hermitian four-manifold (M, g, J) is spinnable if and only if there exists a square root of the determinant line bundle, i.e. a bundle L such that $L^2 = \Lambda^{2,0}TM$.*

Proof. If M is spinnable then $\epsilon_{\alpha\beta\gamma} = 1$ and hence $\zeta_{\alpha\beta\gamma} = 1$. Hence the transition functions $z_{\alpha\beta}$ define a line bundle L which squares to the determinant line bundle.

Conversely suppose there is a line bundle L which squares to the determinant line bundle. Let the transition functions of L be given by $w_{\alpha\beta}$. We know that $w_{\alpha\beta} = \pm z_{\alpha\beta}$. Hence we can pick $z_{\alpha\beta}$ such that $\zeta_{\alpha\beta\gamma} = 1$ for all $\alpha\beta\gamma$. This implies that $\epsilon_{\alpha\beta\gamma} = 1$ for all $\alpha\beta\gamma$ and thus this defines a spin structure for M. □

Example 3.9. Let U be a two-dimensional complex Hermitian vector space with a determinant form $\theta = dz_1 \wedge dz_2$. Then there is a canonical spin structure on U with the trivial bundles $S^+ = \mathbb{C} \oplus \Lambda^2_{\mathbb{C}} U$, $S^- = U$ and

$$\gamma_u = \delta_u + \delta_u^* : S^+ \to S^-.$$

The metric on S^- is chosen such that the norm of θ is 1. The structure group of U is $SU(2)$. We get a spin structure on U by letting the double cover $\rho : SU(2) \times SU(2) \to SU(2)$ be the projection on the second factor.

Note that there is a natural identification $\Gamma(S^+) \cong \Omega_U^{0,0} \oplus \Omega_U^{0,2}$ and $\Gamma(S^-) \cong \Omega_U^{0,1}$. Under this identification the Dirac operator becomes

$$D^+ = 2(\overline{\partial} - \overline{\partial}^*) : \Omega_U^{0,0} \oplus \Omega_U^{0,2} \to \Omega_U^{0,1}.$$

3.1.3 The spin connection

For a Riemannian four-manifold (M, g) we have a canonical connection on the tangent bundle TM, the Levi-Civita connection. The Levi-Civita connection extends to a connection on $\bigwedge^k TM \otimes \mathbb{C}$ for each k by the Leibniz rule $\nabla(\alpha \wedge \beta) = \nabla(\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \nabla(\beta)$ and extending scalars. This connection will also be called the Levi-Civita connection. This gives us a connection on

$$\mathrm{End}(S) = \bigoplus_{i=1}^{4} \bigwedge^i TM \otimes \mathbb{C}.$$

A connection on S is called a Spin(4)-connection if locally it can be expressed as

$$\mathrm{d}_A s = \mathrm{d}s + As,$$

where A is a one-form with values in $\mathfrak{su}_+(2) \times \mathfrak{su}_-(2) \cong \bigwedge^2 TM$, the Lie algebra of $Spin(4)$. Remember that $\bigwedge^2 TM \subset \mathrm{End}(S)$ is generated by $e_i e_j$ for $i < j$. This allows us to express A as

$$A = \sum_{i,j} A_{ij} e_i e_j \qquad A_{ij} = -A_{ji},$$

where A_{ij} is a real valued 1-form .

Let $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ be locally orthogonal sections of $S = S^+ \oplus S^-$ on some open $U \subset M$ which are in a trivialization $\tilde{\psi}$ given by

$$(\tilde{\psi} \circ \epsilon_1)(p) = \left(p, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right), \ldots, (\tilde{\psi} \circ \epsilon_4)(p) = \left(p, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

The trivialization $\tilde{\psi}$ determines a trivialization ψ of $\mathrm{End}(S)$ and thus also of TM over U.

Let (e_1, e_2, e_3, e_4) be the orthonormal sections of $TM|_U \subset \mathrm{End}(S)|_U$ defined by

$$(\psi \circ e_1)(p) = (p, e_1), \ldots, (\psi \circ e_4)(p) = (p, e_4),$$

where e_1, e_2, e_3, e_4 are as in (3.1.3)

Remember that a connection d_A on S, induces a unique connection d_A on $\mathrm{End}(S)$ which satisfies the Leibniz rule:

$$d_A(\omega s) = (d_A\omega)s + \omega d_A s, \qquad \text{for } \omega \in \Gamma(\mathrm{End}(S)),\ s \in \Gamma(S).$$

Theorem 3.10. *Suppose that M is a four-dimensional oriented Riemannian manifold with a spin structure. Then there is a a one-to-one correspondence between $Spin(4)$-connection on S and $SO(4)$-connections on TM.*

Proof. Let d_A be a $Spin(4)$-connection on S given locally by

$$\mathrm{d}_A = \mathrm{d} + \sum_{i,j=1}^{4} A_{ij} e_i e_j, \qquad A_{ij} = -A_{ji}.$$

Then the Leibniz rule gives us for the connection on $\mathrm{End}(E)$ that

$$\begin{aligned}
(\mathrm{d}_A e_k)(s) &= \mathrm{d}_A(e_k s) - e_k \mathrm{d}_A s \\
&= e_k \mathrm{d}s + \sum_{i,j=1}^{4} A_{ij} e_i e_j e_k s - e_k \mathrm{d}s - e_k \sum_{i,j=1}^{4} A_{ij} e_i e_j s \\
&= \sum_{i,j=1}^{4} A_{ij}(e_i e_j e_k - e_k e_i e_j)s \\
&= -4 \sum_{i=1}^{4} A_{ik} e_k s,
\end{aligned}$$

where for the last inequality only the terms with $i = k$ or $j = k$ are nonzero. This is an $SO(4)$-connection because $A_{ik} = -A_{ki}$.

Let ∇ be a $SO(4)$-connection on TM given by

$$\nabla e_k = \sum_{i=1}^{4} \phi_{ik} e_i.$$

If we define $A_{ij} = -\frac{1}{4}\phi_{ij}$. Then

$$\mathrm{d}_A = \mathrm{d} + A = \mathrm{d} - \frac{1}{4} \sum_{i,j=1}^{4} \phi_{ij} e_i e_j$$

defines a Spin(4)-connection which induces the $SO(4)$-connection on $\mathrm{End}(S)$. □

In particular the Levi-Civita connection induces a $Spin(4)$-connection on S. From now on this will be the connection we use on S.

3.1.4 The Dirac operator

Let d be the spin-connection on S and $\{e_1, e_2, e_3, e_4\}$ a local orthogonal frame of TM.

Definition 3.11. The *Dirac operator* $D : \Gamma(S) \to \Gamma(S)$ is defined by

$$D(s) = \sum_{i=1}^{4} \gamma(e_i) \cdot \mathrm{d}s(e_i),$$

where γ is the Clifford multiplication from (3.1.4).

For the Dirac operator to be well-defined we need to check that the definition is independent of a choice of framing. Suppose that $\{e_1', e_2', e_3', e_4'\}$ is another local orthogonal frame of TM related by

$$e_i' = \sum_{j=1}^{4} g_i^j e_j.$$

Then we have that

$$\begin{aligned} D(s) &= \sum_{i=1}^{4} \gamma(e_i') \cdot (\mathrm{d}s)(e_i') = \sum_{i=1}^{4} \sum_{j,k=1}^{4} \gamma(g_i^j e_i) \cdot (\mathrm{d}s)(g_i^k e_i) \\ &= \sum_{i=1}^{4} \sum_{j,k=1}^{4} g_i^j g_i^k \gamma(e_i) \cdot (\mathrm{d}s)(e_i) \\ &= \sum_{i=1}^{4} \gamma(e_i) \cdot (\mathrm{d}s)(e_i), \end{aligned}$$

where the last equality follows from the fact that g is orthogonal, so $\sum_{i=1}^{4} g_i^j g_i^k = \delta^{jk}$.

Suppose that M is a four-dimensional Riemannian manifold with a spin structure. Let E be a Hermitian vector bundle over M with a Hermitian connection d_A. The connection d_A and the connection on S give an induced Hermitian connection ∇_A on $E \otimes S$. Let $\{e_1, e_2, e_3, e_4\}$ be a local frame of TM.

Definition 3.12. The *Dirac operator* $D_A : \Gamma(E \otimes S) \to \Gamma(E \otimes S)$ is defined by

$$D_A(s) = \sum_{i=1}^{4} \gamma(e_i) \nabla_A s(e_i),$$

where $\gamma(e_i)$ only acts on the spinor factor. Note that D_A splits as two operators $D_A^+ : \Gamma(E \otimes S_+) \to \Gamma(E \otimes S_-)$ and $D_A^- : \Gamma(E \otimes S_-) \to \Gamma(E \otimes S_+)$.

The Dirac operator is (essentially) self-adjoint.[1] This implies that $\left(D_A^+\right)^* = D_A^-$.

Theorem 3.13. *(Weitzenböck formula) The Dirac Laplacian $D_A^- D_A^+$ is related to the trace Laplacian $\nabla_A^* \nabla_A$ via*

$$D_A^- D_A^+ = \nabla_A^* \nabla_A - F_A^+ + \frac{1}{4} R,$$

where R is the scalar curvature of the base manifold.

[1] Essentially self-adjoint means that an extension of D_A to a completion of $\Gamma(E \otimes S)$ is self-adjoint.

3.1.5 Connections and projections

Let $F \to X$ be a trivial vector bundle with the product connection $\underline{\mathrm{d}}$. If E is a subbundle of F with a smooth bundle projection $p : F \to E$, which is a left-inverse of the inclusion $i : E \hookrightarrow F$ then we get a induced connection ∇ on E via

$$\nabla = p\underline{\mathrm{d}}i.$$

If F has a metric then we can take the orthogonal projection for p.

Suppose that X is a complex manifold with holomorphic bundle maps

$$\underline{K_0} \xrightarrow{\alpha} \underline{K_1} \xrightarrow{\beta} \underline{K_2},$$

where $\underline{K_0}$, $\underline{K_1}$, $\underline{K_2}$ are trivial bundles and $\beta\alpha = 0$. Suppose that α is fiberwise injective and β is fiberwise surjective. Then the cohomology bundle defines a holomorphic vector bundle with fibers

$$\mathcal{E}_x := \ker \beta_x / \mathrm{Im}\ \alpha_x.$$

This follows from Remark 2.17 and 2.21.

Proposition 3.14. *Let $\mathcal{E}$ be a Hermitian holomorphic vector bundle with connection ∇ which is compatible with both structures. Suppose that $\mathcal{F} \subset \mathcal{E}$ is a holomorphic subbundle. The connection obtained from projecting onto $\mathcal{F}$ is compatible with the Hermitian metric and holomorphic structure.*

Proof. Let $p : \mathcal{E} \to \mathcal{F}$ be the orthogonal projection. Let $s,\ s'$ be sections of $\mathcal{F}$ and thus also of $\mathcal{E}$ then

$$\begin{aligned} \mathrm{d}\left\langle s, s'\right\rangle &= \left\langle \nabla s, s'\right\rangle + \left\langle s, \nabla s'\right\rangle \\ &= \left\langle \nabla s, ps'\right\rangle + \left\langle ps, \nabla s'\right\rangle \\ &= \left\langle p\nabla s, s'\right\rangle + \left\langle s, p\nabla s'\right\rangle, \end{aligned}$$

which shows that $p\nabla$ is compatible with the Hermitian metric on $\mathcal{F}$. Let s be a section of $\mathcal{F}$ then

$$\overline{\partial}_{p\nabla} s = p\overline{\partial}_{\nabla} s = p\overline{\partial} s = \overline{\partial} s,$$

where the last equality is because $\mathcal{F} \subset \mathcal{E}$ is a holomorphic subbundle. □

So the projected connection from $\underline{K_1}$ to $\mathcal{E}$ is compatible with the holomorphic structure.

Example 3.15. Suppose we are given a Riemannian manifold with a spin structure and a complex Hermitian vector bundle E over M. Suppose that we are given a smooth family of connections $\{A_t\}_{t\in T}$ in E. We have the coupled Dirac operators

$$\begin{aligned} D^+_{A_t} : \Gamma(E\otimes S^+) &\rightarrow \Gamma(E\otimes S^-), \\ D^-_{A_t} : \Gamma(E\otimes S^-) &\rightarrow \Gamma(E\otimes S^+). \end{aligned}$$

Suppose that $\ker D^+_{A_t} = \{0\}$ for all $t \in T$. Since the Dirac operator is elliptic by Example A.21 we get from Theorem A.18 that the index of $D^-_{A_t}$ is finite. So we have a surjective bundle map

$$\begin{array}{ccc} T\times\Gamma(E\otimes S^-) & \xrightarrow{D^-_A} & T\times\Gamma(E\otimes S^+) \\ \downarrow & & \downarrow \\ T & = & T. \end{array}$$

A splitting of $D^-_{A_t}$ is given by $P_t = G_t \circ D_{A_t}$, where $G_t = (D^-_{A_t} D^+_{A_t})^{-1}$ is the Green's operator for the Dirac Laplacian. Hence we have by Proposition 2.18 that $\ker D^-_A \to T$ is a finite-dimensional smooth vector bundle. If T is a complex manifold and $\{A_t\}_{t\in T}$ a holomorphic family of connections then it is easy to see that D^-_A is a holomorphic bundle map. We can conclude from Remark 2.17 that in this case $\ker D^-_A \to T$ is a holomorphic vector bundle.

3.2 General formalism of the Nahm transform

We will begin with a more general version of the Nahm transform which is introduced in [13]. The Nahm transform takes the following data:

A Riemannian spin four-manifold (M, g) with non-negative scalar curvature; a Hermitian vector bundle $E \to M$ with an ASD connection d_A; a smooth manifold T parametrizing a family $\{\mathrm{d}_{B_t}\}_{t\in T}$ of ASD connections on a complex vector bundle $F \to M$.

From this it produces a Hermitian bundle $\hat{E} \to T$ with a unitary connection $\mathrm{d}_{\hat{A}}$.

The transformation works as follows. On the bundle $E \otimes F$ we have the twisted ASD connections $\mathrm{d}_{A_t} = \mathrm{d}_A \otimes 1_F + 1_E \otimes \mathrm{d}_{B_t}$. We assume that this bundle is 1-irreducible, in the sense that:

$$\mathrm{d}_{A_t} s = 0 \Rightarrow s = 0 \qquad \forall t \in T.$$

We consider the coupled Dirac operators

$$D_{A_t} : H_{k,loc}(E \otimes F \otimes S^+) \to H_{k-1,loc}(E \otimes F \otimes S^-).$$

From the Weitzenböck formula it follows that $\ker(D^+_{A_t}) = \{0\}$ for all $t \in T$. Example A.21 gives us that $\hat{E} = \operatorname{coker}(D^+_{A_t})$ is a vector bundle over T. Let $i : \hat{E} \hookrightarrow H_{k-1,loc}(E \otimes F \otimes S^-)$ be the inclusion and $P : H_{k-1,loc}(E \otimes F \otimes S^-) \to \hat{E}$ the projection. We get a connection on $\hat{E}$ by

$$\mathrm{d}_{\hat{A}} = P\underline{\mathrm{d}}i,$$

where $\underline{\mathrm{d}}$ denotes the trivial covariant derivative on $H_{k-1,loc}(E \otimes F \otimes S^-)$. The map which sends the pair (E, d_A) to $(\hat{E}, \mathrm{d}_{\hat{A}})$ is called the Nahm transform.

In general it is hard to answer questions about the curvature of $F_{\mathrm{d}_{\hat{A}}}$. An important question is whether $\mathrm{d}_{\hat{A}}$ is again an ASD connection. An easy example for which this is the case is the following.

Theorem 3.16. *Let M and T are four-dimensional hyperkähler manifolds then the transformed connection $\mathrm{d}_{\hat{A}}$ is ASD.*

Proof. We have seen in Example 3.15 that $\hat{E}$ is a holomorphic subbundle. Since $\underline{\mathrm{d}}$ is flat we get from Proposition 3.14 that $\mathrm{d}_{\hat{A}}$ is compatible with all complex structures on T. It follows from Lemma 2.33 that $\mathrm{d}_{\hat{A}}$ is ASD. □

Lemma 3.17. *If d_A and $d_{A'}$ are gauge equivalent connections on a vector bundle E then $d_{\hat{A}}$ and $d_{\hat{A}'}$ are gauge equivalent on the transformed bundle $\hat{E}$.*

Proof. There is a gauge transformation g such that $\mathrm{d}_{A'} = g^{-1}\mathrm{d}_A g$. Let $h = g \otimes 1$, so that $\mathrm{d}_{A'_t} = h^{-1}\mathrm{d}_{A_t}h$ for all t. This gives us that $D^-_{A'_t} = g^{-1}D^-_{A_t}g$. Since g does not depend on t we get that $\underline{\mathrm{d}}g = 0$ and thus that

$$\mathrm{d}_{\hat{A}'} = P'\underline{\mathrm{d}}i' = (g^{-1}Pg)\underline{\mathrm{d}}(g^{-1}ig) = g^{-1}P\underline{\mathrm{d}}ig = g^{-1}\mathrm{d}_{\hat{A}}g.$$

□

3.3 Nahm transform on a flat Riemannian four-torus

Most of the theory in this section can be found in [8]. A different approach to the Nahm transform over the torus can be found in [4].

In this section we discuss the Nahm transform from the flat Riemannian four-torus T to a dual torus T^*. We will show that this is a transformation from so-called without flat factors (WFF) ASD connections on T to WFF ASD connections on T^*. Moreover this transformation is an involution.

A flat Riemannian four-torus is given by $T = \mathbb{R}^4/\Lambda$, where $\mathbb{R}^4$ has the canonical metric and Λ is a maximal lattice in $\mathbb{R}^4$. The torus has the unique Riemannian metric which makes the quotient $\mathbb{R}^4 \to T$ a Riemannian submersion.

Since conjugacy classes of representations $\pi_1(T) \to U(1)$ are in one-to-one correspondence with equivalence classes of flat $U(1)$-connections on T. Since $\pi_1(T) \cong \Lambda$ and $U(1) \cong \mathbb{R}/\mathbb{Z}$ we get that equivalence classes of flat $U(1)$-connections on T are parametrized by a dual torus T^*.

These are described as follows. The quotient $\mathbb{R}^4 \to T$ is the universal cover. Let $\chi : \Lambda \to U(1)$ be a character given by $\chi(n) = e^{i\langle \xi, n\rangle}$ for some $\xi \in (\mathbb{R}^4)^*$. Consider the complex trivial line bundle K over $\mathbb{R}^4$. The lattice Λ acts on this bundle by

$$n(x, v) = (x + n, \chi(n)v).$$

Taking the quotient of this action gives a bundle $L_\xi \to T$. Moreover the product connection on K induces a connection on the quotient. The holonomy of this connection is given by χ.
Equivalently we can describe the bundles L_ξ as follows. Let Λ^* be the dual lattice given by

$$\Lambda^* = \{\xi \in V^* : \xi(\Lambda) \subset \mathbb{Z}\}.$$

Any $\xi \in V^*$ defines a one-form on T. This gives us a flat $U(1)$ connection on the trivial line bundle over T with connection matrix $B_\xi = -i\xi$. Let w be a constant section of the trivial bundle and define a new section by

$$s(x) = e^{i\langle \xi, x\rangle} w,$$

This is a parallel section since

$$(\nabla_\xi s)v_x = (i\langle \xi, v_x\rangle - i\langle \xi, v_x\rangle)\, e^{i\langle \xi, x\rangle} w = 0.$$

This gives us that the holonomy of a loop associated with a lattice point $n \in \Lambda$ of this connection is given by $e^{i\langle \xi, n\rangle}$. Hence this bundle is isomorphic to L_ξ.

Since T^* is also a flat Riemannian four-torus and T is a dual torus of T^* we get that T parametrizes equivalence classes of flat $U(1)$-connections on T^*. Similarly as before we have that a point $x \in V$ yields a character

$$\chi_x : 2\pi\Lambda^* \to U(1), \qquad \chi_x(\xi) = e^{2\pi i\langle \xi, x\rangle}.$$

This gives a flat line bundle $\tilde{L}_x$ over T^* with holonomy given by χ_x.

The family of flat connections $\{\mathrm{d}_{B_\xi}\}_{\xi\in T^*}$ introduced above will be used to twist an ASD connection A on E. For the Nahm transform we required the smooth family $\mathrm{d}_{A_\xi} = \mathrm{d}_A \otimes 1 + 1 \otimes \mathrm{d}_{B_\xi}$ to be 1-irreducible. The following property of d_A will guaranty that this is the case.

Definition 3.18. A connection d_A on a bundle E is *WFF* (*without flat factors*) if there is no splitting $E = E' \oplus L_\xi$ compatible with d_A for any $\xi \in T^*$.

The family of connections $\mathrm{d}_{A_\xi} = \mathrm{d}_A \otimes 1 + 1 \otimes \mathrm{d}_{B_\xi}$ is 1-irreducible. Suppose that s is a nonzero covariantly constant section of d_{A_ξ}. This gives a splitting $E \otimes L_\xi = E' \oplus L_\nu$. Hence we have that

$$E = (E' \oplus L_\nu) \otimes L_\xi^* = E'' \oplus (L_\xi \otimes L_\nu^*) = E'' \oplus L_{\xi-\nu}.$$

This contradicts the fact that the connection d_A is WFF. So we can apply the Nahm transform. This gives us a map from pairs (E, d_A) of ASD connections on Hermitian bundles over T to pairs $(\hat{E}, \mathrm{d}_{\hat{A}})$ of Hermitian connections on Hermitian bundles over T^*.

The tangent space of T is a trivial bundle $T(T) = T \times \mathbb{R}^4$, so a spin structure on $\mathbb{R}^4$ induces a spin structure on T. If we identify the bundles L_ξ with the trivial line bundle with connection matrix $i\xi$, then the Dirac operators

$$D_\xi^+ : \Gamma(E \otimes L_\xi \otimes S^+) \to \Gamma(E \otimes L_\xi \otimes S^-)$$

are identified with

$$D_\xi^+ = D_A^+ - i\gamma(\xi) : \Gamma(E \otimes S^+) \to \Gamma(E \otimes S^-)$$

and similarly for the adjoint operators $D_\xi^- = D_A^- + i\gamma^*(\xi)$, where $\gamma : T(T) \to \mathrm{Hom}(S^+, S^-)$ is Clifford multiplication and we identify $T(T)$ with $T^*(T)$ via the Riemannian metric.

Fix a complex structure I on V and on V^* we take the complex structure $-I^*$. Then T and T^* become complex tori. We know from Subsection 2.2.4 that an ASD connection on $E \otimes L_\xi$ defines a holomorphic structure. We will denote this holomorphic bundle by $\mathscr{E}$ and write the corresponding $\overline{\partial}$ operators on $\mathscr{E}$ as $\overline{\partial}_\xi$. We have the following Dolbeault complex:

$$\Omega_T^{0,0}(\mathscr{E}) \xrightarrow{\overline{\partial}_\xi} \Omega_T^{0,1}(\mathscr{E}) \xrightarrow{\overline{\partial}_\xi} \Omega_T^{0,2}(\mathscr{E}),$$

Let us denote the cohomology of this complex by $H^p(\mathscr{E} \otimes L_\xi)$. We have already seen that this gives us a holomorphic bundle $\hat{\mathscr{E}}$ with fibers

$$\hat{\mathscr{E}}_\xi = H^1(\mathscr{E} \otimes L_\xi).$$

By Hodge theory we get that

$$\begin{aligned} H^1(\mathscr{E} \otimes L_\xi) &\cong \ker(\overline{\partial}_\xi^* \oplus -\overline{\partial}_\xi) \\ &\cong \ker(D_\xi^-). \end{aligned}$$

Furthermore we get that $H^0(\mathscr{E} \otimes L_\xi) \cong \ker(\overline{\partial}_\xi)$ and $H^2(\mathscr{E} \otimes L_\xi) \cong \ker(\overline{\partial}_\xi^*)$ and thus

$$H^0(\mathscr{E} \otimes L_\xi) \oplus H^2(\mathscr{E} \otimes L_\xi) \cong \ker(D_\xi^+).$$

Remember that the family A_ξ is 1-irreducible if and only if $\ker(D_\xi^+) = \ker(\nabla_{A_\xi}) = 0$. Hence we get that being 1-irreducible is equivalent to the condition

$$H^0(\mathscr{E} \otimes L_\xi) \oplus H^2(\mathscr{E} \otimes L_\xi) = 0, \qquad \forall \xi \in T^*.$$

This gives us a natural isomorphism between the fibers of $\hat{E}$ and $\hat{\mathscr{E}}$.

Theorem 3.19. *The transformed connection $\mathrm{d}_{\hat{A}}$ is ASD with respect to the flat Riemannian metric on T^*.*

Proof. We have seen that if we choose a complex structure on T then the Dirac operator D_ξ^- is holomorphic in the ξ variable with respect to the dual complex structure on T^*. By Proposition 3.14 we get that the connection $\mathrm{d}_{\hat{A}}$ is compatible with the holomorphic structure. This holds for all complex structures. Since T^* is hyperkähler we get by Lemma 2.33 that $\mathrm{d}_{\hat{A}}$ is ASD. $\square$

3.3.1 The inverse Nahm transform

We saw how T^* parametrizes the unitary flat connections on the trivial line bundle over T. Similarly we have that T parametrizes the unitary flat connections on the trivial line bundle over T^*.

Now we will prove that if d_A is ASD and WFF then $\mathrm{d}_{\hat{A}}$ is also WFF. Since T parametrizes equivalence classes of flat $U(1)$ connections on T^* we can apply the Nahm transform again to obtain a connection $\mathrm{d}_{\check{\hat{A}}}$ on $\check{\hat{E}}$ over T, which is ASD and WFF. We will show that these two transformations are inverse of each other, i.e. there is an isomorphism $\omega : \check{\hat{E}} \to E$ such that $\omega^*(\mathrm{d}_A) = \mathrm{d}_{\check{\hat{A}}}$.

Definition 3.20. Let $\underline{\mathbb{C}}$ be the trivial line bundle over $T \times V^*$ with a connection matrix given by

$$\mathbb{A}_{(x,\xi)} = -i\left(\sum_{i=1}^{4} \xi_i dx_i\right),$$

where x_i and ξ_i are dual linear coordinates on V and V^* respectively. We have for a fixed $\eta \in 2\pi\Lambda^*$ that $\mathbb{A}_{(x,\xi)}$ and $\mathbb{A}_{(x,\xi+\eta)}$ are gauge equivalent by the gauge transformation $g_\eta(x,\xi) = e^{-i\langle \eta, x\rangle}$:

$$\mathbb{A}_{(x,\xi+\eta)} = -i\left(\sum_{i=1}^{4} \xi_i dx_i + \eta_i dx_i\right) = \mathbb{A}_{(x,\xi)} + dg_\eta \cdot g_\eta^{-1} = g_\eta(\mathbb{A}_{(x,\xi)}).$$

This gives a quotient bundle $\mathbb{P} \to T \times T^*$, called the *Poincaré bundle*, with connection $\mathrm{d}_{\mathbb{A}}$.

If we restrict the Poincaré bundle to the slices $T_\xi = T \times \{\xi\}$, the connection restricts to the connection on L_ξ, hence $\mathbb{P}|_{T_\xi} = L_\xi$. The restriction of $\mathbb{P}$ to the slices $T_x^* = \{x\} \times T^*$ is obtained from the trivial bundle with the product connection on V^* and then divided out by the same action of $2\pi\Lambda^*$ as in the definition of $\tilde{L}_{-x}$, hence $\mathbb{P}|_{T_x^*} = \tilde{L}_{-x}$.

The curvature of the Poincaré bundle is given by

$$F_{\mathbb{A}} = \mathrm{d}\mathbb{A} = -i\sum_{i=1}^{4} \mathrm{d}\xi_i \wedge \mathrm{d}x_i.$$

If we fix a complex structure I on T and take the dual complex structure $-I^*$ on T^* then if $I\mathrm{d}x_i = \epsilon \mathrm{d}x_j$ we have $(-I)^*\mathrm{d}\xi_i = \epsilon \mathrm{d}\xi_j$, where $\epsilon = \pm 1$. This gives us that $JF_{\mathbb{A}} = F_{\mathbb{A}}$ and since the $+1$ eigenspace of a complex structure acting on two-forms is precisely the forms of type $(1,1)$ we get that $F_{\mathbb{A}}$ is of type $(1,1)$. Since $\mathbb{A}$ takes values in $\mathfrak{u}(1)$ it is also unitary. Hence it follows from Theorem 2.30 that the connection $\mathrm{d}_{\mathbb{A}}$ define a holomorphic structure on $\mathbb{P}$.

3.3.2 Spectral sequence

In this section we assume that $\mathscr{E} \to T$ is a holomorphic bundle which is WFF.

We introduce the holomorphic vector bundle

$$\begin{array}{c} \mathscr{G} := \mathrm{pr}_1^*(\mathscr{E}) \otimes \mathbb{P} \\ \big\downarrow \\ T \times T^*. \end{array}$$

We will compute the cohomology of the bundle $\mathscr{G}$ in two different ways, using spectral sequences (See Appendix C). From this we will obtain a natural isomorphism $(\hat{\mathscr{E}}\check{\ })_0 \to \mathscr{E}_0$, which naturally extends to a bundle isomorphism $\hat{\mathscr{E}}\check{\ } \to \mathscr{E}$.

We define a double complex of Abelian groups $(\mathscr{C}^{**}, \overline{\partial}_1, \overline{\partial}_2)$ by

$$\mathscr{C}^{p,q} = \{s(z,\zeta)\mathrm{d}\overline{z}_I \mathrm{d}\overline{\zeta}_J : s \in \Gamma(\mathscr{G}),\ |I| = p,\ |J| = q\},$$

where z_1, z_2 are complex coordinates on T and ζ_1, ζ_2 are dual coordinates. The homomorphism $\overline{\partial}_1 : \mathscr{C}^{p,q} \to \mathscr{C}^{p+1,q}$ is the $\overline{\partial}$ operator on T and $\overline{\partial}_2 : \mathscr{C}^{p,q} \to \mathscr{C}^{p,q+1}$ is the $\overline{\partial}$ operator on T^*. The total complex is introduced as

$$C_n = \bigoplus_{p+q=n} \mathscr{C}^{p,q},$$

with differential

$$\overline{\partial}_1 + \overline{\partial}_2 : C_n \to C_{n+1}$$

is given by the $\overline{\partial}$ operator of $\mathscr{G}$.

We denote the spectral sequence which first takes the $\overline{\partial}_1$ cohomology of $\mathscr{C}^{p,q}$ by (E_r, d_r) and the spectral sequence which first takes the $\overline{\partial}_2$ cohomology by $(\tilde{E}_r, \tilde{d}_r)$.

Lemma 3.21. *Suppose that we are given a complex of vector bundles over M*

$$\underline{K_0} \xrightarrow{\alpha} \underline{K_1} \xrightarrow{\beta} \underline{K_2},$$

with α_x injective and β_x surjective for all $x \in M$. Suppose we are given a splitting $P : \underline{K_2} \to \underline{K_1}$ of β. Then the sections of these bundle give a complex of vector spaces

$$\Gamma\left(\underline{K_0}\right) \xrightarrow{\alpha_*} \Gamma\left(\underline{K_1}\right) \xrightarrow{\beta_*} \Gamma\left(\underline{K_2}\right),$$

with α_* injective and β_* surjective. The cohomology of this complex is isomorphic to the space of sections of the cohomology bundle H, i.e.

$$\Gamma(H) \cong \frac{\ker\,\beta_*}{\mathrm{im}\,\alpha_*}.$$

Proof. Clearly α_* is injective and since $\beta \circ P = \mathrm{id}_{\underline{K_2}}$ we have that $\beta_* \circ P_* = \mathrm{id}_{\Gamma(\underline{K_2})}$, so β_* is surjective. Note that $\Gamma(\ker\,\beta) \cong \ker\,\beta_*$ and $\Gamma(\mathrm{im}\,\alpha) \cong \mathrm{im}\,\alpha_*$. It is immediate that the map

$$\Gamma(H) \ni s \mapsto s + \mathrm{im}\,\alpha_*$$

is an isomorphism. □

Proposition 3.22. *The E_2-page of the double complex $\mathscr{C}^{**}$ is given by*

$$E_2\colon \quad \begin{array}{c|ccc} q & 0 & H^2(T^*,\hat{\mathscr{E}}) & 0 \\ & & & \\ & 0 & H^1(T^*,\hat{\mathscr{E}}) & 0 \\ & & & \\ & 0 & H^0(T^*,\hat{\mathscr{E}}) & 0 \\ \hline & & & p. \end{array}$$

Proof. The bundles $\Lambda^{0,p}T^*(T)$ and $\Lambda^{0,q}T^*(T^*)$ are trivial. Hence we can identify the groups $\mathscr{C}^{p,q}$ with functions on T^* with values in $\Omega_T^{0,p}(\mathscr{E}) \otimes \Lambda^{0,q}T^*(T^*)$, where we identified $\mathscr{G}|_{T_\xi} = \mathscr{E} \otimes L_\xi$ with $\mathscr{E}$. Under this identification we have that the operator $\overline{\partial}_1$ acts as

$$(\overline{\partial}_1 f)(\xi) = \overline{\partial}_\xi f(\xi).$$

By the above lemma we can first take the cohomology of the complex

$$\Omega_T^{0,0}(\mathscr{E}) \otimes \Lambda^{0,q}T^*(T^*) \xrightarrow{\overline{\partial}_\xi} \Omega_T^{0,1}(\mathscr{E}) \otimes \Lambda^{0,q}T^*(T^*) \xrightarrow{\overline{\partial}_\xi} \Omega_T^{0,2}(\mathscr{E}) \otimes \Lambda^{0,q}T^*(T^*).$$

Since $\mathscr{E}$ is WFF we have that the only nonzero cohomology is in dimension one. By definition we have

$$\hat{\mathscr{E}}_\xi = H^1(\mathscr{E} \otimes L_\xi).$$

So the cohomology of the above complex is given in dimension one by $\hat{\mathscr{E}}_\xi \otimes \Lambda^{0,q}T^*(T^*)$. If we now take the space of sections of these cohomology bundles we get that the E_1-page looks like

$$E_1\colon \quad \begin{array}{c|ccc} q & 0 & \Omega^{0,2}(T^*,\hat{\mathscr{E}}) & 0 \\ & & & \\ & 0 & \Omega^{0,1}(T^*,\hat{\mathscr{E}}) & 0 \\ & & & \\ & 0 & \Omega^{0,0}(T^*,\hat{\mathscr{E}}) & 0 \\ \hline & & & p. \end{array}$$

Taking the $\overline{\partial}_2$ cohomology of this gives us the desired result. □

We will now show that the groups $\tilde{E}_*^{**}$ are isomorphic to $\mathrm{germ}(\tilde{E}_*^{**})$.

Lemma 3.23. *The cohomology groups of T^* with values in $\tilde{L}_x$ are given by:*

$$H^i(T^*, \tilde{L}_x) \cong \begin{cases} 0 & \text{if } x \notin \Lambda \\ \Lambda^i U & \text{if } x \in \Lambda \end{cases}.$$

Proof. We view the bundle $\tilde{L}_x$ as the trivial bundle with $\overline{\partial}$ operator $\overline{\partial}_x = \overline{\partial} - i\delta_x$, where x_1 and x_2 are complex coordinates of T. We use the Fourier transform to decompose the functions on T^* into its Fourier components. For $f : T^* \to \mathbb{C}$ smooth and $n \in \Lambda$ let

$$c_n := \frac{1}{a} \int_{T^*} f(\xi) e^{-i\langle n,\xi\rangle} \, \mathrm{d}\xi$$

be the nth Fourier coefficient of f. Then we have that

$$f(\xi) = \sum_{n\in\Lambda} c_n e^{i\langle n,\xi\rangle},$$

where the coefficients c_n are rapidly decaying. So we have that the functions $e_n(\xi) := e^{i\langle n,\xi\rangle}$ form a basis for the smooth functions. Let $\xi_1,\ \xi_2$ be complex coordinates on T^*, dual to the coordinates $x_1,\ x_2$ on T. Since the cotangent space of T^* is trivial we also have that the one-forms are spanned by $e_n \otimes \mathrm{d}\overline{\xi}_1,\ e_n \mathrm{d}\overline{\xi}_2$ and the two forms by $e_n \otimes \mathrm{d}\overline{\xi}_1 \mathrm{d}\overline{\xi}_2$. In these bases the $\overline{\partial}_x$ operator acts as

$$\begin{aligned} \overline{\partial}_x(e_n) &= i(n_1 - x_1)e_n \otimes \mathrm{d}\overline{\xi}_1 + i(n_2 - x_2)e_n \mathrm{d}\overline{\xi}_2 \\ \overline{\partial}_x(e_n \otimes \mathrm{d}\overline{\xi}_1) &= -i(n_2 - x_2)e_n \otimes \mathrm{d}\overline{\xi}_1 \mathrm{d}\overline{\xi}_2 \\ \overline{\partial}_x(e_n \otimes \mathrm{d}\overline{\xi}_2) &= i(n_1 - x_1)e_n \otimes \mathrm{d}\overline{\xi}_1 \mathrm{d}\overline{\xi}_2. \end{aligned}$$

So we see that in this basis the $\overline{\partial}_x$ complex decomposes as a direct sum of complexes $\oplus(D_{n,x}, \delta_{n-x})$, where

$$D_{n,x} : \qquad \mathbb{C}e_n \overset{\delta_{n-x}}{\longrightarrow} \mathbb{C}e_n \otimes \mathrm{d}\overline{\xi}_1 \oplus \mathbb{C}e_n \otimes \mathrm{d}\overline{\xi}_2 \overset{\delta_{n-x}}{\longrightarrow} \mathbb{C}e_n \otimes \mathrm{d}\overline{\xi}_1 \mathrm{d}\overline{\xi}_2.$$

If $n - x$ is nonzero then the complex $D_{n,x}$ is exact. If $n - x = 0$ then all the differentials are zero and we get a natural identification between the cohomology and $\Lambda^i U$. □

Proposition 3.24. *The total cohomology of $\mathscr{C}^{**}$ is given by*

$$H^i(C^*, \overline{\partial}_1 + \overline{\partial}_2) = \begin{cases} 0 & \text{if } i \neq 2 \\ \mathscr{E}_0 & \text{if } i = 2 \end{cases}.$$

Proof. The groups $\tilde{E}_*^{**}$ only depend on an arbitrary neighbourhood of T_0^* in $T \times T^*$. Take a neighbourhood N of 0 in T. The family of operators $\overline{\partial}_{2,x}^* \overline{\partial}_{2,x}$ is smooth in the x variable and it is invertible away from $x = 0$, by Lemma 3.23. Hence the family of greens operators $G_x := \left(\overline{\partial}_{2,x}^* \overline{\partial}_{2,x}\right)^{-1}$ is a smooth family of operators away from $x = 0$. Let $[\alpha] \in \tilde{E}_1^{p,q}$ and suppose that $x \mapsto \alpha_x$ is supported away from zero. We have that $\overline{\partial}_2 \alpha = 0$ and thus by the above lemma we have that $\alpha_x = \overline{\partial}_{2,x} \beta_x$ for some β_x. Hodge theory gives us that we can pick $\beta_x := \overline{\partial}_{2,x}^* G_x \alpha_x$. Hence there is a smooth form β such that $\overline{\partial}_2 \beta = \alpha$. This tells us that $[\alpha] = 0$. Hence, the restriction of forms $\tilde{E}_1^{p,q} \to \mathrm{germ}(\tilde{E}_1^{p,q})$ to an arbitrary small neighbourhood of $\{0\} \times T^*$ and identifying them if they agree on some neighbourhood, is an isomorphism. Similarly we have a restriction map $\mathscr{C}^{p,q} \to \mathrm{germ}(\mathscr{C}^{p,q})$ which obviously commutes with all the differentials. Since the first page of the $\overline{\partial}_2$ spectral sequences of these double complexes agree we get that the total cohomology of $\mathscr{C}^{p,q}$ and $\mathrm{germ}(\mathscr{C}^{p,q})$ are isomorphic.

So we can also first take the $\overline{\partial}_1$ spectral sequence of $\mathrm{germ}(\mathscr{C}^{p,q})$. We fix a trivialization of $\mathscr{E}$ over a small neighbourhood $N \times T^*$ and identify the fibers with $\mathscr{E}_0$. We have the following isomorphism

$$\mathscr{C}^{p,q}|_{N \times T^*} = \Gamma(\mathscr{G}|_{N \times T^*}) \otimes \Omega_N^{0,p} \otimes \Omega_{T^*}^{0,q} \cong \mathscr{E}_0 \otimes \Omega_N^{0,p} \otimes \Omega_{T^*}^{0,q}.$$

We can decompose the factor $\Omega_{T^*}^{0,q}$ into its Fourier components as in Lemma 3.23

$$\Omega_{T^*}^{0,q} = \bigoplus_{n \in \Lambda^*} e_n \otimes \Lambda^{0,q} T^*(T^*) \equiv \bigoplus_{n \in \Lambda^*} R_n^q.$$

Here one should keep in mind that elements of the Fourier coefficients are rapidly decaying. The $\overline{\partial}_1$ operator only acts on the factor $\Omega_N^{0,p}$. The Poincaré lemma gives that the cohomology is only nonzero in dimension zero. In dimension zero it is given by holomorphic functions on N. This gives us that $\mathrm{germ}(E_1^{p,q})$ is given by the germs of holomorphic maps on N with values in $\mathscr{E}_0 \otimes \bigoplus_{n \in \Lambda^*} R_n^q$. The $\overline{\partial}_2$ cohomology of this is the same as that of Lemma 3.23 with values in $\mathscr{E}_0$. So the only contribution is from R_0^q and this yields the desired result. □

Combining Proposition 3.22 and 3.24 we get that

$$H^0(T^*; \hat{\mathscr{E}}) = H^2(T^*; \hat{\mathscr{E}}) = 0$$

and

$$(\hat{\mathscr{E}}\,\check{}\,)_0 = H^1(T^*; \hat{\mathscr{E}}) \cong \mathscr{E}_0.$$

The isomorphism $\omega_I : (\hat{\mathscr{E}}\,\check{}\,)_0 \to \mathscr{E}_0$ can be explicitly given by as follows. An element of $H^1(T^*, \hat{\mathscr{E}})$ is represented by an element $\alpha \in \mathscr{C}^{1,1}$ by Proposition 3.22. We know that $\overline{\partial}_2\alpha = 0 \bmod(\mathrm{im}\,\overline{\partial}_1)$. Let β be an element of $\mathscr{C}^{0,2}$ such that $\overline{\partial}_1\beta = -\overline{\partial}_2\alpha$. We restrict β to T_0^* to get a form $r_0(\beta) := \beta|_{T_0^*}$. Then we take the cohomology class of $r_0(\beta)$ in $H^2(T^*, \tilde{L}_0) \otimes \mathscr{E}_0 \cong \Lambda^2 U \otimes \mathscr{E}_0 \cong \mathscr{E}_0$. More concretely this map is given by

$$\omega_I([\alpha]) = \int_{T^*} r_0(\beta) \wedge \theta.$$

We will now extend the map ω_I to a bundle map $\omega_I : (\hat{\mathscr{E}}\,\check{}\,) \to \mathscr{E}$.

It is easy to see that the Poincaré bundle $\mathbb{P}$ is the unique holomorphic bundle over $T \times T^*$ with $\mathbb{P}|_{T_\xi} = L_\xi$ and $\mathbb{P}|_{T_0^*}$ trivial. For any bundle which satisfies these properties we can construct an isomorphism with $\mathbb{P}$ using parallel transport.

Let $t_y : T \to T$ be the given by translation along y, i.e. $t_y(x) = x + y$.

Lemma 3.25. *Let y be some point in T. Any holomorphic vector bundle $\mathscr{E} \to T$ is WFF if and only if $t_y^*\mathscr{E}$ is WFF. Moreover in this case there is a natural isomorphism between $\left(t_y^*\mathscr{E}\right)\hat{}$ and $\hat{\mathscr{E}} \otimes \tilde{L}_y$.*

Proof. From the above characterization of the Poincaré bundle we get that

$$t_y^*\mathbb{P} \otimes p_2^*\tilde{L}_y \cong \mathbb{P}.$$

Using this we get the following natural isomorphisms

$$\begin{aligned} H^i\left((p_1^*t_y^*\mathscr{E} \otimes \mathbb{P})|_{T_\xi}\right) &\cong H^i\left(\left(t_y^*(p_1^*\mathscr{E} \otimes \mathbb{P}) \otimes p_2^*\tilde{L}_y\right)_{T_\xi}\right) \\ &\cong H^i\left((p_1^*\mathscr{E} \otimes \mathbb{P})_{T_\xi}\right) \otimes \left(\tilde{L}_y\right)_\xi. \end{aligned}$$

For $i = 0, 2$ we get the first statement. For $i = 1$ the left-hand side is isomorphic to $\left(t_y^*\mathscr{E}\right)_\xi\hat{}$ and the right-hand side is isomorphic to $\hat{\mathscr{E}}_\xi \otimes \left(\tilde{L}_y\right)_\xi$. Moreover, this is holomorphic in ξ. □

Combining the above results gives

$$\left(\hat{\mathscr{E}}\,\check{}\,\right)_y = H^1(T^*; \hat{\mathscr{E}} \otimes \tilde{L}_y) \cong H^1(T^*; (t_y^*\mathscr{E})\,\hat{}\,) \cong (t_y^*\mathscr{E})_0 = \mathscr{E}_y.$$

These isomorphisms fit together to form a holomorphic bundle isomorphism

$$\omega_I : \hat{\mathscr{E}}^{\vee} \to \mathscr{E}.$$

Identifying $\hat{\mathscr{E}}^{\vee}$ with $\hat{E}^{\vee}$ and $\mathscr{E}$ with E we get an isomorphism $\omega_I : \hat{E}^{\vee} \to E$. We want to show that $\omega_I^*(\mathrm{d}_A) = \mathrm{d}_{\hat{A}^{\vee}}$. Since ω_I is holomorphic we know that $\omega_I^*(\overline{\partial}_A) = \overline{\partial}_{\hat{A}^{\vee}}$. It turns out that ω_I is independent of the complex structure I. In particular for the complex structure $-I$ we get that ∂ and $\overline{\partial}$ are interchanged. This implies that

$$\omega_I^*(\partial_A) = \omega_{-I}^*(\overline{\partial}_A) = \overline{\partial}_{\hat{A}^{\vee},-I} = \partial_{\hat{A}^{\vee}},$$

hence $\omega_I^*(\mathrm{d}_A) = \mathrm{d}_{\hat{A}^{\vee}}$. It remains to show that ω_I is independent of the complex structure I.

Proposition 3.26. *The map $\omega_I : \hat{E}^{\vee} \to E$ is independent of the complex structure I.*

Proof. Remember that the map ω_I is given by

$$\omega_I([\alpha]) = \int_{T^*} r_0(\beta) \wedge \theta.$$

To define ω_I we had to find a β such that $\overline{\partial}_1\beta = \overline{\partial}_2\alpha$. Such a solution is provided by Hodge theory:

$$\beta = G_I(\overline{\partial}_1^*\overline{\partial}_2\alpha),$$

where G_I is the Green's operator $(\overline{\partial}_1^*\overline{\partial}_1)^{-1}$. We now want to show that β doesn't depend on the complex structure I. We do this by showing that both G_I and $\overline{\partial}_1^*\overline{\partial}_2\alpha$ do not depend on the complex structure I. The Weitzenböck formula gives us that

$$\overline{\partial}_1^*\overline{\partial}_1 = \frac{1}{4}\nabla_1^*\nabla_1,$$

where ∇_1 is the covariant derivative which on the slice $T \times \{\xi\}$ is given by the connection ∇_ξ. Since ∇_1 is independent of the complex structure we get that G_I is independent of I.

To show that $\overline{\partial}_1^*\overline{\partial}_2\alpha$ is independent on the complex structure we define the operator

$$\kappa : \Gamma(\mathscr{G} \otimes \Lambda^{0,1}T^*(T) \boxtimes \Lambda^{0,1}T^*(T^*)) \to \Gamma(\mathscr{G} \otimes \Lambda^{0,2}T^*(T))$$

by identifying the cotangent spaces of T and T^* via the Riemannian metric and taking the wedge product. Explicitly it is given by

$$\kappa\left(\sum_{i,j} f_{ij}\mathrm{d}\overline{z}_i \wedge \mathrm{d}\overline{\zeta}_j\right) = \sum_{i,j} f_{ij}\mathrm{d}\overline{\zeta}_i \wedge \mathrm{d}\overline{\zeta}_j = (f_{12}-f_{21})\mathrm{d}\overline{\zeta}_1 \wedge \mathrm{d}\overline{\zeta}_2.$$

By an explicit computation in complex coordinates one can see that the commutator $[\overline{\partial}_1^*, \overline{\partial}_2]$ equals κ.

We can always represent $[\alpha]$ by an element α such that $\overline{\partial}_1\alpha = \overline{\partial}_1^*\alpha = 0$ by replacing α by $\alpha' = \alpha - \overline{\partial}_1 G_I(\overline{\partial}_1^*\alpha)$. This gives us that $\overline{\partial}_1^*\overline{\partial}_2\alpha = \kappa(\alpha)$, which is independent of the complex structure. □

We have now proven that the Nahm transform is a map from WFF ASD connections on a bundle $E \to T$ to WFF ASD connections on a bundle $\hat{E} \to T^*$. Moreover we have shown that this map is an involution.

About the topological type of the bundle $\hat{E}$ we can say the following. From the Atiyah-Singer index theorem we get that

$$\operatorname{rank}(\hat{E}) = \operatorname{ind}(D_A^-) = -\left(\int_T -\frac{\operatorname{rank}(E)}{24}p_1(T(T)) + \frac{c_1(E)^2}{2} - c_2(E)\right) = c_2(E) - \frac{c_1(E)^2}{2}.$$

where we used that $\int_M \frac{p_1(TM)}{24} = \frac{\sigma(M)}{8}$ and the fact that the signature of a four-manifold which is the boundary of some oriented five-manifold is zero. The same formula gives us that

$$\operatorname{rank}(E) = c_2(\hat{E}) - \frac{c_1(\hat{E})^2}{2}.$$

Furthermore, the Atiyah-Singer index theorem gives that

$$c_1(\hat{E}) = \sigma(c_1(E)),$$

where $\sigma : H^2(T) \to H^2(T^*)$ is the composition of the isomorphism $H^2(T) \to H_2(T^*)$ and the Poincaré duality isomorphism. As a corollary, we have:

Corollary 3.27. *For an $SU(2)$-bundle E over T there is no ASD connection of charge* 1.

Proof. Since $c_2(E) \neq 0$ any connection on E has to be WFF. So we can apply the Nahm transform. For an $SU(2)$-bundle we have $c_1(E) = 0$. From the above we get that $\operatorname{rank}(\hat{E}) = 1$ and $c_2(\hat{E}) = 2$, which is a contradiction. □

Chapter 4

ADHM construction

In this chapter we will describe the ADHM construction, by Atiyah, Drinfeld, Hitchin and Manin. The ADHM construction gives a 1-1 correspondence between irreducible charge k ASD connections over S^4 for the structure group $SU(n)$ and the solutions of some algebraic equations indexed by the integer k. We will describe how this algebraic data can be described in the holomorphic category in the presence of a complex structure. Then we will show how to construct an ASD connection from this data and vice versa.

4.1 ADHM description of instantons

We can restrict a connection on S^4 to $\mathbb{R}^4 = S^4\backslash\{\infty\}$ together with the pullback metric. This will give a connection d_A on $\mathbb{R}^4$ with finite Yang-Mills functional

$$\int_{\mathbb{R}^4} |F_A|^2 \, \mathrm{d}\mu < \infty.$$

Conversely a result by Uhlenbeck tells us that a connection on $\mathbb{R}^4$ with finite Yang-Mills functional can be extended to S^4. More precisely we have the following.

Theorem 4.1. *Let d_A be a unitary ASD connection over the punctured ball $B^4\backslash\{0\}$ with finite energy*

$$\int_{B^4} |F_A|^2 \, d\mu < \infty,$$

then there is a smooth ASD connection over B^4 which is gauge equivalent to d_A over $B^4\backslash\{0\}$.

It has several advantages to work over $\mathbb{R}^4$. One of the advantages is that $\mathbb{R}^4$ is a hyperkähler manifold. This means that we can use results from Chapter 3. We will identify $\mathbb{R}^4$ with the quaternions and let $S = S^+ \oplus S^-$ be the spin structure obtained from the quaternionic structure.

The algebraic data for the ADHM construction is the following:

Definition 4.2. (ADHM data) $(\mathscr{H},\ E_\infty,\ T,\ P)$

1. A k-dimensional complex vector space $\mathscr{H}$ with a Hermitian metric.
2. An n-dimensional complex vector space E_∞, with a Hermitian metric and determinant form.
3. Self-adjoint linear maps $T_i : \mathscr{H} \to \mathscr{H}$ for $i = 1, 2, 3, 4$.
4. A linear map $P : E_\infty \to \mathscr{H} \otimes S^+$.

We will denote this data by the pair of maps (T, P).

With this data we can define a map

$$R_x : \mathscr{H} \otimes S^- \oplus E_\infty \to \mathscr{H} \otimes S^+$$

by

$$R_x = \left(\sum_{i=1}^{4} (T_i - x_i) \otimes \gamma(e_i)^* \right) \oplus P.$$

We have that $PP^* \in \mathrm{End}(\mathscr{H} \otimes S^+) \cong \mathrm{End}(\mathscr{H}) \otimes \mathrm{End}(S^+)$. In the standard basis of $\mathbb{R}^4$ we have that the anti-self-dual forms are $e_1 \wedge e_2 - e_3 \wedge e_4,\ e_1 \wedge e_4 - e_2 \wedge e_3,\ e_1 \wedge e_3 - e_4 \wedge e_2$. Using the matrices from (3.1.1) one easily checks that these anti-self-dual two forms act as $-2\mathbf{i},\ -2\mathbf{j}$ and $-2\mathbf{k}$, respectively. One also sees that the self-dual-forms act trivially. We will denote the component of PP^* along $\mathbf{i},\ \mathbf{j}$ or $\mathbf{k}$ by $(PP^*)_\mathbf{i},\ (PP^*)_\mathbf{j}$ or $(PP^*)_\mathbf{k}$, respectively

An ASD connections corresponds to ADHM data which satisfies the following properties.

Definition 4.3. (System of ADHM data)

1. The ADHM equations:

$$\begin{aligned}
[T_1, T_2] + [T_3, T_4] &= (PP^*)_\mathbf{i} \\
[T_1, T_3] + [T_4, T_2] &= (PP^*)_\mathbf{j} \\
[T_1, T_4] + [T_2, T_3] &= (PP^*)_\mathbf{k}
\end{aligned}$$

2. For each $x \in \mathbb{R}^4$ the map R_x has full rank.

Two such systems (T, P) and (T', P') are equivalent if there are maps $v : \mathscr{H} \to \mathscr{H}'$ and $u : E_\infty \to E_\infty$ respecting the structures and intertwining the maps

$$T' = vTv^{-1}, \qquad P' = vPu^{-1}.$$

We denote the space of equivalence classes of ADHM data by $\hat{M}_k$. The main result is the following.

Proposition 4.4. *There exists a bijection between $\hat{M}_k$ and the moduli space of $SU(n)$ instantons of charge k on $\mathbb{R}^4$.*

We will first have a look at the simpler case of $SU(2)$ instantons of charge 1 on $\mathbb{R}^4$. We identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$. Remember that $SU(2)$ can be identified with $\{x \in \mathbb{H} : |x| = 1\}$ and $\mathfrak{su}(2)$ with the pure imaginary quaternions $\mathrm{Im}(\mathbb{H})$. A $SU(2)$-connection on $\mathbb{R}^4$ is given by

$$A(x) = \sum_{\mu=1}^{4} A_\mu(x) dx^\mu, \qquad \text{with } A_\mu(x) \in \mathrm{Im}(\mathbb{H}).$$

Let dx be the quaternionic differential given by

$$dx = dx^0 + dx^1 i + dx^2 j + dx^3 k.$$

For a function $f : \mathbb{H} \to \mathbb{H}$ we get a connection by

$$A(x) = \mathrm{Im}(f(x)dx) = \frac{1}{2}(f(x)dx - d\overline{x}\,\overline{f(x)}).$$

We note that

$$\begin{aligned} \frac{1}{2} d\overline{x} \wedge dx &= (dx^0 \wedge dx^1 - dx^2 \wedge dx^3)i + (dx^0 \wedge dx^2 - dx^3 \wedge dx^1)j \\ &+ (dx^0 \wedge dx^3 - dx^1 \wedge dx^2)k \end{aligned}$$

is anti-self dual. The basic instanton of charge 1 has connection matrix

$$A(x) = \mathrm{Im}\left(\frac{\overline{x}dx}{1 + |x|^2}\right) \in \Omega^1(\mathbb{H}; \mathfrak{su}(2))$$

and its curvature is given by

$$F_A = \frac{d\overline{x} \wedge dx}{(1 + |x|^2)^2}.$$

The asymptotic behavior of A at infinity is

$$A(x) \sim \mathrm{Im}\left(\frac{\overline{x}}{|x|^2}dx\right) = g(x)^{-1}dg(x), \qquad \text{with } g(x) = x/|x|.$$

The gauge transformation g restricted to S^3 has degree 1, hence the connection A has charge 1. Alternatively we can compute the integral

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}(F_A^2),$$

which is equal to the second Chern class c_2 by the general Chern-Weil theory. The curvature density is given by

$$\begin{aligned}\mathrm{Tr}(F_A^2) &= \frac{4}{(1+|x|^2)^4}\big|(dx^0\wedge dx^1 - dx^2\wedge dx^3)i + (dx^0\wedge dx^2 - dx^3\wedge dx^1)j \\ &\quad + (dx^0\wedge dx^3 - dx^1\wedge dx^2)k\big|^2 \\ &= \frac{48}{(1+|x|^2)^4}.\end{aligned}$$

This gives us that the charge is given by

$$\begin{aligned}c_2 &= \frac{1}{8\pi^2}\int_{\mathbb{R}^4}\frac{48}{(1+|x|^2)^4}\,\mathrm{d}x = \frac{1}{8\pi^2}2\pi^2\int_0^\infty \frac{48r^3}{(1+r^2)^4}\,\mathrm{d}r \\ &= 1.\end{aligned}$$

We can transform the basic charge 1 instanton by conformal automorphisms of $\mathbb{R}^4$ and since the ASD equations are conformally invariant this will still be an ASD connections. Consider the conformal transformations

$$\varphi_{\lambda,c}(x) = \lambda(x-c).$$

The pulled back connection has curvature density given by

$$\mathrm{Tr}(F^2_{\varphi^*(A)}) = \frac{48\lambda^2}{(1+\lambda^2|x-c|^2)^4}.$$

Since the curvature density is invariant under conformal transformations we get that non of the connections $\varphi^*_{\lambda,c}(\mathrm{d}_A)$ are gauge equivalent. It turns out that these are in fact all charge 1 ASD connections.

4.2 Going from ADHM data to instanton

Suppose that we are given a set of ADHM data (T, P). From this we can construct a Hermitian bundle with an ASD connection as follows

Let F be the trivial Hermitian bundle

$$\begin{array}{c} \mathbb{R}^4 \times (\mathscr{H} \otimes S^- \oplus E_\infty) \\ \downarrow \\ \mathbb{R}^4 \end{array}$$

Since R_x is non-degenerate we get from Subsection 2.1.3 that $\ker(R)$ is a subbundle of F. Using the metric on F we can project the product connection on F down to $\ker(R)$, which we will denote by $A(T, P)$. Now we will rephrase this in the holomorphic setting.

Pick complex coordinates (z_1, z_2) on $\mathbb{R}^4$ and denote $\mathbb{R}^4$ with this complex structure by U. This gives us new maps

$$\tau_1 = T_1 + iT_2, \qquad \tau_2 = T_3 + iT_4.$$

This complex structure on $\mathbb{R}^4$ decomposes S^+ into two factors $S^+ = \mathbb{C} \oplus \Lambda^2 U$ just as in Example 3.9. This allows us to decompose P into these two factors as

$$P = \pi^* \otimes \langle 1 \rangle + \sigma \otimes \langle \theta \rangle ,$$

so π and σ are maps $\pi : \mathscr{H} \to E_\infty$ and $\sigma : E_\infty \to \mathscr{H}$.

Now the ADHM equations take the form

$$\begin{aligned} [\tau_1, \tau_2] + \sigma\pi &= 0 \\ [\tau_1, \tau_1^*] + [\tau_2, \tau_2^*] + \sigma\sigma^* - \pi^*\pi &= 0. \end{aligned}$$

We can define a complex

$$\mathscr{H} \xrightarrow{\alpha} \mathscr{H} \otimes U \oplus E_\infty \xrightarrow{\beta} \mathscr{H},$$

where

$$\alpha = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \pi \end{pmatrix} \qquad \beta = \begin{pmatrix} -\tau_2 & \tau_1 & \sigma \end{pmatrix}.$$

This defines a complex because $\beta\alpha = [\tau_1, \tau_2] + \sigma\pi = 0$. If we identify S^- with U then we see that $R_0 = \alpha^* \oplus \beta$. We can translate this complex by

$x = (z_1, z_2)$, since the ADHM equations remain true if we replace τ_i by $\tau_i - z_i$. This gives us a family of complexes

$$\mathscr{H} \xrightarrow{\alpha_x} \mathscr{H} \otimes U \oplus E_\infty \xrightarrow{\beta_x} \mathscr{H}.$$

Furthermore, these complexes are holomorphic in x. It is easy to see that we have the identification $R_x = \alpha_x^* \oplus \beta_x$.

If we pick a complex structure on $\mathbb{R}^4$ then we know from Proposition 3.14 that $A(T, P)$ will be compatible with the Hermitian structure on F and the holomorphic structure on F. By Lemma 2.33 we now know that $A(T, P)$ is an ASD connection. This is the map from the ADHM data to an ASD connection over $\mathbb{R}^4$. The next thing we will show is that this connection has finite and thus extends to S^4.

If we identify $\mathbb{R}^4$ with the quaternions by $(x_1, x_2, x_3, x_4) = x_1 - x_2\mathbf{i} - x_3\mathbf{j} - x_4\mathbf{k}$ then $\gamma^*(x)$ becomes left quaternion multiplication by x. The spin spaces $S^\pm$ can also be identified with $\mathbb{H}$ in a natural way. So we can write the map R_x as

$$R_x = (T - x) \oplus P : \mathbb{H}^k \oplus \mathbb{C}^n \to \mathbb{H}^k.$$

Proposition 4.5. *The connection $A(T, P)$ has charge equal to k.*

Proof. Let $U_0,\ U_1 \subset \mathbb{H}P^1$ be the open neighbourhoods

$$\begin{aligned} U_0 &= \{(X_0 : X_1) \in \mathbb{H}P^1 | X_0 \neq 0\} \\ U_1 &= \{(X_0 : X_1) \in \mathbb{H}P^1 | X_1 \neq 0\}. \end{aligned}$$

Let $L \to \mathbb{H}P^1$ be the line bundle with transition function from U_0 to U_1 is given by

$$g : U_0 \cap U_1 \to \mathbb{H}, \qquad (X_0 : X_1) \mapsto \frac{X_0}{X_1}.$$

Note that L^{-1} is the tautological line bundle over $\mathbb{H}P^1$. Let τ_0 and τ_1 be trivializations over U_0 and U_1 respectively

$$\begin{aligned} \tau_0 : \mathbb{H} \times \mathbb{H} &\to L_{U_0} \\ \tau_1 : \mathbb{H} \times \mathbb{H} &\to L_{U_1}. \end{aligned}$$

So we have that $(\tau_1^{-1} \circ \tau_0)(X_1, q) = (X_1^{-1}, X_1^{-1} q)$. We define a bundle map

$$R : \underline{\mathbb{H}^k \oplus \mathbb{C}^n} \to L^k$$

by

$$\begin{aligned} (\tau_0^k)^{-1} \circ R_{(1:X_1)} &= (T - X_1) \oplus P \\ (\tau_1^k)^{-1} \circ R_{(X_0:1)} &= (X_0 T - 1) \oplus X_0 P. \end{aligned}$$

One sees that this is well-defined by applying the transition function. Note that if we embed $\mathbb{H}$ in $\mathbb{H}P^1$ as $X_1 \mapsto (1 : X_1)$ then this bundle map R agrees with our map R over $\mathbb{H}$. For the point at infinite $(0 : 1)$ the bundle map is the projection $\mathbb{H}^k \oplus \mathbb{C}^n \to \mathbb{H}^k$. So the bundle map R has full rank. This implies that $\ker(R)$ defines a bundle over $\mathbb{H}P^1$. In particular this tells us that our connection has finite energy. Moreover we can compute the charge. We have an exact sequence of bundles $\ker(R) \hookrightarrow \underline{\mathbb{H}^k \oplus \mathbb{C}^n} \to L^k$, hence $c_2(\ker(R)) = -k \cdot c_2(L) = k$. □

4.3 Going from instanton to ADHM data

Since S^4 is compact and D_A is elliptic we know that the kernel and cokernel of D_A are finite-dimensional.

Proposition 4.6. *For an ASD connection d_A over S^4 we have that* $\ker(D_A^+) = \{0\}$.

Proof. From Theorem 3.13 we get that

$$D_A^- D_A^+ = \nabla_A^* \nabla_A - F_A^+ + \frac{1}{4} R.$$

The scalar curvature of the round four-sphere is 12 and $F_A^+ = 0$. So we get that $D_A^- D_A^+ = \nabla_A^* \nabla_A + 3$. This gives us that

$$\begin{aligned} \left\langle D_A^+ s, D_A^+ s \right\rangle &= \left\langle D_A^- D_A^+ s, s \right\rangle = \left\langle \nabla_A^* \nabla_A s, s \right\rangle + 3 \left\langle s, s \right\rangle \\ &= \left\langle \nabla_A s, \nabla_A s \right\rangle + 3 \left\langle s, s \right\rangle . \end{aligned}$$

Since the right-hand-side vanishes if and only $s = 0$ we conclude that $\ker(D_A^+) = \{0\}$. □

We denote the kernel of

$$D_A^- : \Gamma(E \otimes S^-) \to \Gamma(E \otimes S^+)$$

by $\mathscr{H}_A$. Note that all elements of $\mathscr{H}_A$ are smooth sections by the elliptic regularity Theorem A.19. From the Atiyah-Singer index theorem we get that

$$\dim \mathscr{H}_A = \operatorname{ind} D_A^- = -\left(\int_{S^4} -\frac{\dim(E)}{24} p_1(TS^4) + \frac{c_1(E)^2}{2} - c_2(E) \right) = c_2(E) = k.$$

where we used that $\int_{S^4} \frac{p_1(TS^4)}{24} = \frac{\sigma(S^4)}{8} =0$.

We have an evaluation map at ∞:

$$\mathrm{ev} : \mathscr{H}_A \to S^-_\infty \otimes E_\infty.$$

We can identify S^-_∞ with S^+ by using the natural orientation-reversing map $-1 : S^4 \to S^4$. Taking the adjoint of this map gives us a complex linear map

$$\mathrm{ev}^* : S^+ \otimes E_\infty \to \mathscr{H}_A.$$

Given an instanton d_A on an $SU(n)$-bundle $E \to S^4$ then we obtain a set of ADHM data by choosing:

1. The complex Hermitian vector space $\mathscr{H} = \mathscr{H}_A$
2. The complex Hermitian vector space with determinant form, E_∞, is the fiber at infinity of E.
3. The complex linear map $P_A : E_\infty \to \mathscr{H}_A \otimes S^+$ is ev^* after identifying S^+ with its dual.
4. The self-adjoint linear maps T_i, $i = 1, 2, 3, 4$, are defined by

$$(T_i\psi, \varphi) = \int_{\mathbb{R}^4} (x_i\psi, \varphi) d\mu \quad \psi, \varphi \in \mathscr{H}_A.$$

In [8, p.108-115] it is shown that for a finite energy ASD connection this data is a system of ADHM data. The prove mimics that of the Nahm transform over the torus. So it starts of with constructing a double complex and computing its total cohomology by two different spectral sequences. Computing the spectral sequences is now the more difficult part.

Remark 4.7. For the reader, who is interested in reading the proof of the ADHM construction in [8], we should point out that we found some technical mistakes which we were not able to solve.

Outlook

There are still many aspects of the Nahm transform that are worth further investigating. We point a few of them out.

- There are many other versions of the Nahm transform which are well understood and which are worth looking into. Among these are the paper of Hitchin on the construction of monopoles [11] and doubly-periodic instantons by Jardim [12].
- Instead of the differential-geometric approach there is an algebro-geometric approach, called the Fourier–Mukai transform. We did not have enough time to look into this.
- It is also possible to generalize the Nahm transform on ALE spaces.
- The ASD condition generalizes for hyperkähler manifolds as we noted in Subsection 2.2.4. Also the Nahm transform generalizes to this setting, as explained in [3].

Appendix A

Sobolev spaces and partial differential operators

Distributions

We will give a quick recap of the main function spaces on any open $X \subset \mathbb{R}^n$. A multi-index is a tuple

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}.$$

For any multi-index we write

$$\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}, \qquad \text{where } \partial_i = \frac{\partial}{\partial x_i}.$$

The order of a multi-index is defined as

$$|\alpha| = \sum_{i=1}^n \alpha_i.$$

Let $K \subset X$ be compact. On the space $C^\infty(X)$ of smooth functions on X we have the following family of seminorms

$$\|f\|_{r,K} = \sup\{|\partial^\alpha f(x)| : |\alpha| \leq r,\ x \in K\}.$$

We endow the space $C^\infty(X)$ with the locally convex topology induced by this family of seminorms. For each $K \subset X$ we will denote by $C^\infty_K(X)$ the space of smooth functions with support inside K.

One defines

$$\mathcal{D}(X) := C^\infty_c(X)$$

to be the space of smooth functions with compact support. As vector spaces we have

$$\mathcal{D}(X) = \bigcup_{K \subset X} C_K^\infty(X).$$

We endow $\mathcal{D}(X)$ with the inductive limit topology.

The space of distributions on X is defined as the topological dual of $\mathcal{D}(X)$, i.e.

$$\mathcal{D}'(X) := (\mathcal{D}(X))^*,$$

with the strong dual topology.

It is an easy exercise to show that the assignment

$$X \mapsto \mathcal{D}'(X)$$

defines a sheaf.

Sobolev spaces

Let $L^2(X)$ be the square integrable functions on X. The pairing

$$L^2(X) \times C_c^\infty(X) \to \mathbb{C}$$

given by integration defines an inclusion $L^2(X) \hookrightarrow \mathcal{D}'(X)$.

Definition A.1. The *Sobolev space* on X of order k is defined as

$$H_k(X) := \{u \in \mathcal{D}'(X) : \partial^\alpha u \in L^2(X) \text{ whenever } |\alpha| \leq k\}$$

with the inner product

$$\langle u, u' \rangle = \sum_{|\alpha| \leq r} \langle \partial^\alpha u, \partial^\alpha u' \rangle_{L^2}.$$

For $K \subset \mathbb{R}^n$ a compact subset and $l \in \mathbb{N}$, we define

$$H_{l,K}(\mathbb{R}^n) = \{u \in H_l(\mathbb{R}^n) : \text{supp } u \subset K\}.$$

The following three lemmas are standard results about Sobolev spaces.

Lemma A.2. *(Sobolev embedding theorem) Let $l \in \mathbb{N}$ and let $k > l + \frac{n}{2}$. Then there is a natural continuous linear inclusion*

$$H_k(\mathbb{R}^n) \hookrightarrow C_b^l(\mathbb{R}^n),$$

where $C_b^l(\mathbb{R}^n)$ are the l-times differentiable functions on $\mathbb{R}^n$ with all derivatives of order less or equal to l bounded.

Hence a function which lies in $H_k(\mathbb{R}^n)$ for all k is smooth. The following multiplication lemma is a very simple version. For a more general version we refer the reader to [25, p.183].

Lemma A.3. *(Sobolev multiplication) Assume that X is compact, let $k \in \mathbb{N}$ be such that $n < 2k$. Then there is a constant C such that for all $f \in H_k(X)$ and $g \in H_k(X)$ the product fg lies in $H_k(X)$ and satisfies*

$$\|fg\| \leq C\|f\|\|g\|.$$

Lemma A.4. *(Rellich lemma) Let $k < l$. Then the inclusion $H_{l,K}(\mathbb{R}^n) \hookrightarrow H_k(\mathbb{R}^n)$ is compact.*

We want to define the Sobolev spaces on vector bundles over manifolds. However, the Sobolev spaces H_k are not well behaved enough to be extended to manifolds. We need to replace the Sobolev spaces by their local version $H_{k,loc}$.

The Sobolev spaces are functional spaces, which means that

1. The inclusions $\mathcal{D}(X) \subset H_k(X) \subset \mathcal{D}'(X)$ are continuous linear.
2. For all $\phi \in \mathcal{D}(X)$ the map $u \mapsto \phi u$ from $H_k(X)$ to $H_k(X)$ is continuous linear.

This allows us to define the localization of H_k by

$$H_{k,loc}(X) = \{u \in \mathcal{D}'(X) : \forall \phi \in \mathcal{D}(X) \quad \phi u \in H_k(X)\}.$$

We give $H_{k,loc}$ the smallest topology which makes all the multiplication maps

$$H_{k,loc} \to H_k, \qquad u \mapsto \phi u$$

continuous. It can be shown that $H_{k,loc}$ is invariant under diffeomorphism, i.e. for any diffeomorphism $f : X \to Y$ the map $f_* : H_{k,loc}(X) \to H_{k,loc}(Y)$ is a continuous linear isomorphism. In general local invariant functional spaces induce functional spaces over manifolds.

Generalized sections

Similarly we have for a vector bundle $E \to M$ the space of smooth compactly supported sections

$$\mathcal{D}(M, E) := \bigcup_K \Gamma_K^\infty(E)$$

with the inductive limit topology.

We want to define distributional sections or generalized sections. In order to define a dual of $\mathcal{D}(M,E)$ we would like to be able to integrate over M. Let D be the density bundle on M. This is a trivial line bundle over M which has as fiber over a point $x \in M$ the set of all maps $\omega : \Lambda^{\max} T_x M \to \mathbb{R}$ which satisfy

$$\omega(\lambda\theta) = |\lambda|\omega(\theta), \qquad \forall\theta \in \Lambda^{\max} T_x M.$$

Intuitively a density is a way to measure the volume of hypercubes.

We define the 'functional dual' of E by

$$E^\vee := E^* \otimes D.$$

This bundle comes with a natural pairing

$$\langle -, - \rangle : \Gamma(E^\vee) \times \Gamma(E) \to \Gamma(D).$$

If we restrict the first factor of the domain to compactly supported sections, we get a pairing by composing this map with integration

$$(-,-) : \Gamma_c(E^\vee) \times \Gamma(E) \to \mathbb{R}, \qquad (s,t) = \int_M \langle s, t \rangle .$$

We define the dual of $\mathcal{D}(M,E)$ by

$$\mathcal{D}'(M,E) := \big(D(M,E^\vee)\big)^*$$

with the strong dual topology. The way $\mathcal{D}'(M,E)$ is defined allows for an inclusion

$$\Gamma^\infty(E) \hookrightarrow \mathcal{D}'(M,E), \qquad s \mapsto (-,s).$$

Let (U,κ) be a chart of M over which E is trivial. Then this induces an isomorphism

$$h_\kappa : \mathcal{D}'(U, E_U) \to \mathcal{D}'(\kappa(U))^r, \qquad \text{where } r = \operatorname{rank}(E).$$

Definition A.5. The *localized Sobolev space* $H_{k,loc}(M,E)$ is the space of all $u \in \mathcal{D}'(M,E)$ such that for every trivializing chart (U,κ) we have

$$h_\kappa(u) \in H_{k,loc}(U)^r.$$

Remark A.6. This definition applies to any functional space which is local and invariant.

The Rellich lemma extends for the local Sobolev spaces $H_{k,loc}(M,E)$. If M is compact then we define the k-Sobolev space of sections on E by

$$H_k(M,E) := H_{k,loc}(M,E).$$

Lemma A.7. *Let $k < l$. Then the natural map $H_{l,loc}(M,E) \hookrightarrow H_{k,loc}(M,E)$ is a continuous inclusion. Moreover if M is compact then the inclusion is compact.*

The Sobolev embedding theorem also extends to bundles.

Lemma A.8. *Let E be a vector bundle over a compact manifold M. Let $l \in \mathbb{N}$ and let $k > l + \frac{n}{2}$. Then there is a natural continuous linear inclusion*

$$H_k(M,E) \hookrightarrow C^l(M,E),$$

where $C^l(M,E)$ are the l-times differentiable sections of E.

Let $E \to M$ be a Hermitian vector bundle over a compact Riemannian manifold then $H_k(M,E)$ carries a Hilbert topology. The inner product is given on the dense subset $\Gamma^\infty(M,E) \subset H_k(M,E)$ by

$$\langle s,t\rangle = \int_X (s(x),t(x)) \;\mathrm{d}\mu(x),$$

and extends to $H_k(M,E)$. Note that if k is large enough such that $H_k(M,E)$ embeds in the continuous sections then this formula is well-defined for all elements of $H_k(M,E)$.

Lastly we would like to remark that also for bundle-valued Sobolev spaces we have multiplication theorems if there is a fiberwise multiplication. Of course one has to check that in local coordinates this is given by ordinary multiplication of functions to be able to apply Lemma A.3.

Partial differential operators

We will now define partial differential operators, elliptic differential operators and the elliptic regularity theorem.

Definition A.9. A *differential operator* on $X \subset \mathbb{R}^n$ of order at most $k \in \mathbb{Z}_{\geq 0}$ is an endomorphism $P \in \mathrm{End}(C^\infty(X))$ of the form

$$P = \sum_{|\alpha| \leq k} c_\alpha \partial^\alpha,$$

where $c_\alpha \in C^\infty(X)$.

Let us denote the space of all differential operators on X of order at most k by $D_k(X)$ and define $D(X) = \cup_k D_k(X)$.

It is easy to see that any differential operator $P \in D_k(X)$ defines a continuous linear map

$$P : H_l(X) \to H_{l-k}(X), \qquad l \geq k.$$

Next, we introduce the symbol of a differential operator.

Definition A.10. The *full symbol* of a differential operator $P \in D_k(X)$ is the function $\sigma(P) : T^*X \to \mathbb{C}$ defined by

$$\sigma(P)(x,\xi) = \sum_{|\alpha| \leq k} c_\alpha(x)(i\xi)^\alpha.$$

The principal symbol of P is defined as

$$\sigma_k(P)(x,\xi) = \sum_{|\alpha| = k} c_\alpha(x)(i\xi)^\alpha.$$

Let $X, Y \subset \mathbb{R}^n$ be open and $h : X \to Y$ a diffeomorphism. This induces a bijection $h^* : C^\infty(Y) \to C^\infty(X)$ by pulling back. In turn this gives us a bijection $h_* : D_k(X) \to D_k(Y)$ given by

$$h_*(P) = (h^*)^{-1} \circ P \circ h^*.$$

This shows that D_k is invariant under diffeomorphisms. It is an easy fact that differential operators are local operators. These two properties allow us to extend differential operators to manifolds in the obvious way.

Definition A.11. Let M be a smooth manifold. A *differential operator* on M of order at most $k \in \mathbb{Z}_{\geq 0}$ is a local linear operator $P \in \mathrm{End}(C^\infty(M))$ such that for each coordinate chart (U, κ) we have $\kappa_* P|_U \in D_k(\kappa(U))$.

In order to extend the definition of the symbol to manifolds one needs that it is invariant under diffeomorphisms. This is not the case for the full symbol, but for the principal symbol it is.

Let $X, Y \subset \mathbb{R}^n$ be open and $h : X \to Y$ a diffeomorphism. Let $T^*h : T^*X \to T^*Y$ be the diffeomorphism given by

$$(T^*h)(x,\xi) = (h(x), \xi \circ \mathrm{d}_x h).$$

This induces a map $h_* : C^\infty(T^*X) \to C^\infty(T^*Y)$ given by

$$h_*\sigma = \sigma \circ (T^*h)^{-1}.$$

Lemma A.12. *For any differential operator $P \in D_k(X)$ we have*

$$\sigma_k(h_* P) = h_* \sigma_k(P).$$

Now we can define the principal symbol of a differential operator on a smooth manifold M as follows.

Definition A.13. The *principal symbol* of any differential operator $P \in D_k(M)$ is defined as the smooth function $\sigma_k(P) : T^*M \to \mathbb{C}$ with the property that for every chart (U, κ) we have

$$\kappa_* \left(\sigma_k(P)|_{T^*U}\right) = \sigma_k(\kappa_* P|_U) : T^*\kappa(U) \to \mathbb{C}.$$

We want to define differential operators on sections of smooth vector bundles. Suppose that $E,\ F$ are trivial vector bundles over $X \subset \mathbb{R}^n$. Then we define the space $D_k(X; E, F)$ of differential operators of order at most $k \in \mathbb{Z}_{\geq 0}$ from E to F to be the operators of the form

$$P = \sum_{|\alpha| \leq k} C_\alpha \circ \partial^\alpha,$$

where $C_\alpha \in \mathrm{Hom}(E, F)$ is a smooth bundle map. The symbol of P is defined as

$$\sigma(P)(x, \xi) = \sum_{|\alpha| \leq k} C_\alpha(x)(i\xi)^\alpha \in \mathrm{Hom}(E_x, F_x)$$

This extends again in the obvious way to non-trivial bundles.

Definition A.14. Let $E,\ F$ be smooth vector bundles over a smooth manifold M. A *differential operator* of order at most $k \in \mathbb{Z}_{\geq 0}$ from E to F is a local operator

$$P : \Gamma(E) \to \Gamma(F)$$

such that for every chart (U, κ) on which E and F are trivial we have $\kappa_* P \in D_k(\kappa(U); E_\kappa, F_\kappa)$, where E_κ is the trivialization of E over U. We will denote the space of all differential operators from E to F of order at most k by $D_k(E, F)$.

Remark A.15. Of course there is no real need for the vector bundles E and F to be over the same manifold.

Let $\pi : T^*M \to M$ be the cotangent bundle.

Definition A.16. The *principal symbol* of any differential operator $P \in D_k(E,F)$ is defined as the smooth section of $\pi^*\mathrm{Hom}(E,F)$ with the property that for every chart (U,κ) over which E and F are trivial we have

$$\kappa_*\left(\sigma_k(P)|_{T^*U}\right) = \sigma_k(\kappa_* P|_U) \in \mathrm{Hom}(E_\kappa, F_\kappa).$$

Elliptic operators generalize the classical Laplace operator.

Definition A.17. A differential operator $P \in D_k(E,F)$ is said to be *elliptic* if

$$\sigma_k(P)(x,\xi) : E_x \to F_x$$

is invertible for all $x \in M$ and $\xi \in T_x^*M \backslash \{0\}$.

An important result of pseudo-differential operators is that Elliptic operators have pseudo-inverses[1]. The pseudo-inverse of the Laplacian is the Green's operator.

A version of the Atiyah-Singer index theorem is

Theorem A.18. *Let M be a compact manifold and let $P \in D_k(E,F)$ be an elliptic differential operator. Then P is Fredholm, i.e.* $\ker(P)$ *and* $\mathrm{coker}(P)$ *are finite-dimensional. The index*

$$\mathrm{ind}(P) = \dim(\ker(P)) - \dim(\mathrm{coker}(P))$$

depends only on the principal symbol $\sigma_k(P)$. Moreover the index only depends on topological data of $\sigma_k(P)$ and is given by[2]

$$\mathrm{ind}(P) = (-1)^{\dim(M)} \int_{TM} ch(\sigma_k(P)) Td(TM \otimes \mathbb{C}).$$

Theorem A.19. *(Elliptic regularity) For any elliptic differential operator $P \in D_k(E,F)$ and any $s \in \mathcal{D}'(E)$ we have*

$$\text{singsupp } s = \text{singsupp } Ps.$$

Remark A.20. In particular if Ps is smooth then we know that s has to be smooth already.

[1] A pseudo-inverse of a differential operator $P : \Gamma(E) \to \Gamma(F)$ is an operator $Q : \Gamma(F) \to \Gamma(E)$ such that $PQ - \mathrm{id}$ and $QP - \mathrm{id}$ are smoothing operators.

[2] $ch(\sigma_k(P))$ is the Chern character of $\sigma_k(p)$, where $\sigma_k(p)$ is interpreted as an element of the K-theory of T^*M with compact support. And $Td(TM \otimes \mathbb{C})$ is the total Todd class.

Example A.21. The Dirac operator is elliptic. The principal symbol of D_A is given by

$$\sigma(D_A)(x,\xi) = i\sum_{i=1}^{4}\gamma(\xi_i e_i).$$

This is clearly invertible for nonzero ξ.

Appendix B

Principal bundles

In the following X will be a smooth manifold and G a smooth Lie group.

Definition B.1. A *principle fiber bundle* over X with structure group G is a pair (P, π) consisting of a smooth map $\pi : P \to X$ together with a right G-action on P with the following property. For every $x \in X$ there exists a neighbourhood V of x and a local equivariant trivialization

$$\tau : \pi^{-1}(V) \to V \times G$$

such that $\pi = \mathrm{pr}_1 \circ \tau$, where $\mathrm{pr}_1 : V \times G \to V$ is the projection.

Suppose that the group G acts on another space F via automorphisms. For example if F is a vector space then this is a representation of G. The data of a principal bundle allows us to construct an associated bundle with fibers F.

Definition B.2. Let (P, π) be a principal G-bundle and $\rho : G \to \mathrm{Aut}(F)$ an action for some set F. We can construct an *associated fiber bundle* with fiber F by

$$P \times_G F := (P \times F)/G,$$

where the action of G is defined by $g : (p, f) \mapsto (p \cdot g, \rho(g^{-1})f)$. The action is proper and free, because the action of G on P is proper and free. Hence $P \times_G F$ is a smooth manifold. We can define $\pi_F : P \times_G F \to X$ by $\pi_F([p, f]) := \pi(p)$, this is well-defined because $\pi(p \cdot g) = \pi(p)$. The fibers of π_F are isomorphic to F.

Let $\varphi_U : P_U \to U \times G$ be an equivariant trivialization of P over U, and denote the composition of φ_U with the projection on the second factor by

$\varphi_{U,2}$. A trivialization of $P \times_G F$ over U is now given by

$$[p, f] \mapsto (\pi(p), \rho(\varphi_{u,2}(p))f).$$

The coordinate transformations for $P \times_G F$ are given by

$$(x, f) \mapsto (x, \rho(\varphi_{V,2}(p) \cdot \varphi_{U,2}(p))f).$$

Let $\psi : P \to F$ be a map, which transforms as $\psi(p \cdot g) = \rho(g)^{-1}\psi(p)$. Such a map defines a section of $P \times_G F$ by

$$x \mapsto [p, \psi(p)],$$

where $\pi(p) = x$. Conversely a section s of $P \times_G F$, whose value in x we denote by $s(x) = [p_x, f_x]$, defines a function $\psi : P \to F$ by $\psi(p_x) = f_x$. This function transforms as $\psi(p \cdot g) = \rho(g)^{-1}\psi(p)$.

Appendix C

Spectral sequences

In this appendix we give a very brief introduction to spectral sequences. The theory of spectral sequences is much broader then we sketch here. This short exposition will suffice for our applications. The main reference for this appendix is [18].

Definition C.1. A *spectral sequence* is a collection of bigraded differential k-vector space $(E_r^{p,q}, d_r)$, $r = 0, 1, 2, \ldots$, where $d_r : E_r^{p,q} \to E_r^{p-r+1,q+r}$, with

$$E_{r+1}^{p,q} \cong \frac{\ker(d_r) \cap E_r^{p,q}}{\operatorname{im}(d_r) \cap E_r^{p,q}}.$$

Remark C.2. In general $E_r^{p,q}$ can also be R-modules, or k-algebras, or some other object.

We will only consider the case that $E_r^{p,q} = 0$ unless $p, q \geq 0$. Note that the differentials $d_r : E_r^{p,q} \to E_r^{p-r+1,q+r}$ are zero for $r > p+1$. This implies that the spectral sequence stabilizes, i.e. $E_r^{p,q} = E_{r+1}^{p,q}$ for sufficiently large r. We will denote the limits by $E_\infty^{p,q}$.

A double complex $(C^{*,*}, \delta_1, \delta_2)$ is a collection of k-vector spaces $C^{p,q}$, indexed by positive integers p, q, and homomorphisms

$$\delta_1 : C^{p,q} \to C^{p+1,q}, \quad \delta_2 : C^{p,q} \to C^{p,q+1},$$

such that $\delta_1^2 = \delta_2^2 = \delta_1\delta_2 + \delta_2\delta_1 = 0$. The total complex is introduced as

$$\delta := \delta_1 + \delta_2 : C^k \to C^{k+1},$$

where $C^k = \bigoplus_{p+q=k} C^{p,q}$. The goal is to compute the cohomology H^* of the complex (C^*, δ). A cocycle α of the differential δ is of the form

$$\alpha^{k,0} + \alpha^{k-1,1} + \cdots + \alpha^{0,k},$$

which satisfies the system of equations

$$\begin{aligned} \delta_1\alpha^{k,0} &= 0 \\ \delta_1\alpha^{k-1,1} &= -\delta_2\alpha^{k,0} \\ &\vdots \\ \delta_1\alpha^{0,k} &= -\delta_2\alpha^{1,k-1} \\ 0 &= -\delta_2\alpha^{0,k}. \end{aligned}$$

A natural approach would be to solve this system by induction. Suppose that $\alpha^{p,q}$ is the first nonzero term of α. The first equation leads us to consider the space

$$E_1^{p,q} := \frac{\ker(\delta_1) \cap C^{p,q}}{\operatorname{im}(\delta_1) \cap C^{p,q}}.$$

The differential δ_2 induces a differential $d_1 : E_1^{p,q} \to E_1^{p,q+1}$. A representative of a class in $E_1^{p,q}$ solves the second equation if and only if it is closed under the differential d_1. This leads us to consider

$$E_2^{p,q} := \frac{\ker(d_1) \cap E_1^{p,q}}{\operatorname{im}(d_1) \cap E_1^{p,q}}.$$

Iterating this idea will lead to a spectral sequence (E_r, d_r), with $d_r : E_r^{p,q} \to E_r^{p-r+1,q+r}$ and with E_{r+1} the cohomology of E_r. This process can be done as follows.

Define the vertical filtration of $C^{*,*}$ by

$$F^q(C) := \bigoplus_{i\geq 0, j\geq q} C^{i,j},$$

and set $F^q(C^k) := F^q(C) \cap C^k$. Since δ maps $F^q(C)$ into $F^q(C)$ we get a filtration for H^*

$$F^0(H^k) \supset F^1(H^k) \supset F^2(H^k) \supset \dots.$$

A filtration can be collapsed into another graded vector space called the associated graded vector space and defined by

$$\bigoplus_i F^i(H^k)/F^{i+1}(H^k).$$

Note that if H^* is a finite-dimensional vector space then we can recover H^* up to isomorphism from this associated graded vector space.

Elements of C^{p+q} which start at (p,q) and solve the first r equations of our system of equations are given by

$$Z_r^{p,q} := \{\alpha \in F^q(C^{p+q}) | \delta\alpha \in F^{q+r}(C^{p+q+1})\}.$$

Define a submodule by

$$B_r^{p,q} := Z_r^{p-1,q+1} + \delta(Z_{r-1}^{p+r-2,q-r+1}).$$

Note that $\delta(Z_{r-1}^{p+r-2,q-r+1}) \subset F^q(C^{p+q})$, so $B_r^{p,q}$ is a submodule of $Z_r^{p,q}$. Let

$$E_r^{p,q} := Z_r^{p,q}/B_r^{p,q}.$$

For $\alpha \in Z_r^{p,q}$ we obviously have that $\delta\alpha \in Z_{r+1}^{p-r+1,q+r}$ and since

$$\delta(B_r^{p,q}) = \delta(Z_r^{p-1,q+1}) \subset B_{r+1}^{p-r+1,q+r}$$

the differential descends to a map $d_r : E_r^{p,q} \to E_r^{p-r+1,q+r}$. It is straight forward that $d_r^2 = 0$. By a tedious computation one can check that the cohomology of E_r is E_{r+1}. Elements of $E_\infty^{p,q}$ are by construction represented by elements that start at (p,q) and are cocycles for δ. Note that $Z_\infty^{p,q} = F^q(H^{p+q})$ and

$$E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q} = Z_\infty^{p,q}/Z_\infty^{p-1,q+1} = F^q(H^{p+q})/F^{q+1}(H^{p+q}).$$

So the spectral sequence gives us the associated graded vector space of the filtration of the cohomology H^* of the total complex.

Instead of taking the vertical filtration we can just as well take the horizontal filtration. This will give us another spectral sequence and another associated graded vector space of the cohomology H^*.

There is much more to say about spectral sequences, but we will mainly use it for computing the total cohomology in two different ways. These two computation will give us an isomorphism between the two associated graded vector spaces.

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